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p-Analytic and p-quasi-analytic vectors

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Abstract. For every symmetric operator acting in a Hilbert space, we introduce the families of p-analytic and p-quasi-analytic vectors ($p > 0$), indexed by positive numbers. We prove various properties of these families. We make use of these families to show that certain results guaranteeing essential selfadjointness of an operator with sufficiently large sets of quasi-analytic and Stieltjes vectors are optimal.

0. Introduction. In [S], B. Simon has made the following remark:

"ϕ can only be quasi-analytic (Stieltjes) without being analytic (semi-analytic) if $\|A^n\phi\|$ does not have fairly regular growth".

In this paper, we present examples of quasi-analytic and Stieltjes vectors ϕ for an operator A such that $\|A^n\phi\|$ has regular growth. The vectors ϕ in those examples are neither analytic nor semi-analytic.

We construct an example of a symmetric operator that has a dense set of quasi-analytic vectors without having a non-zero semi-analytic vector. The corresponding quasi-analytic vectors have an irregular growth.

We also show that the theorems guaranteeing selfadjointness of operators with sufficiently large sets of quasianalytic and Stieltjes vectors are optimal.

For any $p > 0$, and for any $q \in \mathbb{N}$ the classes of q-analytic and p-quasi-analytic vectors were introduced in [I]. p-quasi-analytic vectors were also considered in [E] and used for analytic semigroups of operators. We also introduce the classes of q-analytic vectors for any real $q > 0$. Our examples demonstrate connections between those classes. We also show that an operator can have a dense set of p-analytic (respectively p-quasi-analytic) vectors and have no non-zero p'-analytic (p'-quasi-analytic) vector with $p' < p$.

1. Analytic and quasi-analytic vectors. Let A be a symmetric operator acting in a Hilbert space H. Denote by $C^\infty(A)$ the space $\bigcap_{n\in\mathbb{N}} D(A^n)$.

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DEFINITION 1.1. An element $x \in C^\infty(A)$ is called an *analytic vector* for A if

$$\sum_{n=0}^{\infty} \frac{\|A^n x\|}{n!} t^n < \infty \tag{1.1}$$

for some $t > 0$.

DEFINITION 1.2. An element $x \in C^\infty(A)$ is said to be a *quasi-analytic vector* for A if

$$\sum_{n=0}^{\infty} \|A^n x\|^{-1/n} = \infty. \tag{1.2}$$

The following two theorems hold.

THEOREM A (Nelson [Ne]). *Let A be a symmetric operator on a Hilbert space. If A has a dense set of analytic vectors, then A is essentially selfadjoint, that is, its closure is selfadjoint.*

THEOREM QA (Nussbaum [Nu1]). *Let A be a symmetric operator on a Hilbert space. If the set of quasi-analytic vectors for A has a dense linear span, then A is essentially selfadjoint.*

Let l_2 be the Hilbert space (real or complex) of all sequences $x = (x_1, x_2, \ldots)$ such that $\sum_{n=1}^{\infty} |x_n|^2 < \infty$, with the scalar product given by

$$(x \mid y) = ((x_1, x_2, \ldots) \mid (y_1, y_2, \ldots)) = \sum_{i=1}^{\infty} x_i \overline{y}_i.$$

Let e_k be the standard unit vectors in l_2.

The linear space m_0 spanned by $\{e_1, e_2, \ldots\}$ is the space of all finite sequences. Of course m_0 is dense in l_2.

Given $p, q \geq 0$, let $A_{p,q}$ be the operator defined on m_0 by setting

$$A_{p,q} e_1 = a_1 e_2,$$
$$A_{p,q} e_k = a_{k-1} e_{k-1} + a_k e_{k+1} \quad (k = 2, 3, \ldots),$$

where $a_k = k^p (\ln(k+1))^q$. The matrix form of A is as follows:

$$\begin{bmatrix} 0 & a_1 & 0 & 0 & 0 & \ldots \\ a_1 & 0 & a_2 & 0 & 0 & \ldots \\ 0 & a_2 & 0 & a_3 & 0 & \ldots \\ 0 & 0 & a_3 & 0 & a_4 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix}.$$

It is obvious that $A_{p,q}$ is symmetric.

First we show the following

LEMMA 1.1. *For any $k, n \in \mathbb{N}$,*

$$A_{p,q}^n e_k = \sum_{i=1}^{n+k} \alpha_i^{k,n} e_i,$$

where $\alpha_i^{k,n}$ are positive numbers such that

$$\begin{aligned} \alpha_{k+n}^{k,n} &= a_k a_{k+1} \ldots a_{k+n-1} \\ &= \left(\frac{(n+k-1)!}{(k-1)!}\right)^p (\ln(k+1) \ldots \ln(k+n))^q, \end{aligned} \tag{1.3}$$

and, for all k, n and all i with $1 \leq i \leq k+n$,

$$\alpha_i^{k,n} < 2^n \left(\frac{(n+k-1)!}{(k-1)!}\right)^p (\ln(n+k))^{nq}. \tag{1.4}$$

Proof. We proceed by induction on n. For $n = 1$ the conclusion results immediately from the definition of $A_{p,q}$. Assume that (1.3) and (1.4) are true for some n and all $i \leq k+n$. We have

$$\begin{aligned} A_{p,q}^{n+1} e_k &= A_{p,q}(A_{p,q}^n e_k) = A_{p,q}\Big(\sum_{i=1}^{n+k} \alpha_i^{k,n} e_i\Big) \\ &= \sum_{i=1}^{n+k} \alpha_i^{k,n} (a_{i-1} e_{i-1} + a_i e_{i+1}). \end{aligned} \tag{1.5}$$

Using the inductive hypothesis we have

$$\begin{aligned} \alpha_{k+n+1}^{k,n+1} &= \alpha_{k+n}^{k,n} a_{k+n} = a_k a_{k+1} \ldots a_{k+n-1} a_{k+n} \\ &= \left(\frac{(n+k-1)!}{(k-1)!}\right)^p \\ &\quad \times (\ln(k+1) \ldots \ln(k+n))^q (k+n)^p (\ln(k+n+1))^q. \end{aligned}$$

This implies (1.3) for $n+1$.

