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Harmonic extensions and the Böttcher–Silbermann conjecture

by

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Abstract. We present counterexamples to a conjecture of Böttcher and Silbermann on the asymptotic multiplicity of the Poisson kernel of the space $L^\infty(\partial D)$ and discuss conditions under which the Poisson kernel is asymptotically multiplicative.

In this paper, we let L^p denote the space of Lebesgue measurable functions on the unit circle ∂D such that

$$\int_{\partial D} |f(e^{i\theta})|^p \, d\theta < \infty$$

for $1 \le p < \infty$ and L^∞ denote the space of essentially bounded Lebesgue measurable functions on the unit circle. Each function f in L^1 has a natural harmonic extension into D via the Poisson formula, and it is the harmonic extension that we are interested in studying in this paper. We let $\widehat{f}(z)$ denote the harmonic extension at z.

For $1 \le p \le \infty$, let H^p denote the Hardy space on the unit circle; that is, the subspace of L^p consisting of those functions which are analytic in D. Another space that will be of interest to us is the space $H^\infty + C$ consisting of sums of functions in H^∞ and continuous functions on the unit circle. Sarason [27] showed that this is a closed subalgebra of L^∞. In what follows we will denote the closed subalgebra of L^∞ generated by H^∞ and a function f in L^∞ by $H^\infty[f]$.

We are interested in studying the asymptotic multiplicative properties of the harmonic extension. Recall that

$$\widehat{f}(z) = \frac{1}{2\pi} \int_{\partial D} f(e^{i\theta}) P(z,\theta)\, d\theta$$

where $P(z,\theta)$ is the Poisson kernel

$$P(z,\theta) = \frac{1-|z|^2}{|1-\overline{z}e^{i\theta}|^2}.$$

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The mapping $f \to \widehat{f}$ from L^∞ to $C(D)$ plays an important role in the theory of Toeplitz operators and Hankel operators on the Hardy space H^2 (see [6], [10]), but it is not multiplicative. On the other hand, it was shown (see [10], p. 169) that the mapping is asymptotically multiplicative on the Douglas algebra $H^\infty + C$, that is, for two functions f and g in $H^\infty + C$,

$$\widehat{fg}(z) - \widehat{f}(z)\widehat{g}(z) \to 0$$

as $z \to \partial D$.

In this paper we will consider the problem of when the harmonic extension is asymptotically multiplicative for two given functions f and g in L^∞. This problem was studied in [6] and it is related to the so-called rth *Abel–Poisson mean* $h_r(f)$ $(0 < r < 1)$ defined by

$$(h_r f)(e^{i\theta}) = \sum_{k=-\infty}^{\infty} r^{|k|} a_k e^{ik\theta}$$

where a_k is the Fourier coefficient of f.

Motivated by the Axler–Chang–Sarason–Volberg Theorem ([1], [30]) on the compactness of the semi-commutator of two Toeplitz operators, Böttcher and Silbermann ([6], Section 7.18), ([7], p. 178) made the following conjecture.

BÖTTCHER–SILBERMANN CONJECTURE. *Let f and g be in L^∞. Then $\widehat{fg}(z) - \widehat{f}(z)\widehat{g}(z) \to 0$ as $z \to \partial D$ if and only if $[H^\infty[\bar{f}] \cap H^\infty[g]] \cup [H^\infty[f] \cap H^\infty[\bar{g}]] \subseteq H^\infty + C$.*

In the first section of the paper we look at a related question; then we answer Böttcher and Silbermann's question negatively.

We begin the paper with a look at the following problem: If $f \in L^\infty$ and $\widehat{fg}(z) - \widehat{f}(z)\widehat{g}(z) \to 0$ as $z \to \partial D$ for all $g \in H^\infty$, what can we say about f? This result has a connection to a well-known theorem of Sarason [27] which tells us that if we know that the statement $\widehat{f^n g}(z) - \widehat{f^n}(z)\widehat{g}(z) \to 0$ holds for all positive integers n and all functions $g \in H^\infty + C$ then f is actually in the algebra $H^\infty + C$. The point to this result as well as the Böttcher–Silbermann conjecture is that the asymptotic multiplicity condition is not a condition on an algebra of functions, whereas the other condition is. We will answer the question mentioned above in Section 2 and present some related results.

The remainder of the paper is devoted to certain counterexamples to the Böttcher–Silbermann conjecture as well as studying conditions under which two functions do have the asymptotic multiplicative property.

In Section 3, we will give conditions that ensure that $\widehat{fg}(z) - \widehat{f}(z)\widehat{g}(z) \to 0$ as $z \to \partial D$. Although these conditions are not both necessary and sufficient, they do provide us with a simple example showing that the conjecture above

is false. We study the Böttcher–Silbermann condition

$$[H^\infty[\bar{f}] \cap H^\infty[g]] \cup [H^\infty[f] \cap H^\infty[\bar{g}]] \subseteq H^\infty + C$$

and consequences thereof, and we show that this condition is sufficient but not necessary for the asymptotic multiplicative property.

In Section 4, we work from the other direction and give conditions that must hold whenever $\widehat{fg}(z) - \widehat{f}(z)\widehat{g}(z) \to 0$ as $z \to \partial D$. We will show that these conditions are necessary and sufficient for a broad class of functions to satisfy the asymptotic multiplicity property. The ideas of this proof allow us to create a class of examples for which the conjecture above must fail.

We conclude the paper in the fifth section with some comments on operator-theoretic conditions closely related to this problem.

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1. Preliminaries. Let B denote a *Douglas algebra*, that is, a closed subalgebra of L^∞ containing H^∞, and let $M(B)$ denote the maximal ideal space of B. The Chang–Marshall Theorem ([9], [24]) asserts that any Douglas algebra B is generated by H^∞ together with the conjugates of the interpolating Blaschke products invertible in B. Recall that $M(H^\infty + C) = M(H^\infty) - D$. For each $m \in M(H^\infty + C)$, the support of the unique representing measure for m is denoted by $\operatorname{supp} m$. Hoffman ([21], [22]) has shown that m has a unique Hahn–Banach extension to L^∞, which is given by

$$m(f) = \int_{\operatorname{supp} m} f\, dm$$

for $f \in L^\infty$.

Let $M(H^\infty)$ be the maximal ideal space of H^∞. With the weak-star topology, $M(H^\infty)$ is a compact Hausdorff space. If z is a point in the unit disc D, then point evaluation at z is a multiplicative linear functional on H^∞ and thus we can think of z as an element of $M(H^\infty)$. As usual, we will think of the unit disc D as a subset of $M(H^\infty)$. Carleson's Corona Theorem states that D is dense in $M(H^\infty)$.

