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Department of Mathematics
Bucknell University
Lewisburg, Pennsylvania 17837
U.S.A.
E-mail: pgorkin@bucknell.edu

Department of Mathematics
Vanderbilt University
Nashville, Tennessee 37240
U.S.A.
E-mail: zheng@math.vanderbilt.edu

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Existence and uniqueness results for solutions of nonlinear equations with right hand side in L^1

by

A. FIORENZA and C. SBORDONE (Napoli)

Abstract. We prove an existence and uniqueness theorem for the elliptic Dirichlet problem for the equation $\operatorname{div} a(x, \nabla u) = f$ in a planar domain Ω. Here $f \in L^1(\Omega)$ and the solution belongs to the so-called *grand Sobolev space* $W_0^{1,2)}(\Omega)$. This is the proper space when the right hand side is assumed to be only L^1-integrable. In particular, we obtain the exponential integrability of the solution, which in the linear case was previously proved by Brezis–Merle and Chanillo–Li.

1. Introduction. We consider the Dirichlet problem on a bounded open set $\Omega \subset \mathbb{R}^2$ with C^1 boundary,

$$\begin{cases} Au = f & \text{in } \Omega \subset \mathbb{R}^2, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $f \in L^1(\Omega)$ and A is a differential operator defined by

$$Au = \operatorname{div} a(x, \nabla u). \tag{1.2}$$

Here $a : \Omega \times \mathbb{R}^2 \to \mathbb{R}^2$ is a mapping such that

$$\begin{cases} x \to a(x,\xi) & \text{is measurable for all } \xi \in \mathbb{R}^2, \\ \xi \to a(x,\xi) & \text{is continuous for almost every } x \in \Omega. \end{cases} \tag{1.3}$$

Furthermore, we assume that there exists $m \ge 1$ such that for almost every $x \in \Omega$ we have

$$\begin{aligned} &\text{(i) } |a(x,\xi) - a(x,\eta)| \le m|\xi - \eta| && \text{(Lipschitz continuity)},\\ &\text{(ii) } \frac{1}{m}|\xi-\eta|^2 \le \langle a(x,\xi) - a(x,\eta), \xi - \eta\rangle && \text{(strong monotonicity)},\\ &\text{(iii) } a(x,0) = 0, \end{aligned} \tag{1.4}$$

where ξ, η are arbitrary vectors in $\mathbb{R}^2$ ([LL]).

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We would like to point out that the linear growth of $a(x,\xi)$ with respect to ξ is absolutely essential for the results in the sequel. The main difficulty with the p-harmonic type equations ($p \neq 2$) is due to the lack of uniqueness results for very weak solutions.

We shall work with functions u of Sobolev class $W_0^{1,1}(\Omega)$ whose gradient satisfies

$$\sup_{1<s<2}\Big[(2-s)\fint_\Omega |\nabla u|^s\,dx\Big]^{1/s} = \|u\|_{W_0^{1,2)}} < \infty. \tag{1.5}$$

The space of such functions, denoted by $W_0^{1,2)}(\Omega)$, will be called the *grand Sobolev space* because it is slightly larger than $W_0^{1,2}(\Omega)$. Note that (1.5) defines a norm in which $W_0^{1,2)}(\Omega)$ becomes a Banach space (see Section 2).

By a solution of problem (1.1) we understand here a function $u \in W_0^{1,2)}(\Omega)$ such that

$$\int_\Omega a(x,\nabla u)\nabla\varphi\,dx = \int_\Omega f\varphi\,dx \quad \forall\varphi\in C_0^\infty(\Omega).$$

An existence theorem for the Dirichlet problem in the space $\bigcap_{q<2} W_0^{1,q}(\Omega)$ was established by Boccardo–Gallouët in [BG]. In order to obtain uniqueness, supplementary conditions were imposed on u. The so-called *entropy solutions* [BB], *transposition solutions* [M] and *renormalized solutions* [LM] were introduced for that purpose.