Now assume that (1.4) is true for some n and all k, i. From (1.5) and (1.4) we have

$$\begin{aligned} \alpha_i^{k,n+1} &= \alpha_{i-1}^{k,n} a_{i-1} + \alpha_{i+1}^{k,n} a_i \\ &\leq 2^n \left(\frac{(n+k-1)!}{(k-1)!}\right)^p (\ln(n+k))^{nq} [(i-1)^p (\ln i)^q + i^p (\ln(i+1))^q]. \end{aligned}$$

For $i < k+n$ we obtain

$$\begin{aligned} \alpha_i^{k,n+1} &\leq 2 \cdot 2^n \left(\frac{(n+k-1)!}{(k-1)!}\right)^p (\ln(n+k))^{nq} (k+n)^p (\ln(k+n+1))^q \\ &= 2^{n+1} \left(\frac{(n+k)!}{(k-1)!}\right)^p (\ln(n+k+1))^{(n+1)q}. \end{aligned}$$

Thus (1.4) is also proved for $n+1$.

In the sequel we shall use several times the following simple consequence of Stirling's formula [K]:

$$\text{(S)}\qquad \sqrt{2\pi n}\,(n/e)^n < n! < 2\sqrt{2\pi n}\,(n/e)^n.$$

COROLLARY 1.1. *Each vector e_k is analytic for $A_{p,0}$ if $0 < p \le 1$ and, consequently, also for $A_{p,q}$ (with $0 < p < 1$ and q arbitrary).*

Proof. Obviously it is sufficient to consider the case $p = 1$. Let

$$q_n = \sqrt[n]{\|A^n_{1,0}e_k\|/n!}.$$

By Lemma 1.1 and (S) we have

$$q_n \le \Big(\frac{1}{n!}\sum_{i=1}^{n+k}|\alpha_i^{k,n}|\Big)^{1/n} \le \Big[\frac{n+k}{n!}\cdot 2^n\Big(\frac{(n+k-1)!}{(k-1)!}\Big)^p\Big]^{1/n}$$
$$\le 2\frac{n+k-1}{n}\Big(\frac{n+k-1}{k-1}\Big)^{(k-1)/n}\Big(\frac{2\sqrt{2\pi(n+k-1)}}{\sqrt{2\pi(k-1)}\sqrt{2\pi n}}\Big)^{1/n}.$$

From this we can see that $\limsup_{n\to\infty} q_n \le 2$ and, consequently, that e_k is analytic for $A_{1,0}$.

COROLLARY 1.2. *No non-zero vector in m_0 is quasi-analytic for $A_{p,0}$ if $1 < p$ and, consequently, no non-zero vector in m_0 is quasi-analytic for $A_{p,q}$ (q arbitrary).*

Proof. Let $v \in m_0$, $v = c_1e_1 + \ldots + c_ke_k$ ($c_k \ne 0$). By Lemma 1.1 we have

$$\text{(1.6)}\qquad A^n_{p,0}v = \sum_{j=1}^{k}\sum_{i=1}^{j+n} c_j\alpha_i^{j,n}e_i.$$

Thus

$$\text{(1.7)}\qquad \|A^n_{p,0}v\| > |c_k|\alpha^{k,n}_{n+k}.$$

Let $s_n = \sqrt[n]{\|A^n_{p,0}v\|}$. By Lemma 1.1 and (S) we have

$$s_n \ge \sqrt[n]{|c_k|\alpha^{k,n}_{n+k}} = \sqrt[n]{|c_k|}\Big(\frac{(n+k-1)!}{(k-1)!}\Big)^{p/n}$$
$$\ge \sqrt[n]{|c_k|}\frac{(n+k-1)^p}{e^p}\Big(\frac{n+k-1}{k-1}\Big)^{(k-1)p/n}\Big(\frac{\sqrt{2\pi(n+k-1)}}{2\sqrt{2\pi(k-1)}}\Big)^{p/n}$$
$$\ge M(n+k-1)^p,$$

for some constant M. Thus the series $\sum_{n=1}^{\infty} 1/\sqrt[n]{\|A^n_{p,0}v\|}$ is convergent if $p > 1$ and, consequently, the vector v is not quasi-analytic for $A_{p,0}$.

Now we make use of the operator $A_{1,1}$ to demonstrate that a symmetric operator may have no non-zero analytic vector and yet may have a dense set of quasi-analytic vectors.

COROLLARY 1.3. *No non-zero vector $v \in m_0$ is analytic for $A_{1,1}$.*

Proof. Let $v = c_1e_1 + \ldots + c_ke_k$. Using a similar argument to that in the previous corollary, we see that it suffices to show that the upper limit of the sequence

$$q_n = \sqrt[n]{\|A^n_{1,1}e_k\|/n!}$$

is equal to ∞. By (1.5) we have

$$q_n = \Big(\frac{\|\sum_{i=1}^{n+k}\alpha_i^{k,n}e_i\|}{n!}\Big)^{1/n} \ge \Big(\frac{\|\alpha^{k,n}_{n+k}e_{n+k}\|}{n!}\Big)^{1/n}$$
$$= \Big(\frac{(n+k-1)!}{n!(k-1)!}\ln(k+1)\ldots\ln(k+n)\Big)^{1/n}$$
$$\ge \Big(\frac{\ln 2\ldots\ln(1+n)}{(k-1)!}\Big)^{1/n}.$$

Since $\sqrt[n]{(k-1)!}$ tends to 1 and $\sqrt[n]{\ln 2\ldots\ln(1+n)}$ diverges to infinity as $n\to\infty$, we see that $q_n \to \infty$. Thus v is not an analytic vector for A.