On $M(H^\infty)$ the pseudo-hyperbolic distance $\varrho(m,x)$ is defined by

$$\varrho(m,x) = \sup\{|f(m)| : f \in H^\infty,\ \|f\|_\infty \le 1 \text{ and } f(x) = 0\}.$$

We can partition $M(H^\infty)$ into equivalence classes known as *Gleason parts*, calling x and y *equivalent* provided $\varrho(x,y) < 1$. The Gleason part of m is denoted by $P(m)$. Wermer has shown that each Gleason part $P(m)$ is either trivial (contains only one point, called *trivial*) or an analytic disc (in this case, the point m is called *nontrivial*). When $P(m)$ is an analytic disc, Hoffman [22] constructed a bijective map L_m such that $f \circ L_m$ is holomorphic for all f in H^∞ and $L_m(0) = m$. If

$$L_\alpha(z) = \frac{z+\alpha}{1+\overline{\alpha}z},$$

then L_{z_α} converges pointwise to L_m on D provided the net z_α converges to m.

In this paper we will need more information about the Gleason parts in the maximal ideal space of H^∞. The parts that are most easily studied are the so-called thin parts. A *thin part* is a part such that some (and hence every) point in the part is in the closure of an interpolating sequence for the algebra $QA = H^\infty \cap \overline{H^\infty + C}$. Sundberg and Wolff [29] showed that this is equivalent to requiring the H^∞ interpolating sequence $\{z_n\}$ in the disc D to satisfy the following condition:

$$\lim_{k\to\infty} \prod_{n\neq k} \frac{|z_n - z_k|}{|1-\overline{z}_n z_k|} = 1.$$

Such an interpolating sequence is called a *thin sequence* and the associated Blaschke product b is called a *thin Blaschke product*. This function b has the property that $|b| = 1$ on every trivial point. Mortini and Tolokonnikov [25] and Guillory and Izuchi [19] have studied such Blaschke products, and in their papers they give some characterizations of the interpolating Blaschke products that have modulus one on all trivial parts.

2. An algebraic condition. We begin with a related question: If $f \in L^\infty$ is such that $\widehat{fg}(z) - \widehat{f}(z)\widehat{g}(z) \to 0$ as $|z| \to 1$ for all $g \in H^\infty$, what can we say about f? This question is related to the Chang–Marshall theorem (as we shall show below) in that it is equivalent to assuming that $m(fg) = m(f)m(g)$ for all $m \in M(H^\infty + C)$ and all $g \in H^\infty + C$. However, since we are not assuming that every power of f has this property, it is not a statement about the maximal ideal space of the algebra generated by H^∞ and f. We will show that f must be in $H^\infty + C$ by showing that our apparently weaker assumption implies the stronger algebra statement and we will use Sarason's theorem [28] (see also [13], p. 378) to obtain this conclusion. The next lemma is well known when both functions are assumed to be in $H^\infty + C$. It is a little less well known in the generality stated below.

LEMMA 1. *Let f and g be functions in L^∞. Then $\widehat{fg}(z) - \widehat{f}(z)\widehat{g}(z) \to 0$ as $z \to \partial D$ if and only if $m(fg) = m(f)m(g)$ for all $m \in M(H^\infty + C)$.*

Proof. Suppose that $\widehat{fg}(z) - \widehat{f}(z)\widehat{g}(z) \to 0$ as $z \to \partial D$. Let $m \in M(H^\infty + C)$. By the Corona Theorem, there is a net (z_α) of points with $z_\alpha \to m$. By Hoffman's work [22], $\widehat{fg}, \widehat{f}$ and $\widehat{g}$ are all continuous functions on $M(H^\infty)$. Taking limits, we obtain $m(fg) = m(f)m(g)$.

For the other direction suppose that $m(fg) = m(f)m(g)$ for all $m \in M(H^\infty + C)$. Therefore, given $\varepsilon > 0$ there exists r with $0 < r < 1$ such that $|\widehat{fg}(z) - \widehat{f}(z)\widehat{g}(z)| < \varepsilon$ for $|z| > r$. This completes the proof of this lemma.

Now let X denote a compact Hausdorff space and let A denote a closed logmodular subalgebra of $C(X)$. Hoffman made an in-depth study of such algebras which we use to study the Böttcher–Silbermann conjecture.

For any multiplicative linear functional m on A, Hoffman [21] defines $H^2(m)$ to be the closure of A in the Banach space $L^2(m)$. Letting $H_0^2(m)$ denote the set of $H^2(m)$ functions annulled by m, Hoffman ([21], Theorem 5.4) shows that

$$L^2(m) = H^2(m) \oplus \overline{H_0^2(m)}.$$

Let A_m denote the space of functions in A which are annulled by m. In particular, a function $f \in L^2(m)$ belongs to $H^2(m)$ if and only if $\int fg\,dm = 0$ for all $g \in A_m$. We will use this result to prove the theorem below. We note that if (as in Hoffman) one defines $H^\infty(m) = H^2(m) \cap L^\infty(m)$, then ([23], p. 123) $H^\infty(m)$ is a Banach algebra.

In our situation, we take $X = M(L^\infty)$ and identify a function $f \in L^\infty$ with its Gelfand transform in $C(X)$. We take the logmodular algebra $A = H^\infty$.

THEOREM 2. *Let $f \in L^\infty$. Suppose that for every $m \in M(H^\infty + C)$, $m(fg) = m(f)m(g)$ for all $g \in H^\infty$. Then for every $m \in M(H^\infty + C)$ and every positive integer n,*

$$m(f^n g) = m(f^n)m(g) = m(f)^n m(g)$$

for all $g \in H^\infty$.

Proof. Let f be as above and let $m \in M(H^\infty + C)$. Since f satisfies $m(fg) = m(f)m(g)$ for all $g \in H^\infty$, we see that $m(fg) = 0$ for all $g \in H^\infty$ such that $m(g) = 0$. Since such g are dense in $H_0^2(m)$ the comments preceding this theorem show that $f \in H^2(m)$. Therefore $f \in H^\infty(m)$, and consequently, $f^n \in H^\infty(m)$ for all positive integers n. Now let $g \in H^\infty$. Then again the comments preceding this theorem show that for any positive integer n, we have $m(f^n(g - m(g))) = \int (f^n(g - m(g)))\,dm = 0$. Consequently, $m(f^n g) = m(f^n)m(g)$. To complete the proof, note that since $f \in H^\infty(m)$, the function f is an $L^2(m)$-limit of functions in H^∞. Therefore, if we let h_j denote a sequence of bounded analytic functions converging to f in $L^2(m)$, we see that $m(f^2) = \int f^2\,dm = \lim \int fh_j\,dm$, since f is bounded. Therefore, $m(f^2) = \lim m(f)m(h_j)$ and since $\lim m(h_j) = \lim \int h_j\,dm = \int f\,dm = m(f)$, we conclude that $m(f^2) = m(f)^2$. The proof for an arbitrary integer n can be completed using induction.

The argument above is actually more general than what we have stated, but the corollary below is not, because it depends on Sarason's result stating that if a closed subalgebra A of L^∞ containing H^∞ has the property $M(A) = M(H^\infty + C)$, then $A = H^\infty + C$. However, it is possible to use the full strength of the Chang–Marshall theorem to obtain a similar result for an arbitrary Douglas algebra A in place of the algebra $H^\infty + C$.