In this paper we take another approach and prove the following existence and uniqueness theorem:

THEOREM A. *Under the assumptions* (1.3) *and* (1.4), *for any* $f\in L^1(\Omega)$ *there exists a unique solution* $u\in W_0^{1,2)}(\Omega)$ *of problem* (1.1). *Moreover*,

$$\|u\|_{W_0^{1,2)}} \le c\|f\|_{L^1(\Omega)}.$$

Actually, we prove a slightly stronger result, namely

$$\lim_{\varepsilon\to 0}\varepsilon\fint_\Omega |\nabla u|^{2-\varepsilon}\,dx = 0.$$

Consequently, by a Sobolev-type imbedding theorem due to Fusco–Lions–Sbordone [FLS], we deduce that $u\in\exp(\Omega)$. This means that

$$\fint_\Omega e^{\lambda|u|} < \infty \quad \forall\lambda>0.$$

This result, in the case of a linear operator, was previously proved by Brezis–Merle [BM] and Chanillo–Li [CL]. Crucial for our proof of Theorem A will be the notion of a very weak solution $u\in W_0^{1,2-\varepsilon}(\Omega)$ of the equation with right hand side in divergence form,

$$\operatorname{div} a(x,\nabla u) = \operatorname{div} F$$

with $F\in L^{2-\varepsilon}(\Omega,\mathbb{R}^2)$. Such solutions were introduced in [IS2]. For the sake of completeness we shall discuss briefly this theory in Section 3.

2. The space grand-L^q. For any $q>1$ the function space

$$L^{q)}(\Omega) = \Big\{f\in L^1(\Omega): \|f\|_{L^{q)}} = \sup_{0<\varepsilon\le q-1}\Big(\varepsilon\fint_\Omega |f|^{q-\varepsilon}\Big)^{1/(q-\varepsilon)} < \infty\Big\}$$

was introduced by T. Iwaniec and C. Sbordone [IS1] in connection with their study of integrability properties of the Jacobian determinant (see also [GIS] and [G]). Note that $\|f\|_{L^{q)}}$ is a norm and $L^{q)}(\Omega)$ is a Banach space. The inclusion $L^q(\Omega)\subset L^{q)}(\Omega)$ is obvious, and we know also that $C_0^\infty(\Omega)$ is not dense in $L^{q)}(\Omega)$. Its closure consists of the functions $f\in L^{q)}(\Omega)$ such that

$$\lim_{\varepsilon\downarrow 0}\varepsilon\fint_\Omega |f|^{q-\varepsilon}\,dx = 0.$$

It contains the Zygmund space $L^q\log^{-1}L(\Omega)$, i.e. the functions f such that

$$\fint_\Omega |f|^q\log^{-1}(e+|f|)\,dx < \infty.$$

In [IS1] it is noticed, in particular, that weak-$L^q(\Omega)\subset L^{q)}(\Omega)$. We will call $L^{q)}(\Omega)$ the *grand-$L^q(\Omega)$ space*.

Similarly, in [GIS] the *grand Sobolev space* $W_0^{1,q)}(\Omega)$ has been introduced as the space of all functions $u\in\bigcap_{0<\varepsilon\le q-1}W_0^{1,q-\varepsilon}(\Omega)$ such that $\nabla u\in L^{q)}(\Omega)$. Again, $W_0^{1,q)}(\Omega)$ is a Banach space and the inclusion $W_0^{1,q}(\Omega)\subset W_0^{1,q)}(\Omega)$ is obvious. Imbedding theorems of the Sobolev type for these grand Sobolev spaces have recently been proved in [FLS].

3. Very weak solutions of monotone operators. The results we are going to formulate here are true in all dimensions. Therefore, for the purpose of this section we assume $a(x,\xi)$ to be defined on $\Omega\times\mathbb{R}^N$, where the conditions (1.3) and (1.4) hold for $x\in\Omega\subset\mathbb{R}^N$ and $\xi,\eta\in\mathbb{R}^N$.

We prove the following

THEOREM 3.1. *There exists* $\varepsilon_0=\varepsilon_0(m)>0$ *such that for* $|\varepsilon|\le\varepsilon_0$ *and* $F,G\in L^{2-\varepsilon}(\Omega,\mathbb{R}^N)$, *each of the two problems*

$$\begin{cases}\operatorname{div} a(x,\nabla u) = \operatorname{div} F,\\ u\in W_0^{1,2-\varepsilon}(\Omega),\end{cases} \tag{3.1}$$

$$\begin{cases}\operatorname{div} a(x,\nabla v) = \operatorname{div} G,\\ v\in W_0^{1,2-\varepsilon}(\Omega),\end{cases} \tag{3.2}$$

has a unique solution and

$$\|u-v\|_{W_0^{1,2-\varepsilon}(\Omega)} \le c(m)\|F-G\|_{L^{2-\varepsilon}(\Omega,\mathbb{R}^N)}. \tag{3.3}$$

Proof. We mimic the arguments from the proof of Theorem 5.1 in [IS2]. Suppose we are given $F, G \in L^{2-\varepsilon}(\Omega, \mathbb{R}^N)$ and the corresponding solutions u, v to (3.1), (3.2). We wish to prove that (3.3) holds for $|\varepsilon|$ sufficiently small.