COROLLARY 1.4. *Each vector $v \in m_0$ is quasi-analytic for $A_{1,1}$.*

Proof. Let $v = c_1e_1 + \ldots + c_ke_k$ and let $M = \max(|c_1|, \ldots, |c_k|)$. Using Lemma 1.1, (1.6) and (S) we have (for large n)

$$\sqrt[n]{\|A^nv\|} = \Big\|\sum_{j=1}^{k}c_j\sum_{i=1}^{n+j}\alpha_i^{j,n}e_i\Big\|^{1/n} \le \Big(M\sum_{j=1}^{k}\sum_{i=1}^{n+j}|\alpha_i^{k,n}|\Big)^{1/n}$$
$$\le \Big(Mk(n+k)\cdot 2^n\frac{(n+k-1)!(\ln(n+k))^n}{(k-1)!}\Big)^{1/n}$$
$$\le 2\ln(n+k)\sqrt[n]{Mk(n+k)!}$$
$$\le 2\ln(n+k)(2Mk(n+k)^{n+k}e^{-n-k}\sqrt{2\pi n})^{1/n}$$
$$\le 2\ln(n+k)(n+k)e^{-1}\sqrt[n]{2kM(n+k)^ke^{-k}\sqrt{2\pi n}}.$$

Since the nth root above tends to 1 as $n\to\infty$, there exists $n_0 \in \mathbb{N}$ such that the root is smaller than 2 for $n > n_0$. Then for $n > n_0$,

$$\sqrt[n]{\|A^nv\|} < 4(n+k)\ln(n+k)e^{-1}$$

and hence

$$\frac{1}{\sqrt[n]{\|A^ne_k\|}} > \frac{e}{4(n+k)\ln(n+k)}.$$

Since

$$\sum_{n=1}^{\infty} \frac{1}{(n+r)\ln(n+r)} = \infty,$$

the series (1.2) diverges to infinity. Thus e_k is quasi-analytic for $A_{1,1}$.

Now we can see that we cannot use Theorem A to show that the operator $A_{1,1}$ is essentially sefadjoint on m_0. But from Theorem QA the conclusion follows immediately.

2. Semi-analytic and Stieltjes vectors. Let A be a symmetric operator acting in a Hilbert space H.

DEFINITION 2.1. An element $x \in C^\infty(A)$ is called a *semi-analytic vector* for A if

(2.1)
$$\sum_{n=0}^{\infty} \frac{\|A^n x\|}{(2n)!} t^n < \infty$$

for some $t > 0$.

DEFINITION 2.2. An element $x \in C^\infty(A)$ is called a *Stieltjes vector* for A if

(2.2)
$$\sum_{n=0}^{\infty} \|A^n x\|^{-1/(2n)} = \infty.$$

It will be proved that the set of Stieltjes vectors is larger than that of analytic vectors (Lemma 2.2).

Obviously the set of analytic vectors for A is contained in the sets of semi-analytic vectors and of quasi-analytic vectors for A and the last two sets are each contained in the set of Stieltjes vectors.

So we have the following diagram ([S], [I]):

$$\begin{array}{ccc} \text{analytic} & \subset & \text{quasi-analytic} \\ \cap & & \cap \\ \text{semi-analytic} & \subset & \text{Stieltjes} \end{array}$$

We come back to the operators $A_{p,q}$ to state the following

COROLLARY 2.1. *If $p < 2$, then each e_k is a semi-analytic* (*and consequently Stieltjes*) *vector for $A_{p,0}$.*

The proof is identical to that of Corollary 1.1.

This result together with Corollary 1.2 (with $1 < p < 2$) shows that the set of semi-analytic vectors can be essentially larger than that of quasi-analytic vectors.

The following theorems are stronger than Theorems A and QA respectively, but for a more restricted class of operators.

THEOREM SA (Simon [S]). *Let A be a semibounded symmetric operator on a Hilbert space. If A has a dense set of semi-analytic vectors, then A is essentially selfadjoint.*

THEOREM S (Nussbaum [Nu1]). *Let A be a semibounded symmetric operator on a Hilbert space. If the set of Stieltjes vectors for A has a dense span, then A is essentially selfadjoint.*

We will see shortly that for positive operators the set of Stieltjes vectors can be essentially larger than the sets of semi-analytic and quasi-analytic vectors.

Let H, e_k and m_0 be as in Section 1. Given $p, q \geq 0$, let $B_{p,q}$ be the operator defined on m_0 as follows:

$$B_{p,q} e_1 = a_1 e_2 + b_1 e_1,$$
$$B_{p,q} e_k = a_{k-1} e_{k-1} + b_k e_k + a_k e_{k+1} \quad (k = 2, 3, \ldots),$$

where $a_1 = 1$, $a_k = k^p (\ln(k+1))^q$ for $k > 1$ and $b_k = 2a_k$. The matrix form of $B_{p,q}$ is as follows:

$$\begin{bmatrix} b_1 & a_1 & 0 & 0 & 0 & \ldots \\ a_1 & b_2 & a_2 & 0 & 0 & \ldots \\ 0 & a_2 & b_3 & a_3 & 0 & \ldots \\ 0 & 0 & a_3 & b_4 & a_4 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix}.$$

Thus $B_{p,q}$ is symmetric.