COROLLARY 3. *Let $f \in L^\infty$ and suppose that $m(fg) = m(f)m(g)$ for all $m \in M(H^\infty + C)$ and all $g \in H^\infty$. Then $f \in H^\infty + C$.*

Proof. Let $A = H^\infty[f]$ denote the closed subalgebra of L^∞ generated by H^∞ and the function f. By Theorem 2 we know that $m(f^n g) = m(f^n)m(g) = m(f)^n m(g)$ for all positive integers n and all H^∞ functions g. Since ([13], p. 375)

$$M(A) = \{m \in M(H^\infty + C) : m(gh) = m(g)m(h) \text{ for all } g, h \in A\}$$

we see that $M(A) = M(H^\infty + C)$ and consequently (see [13], Chapter IX) $A = H^\infty + C$.

The results above arose in connection with a question about Hankel type operators on $H^\infty(U)$, the algebra of bounded analytic functions on a bounded planar domain U in the complex plane. For $f \in L^\infty(U)$ define the Hankel type operator $S_f : H^\infty(U) \to L^\infty(U)/H^\infty(U)$ by $S_f(g) = gf + H^\infty(U)$. The symbols f that yield compact, weakly compact or completely continuous operators have been studied by many people (for a detailed survey of these operators see [15]). In many cases one can pass to a boundary algebra and ask the same question about symbols on the boundary. In virtually all such studies, new techniques were required to move from the boundary to the interior or the interior to the boundary. In looking for a natural way of passing from a domain to its boundary and maintaining compactness the following question arose: For $f \in L^\infty(\partial D)$ define the operator $B_f : H^\infty(\partial D) \to L^\infty(D)/H^\infty(D)$ by $B_f(g) = \widehat{fg} - \widehat{f}\widehat{g} + H^\infty(D)$. When is B_f compact?

COROLLARY 4. *Let $f \in L^\infty(\partial D)$. Then the operator B_f is compact if and only if $f \in H^\infty + C$.*

Proof. First suppose that $f \in H^\infty + C$. Then there exist functions $h \in H^\infty$ and $c \in C$ such that $f = h + c$. Now $B_f = B_c$, so we may as well assume that f is of the form $\bar{z}^n$ for some positive integer n. Let h_m be a uniformly bounded sequence of bounded analytic functions. Without loss of generality, we may assume that $\{h_m\}$ converges to zero uniformly on compacta. The computations on page 169 of [10] show that if $h_m = \sum_{n=0}^\infty a_{m,n} z^n$ is the Fourier expansion of h_m, then for $w = re^{it}$ we obtain

$$|\widehat{\bar{z}^n h_m}(w) - \widehat{\bar{z}^n}(w)\widehat{h}_m(w)| \le \sum_{k=0}^{n} |r^{|k-n|} - r^{|k+n|}| \cdot |a_{m,k}| + (r^{-n} - r^n)\Big\| \sum_{k=n+1}^\infty a_{m,k} z^k \Big\|_\infty.$$

Since h_m converge uniformly to zero on compacta, both terms can be made small as long as $|1 - r|$ is sufficiently small for m sufficiently large. Since $D_r = \{|z| \le r\}$ is a compact set, we can choose M large enough to make $\sup_{w \in D_r} |h_m(w)|$ small. The formula for the Poisson extension now shows that $\|B_f(h_m)\|$ will also be small for $m \ge M$. Therefore B_f is compact.

Now suppose that B_f is compact. We claim that f satisfies the assumptions of Theorem 2. Let $g \in H^\infty$ and $m \in M(H^\infty + C)$. We may assume that $m(z) = 1$. Since B_f is compact, there is a sequence $\{n_k\}$ of positive integers and an element $h \in L^\infty$ such that $\|B_f(gz^{n_k}) + h + H^\infty\| \to 0$. The Riemann–Lebesgue Lemma shows that, in fact, $h \in H^\infty$. Therefore, we may choose $h_k \in H^\infty$ with

$$\sup_{w \in D} |\widehat{fgz^{n_k}}(w) - \widehat{f}(w)\widehat{gz^{n_k}}(w) + h_k(w)| \to 0.$$

Taking radial limits shows that $\|h_k\| \to 0$ and so for any $\varepsilon > 0$ we can find an integer N such that for $n_k \ge N$,

$$\sup_{w \in D} |\widehat{fgz^{n_k}}(w) - \widehat{f}(w)\widehat{gz^{n_k}}(w)| < \varepsilon.$$

Since $m \in M(H^\infty + C)$ we can use the Corona Theorem to conclude that for $n_k \ge N$ we have $|m(fgz^{n_k}) - m(f)m(gz^{n_k})| < \varepsilon$.

But m has a representing measure supported on the fiber over the point $z = 1$ and so for all integers n we know that $m(fgz^n) = m(fg)$ and $m(gz^n) = m(g)$.

Putting all this together tells us that for $n_k \ge N$ we have

$$|m(fg) - m(f)m(g)| = |m(fgz^{n_k}) - m(f)m(gz^{n_k})| < \varepsilon.$$

Therefore f satisfies the assumptions of Corollary 3 and consequently is in $H^\infty + C$.

3. The condition $[H^\infty[\bar{f}] \cap H^\infty[g]] \cup [H^\infty[f] \cap H^\infty[\bar{g}]] \subseteq H^\infty + C$.

In this section, we use some well-known results on Douglas algebras to obtain some conditions equivalent to those of Böttcher and Silbermann. We first state the lemmas that we will need. The first is due to Sarason; the rest are known. For $f \in L^\infty$ recall that $m(f)$ denotes the unique Hahn–Banach extension of m applied to f, and that Hoffman's results tell us that $f \in C(M(H^\infty))$.

LEMMA 5. *Let A and B be Douglas algebras. Then*

$$M(A \cap B) = M(A) \cup M(B).$$

Proof. See [14] or [18].

The next lemma is a well-known consequence of the Chang–Marshall theorem. (For a detailed proof see [18].)

LEMMA 6. *Let $f \in L^\infty$. Then for $m \in M(H^\infty + C)$, $f|\operatorname{supp} m \in H^\infty|\operatorname{supp} m$ if and only if $m \in M(H^\infty[f])$.*

The next lemma is also known, but we include it here for easy reference.

LEMMA 7. *Let u be an inner function and let $m \in M(H^\infty + C)$. Then the following are equivalent*:

(1) $\overline{u}|\operatorname{supp} m \in H^\infty|\operatorname{supp} m$.
(2) $|m(u)| = 1$.
(3) $|u| = 1$ *on* $P(m)$.

Proof. Suppose (1) holds. Then $1 = m(1) = m(u\overline{u}) = m(u)\overline{m(u)}$, where the last equality holds because m is given by integration against a positive measure. Therefore (2) follows.