By the stability of Hodge decomposition stated in [IS2] we have

$$|\nabla u - \nabla v|^{-\varepsilon}(\nabla u - \nabla v) = \nabla\varphi + h$$

where h is divergence free and

$$\begin{aligned} \|h\|_{(2-\varepsilon)/(1-\varepsilon)} &\le c|\varepsilon| \cdot \|\nabla u - \nabla v\|_{2-\varepsilon}^{1-\varepsilon}, \\ \|\nabla\varphi\|_{(2-\varepsilon)/(1-\varepsilon)} &\le c\|\nabla u - \nabla v\|_{2-\varepsilon}^{1-\varepsilon}. \end{aligned} \tag{3.4}$$

From our assumptions we have

$$\int_\Omega \langle (a(x,\nabla u) - F) - (a(x,\nabla v) - G), \nabla\varphi\rangle\,dx = 0,$$

that is,

$$\int_\Omega \langle (a(x,\nabla u) - F) - (a(x,\nabla v) - G), |\nabla u - \nabla v|^{-\varepsilon}(\nabla u - \nabla v)\rangle\,dx = \int_\Omega \langle (a(x,\nabla u) - F) - (a(x,\nabla v) - G), h\rangle\,dx.$$

We have

$$\begin{aligned} &\int_\Omega \langle (a(x,\nabla u) - F) - (a(x,\nabla v) - F), |\nabla u - \nabla v|^{-\varepsilon}(\nabla u - \nabla v)\rangle\,dx \\ &\quad + \int_\Omega \langle (a(x,\nabla v) - F) - (a(x,\nabla v) - G), |\nabla u - \nabla v|^{-\varepsilon}(\nabla u - \nabla v)\rangle\,dx \\ &= \int_\Omega \langle (a(x,\nabla u) - F) - (a(x,\nabla v) - G), h\rangle\,dx. \end{aligned}$$

By our assumptions,

$$\begin{aligned} \frac{1}{m}\int_\Omega |\nabla u - \nabla v|^{2-\varepsilon}\,dx &\le \int_\Omega \langle G - F, |\nabla u - \nabla v|^{-\varepsilon}(\nabla u - \nabla v)\rangle\,dx \\ &\quad + \int_\Omega m|\nabla u - \nabla v|\cdot|h|\,dx + \int_\Omega |F - G|\cdot|h|\,dx \\ &\le \int_\Omega |F - G|\cdot|\nabla u - \nabla v|^{1-\varepsilon}\,dx \\ &\quad + m\|\nabla u - \nabla v\|_{2-\varepsilon}\|h\|_{(2-\varepsilon)/(1-\varepsilon)} + \|F - G\|_{2-\varepsilon}\|h\|_{(2-\varepsilon)/(1-\varepsilon)} \\ &\le \Big(\int_\Omega |\nabla u - \nabla v|^{2-\varepsilon}\,dx\Big)^{(1-\varepsilon)/(2-\varepsilon)}\Big(\int_\Omega |F - G|^{2-\varepsilon}\,dx\Big)^{1/(2-\varepsilon)} \\ &\quad + (m\|\nabla u - \nabla v\|_{2-\varepsilon} + \|F - G\|_{2-\varepsilon})\|h\|_{(2-\varepsilon)/(1-\varepsilon)}. \end{aligned}$$

From (3.4) it follows that

$$\begin{aligned} \frac{1}{m}\|\nabla u - \nabla v\|_{2-\varepsilon}^{2-\varepsilon} &\le \|\nabla u - \nabla v\|_{2-\varepsilon}^{1-\varepsilon}\|F - G\|_{2-\varepsilon} \\ &\quad + c|\varepsilon|\cdot\|\nabla u - \nabla v\|_{2-\varepsilon}^{2-\varepsilon} \\ &\quad + c|\varepsilon|\cdot\|\nabla u - \nabla v\|_{2-\varepsilon}^{1-\varepsilon}\|F - G\|_{2-\varepsilon} \end{aligned}$$

and

$$\Big(\frac{1}{m} - c|\varepsilon|\Big)\|\nabla u - \nabla v\|_{2-\varepsilon} \le (1 + c|\varepsilon|)\|F - G\|_{2-\varepsilon},$$

i.e. inequality (3.3) holds for $|\varepsilon| \le \varepsilon_0$ and $\varepsilon_0 = \varepsilon_0(m) > 0$. Inequality (3.3) implies the uniqueness in Theorem 3.1.