We now show that $B_{p,q} \geq 0$. Let B_k be the $k \times k$ matrix formed by the first k columns and k rows of $B_{p,q}$. It is sufficient to show that $\det B_k > 0$ for all k. We shall show that

$$\det B_k > 0 \quad \text{and} \quad \det B_k > a_k \det B_{k-1}.$$

For $k = 2$,

$$B_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \cdot 2^p (\ln 3)^q \end{bmatrix}, \qquad \det B_2 = 4 \cdot 2^p (\ln 3)^q - 1 > 3 > 2 = \det B_1.$$

Suppose that the assertion is true for some k. We have

$$\det B_{k+1} = b_{k+1} \cdot \det B_k - a_k^2 \cdot \det B_{k-1}.$$

Thus

$$\det B_{k+1} \geq b_{k+1} \det B_k - a_k \det B_k \geq \det B_k \cdot (b_{k+1} - a_k) > \det B_k \cdot a_{k+1},$$

since $b_{k+1} = 2a_{k+1} > a_{k+1} + a_k$.

Now we prove the following

LEMMA 2.1. *For fixed p,q and for any $k,n\in\mathbb{N}$,*

$$B_{p,q}^n e_k=\sum_{i=1}^{n+k}\beta_i^{k,n}e_i,$$

where $\beta_i^{k,n}$ are positive numbers such that

$$\beta_{k+n}^{k,n}=a_k a_{k+1}\dots a_{k+n-1}=\frac{((n+k-1)!)^p}{((k-1)!)^p}(\ln(k+1)\dots\ln(k+n))^q \tag{2.3}$$

and for all i,

$$\beta_i^{k,n}<6^n\left(\frac{(n+k)!}{k!}\right)^p(\ln(n+k+1))^{nq}. \tag{2.4}$$

Proof. We proceed by induction on n. For $n=1$ the conclusion results immediately from the definition of A. Assume that (2.3) and (2.4) are true for some n and all $i\le k+n$. We have

$$B_{p,q}^{n+1}e_k=B_{p,q}(B_{p,q}^n e_k)=B_{p,q}\Big(\sum_{i=1}^{n+k}\beta_i^{k,n}e_i\Big)=\sum_{i=1}^{n+k}\beta_i^{k,n}(a_{i-1}e_{i-1}+b_ie_i+a_ie_{i+1}). \tag{2.5}$$

Using the inductive hypothesis for (2.3) we have

$$\begin{aligned}\beta_{k+n+1}^{k,n+1}&=\beta_{k+n}^{k,n}a_{k+n}\\&=\frac{((n+k-1)!)^p}{((k-1)!)^q}\\&\quad\times(\ln(k+1)\dots\ln(k+n))^q(k+n)^p(\ln(k+n+1))^q.\end{aligned}$$

Thus (2.3) is proved for $n+1$.

Now assume that (2.4) is true for some n and all k,i. From (2.5) and (2.4) we have

$$\begin{aligned}\beta_i^{k,n+1}&=\beta_{i-1}^{k,n}a_{i-1}+\beta_i^{k,n}b_i+\beta_{i+1}^{k,n}a_i\\&\le 6^n\left(\frac{(n+k)!}{k!}\right)^p(\ln(n+k+1))^{nq}\\&\quad\times[(i-2)^p(\ln(i-1))^q+(i+1)^p(\ln(i+2))^q+i^p(\ln(i+1))^q].\end{aligned}$$

Using (2.4) and the estimates $a_i<b_{n+k}$ and $b_i\le b_{n+k}$ for all i with $i\le k+n$ we have

$$\begin{aligned}\beta_i^{k,n+1}&\le 3\beta_i^{n,k}b_{n+k}<3\cdot 6^n\left(\frac{(n+k)!}{k!}\right)^p 2(n+k)^p(\ln(n+k+1))^q\\&=6^{n+1}\left(\frac{(n+k+1)!}{k!}\right)^p(\ln(n+k+2))^{nq}.\end{aligned}$$

Thus also (2.4) is proved for $n+1$.

Similarly to Chapter 1 one can prove the following

COROLLARY 2.2. *If $p<2$, then each vector $v\in m_0$ is semi-analytic for $B_{p,q}$, and if $p>2$, then no non-zero vector $v\in m_0$ is Stieltjes for $B_{p,q}$ (q arbitrary).*

Now we consider the special case $p=2$, $q=1$.

COROLLARY 2.3. *No non-zero vector $v\in m_0$ is semi-analytic for $B_{2,1}$.*

Proof. Let $v=c_1e_1+\ldots+c_ke_k$ ($c_k\ne 0$). Because $\|B_{2,1}v\|>|c_k|\cdot|\beta_{k+n}^{k,n}|$ and since $\sqrt[n]{|c_k|}$ tends to 1 as $n\to\infty$, it suffices to show that the sequence

$$q_n=\sqrt[n]{\|B_{2,1}^n e_k\|/(2n)!}$$

diverges to infinity. By (2.3) we have

$$\begin{aligned}q_n&=\left(\frac{\|\sum_{i=1}^{n+k}\beta_i^{k,n}e_i\|}{(2n)!}\right)^{1/n}\ge\left(\frac{\|\beta_{n+k}^{k,n}e_{n+k}\|}{(2n)!}\right)^{1/n}\\&\ge\left(\frac{((n+k)!)^2}{(2n)!(k!)^2}\ln(k+1)\dots\ln(k+n+1)\right)^{1/n}.\end{aligned}$$

Using (S) we obtain

$$q_n>\left(\frac{(n+k)^{2(n+k)}e^{-2(n+k)}2\pi n}{2(2n)^{2n}e^{-2n}\sqrt{2\pi n}\,(k!)^2}\right)^{1/n}p_n$$

where $p_n=\sqrt[n]{\ln(k+1)\dots\ln(k+n+1)}$. Consequently,

$$q_n>\frac{\sqrt[n]{2\pi n}\,p_n}{4\sqrt[n]{e^{2k}(k!)^2}}.$$

Since

$$\sqrt[n]{\frac{\sqrt{2\pi n}}{e^{2k}k!}}\to 1\quad\text{as } n\to\infty$$

and p_n diverges to infinity we see that $q_n\to\infty$. Thus v is not a semi-analytic vector for $B_{2,1}$.