Now, suppose that (2) holds. Then $1 = |m(u)| = |\int_{\operatorname{supp} m} u\, dm|$. Since the representing measure for m is a probability measure on $\operatorname{supp} m$ and $|u| = 1$ on $\operatorname{supp} m$, we see that u must be constant on $\operatorname{supp} m$. This obviously implies (1). Thus (1) and (2) are equivalent.

Now, (3) clearly implies (2). If (1) holds and (3) does not, then $|m(u)| = 1$ while $|x(u)| < 1$ for some $x \in P(m)$. Forming the inner function $b = (u - x(u))/(1 - \overline{x(u)}u)$, we see that $|m(b)| = 1$ while $x(b) = 0$. This contradicts the fact that the pseudohyperbolic distance between the two points is less than 1.

LEMMA 8. *Let u be an inner function and let $f \in L^\infty$. If $m \in M(H^\infty + C)$ and $|m(u)| = 1$, then $m(uf) = m(u)m(f)$.*

Proof. By Lemma 7, u is a constant of modulus one on the support of m. Thus

$$m(uf) = \int_{\operatorname{supp} m} uf\, dm = m(u) \int_{\operatorname{supp} m} f\, dm = m(u)m(f).$$

This completes our list of lemmas. Now we turn to the Böttcher–Silbermann conjecture.

THEOREM 9. *Let f and g belong to L^∞. Then*

$$[H^\infty[\bar{f}] \cap H^\infty[g]] \cup [H^\infty[f] \cap H^\infty[\bar{g}]] \subseteq H^\infty + C$$

if and only if for every $m \in M(H^\infty + C)$ at least one of the following four statements holds:

(a) *$\bar{f}|\operatorname{supp} m$ and $\bar{g}|\operatorname{supp} m$ are elements of $H^\infty|\operatorname{supp} m$.*
(b) *$g|\operatorname{supp} m$ and $f|\operatorname{supp} m$ are elements of $H^\infty|\operatorname{supp} m$.*
(c) *$f|\operatorname{supp} m$ is constant.*
(d) *$g|\operatorname{supp} m$ is constant.*

Furthermore, each of these conditions implies that $m(fg) = m(f)m(g)$ for all $m \in M(H^\infty + C)$.

Proof. We have $[H^\infty[\bar{f}] \cap H^\infty[g]] \cup [H^\infty[f] \cap H^\infty[\bar{g}]] \subseteq H^\infty + C$ if and only if $H^\infty[\bar{f}] \cap H^\infty[g]$ and $H^\infty[\bar{g}] \cap H^\infty[f]$ are both subsets of $H^\infty + C$. Thus, our condition is equivalent to $M(H^\infty + C) \subseteq M(H^\infty[\bar{f}] \cap H^\infty[g])$ and $M(H^\infty + C) \subseteq M(H^\infty[\bar{g}] \cap H^\infty[f])$. But for any two Douglas algebras A and B, $M(A \cap B) = M(A) \cup M(B)$ (see Lemma 5), so $[H^\infty[\bar{f}] \cap H^\infty[g]] \cup [H^\infty[f] \cap H^\infty[\bar{g}]] \subseteq H^\infty + C$ if and only if

$$(3.1) \qquad \begin{aligned} M(H^\infty + C) &\subseteq M(H^\infty[\bar{f}]) \cup M(H^\infty[g]),\\ M(H^\infty + C) &\subseteq M(H^\infty[\bar{g}]) \cup M(H^\infty[f]). \end{aligned}$$

Now suppose

$$[H^\infty[\bar{f}] \cap H^\infty[g]] \cup [H^\infty[f] \cap H^\infty[\bar{g}]] \subseteq H^\infty + C$$

and let $m \in M(H^\infty + C)$. From (3.1) either $m \in M(H^\infty[\bar{f}]) \cap M(H^\infty[\bar{g}])$ and Lemma 6 implies that condition (a) holds, $m \in M(H^\infty[g]) \cap M(H^\infty[f])$ and (b) holds, $m \in M(H^\infty[\bar{f}]) \cap M(H^\infty[f])$ or $m \in M(H^\infty[\bar{g}]) \cap M(H^\infty[g])$. If $m \in M(H^\infty[\bar{f}]) \cap M(H^\infty[f])$, then both $f|\operatorname{supp} m$ and $\bar{f}|\operatorname{supp} m$ are in $H^\infty|\operatorname{supp} m$. In this case, since the support sets are antisymmetric for the algebra ([23], p. 135), condition (c) holds. Finally, if $m \in M(H^\infty[\bar{g}]) \cap M(H^\infty[g])$, then the antisymmetry of support sets shows that (d) holds.

On the other hand, if one of the conditions (a)–(d) holds, then for each $m \in M(H^\infty + C)$, condition (a) holding implies that $m \in M(H^\infty[\bar{f}]) \cap M(H^\infty[\bar{g}])$; condition (b) implies that $m \in M(H^\infty[f]) \cap M(H^\infty[g])$, condition (c) implies $m \in M(H^\infty[\bar{f}]) \cap M(H^\infty[f])$ and condition (d) implies that $m \in M(H^\infty[\bar{g}]) \cap M(H^\infty[g])$. Therefore, for any $m \in M(H^\infty + C)$ we see that $m \in (M(H^\infty[\bar{f}]) \cup M(H^\infty[g])) \cap (M(H^\infty[f]) \cup M(H^\infty[\bar{g}]))$. Again using Lemma 5 we have $m \in M(H^\infty[\bar{f}] \cap H^\infty[g]) \cap M(H^\infty[\bar{g}] \cap H^\infty[f])$. This implies $M(H^\infty + C) \subseteq M(H^\infty[\bar{f}] \cap H^\infty[g])$ and $M(H^\infty + C) \subseteq M(H^\infty[\bar{g}] \cap H^\infty[f])$. Sarason's precursor to the Chang–Marshall theorem tells us that if $M(H^\infty + C)$ is contained in the maximal ideal space of a closed subalgebra A containing H^∞, then the algebra is contained in $H^\infty + C$. Therefore, we know that $H^\infty[\bar{f}] \cap H^\infty[g]$ and $H^\infty[\bar{g}] \cap H^\infty[f]$ are both subsets of $H^\infty + C$, and we have established the equivalence of these two conditions.

To complete the proof of this theorem, we need only show that each of (a), (b), (c), and (d) implies that $m(fg) = m(f)m(g)$ for all $m \in M(H^\infty + C)$. This is easy if we note that since the representing measure for m is a probability measure, $m(\bar{f}) = \overline{m(f)}$ for any $f \in L^\infty$, and use the integral representation of m as follows:

Suppose that (a) holds. Then there exist H^∞ functions f_1 and g_1 such that $\bar{f} = f_1$ on $\operatorname{supp} m$ and $\bar{g} = g_1$ on $\operatorname{supp} m$. Thus

$$m(\bar{f}\bar{g}) = \int_{\operatorname{supp} m} \bar{f}\bar{g}\,dm = \int_{\operatorname{supp} m} f_1 g_1\,dm.$$

But since both f_1 and g_1 are in H^∞ and m is multiplicative on H^∞ we have

$$m(\bar{f}\bar{g}) = \int_{\operatorname{supp} m} f_1\,dm \int_{\operatorname{supp} m} g_1\,dm = \int_{\operatorname{supp} m} \bar{f}\,dm \int_{\operatorname{supp} m} \bar{g}\,dm = m(\bar{f})m(\bar{g}).$$

Taking conjugates yields the result in this case. The proof that (b) implies the multiplication statement is the same as above, and using the integral representation, one sees that (c) and (d) easily imply that $m(fg) = m(f)m(g)$ for all $m \in M(H^\infty + C)$.