It remains to prove existence. To this end, for $F \in L^{2-\varepsilon}(\Omega, \mathbb{R}^N)$ suppose $F_j \in L^2(\Omega, \mathbb{R}^N)$ converges to F in $L^{2-\varepsilon}$ and denote by u_j the solution in $W_0^{1,2}(\Omega)$ of

$$\operatorname{div} a(x, \nabla u_j) = \operatorname{div} F_j.$$

We use inequality (3.3) to get

$$\|u_j - u_k\|_{W_0^{1,2-\varepsilon}(\Omega)} \le c(m)\|F_j - F_k\|_{L^{2-\varepsilon}(\Omega,\mathbb{R}^N)} \tag{3.5}$$

for $j, k = 1, 2, \ldots$, which implies that u_j is a Cauchy sequence in $W_0^{1,2-\varepsilon}$. Let $u \in W_0^{2-\varepsilon}$ be the limit of u_j. Passing to the limit in (3.5) as $k \to \infty$ we obtain

$$\|u_i - u\|_{W_0^{1,2-\varepsilon}} \le c\|F_i - F\|_{L^{2-\varepsilon}}.$$

Passing to the limit in the equation

$$\int_\Omega \langle a(x, \nabla u_j) - F_j, \nabla\varphi\rangle\,dx = 0$$

yields

$$\int_\Omega \langle a(x, \nabla u) - F, \nabla\varphi\rangle\,dx = 0 \quad \forall\varphi \in C_0^1(\Omega)$$

and so u is a solution of equation (3.1). Inequality (3.3) follows by standard arguments.

For this result see also [B].

4. Estimates for solutions to the equation $\operatorname{div} F = f \in L^1$. We assume throughout this section that $\Omega \subset \mathbb{R}^N$ $(N \ge 2)$ is a bounded open set.

Let us prove the following

THEOREM 4.1. *Given $f \in L^1(\Omega)$, there exists $F \in L^{N/(N-1))}(\Omega, \mathbb{R}^N)$ such that* $\operatorname{div} F = f$ *in* Ω *and*

$$\left(\frac{N}{N-1} - s\right) \int_\Omega |F|^s \, dx \le c(N)|\Omega|^{(N-Ns+s)/N} \|f\|^s_{L^1(\Omega)} \quad \forall 1 \le s < N/(N-1).$$

Proof. A solution to the equation $\operatorname{div} F = f$ can be expressed explicitly by the vector Riesz potential:

$$F(x) = \frac{1}{N\omega_N} \int_\Omega \frac{x-y}{|x-y|^N} f(y)\, dy$$

where ω_N is the measure of the unit ball in $\mathbb{R}^N$ (see [GT]).

For every $1 \le s < N/(N-1)$ we can use Minkowski's inequality for integrals to obtain

$$\|F\|_s \le \frac{1}{N\omega_N} \int_\Omega \left\| \frac{1}{|\cdot - y|^{N-1}} \right\|_s |f(y)|\, dy \le \frac{1}{N\omega_N} \sup_{y\in\Omega} \left\| \frac{1}{|\cdot - y|^{N-1}} \right\|_s \|f\|_1.$$

It is standard ([Z]) that the integral over Ω of the function $|\cdot - y|^{(1-N)s}$ is less than or equal to the integral of the same function over the ball centered at y and with the same volume as Ω, therefore

$$\sup_{y\in\Omega} \left\| \frac{1}{|\cdot - y|^{N-1}} \right\|_s \le \frac{\omega_N^{(N-1)/N}}{(N-Ns+s)^{1/s}} |\Omega|^{(N-Ns+s)/(Ns)},$$

from which the theorem follows.