COROLLARY 2.4. *No non-zero vector $v\in m_0$ is quasi-analytic for $B_{p,q}$ if $p>1$ (in particular for $p=2$, $q=1$).*

Proof. Let $d_n = \sqrt[n]{\|B^n_{p,1}v\|}$. Using a similar inequality to one in the previous corollary together with (S) we obtain for sufficiently large n,

$$d_n > \sqrt[n]{|c_k|\beta^{k,n}_{k+n}} = \sqrt[n]{|c_k|\frac{((n+k-1)!)^p[\ln(k+1)\dots\ln(k+n)]^q}{((k-1)!)^p}}$$
$$> \sqrt[n]{|c_k|}\sqrt[n]{(n!)^p} > \frac{n^p}{5},$$

since $\sqrt[n]{|c_k|}$ tends to 1 as $n\to\infty$. Thus the series $\sum 1/d_n$ is convergent and v is not quasi-analytic.

COROLLARY 2.5. *Each vector $v\in m_0$ is a Stieltjes vector for $B_{2,1}$.*

Proof. Let $v = c_1e_1+\dots+c_ke_k$ and let $M=\max(|c_1|,\dots,|c_k|)$. Using (2.4) and (S) we have

$$\sqrt[2n]{\|B^n_{2,1}v\|} = \Big(\sum_{j=1}^{k}\sum_{i=1}^{n+j}\alpha_i^{j,n}e_i\Big)^{1/(2n)} \le \Big(\sum_{j=1}^{k}\sum_{i=1}^{n+j}|\beta_i^{k,n}|\Big)^{1/(2n)}$$
$$\le \Big(Mk\cdot 6^n\frac{((n+k)!)^2(\ln(n+k+1))^n}{k!}\Big)^{1/(2n)}$$
$$\le \sqrt{6\ln(n+k+1)}\,\sqrt[n]{(n+k)!}$$
$$\le \sqrt{6\ln(n+k+1)}((n+k)^{n+k}e^{-n-k}\sqrt{2\pi(n+k)})^{1/n}$$
$$= \sqrt{6\ln(n+k+1)}(n+k)e^{-1}\sqrt[n]{(n+k)^ke^{-k}\sqrt{2\pi(n+k)}}.$$

Since the nth root above tends to 1 as $n\to\infty$, there exists $n_0\in\mathbb{N}$ such that, for some n_0, the root is smaller than 2 whenever $n>n_0$. Hence

$$\sqrt[2n]{\|B^n_{2,1}e_k\|} < 14(n+k)\sqrt{\ln(n+k+1)}\,e^{-1}$$

and consequently

$$\frac{1}{\sqrt[2n]{\|B^n_{2,1}e_k\|}} > \frac{e}{14(n+k)\sqrt{\ln(n+k)}}.$$

Since

$$\sum_{n=1}^{\infty}\frac{1}{(n+r)\sqrt{\ln(n+r)}} = \infty,$$

the series (2.2) is divergent to ∞. Thus v is Stieltjes for $B_{2,1}$.

Now we can see that we cannot use Theorems A, SA and QA to show that the operator $B_{2,1}$ is essentially sefadjoint on m_0. But from Theorem S the result follows immediately.

3. Not essentially selfadjoint operators. In this part we give examples showing that being *quasi-analytic* and respectively *Stieltjes* are in a sense optimal conditions. We shall establish the following.

THEOREM 3.1. *If $p>1/2$, then $A_{2p,0}$ is not essentially selfadjoint on m_0.*

Denote $A_{2p,0}$ by A. We show that $\mathrm{Ran}(A+i)$ is not dense in H by finding a non-zero $x\in H$ orthogonal to $\mathrm{Ran}(A+i)$.

Let $x=\sum_{k=1}^{\infty}x_ke_k$ with $x_1=1$. We calculate x_2 from the equation

$$0 = ((A+i)e_1\mid x) = (a_1e_2+ie_1\mid x_1e_1+x_2e_2) = x_2+ix_1,$$

obtaining $x_2=-i$.

Suppose that we have already found $x_1,\dots,x_k$. We then calculate x_{k+1} from the equation

$$0 = ((A+i)e_k\mid x)$$
$$= (a_{k-1}e_{k-1}+a_ke_{k+1}+ie_k\mid x_{k-1}e_{k-1}+x_ke_k+x_{k+1}e_{k+1})$$
$$= (k-1)^{2p}x_{k-1}+k^{2p}x_{k+1}+ix_k,$$

getting

$$x_{k+1} = -\Big(\frac{k-1}{k}\Big)^{2p}x_{k-1} - \frac{ix_k}{k^{2p}}.$$

We must show that the sequence (x_n) thus defined is in l^2. For this purpose we will consider another sequence defined as follows: $c_1=c_2=1$, and suppose that we have defined $c_1,\dots,c_n$. We define

$$c'_{n+1} = \Big(\frac{n-1}{n}\Big)^{2p}c_{n-1}+\frac{c_n}{n^{2p}}.$$

Let $d'_{n+1}=(n+1)^pc'_{n+1}-1$ (i.e. $c'_{n+1}=\frac{1}{(n+1)^p}(1+d'_{n+1})$).

Let

$$c_{n+1} = \frac{1}{(n+1)^p}(1+|d'_{n+1}|).$$

It is evident that $|x_n|\le c_n$ so it is sufficient to show that the sequence (c_n) is in l^2.