The next theorem is a corollary of the one above. The proof also uses the integral representation argument. It is this theorem that will allow us to give our first counterexample to the Böttcher–Silbermann conjecture.

THEOREM 10. *Suppose that $f, g \in L^\infty$ and for each $m \in M(H^\infty + C)$ one of the following conditions holds*:

(a) *$f|\operatorname{supp} m$ and $g|\operatorname{supp} m$ are elements of $H^\infty|\operatorname{supp} m$.*
(b) *$\bar{g}|\operatorname{supp} m$ and $\bar{f}|\operatorname{supp} m$ are elements of $H^\infty|\operatorname{supp} m$.*
(c) *$f|\operatorname{supp} m$ is constant.*
(d) *$g|\operatorname{supp} m$ is constant.*
(e) *There exist constants α and β, not both zero, and H^∞ functions f_1, g_1 such that $f = \alpha f_1 + \beta\bar{g}_1$ and $g = \alpha f_1 - \beta\bar{g}_1$ on $\operatorname{supp} m$.*

Then $m(fg) = m(f)m(g)$ for all $m \in M(H^\infty + C)$.

Proof. In view of the previous theorem, we only have to show that condition (e) implies $m(fg) = m(f)m(g)$ for all $m \in M(H^\infty + C)$. Using the integral representation as above,

$$m(fg) = \int_{\operatorname{supp} m} fg\,dm = \int_{\operatorname{supp} m} (\alpha f_1 + \beta\bar{g}_1)(\alpha f_1 - \beta\bar{g}_1)\,dm$$
$$= m((\alpha f_1 + \beta\bar{g}_1)(\alpha f_1 - \beta\bar{g}_1)).$$

So $m(fg) = m((\alpha f_1)^2 - (\beta\bar{g}_1)^2) = m((\alpha f_1)^2) - m((\beta\bar{g}_1)^2)$.

As above, the measure we are integrating against, dm, is a probability measure and $g_1 \in H^\infty$ so we see that $m(\bar{g}_1^2) = \overline{m(g_1)^2}$. Again, since f_1 and g_1 are H^∞ functions,

$$m(fg) = m(\alpha f_1)^2 - m(\beta\bar{g}_1)^2 = m(\alpha f_1 + \beta\bar{g}_1)m(\alpha f_1 - \beta\bar{g}_1) = m(f)m(g).$$

We are now in a position to present the first counterexample to the Böttcher–Silbermann conjecture.

EXAMPLE 1. Let b be an infinite Blaschke product. Define $f = (b + \bar{b})/2$ and $g = (b - \bar{b})/2$. Then f and g satisfy condition (e) in Theorem 10 and, by Lemma 1, $\widehat{fg}(z) - \widehat{f}(z)\widehat{g}(z) \to 0$ as $z \to \partial D$.

On the other hand, $H^\infty[\bar{f}] \cap H^\infty[g] = H^\infty[\bar{b}]$ and $H^\infty[\bar{g}] \cap H^\infty[f] = H^\infty[\bar{b}]$. Since b is not invertible in $H^\infty + C$ we see that the inverse of b, namely $\bar{b}$, is not in $H^\infty + C$. Thus $\bar{b} \in ([H^\infty[\bar{f}] \cap H^\infty[g]] \cup [H^\infty[f] \cap H^\infty[\bar{g}]]) \setminus (H^\infty + C)$. So f and g do not satisfy the condition in the Böttcher–Silbermann conjecture.

Thus one might look for conditions on f and g for the conjecture to hold. We will do this in the next section. Before doing so, we give one more example to show that the conjecture need not hold *even if* $f \in H^\infty$ and $g \in \overline{H^\infty}$. Our example will also show that the asymptotic multiplicity condition may hold without conditions (a)–(e) of Theorem 10 holding.

It will be helpful to have the following lemma at hand. This lemma follows from work in [3] and [20].

LEMMA 11. *Let $f \in H^\infty + C$ and let u be an inner function. Then $fH^\infty[\bar{u}] \subseteq H^\infty + C$ if and only if $f = 0$ whenever $|u| < 1$.*

Since u is inner, we can replace the condition $f = 0$ whenever $|u| < 1$ by $f = 0$ on $M(H^\infty + C) \setminus M(H^\infty[\bar{u}])$.

THEOREM 12. *Let $f \in H^\infty + C$ and let $g \in L^\infty$. Suppose $f = 0$ on $M(H^\infty + C) \setminus M(H^\infty[g])$. Then $m(fg) = m(f)m(g)$ for all $m \in M(H^\infty + C)$.*

Proof. Let $f \in H^\infty + C$ satisfy $f = 0$ off $M(H^\infty[g])$. Suppose first that $m \in M(H^\infty[g])$. Then we know that $g|\operatorname{supp} m \in H^\infty|\operatorname{supp} m$ and $f|\operatorname{supp} m \in H^\infty|\operatorname{supp} m$. From Theorem 9(b) we know that $m(fg) = m(f)m(g)$.

Now we will prove the theorem in stages. First suppose that b is a finite product of interpolating Blaschke products invertible in $H^\infty[g]$. Note that since $H^\infty[\bar{b}] \subseteq H^\infty[g]$, we have $M(H^\infty[g]) \subseteq M(H^\infty[\bar{b}])$. Thus by our assumption, $f = 0$ on $M(H^\infty + C) \setminus M(H^\infty[\bar{b}])$. Hence $f = 0$ whenever $|b| < 1$ and from Lemma 11 we know that $f\bar{b} \in H^\infty + C$. If $|m(b)| = 1$, then it follows as above or from Lemma 8 that $m(f\bar{b}) = m(f)m(\bar{b})$. If $|m(b)| < 1$, then $m(f)m(\bar{b}) = 0$. On the other hand, $0 = m(f) = m(f\bar{b}b) = m(f\bar{b})m(b)$. Now if $m(b) \neq 0$, then $m(f\bar{b}) = 0 = m(f)m(\bar{b})$. Hoffman's work [22] tells us that a finite product of interpolating Blaschke products cannot vanish on an open subset of $M(H^\infty + C)$ and hence the set of points for which we have established the asymptotic multiplicity condition ($m \in M(H^\infty + C)$ for which $m(b) \neq 0$) is dense. Therefore $m(f\bar{b}) = m(f)m(\bar{b})$ for all $m \in M(H^\infty + C)$.