5. Proof of Theorem A. Let $g_j = \operatorname{div} F_j$ satisfy $\|g_j - f\|_{L^1} \to 0$. By Theorem 4.1 for $1 < s < 2$ we have the inequality

$$(2-s) \int_\Omega |F_j|^s \, dx \le c\|g_j\|^s_{L^1}. \tag{5.1}$$

Consider the very weak solution $u_j \in W_0^{1,2-\varepsilon}(\Omega)$ of the equation

$$\operatorname{div} a(x, \nabla u_j) = \operatorname{div} F_j = g_j.$$

Theorem 3.1 implies

$$\|u_j - u_k\|_{W_0^{1,s}} \le c\|F_j - F_k\|_{L^s}.$$

Obviously, $F_j - F_k$ satisfies the linear equation

$$\operatorname{div}(F_j - F_k) = g_j - g_k.$$

Hence

$$(2-s) \int_\Omega |\nabla u_j - \nabla u_k|^s \, dx \le (2-s)c \int_\Omega |F_j - F_k|^s \, dx \le c\|g_j - g_k\|^s_{L^1}. \tag{5.2}$$

So (u_j) is a Cauchy sequence. Let u denote the limit of u_j. Passing to the limit in (5.2) with k fixed and j approaching ∞ we find

$$(2-s) \int_\Omega |\nabla u - \nabla u_k|^s \, dx \le c\|f - g_k\|^s_{L^1}. \tag{5.3}$$

Hence $u \in W_0^{1,2)}(\Omega)$ is a weak solution of (1.1). The uniqueness follows from the preceding estimates.

Remark. By density arguments it is easy to deduce, arguing as in [CS] and as in our proof of Corollary 6.1 below, that

$$\lim_{s\uparrow 2} (s-2) \int_\Omega |\nabla u|^s \, dx = 0. \tag{5.4}$$

We also remark that property (5.4) is not true if we replace f by a measure (see e.g. [BB], [BG], [LM], [D]). To see this we consider the following Dirichlet problem in the unit ball B:

$$\begin{cases} \operatorname{div} \nabla u = \delta_0 & \text{in } B, \\ u = 0 & \text{on } \partial B. \end{cases}$$

The function $u(x) = \frac{1}{2\pi} \log|x|$ is a solution and in this case we have

$$(s-2) \int_\Omega |\nabla u|^s \, dx = (2\pi)^{s-1}.$$

6. An application. As an application, we extend the results of Brezis–Merle [BM] and Chanillo–Li [CL].

COROLLARY 6.1. *Let Ω be a bounded region in $\mathbb{R}^2$ and let $A = -\operatorname{div} a(x, \nabla u)$, where a satisfies the conditions of Section 3. Then for $f \in L^1(\Omega)$ the solution of the boundary value problem*

$$\begin{cases} Au = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

belongs to $\exp(\Omega)$. *This means that*

$$\int_\Omega e^{\lambda|u|} < \infty \quad \text{for any } \lambda > 0.$$

Proof. Let $g_k \in L^\infty(\Omega)$ satisfy $\|g_k - f\|_{L^1} \to 0$, and let

$$F_k(x) = \frac{1}{N\omega_N} \int_\Omega \frac{x-y}{|x-y|^2} g_k(y)\, dy$$

be a solution to the equation $\operatorname{div} F_k = g_k$. By well-known potential estimates (see e.g. [GT]) we have $F_k \in L^\infty(\Omega)$ and therefore if $u_k \in W_0^{1,2}(\Omega)$ solves the equation $\operatorname{div} a(x, \nabla u_k) = \operatorname{div} F_k$ then by Theorem 3.1, $u_k \in W_0^{1,2+\varepsilon_0}(\Omega)$ for some $\varepsilon_0 > 0$, from which, using the Sobolev theorem, we conclude that $u_k \in L^\infty(\Omega)$.

By (5.3) we have

$$\|u_k - u\|_{W_0^{1,2)}} \le c_1 \|g_k - f\|_{L^1}.$$

On the other hand, the following inequality has been shown in [FLS]:

$$\|u_k - u\|_{\mathrm{EXP}} \le c_2 \|u_k - u\|_{W_0^{1,2)}}.$$

Consequently,

$$\|u_k - u\|_{\mathrm{EXP}} \le c_3 \|g_k - f\|_{L^1}.$$

Arguing as in [CS], if f belongs to the closure of the bounded functions in L^1-norm then u belongs to $\exp(\Omega)$, which is the closure of $L^\infty(\Omega)$ in $\mathrm{EXP}(\Omega)$, as can be shown (see e.g. [CS]).

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Dipartimento di Matematica e Applicazioni "R. Caccioppoli"
via Cintia
80126 Napoli, Italy
E-mail: fiorenza@matna2.dma.unina.it
sbordone@matna1.dma.unina.it

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