Put $d_n=|d'_n|$. We have

$$c_n = \frac{1}{n^p}(1+d_n).$$

We shall show that the sequence (d_n) is bounded. We have

$$c'_{n+1} = \left(\frac{n-1}{n}\right)^{2p} c_{n-1} + \frac{c_n}{n^{2p}} = \left(\frac{n-1}{n}\right)^{2p} \frac{1+d_{n-1}}{(n-1)^p} + \frac{1+d_n}{n^{3p}}$$
$$= \frac{(n-1)^p}{n^{2p}}(1+d_{n-1}) + \frac{1+d_n}{n^{3p}}$$
$$= \frac{1}{(n+1)^p}\left[\frac{(n^2-1)^p}{n^{2p}}(1+d_{n-1}) + \frac{(n+1)^p}{n^{3p}}(1+d_n)\right].$$

Using Lagrange's formula for the function $f(x) = (1+x)^p$ (put $x = -1/n^2$) we obtain

$$c'_{n+1} = \frac{1}{(n+1)^p}\left[1 + d_{n-1} - \frac{p\theta^{p-1}(1+d_{n-1})}{n^2} + \frac{1}{n^{2p}}\left(1+\frac{1}{n}\right)^p (1+d_n)\right]$$
$$= \frac{1}{(n+1)^p}(1+d'_{n+1})$$

where $1 - 1/n^2 < \theta < 1$. Thus $\theta^{p-1} < 2$ for sufficiently large n and

$$d'_{n+1} = d_{n-1} - \frac{p\theta^{p-1}(1+d_{n-1})}{n^2} + \frac{1}{n^{2p}}\left(1+\frac{1}{n}\right)^p (1+d_n).$$

Thus

$$d_{n+1} \le d_{n-1} + \frac{p\theta^{p-1}(1+d_{n-1})}{n^2} + \frac{1}{n^{2p}}\left(1+\frac{1}{n}\right)^p (1+d_n).$$

We have the following three possibilities:

(i) $d_n \le 1$ and $d_{n-1} < 1$,
(ii) $d_n > 1$ and $d_n \ge d_{n-1}$,
(iii) $d_{n-1} > 1$ and $d_{n-1} \ge d_n$.

In case (i) we have

$$d_{n+1} \le 1 + \frac{4p}{n^2} + \frac{4}{n^{2p}}.$$

In case (ii) we have

$$d_{n+1} \le d_n\left(1 + \frac{2p}{n^2} + \frac{4}{n^{2p}}\right) + \frac{2p}{n^2}.$$

In case (iii) we have

$$d_{n+1} \le d_{n-1}\left(1 + \frac{2p}{n^2} + \frac{4}{n^{2p}}\right) + \frac{2p}{n^2}.$$

From the above inequalities we obtain

$$d_{n+1} \le \max(1, d_n, d_{n-1})\left(1 + \frac{4p}{n^2} + \frac{4}{n^{2p}}\right).$$

Thus

$$d_{n+m} \le \max(1, d_n, d_{n+1}) \prod_{k=n+1}^{n+m} \left[1 + \frac{4p}{k^2} + \frac{4}{k^{2p}}\right]$$

for any $m, n \in \mathbb{N}$.

The above infinite product is convergent since the series

$$\sum_{k=n+1}^{\infty} \left(\frac{4p}{k^2} + \frac{4}{k^{2p}}\right)$$

is convergent [K]. Thus (d_n) is bounded and consequently (c_n) as well as (x_n) belong to l^2. So $x = (x_1, x_2, \ldots)$ is orthogonal to $\mathrm{Ran}(A+i)$. This implies that $A_{p,0}$ is not essentially selfadjoint on m_0.

Now we consider the operator $B = A^2$. It is symmetric semibounded and not essentially selfadjoint since $\mathrm{Ran}(B+1) = \mathrm{Ran}(A+i)(A-i) \subset \mathrm{Ran}(A+i)$ is not dense in H.

The growth of the sequence $\|B^n e_k\|$ (as p tends to 1) is only slighty faster than the growth demanded from a Stieltjes vector. This also shows that Nussbaum's theorem for Stieltjes vectors is optimal.

4. p-Analytic and p-quasi-analytic vectors. Let Γ be the Euler function $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$. In particular, $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$.

Let $p > 0$ be fixed. We consider the following generalization of four kinds of vectors.

DEFINITION 4.1. An element $x \in C^\infty(A)$ is called a *p-analytic vector* for A if

$$\sum_{n=0}^{\infty} \frac{\|A^n x\|}{\Gamma(pn+1)} t^n < \infty$$

for some $t > 0$.

DEFINITION 4.2. An element $x \in C^\infty(A)$ is called a *p-quasi-analytic vector* for A if

$$\sum_{n=0}^{\infty} \|A^n x\|^{-1/(pn)} = \infty. \tag{4.1}$$

For $p = 1$, we obtain analytic and quasi-analytic vectors, and for $p = 2$, we obtain semi-analytic and Stieltjes vectors, respectively.

Obviously if $p_1 > p_2$ then the set of p_1-analytic (respectively p_1-quasi-analytic) vectors is larger than the set of p_2-analytic (p_2-quasi-analytic) vectors.

Moreover, we have the following inclusion.

LEMMA 4.1. *If x is p-analytic then x is also p-quasi-analytic.*

Proof. Suppose that x is p-analytic. Then

$$\limsup \sqrt[n]{\frac{\|A^n x\|}{\Gamma(pn+1)}} < \infty, \quad \text{i.e.} \quad \sqrt[n]{\frac{\|A^n x\|}{\Gamma(pn+1)}} < M.$$

Let $k_n \in \mathbb{N}$ be such that

$$pn \le k_n < pn+1. \tag{4.2}$$

Using (S) and the fact that the function Γ is increasing we have

$$\begin{aligned}\|A^n x\| &\le M^n \Gamma(pn+1) \le M^n \Gamma(k_n+1) \\ &= M^n (k_n)! \le 2M^n k_n^{k_n} e^{-k_n} \sqrt{2\pi k_n} \\ &\le 2M^n (pn+1)^{pn+1} e^{-pn} \sqrt{2\pi(pn+1)}.\end{aligned}$$

Thus

$$\begin{aligned}\|A^n x\|^{1/(pn)} &\le M^{1/p}(pn+1)(pn+1)^{1/(pn)}(\sqrt{2\pi(pn+1)})^{1/(pn)} \\ &< 2M^{1/p}(pn+1),\end{aligned}$$

since $[(pn+1)\sqrt{2\pi(pn+1)}]^{1/(pn)}$ tends to 1 as $n \to \infty$. The lemma follows.