Now let $h \in H^\infty + C$ and let b be as above. Then $fh = 0$ whenever $|b| < 1$. Thus, replacing f by fh above and using the work in the preceding paragraph, we see that $m(fh\bar{b}) = m(fh)m(\bar{b}) = m(f)m(h)m(\bar{b})$. Therefore, if $|m(b)| < 1$, we know that

$$m(fh\bar{b}) = m(f)m(h)m(\bar{b}) = 0 = m(f)m(h\bar{b}).$$

Of course, if $|m(b)| = 1$ then Lemma 8 implies that $m(fh\bar{b}) = m(f)m(h\bar{b})$.

Now let $m \in M(H^\infty + C)$. By the Chang–Marshall theorem, for any $\varepsilon > 0$ there exist $h_1, \ldots, h_n$ in H^∞ and $b_1, \ldots, b_n$ each of which is a finite product of interpolating Blaschke products invertible in $H^\infty[g]$ such that $\|g - \sum_{j=1}^n h_j \bar{b}_j\| < \varepsilon$. By our work above, for any $m \in M(H^\infty + C)$ we know that $m(f\sum_{j=1}^n h_j\bar{b}_j) = \sum m(f)m(h_j\bar{b}_j) = m(f)m(\sum_{j=1}^n h_j\bar{b}_j)$. Now taking limits we obtain $m(f)m(g) = m(fg)$ for all $m \in M(H^\infty + C)$. This completes the proof of the theorem.

The inner functions that do not vanish identically on any Gleason part have been characterized in [17]. In particular, any interpolating Blaschke product satisfies this condition. This leads to another counterexample to the Böttcher–Silbermann conjecture and shows even more: it is possible to have asymptotic multiplicity without any of the conditions (a)–(e) in Theorem 10 holding.

EXAMPLE 2. Let b be any interpolating Blaschke product with zeros $\{z_n\}$. Choose a sequence of positive integers $k_n \to \infty$ such that

$$\sum k_n(1 - |z_n|) < \infty.$$

Let

$$c(z) = \prod \frac{|z_n|}{z_n}\left(\frac{z_n - z}{1 - \bar{z}_n z}\right)^{k_n}$$

denote the corresponding Blaschke product. Now for each positive integer n there exists a finite Blaschke product u and a Blaschke product v such that $c = b^n\bar{u}v$. Since u is a finite Blaschke product, we have $\bar{u}v \in H^\infty + C$. Thus we can conclude that $|m(c)| \le |m(b^n)|$ for all positive integers n. Therefore

$$c = 0 \quad \text{whenever} \quad |b| < 1. \tag{3.2}$$

From Theorem 12 we know that $m(c\bar{b}) = m(c)m(\bar{b})$ for all $m \in M(H^\infty + C)$.

Now we show that (a)–(e) in Theorem 10 do not hold (with $f = c$ and $g = \bar{b}$). For any m such that $|m(b)| < 1$, Lemma 7 and (3.2) imply that neither $\bar{b}|\operatorname{supp} m$ nor $\bar{c}|\operatorname{supp} m$ is in $H^\infty|\operatorname{supp} m$. Note that $H^\infty[\bar{f}] \cap H^\infty[g] = H^\infty[\bar{c}] \cap H^\infty[\bar{b}]$.

Since b is a subproduct of c, we know that $\bar{b} \in H^\infty[\bar{c}] \cap H^\infty[\bar{b}]$. Therefore $[H^\infty[\bar{c}] \cap H^\infty[\bar{b}]] \cup [H^\infty[c] \cap H^\infty[b]]$ is not a subset of $H^\infty + C$. By Theorem 9, conditions (a)–(d) cannot hold. Clearly, (e) cannot hold either, but our work

in the first paragraph shows that $m(c\bar{b}) = m(c)m(\bar{b})$ for all $m \in M(H^\infty + C)$. This completes the second example.

The Blaschke product above is sometimes referred to as the D. J. Newman product.

4. The condition $\widehat{fg}(z) - \widehat{f}(z)\widehat{g}(z) \to 0$ as $z \to \partial D$.

We begin this section with a lemma to be used in the proof of Theorem 14 below (see also [19]). For a Blaschke product b the notation $Z(b)$ denotes the zeros of b in the maximal ideal space of H^∞.

LEMMA 13. *Let $f_1, \ldots, f_p$ be bounded harmonic functions on D, let $m \in M(H^\infty + C)$ be a nontrivial point and $\{z_n\}$ be an interpolating sequence with m in its closure. Then*

(1) *There is a subnet $\{z_{n_\alpha}\}$ such that $f_j \circ L_{z_{n_\alpha}}$ converges uniformly on compact subsets of D to $f_j \circ L_m$ for $j = 1, \ldots, p$.*

(2) *If D_k is an increasing sequence of compact subsets of D with $D = \bigcup D_k$, then there exist points z_{n_k} of the interpolating sequence such that $|z_{n_k}| \to 1$ and $\max_j \|f_j \circ L_{z_{n_k}} - f_j \circ L_m\| < 1/k$ on D_k.*

(3) *If b is any interpolating Blaschke product with zeros chosen from among the z_{n_k} in (2) above, then $f_j \circ L_x = f_j \circ L_m$ for all $x \in Z(b) \setminus D$.*

Proof. We indicate the proof for one function.

For (1) write $f = u + iv$ where u and v are real and harmonic. Let $g = e^{u + i\tilde{u}}$. Since u is a bounded harmonic function, g is an invertible outer function. Let $\{z_\alpha\}$ be a net in D converging to m. Then $g \circ L_{z_\alpha}$ is a bounded family of analytic functions on D. From a well-known result of Hoffman [22] the functions $g \circ L_{z_\alpha}$ converge pointwise to $g \circ L_m$. By Vitali's Theorem $g \circ L_{z_\alpha} \to g \circ L_m$ on compacta. Taking the logarithm of the absolute value of $g \circ L_{z_\alpha}$ we see that $u \circ L_{z_\alpha}$ must converge uniformly on compacta to some function. By Hoffman's results, we know that this function must be $u \circ L_m$. The result for f follows easily.

Statement (2) is a straightforward consequence of the above.

For statement (3), let $x \in Z(b) \setminus D$. Then, by [22], x is in the closure of the zeros of b in D. Let $\{z_{n_\alpha}\}$ be a net from among the points chosen in (2) converging to x. Then (2) implies $f_j \circ L_{z_n}(w) \to f_j \circ L_m(w)$ for all $w \in D$ while Hoffman's theorem tells us that $f_j \circ L_{z_{n_\alpha}} \to f_j \circ L_x$. The result follows from this.

Note that it follows from (1) above that $f \circ L_m$ is a bounded harmonic function on D as long as f is. Now we are ready for the first theorem in this section.

THEOREM 14. *Let $f, g \in L^\infty$. If $\widehat{fg}(z) - \widehat{f}(z)\widehat{g}(z) \to 0$ as $z \to \partial D$, then for each $m \in M(H^\infty + C)$ with nontrivial Gleason part $P(m)$ one of the following holds*:

(1) *$f \circ L_m$ and $g \circ L_m$ are in H^∞.*
(2) *$\bar{f} \circ L_m$ and $\bar{g} \circ L_m$ are in H^∞.*
(3) *There exist constants α, β, not both zero, such that both $(\alpha f + \beta g) \circ L_m$ and $\overline{\alpha f - \beta g} \circ L_m$ are in H^∞.*

Proof. By the Corona Theorem, there is a net $\{z_\alpha\}$ in D converging to m. By Hoffman's work [22] for each $w \in D$ we know that $\widehat{f} \circ L_{z_\alpha}(w)$ converges to $\widehat{f} \circ L_m(w)$. Therefore we have $\widehat{fg} \circ L_{z_\alpha}(z) \to \widehat{fg} \circ L_m(z)$ and $(\widehat{f} \circ L_{z_\alpha})(z)\,(\widehat{g} \circ L_{z_\alpha})(z) \to (\widehat{f} \circ L_m)(z)\,(\widehat{g} \circ L_m)(z)$. So

$$\widehat{fg} \circ L_m(z) = (\widehat{f} \circ L_m)(z)\,(\widehat{g} \circ L_m)(z).$$

By (the remark following) Lemma 13 the left-hand side of the above equation is harmonic in D. So the right-hand side must be harmonic in D. Let Δ be the Laplacian operator $4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}$. Then $\Delta((f \circ L_m)(g \circ L_m)) = 0$. Because $f \circ L_m$ and $g \circ L_m$ are also harmonic on D, an elementary computation implies

$$\frac{1}{4}\Delta((f \circ L_m)(g \circ L_m))(w) = \frac{\partial(f \circ L_m)}{\partial \bar{z}}(w)\frac{\partial(g \circ L_m)}{\partial z}(w) + \frac{\partial(g \circ L_m)}{\partial \bar{z}}(w)\frac{\partial(f \circ L_m)}{\partial z}(w).$$

Hence

$$\frac{\partial(f \circ L_m)}{\partial \bar{z}}(w)\frac{\partial(g \circ L_m)}{\partial z}(w) + \frac{\partial(g \circ L_m)}{\partial \bar{z}}(w)\frac{\partial(f \circ L_m)}{\partial z}(w) = 0.$$

To show that one of statements (1), (2), and (3) holds we argue as in [2].

Write $f \circ L_m = f_1 + \bar{f}_2$ and $g \circ L_m = g_1 + \bar{g}_2$, where $f_1, f_2, g_1, g_2 \in H^2(D)$. Then our computations above imply that

$$\overline{f_2'} g_1' + \overline{g_2'} f_1' = 0.$$

So $f_1' \overline{g_2'} = -\overline{f_2'} g_1'$ on D. If $g_1' = 0$ on D, then either $g_2' = 0$ on D (so g_1 and g_2 are both constant and hence (3) holds with $\alpha = 0$) or $f_1' = 0$ (so f_1 is a constant and (2) holds). The same argument works if $g_2' = 0$. So we will assume that neither g_1' nor g_2' is identically zero. Dividing, we obtain

$$f_1'/g_1' = -\overline{f_2'}/\overline{g_2'}$$

wherever the denominators are nonzero. Since the left-hand side is analytic and the right-hand side is coanalytic, both must be not only equal, but constant. Therefore, there exists a constant γ such that $f_1' - \gamma g_1' = 0$ and $f_2' + \bar{\gamma} g_2' = 0$. So $f_1 = \gamma g_1 + \alpha_1$ and $\bar{f}_2 = -\gamma \bar{g}_2 + \alpha_2$ for some constants α_1, α_2. If $\gamma = 0$, then (3) holds. If $\gamma \neq 0$, then $(f + \gamma g) \circ L_m = 2\gamma g_1 + \alpha_1 + \alpha_2$

and since for any $h \in L^\infty$ we have $\overline{h \circ L_m} = \bar{h} \circ L_m$, we also know that $\overline{f - \gamma g} \circ L_m = \overline{\alpha_1 + \alpha_2 - 2\gamma \bar{g}_2}$. Both are analytic because g_1 and g_2 are, and bounded because $f + \gamma g$ and $f - \gamma g$ are. This completes the proof.

Interestingly, the conditions above can be stated in an equivalent way that looks very much like the Böttcher–Silbermann conjecture. There is one important difference: Our algebras are subalgebras of $C(M(H^\infty))$ where multiplication is on the disc rather than the circle. The setup for the next theorem is the following: Let dA denote the usual normalized area measure on D and $L^\infty(D)$ denote the space of bounded measurable functions on D. For two bounded harmonic functions f and g on the disc let $H^\infty(D)[f,g]$ denote the subalgebra of $L^\infty(D)$ generated by $H^\infty(D), f$ and g. The subalgebra of $C(M(H^\infty))$ consisting of functions analytic on every Gleason part in $M(H^\infty + C)$ is denoted by AOP; that is,

$$AOP = \{u \in C(M(H^\infty)) : u \circ L_m \in H^\infty \text{ for all } m \in M(H^\infty + C)\}.$$

In [4], the authors showed that the conjugate of a thin interpolating Blaschke product b is in the algebra generated by $H^\infty(D)$ and the conjugate of a bounded analytic function f as long as f is nonconstant on any part in $M(H^\infty)$ on which b has a zero. Recently, Younis and Zheng [31] showed that the function f can be replaced by a family of bounded harmonic functions; that is, if $\mathcal{F}$ is a family of bounded harmonic functions such that whenever $x \in Z(b)$ there exists $f \in \mathcal{F}$ with $f \circ L_x \notin H^\infty$, then $\bar{b} \in H^\infty(D)[\mathcal{F}]$. We will use Younis and Zheng's result to give an algebraic formulation of Theorem 14.

THEOREM 15. *Let f and g be bounded harmonic functions on the unit disc. Then*

$$H^\infty(D)[f,g] \cap H^\infty(D)[\bar{f},\bar{g}] \cap \bigcap_{|\alpha|+|\beta|>0} H^\infty(D)[\alpha f + \beta g, \overline{\alpha f - \beta g}] \subseteq AOP$$

if and only if for each $m \in M(H^\infty) \setminus D$ with nontrivial part $P(m)$ one of the following holds:

(1) *$f \circ L_m$ and $g \circ L_m$ are in H^∞.*
(2) *$\bar{f} \circ L_m$ and $\bar{g} \circ L_m$ are in H^∞.*
(3) *There exist constants α, β, not both zero, such that both $(\alpha f + \beta g) \circ L_m$ and $\overline{\alpha f - \beta g} \circ L_m$ are in H^∞.*

Proof. First suppose that one of the conditions numbered (1)–(3) holds. Then for any nontrivial point $m \in M(H^\infty + C)$, any function in the algebra

$$H^\infty(D)[f,g] \cap H^\infty(D)[\bar{f},\bar{g}] \cap \bigcap_{|\alpha|+|\beta|>0} H^\infty(D)[\alpha f + \beta g, \overline{\alpha f - \beta g}]$$

is a uniform limit of functions analytic on the Gleason part of m and so the proof in this direction is complete.