Now we establish the following.

THEOREM 4.1. *Let $p > 0$ be fixed. Each vector $v \in m_0$ is p-analytic for the operator $A_{r,0}$ if $r \le p$ and is not p-quasi-analytic if $r > p$.*

Proof. Let $v \in m_0$, $v = c_1 e_1 + \ldots + c_k e_k$, and let $M = \max(|c_1|, \ldots, |c_k|)$. For the first part it is sufficient to take $r = p$.

Denote $A_{r,0}$ by A and let

$$q_n = \left(\frac{\|A^n v\|}{\Gamma(np+1)}\right)^{1/n}.$$

By Lemma 1.1 and (S) we have

$$\begin{aligned}\|A^n v\|^{1/n} &= \Big(\sum_{j=1}^{k} c_j \sum_{i=1}^{n+j} \alpha_i^{j,n} e_i\Big)^{1/n} \\ &\le \Big(M \sum_{j=1}^{k}\sum_{i=1}^{n+j} |\alpha_i^{k,n}|\Big)^{1/n} \le \left(k(n+k)^p \cdot 2^n \frac{((n+k-1)!)^p}{((k-1)!)^p}\right)^{1/n} \\ &\le [2Mk(n+k)^{p(n+k)} e^{p(-n-k)} (\sqrt{2\pi(n+k)})^p]^{1/n} \\ &\le (n+k)^p e^{-p} [2kM(n+k)^{kp} e^{-kp} \sqrt{2\pi(n+k)}]^{1/n} \le 4e^{-p}(n+k)^p\end{aligned}$$

for sufficiently large n because $[kM(n+k)^{pk} e^{-kp} \sqrt{2\pi(n+k)}]^{1/n}$ tends to 1 as $n \to \infty$.

Let $k_n \in \mathbb{N}$ be such that $pn \le k_n + 1 < pn + 1$. Then

$$\begin{aligned}(\Gamma(pn+1))^{1/n} &\ge (\Gamma(k_n+1))^{1/n} = (k_n!)^{1/n} \ge C(k_n^{k_n} e^{-k_n} \sqrt{2\pi k_n})^{1/n} \\ &\ge C[(pn-1)^{pn-1} e^{-pn-1} \sqrt{2\pi(pn-1)}]^{1/n} \ge C_1 (pn-1)^p\end{aligned}$$

for sufficiently large n since $[(pn-1)^{-1} e^{-pn-1} \sqrt{2\pi(pn-1)}]^{1/n}$ tends to 1 as $n \to \infty$.

Consequently,

$$q_n \le C_2 \frac{(n+k)^p}{(pn-1)^p}$$

for some positive constant C_2. This implies that $\limsup q_n < C_2/p$. Consequently, v is p-analytic for $A_{p,0}$.

Now let $r > p$ and $r_n = \|A_{r,0} v\|^{1/(pn)}$. Using Lemma 1.1 and (S) we obtain

$$\begin{aligned}\|A_{r,0}^n v\| &\ge |c_k| \alpha_{n+k}^{n,k} = |c_k| \frac{((n+k)!)^r}{((k-1)!)^r} \ge C \frac{(n+k)^{r(n+k)} e^{-n-k} \sqrt{2\pi(n+k)}}{((k-1)!)^r} \\ &\ge \frac{n^{rn} n^{rk} e^{-n-k} \sqrt{2\pi(n+k)}}{((k-1)!)^r}.\end{aligned}$$

Since

$$\left(\frac{n^{rk} e^{-n-k} \sqrt{2\pi(n+k)}}{((k-1)!)^r}\right)^{1/(pn)} \to 1$$

we obtain for sufficiently large n,

$$r_n \ge C_2 n^{r/p}.$$

As $r/p > 1$ the series

$$\sum \frac{1}{\|A^n v\|^{1/(pn)}} = \sum \frac{1}{r_n} \le \sum \frac{1}{n^{r/p}}$$

is convergent and v is not p-quasi-analytic for $A_{r,0}$.

Now let $B_r = A_{r,0}^2$. Then B_r is positive and we have the following

THEOREM 4.2. *If $r \le 2p$, then each $v \in m_0$ is $2p$-analytic for B_r, and if $r > 2p$, then no vector $v \in m_0$ is $2p$-quasi-analytic for B_r.*

Proof. The first part results from the equality

$$\limsup_{n\to\infty} \left(\frac{\|B_p^n v\|}{\Gamma(2pn+1)}\right)^{1/n} = \limsup_{n\to\infty} \left(\frac{\|A_{p,0}^{2n} v\|}{\Gamma(2pn+1)}\right)^{1/(2n)}.$$

For the second part, notice that the sequence $\|A^n v\|^{1/n}$ is increasing ([N], [I]). Thus the series $\sum_{n\in\mathbb{N}} 1/\|A^n v\|^{1/(pn)}$ is convergent if and only if the series $\sum_{n\in\mathbb{N}} 1/\|A^{2n} v\|^{1/(2pn)}$ is convergent. The last series is equal to $\sum_{n\in\mathbb{N}} 1/\|(A^2)^n v\|^{1/(2pn)}$.

So any vector x is p-quasi-analytic for A if and only if it is $2p$-quasi-analytic for A^2.

Now Theorems 3.1 and 4.2 give the following

COROLLARY 4.1. *For any $p > 1$ there exists a symmetric operator with dense set of p-analytic vectors which is not essentially selfadjoint and there exists a positive operator with dense set of $2p$-quasi-analytic vectors which is not essentially selfadjoint.*

The examples from this chapter together with Corollary 2.1 also demonstrate that operators must be semibounded in Theorems S and SA.

5. Quasi-analyticity and semi-analyticity. In general the growth of the sequence $\|A^n x\|$ for a quasi-analytic vector x must be slower than for a semi-analytic vector. But in some cases it can happen that the set of quasi-analytic vectors can be larger than the set of semi-analytic vectors. In this section we construct an example demonstrating this.

Let H, m_0 be as in Section 1 and let $A = A_{p,q}$ be an operator as in Section 1 with the sequence (a_n) defined as follows:

Let $k_1 = 1$ and let $k_{n+1} = 3k_n^3$ for $n = 2, 3 \ldots$ Put

$$a_k = \begin{cases} k^3 & \text{if } k_{2n} \le k < k_{2n+1}, \\ 1 & \text{if } k_{2n+1} \le k < k_{2n+2}. \end{cases}$$

We will show that each vector e_k is quasi-analytic for A but is not semi-analytic for A.

Fix k and let n be even such that $k < k_{n+1}^3$. Put $p = k_n$. Let $p^3 \le q < 2p^3$.

LEMMA 5.1. *We have $A^q e_k = \sum_{j=1}^{k+q} b_j^q e_j$, where*

$$b_{q+k}^q = a_k a_{k+1} \ldots a_{k+q}, \tag{5.1}$$

$$|b_j^q| \le 2^q (p+k)^{3q}. \tag{5.2}$$

Proof. (5.1) was proved in Lemma 1.1.

For $q = 1$ we have

$$Ae_k = a_{k-1} e_{k-1} + a_k e_{k+1}.$$

But $|a_{k-1}| \le (k-1)^3 \le (p+k)^3$ and $|a_k| \le k^3 \le (p+k)^3$. Thus (5.2) holds for $q = 1$.

Suppose that (5.2) holds for q. Then

$$A^{q+1} e_k = A \sum_{j=1}^{k+q} b_j^q e_j = \sum_{j=1}^{k+q} a_{j-1} b_j^q e_{j-1} + a_j b_j^q e_{j+1}.$$

Hence $b_j^{q+1} = a_j b_{j+1}^q + a_{j-1} b_{j-1}^q$.

For $j \le 3p^3$, a_j is equal to j^3 if $j \le p$ and $a_j = 1$ if $j > p$. So the largest a_j here is equal to p^3. Thus

$$|b_j^{q+1}| \le 2(p+k)^{3q} p^3 \le 2^{q+1} (p+k)^{3(q+1)}.$$

So the lemma is proved.

From the lemma we have for $p^3 < q < 2p^3$,

$$\sqrt[q]{\|A^q e_k\|} \le \sqrt[q]{(k+q) 2^q p^{3q}}.$$

Therefore

$$\sum_{q=k_n}^{\infty} \frac{1}{\sqrt[q]{\|A^q e_k\|}} \ge \sum_{q=p^3}^{2p^3} \frac{1}{\sqrt[q]{\|A^q e_k\|}} \ge p^3 \frac{1}{\sqrt[q]{2q} \cdot 2 \cdot 2^3 \cdot p^3} \ge \frac{1}{17}.$$

Thus the series $\sum_{q=1}^{\infty} 1/\sqrt[q]{\|A^q e_k\|}$ diverges to infinity and the vector e_k is quasi-analytic for A.

Now let $q = k_{2n+1}$, i.e. $q = 3p^3$ where $p = k_{2n}$.

From Lemma 1.1 we have

$$\|A^q e_k\| > b_{k+q}^q \ge 1 \cdot 1 \cdot \ldots \cdot 1 \cdot a_{p+1} a_{p+2} \ldots a_q = \frac{(q!)^3}{(p!)^3}.$$

Hence

$$\sqrt[q]{\frac{\|A^q e_k\|}{(2q)!}} \ge \sqrt[3p^3]{\frac{1}{(p!)^3}} \sqrt[q]{\frac{(q!)^3}{(2q)^{2q}}}.$$

The first factor tends to 1 and the second is by (S) greater than $q/(4e^3)$. Thus the radius of convergence of the series

$$\sum_{n=0}^{\infty} \frac{\|A^n e_k\|}{(2n)!} t^n$$

is equal to 0. So e_k is not semi-analytic for A.

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Analyticity for some degenerate one-dimensional evolution equations

by

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Abstract. We study the analyticity of the semigroups generated by some degenerate second order differential operators in the space $C([\alpha,\beta])$, where $[\alpha,\beta]$ is a bounded real interval. The asymptotic behaviour and regularity with respect to the space variable are also investigated.

1. Introduction. In this paper we prove the analyticity of the semigroups generated by differential operators of the form

$$A_1 = m(x)[(x-\alpha)(\beta-x)D^2 + b(x)D]$$

or

$$A_2 = m(x)\left[D^2 + \frac{b(x)}{(x-\alpha)(\beta-x)}D\right],$$

where $D = d/dx$, in the space $C([\alpha,\beta])$, with suitable boundary conditions. The functions m and b are real-valued, continuous on the compact interval $[\alpha,\beta]$ and m is strictly positive; moreover, we assume that b satisfies a Hölder condition at the endpoints α and β.

The study of degenerate parabolic problems like

$$\begin{cases} du/dt = Bu, \\ u(0) = u_0, \end{cases} \tag{1.1}$$

where

$$B = a(x)D^2 + b(x)D, \quad x \in I,$$

and I is a real interval, already started in the fifties with the papers by Feller [10] and [11], motivated by some one-dimensional diffusion problems; the subsequent work of Clément and Timmermans (see [7] and [15]) clarified which (necessary and sufficient) conditions on the coefficients a and b guarantee the existence of the semigroup generated by $(B, D(B))$ if $D(B)$

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