Suppose that m is a nontrivial point in $M(H^\infty + C)$ for which (1)–(3) above fail. Choose a sequence of subsets D_n of D and a sequence $\{z_n\}$ as in Lemma 13(2). Passing to a subsequence, we may assume that the sequence is thin and satisfies

$$\max\{\|f\circ L_{z_n} - f\circ L_m\|_{D_n}, \|g\circ L_{z_n} - g\circ L_m\|_{D_n}\} < 1/n.$$

Thus, for any complex numbers α and β,

$$\max\{\|(\alpha f+\beta g)\circ L_{z_n} - (\alpha f+\beta g)\circ L_m\|_{D_n},\ \|\overline{\alpha f-\beta g}\circ L_{z_n} - \overline{\alpha f-\beta g}\circ L_m\|_{D_n}\} < \frac{|\alpha|+|\beta|}{n}.$$

Let b be the thin product with zeros $\{z_n\}$. If $x\in Z(b)\setminus D$, then $f\circ L_x = f\circ L_m$ and $g\circ L_x = g\circ L_m$. Consequently, $(\alpha f+\beta g)\circ L_x = (\alpha f+\beta g)\circ L_m$ and $(\alpha f-\beta g)\circ L_x = (\alpha f-\beta g)\circ L_m$. Now we are assuming that m is a point for which (1)–(3) fail, so if $x\in Z(b)\setminus D$ then (1)–(3) must fail for x as well. In particular, the zeros of b are contained in the set

$$\{y\in M(H^\infty): f\circ L_y\notin H^\infty \text{ or } g\circ L_y\notin H^\infty\}$$
$$\cap\,\{y\in M(H^\infty): \bar f\circ L_y\notin H^\infty \text{ or } \bar g\circ L_y\notin H^\infty\}$$
$$\cap \bigcap_{|\alpha|+|\beta|>0}\{y\in M(H^\infty): (\alpha f+\beta g)\circ L_y\notin H^\infty \text{ or } \overline{\alpha f-\beta g}\circ L_y\notin H^\infty\}.$$

By Theorem 3 of [31],

$$\bar b\in H^\infty(D)[f,g]\cap H^\infty(D)[\bar f,\bar g]\cap \bigcap_{|\alpha|+|\beta|>0} H^\infty(D)[\alpha f+\beta g, \overline{\alpha f-\beta g}].$$

Since $(\bar b\circ L_x)'(0)=1$ for any $x\in Z(b)\setminus D$, we see that b cannot be constant on any part in which it has a zero and so the algebra above cannot be contained in AOP. This completes the proof.

We will now give a class of examples generalizing Example 2. The interesting thing is that we can give necessary and sufficient conditions on functions not in this class for the asymptotic multiplicity condition to be satisfied. To do all this, we need a slightly different version of Lemma 13.

LEMMA 16. *Let c be a Blaschke product and $\lambda\in D$. Suppose that $c=\lambda$ identically on a Gleason part $P(m)$. Then there exists an interpolating sequence $\{z_n\}$ such that c is identically equal to λ on any part in $M(H^\infty+C)$ containing a point in the closure of the sequence.*

Proof. If m is trivial, it follows from [17] that $c=\lambda$ on a nontrivial part as well. Thus we may assume that m is nontrivial. The rest of the lemma follows directly from Lemma 13.

EXAMPLE 3. Let c be a Blaschke product which is a constant of modulus less than one on some Gleason part $P(m)$. Then there exists an infinite Blaschke product b such that $m(b\bar c)=m(b)m(\bar c)$ for all $m\in M(H^\infty+C)$ but $H^\infty[\bar b]\subsetneq (H^\infty[\bar b]\cap H^\infty[\bar c])\cup(H^\infty[b]\cap H^\infty[c])$.

Note that since $H^\infty+C$ does not contain the conjugate of any infinite interpolating Blaschke product, taking $f=b$ and $g=\bar c$ in Theorem 9 we see that if we produce such a Blaschke product b, then conditions (a)–(d) cannot hold; clearly, condition (e) of Theorem 10 does not hold, since $f\in H^\infty$ and $\bar g\in H^\infty$ but neither is constant.

Let m denote a nontrivial point for which $c\circ L_m=\lambda$. (Again, the existence of such a point is guaranteed by [17].) We let b denote the interpolating Blaschke product corresponding to the zeros $\{z_n\}$ given by Lemma 16. Note that since $c\circ L_m$ is constant on D, we know c is the constant λ on any part (other than D) which contains a point in the closure of the zero sequence of b. Thus we see that $c|(Z(b)\setminus D)=\lambda$ and $c-\lambda$ vanishes identically on any part where b has a zero. It follows from [3] or [20] that $c-\lambda$ vanishes identically on any part where $|b|<1$. Thus, on the one hand we know from Theorem 12 that for all $m\in M(H^\infty+C)$ we have $m(b\bar c)=m(b)m(\bar c)$. But from the construction of b we see that if b is not invertible in $H^\infty[\bar c]$, then there exists $y\in M(H^\infty[\bar c])$ with $y(b)=0$. By our choice of b, this would imply that $|y(c)|=|\lambda|<1$, which is a contradiction. Therefore, b is invertible in $H^\infty[\bar c]$ and hence $H^\infty[\bar b]\subseteq H^\infty[\bar c]$. Since no infinite Blaschke products are invertible in $H^\infty+C$ we see that $H^\infty[\bar c]\cap H^\infty[\bar b]$ is not contained in $H^\infty+C$, so (a)–(d) of Theorem 10 cannot hold, and (e) clearly does not hold either. This completes the example.

The functions in Example 3 are those Blaschke products that are a constant of modulus less than one on some nontrivial Gleason part. If c has modulus less than one on some trivial part, then (see [17]) c is constant on a nontrivial part. For those functions that do not have this property, one can actually give necessary and sufficient conditions for $m(b\bar c)=m(b)m(\bar c)$ to hold for all $m\in M(H^\infty+C)$.

In order to show this, we need to make a few remarks. It is well known and follows easily from Hoffman's results that thin points are dense in $M(H^\infty+C)$. (One needs to use the fact that the nontrivial points outside the disc are dense in $M(H^\infty+C)$ and the fact that every sequence contains a thin subsequence.) As mentioned earlier, it follows from a result of Hoffman that any function $f\in L^\infty$ has a continuous extension to $M(H^\infty)$ so we actually know that $f(m_\alpha)\to f(m)$, if m_α is a net converging to m. Thus, if we know that $x(b)x(\bar c)=x(b\bar c)$ for all thin x, and we have a nontrivial point m we can obtain the result for m through a limit argument. We will use a well-known fact that if m is a thin point, then the map L_m has a homeomorphic extension to $M(H^\infty)$, and the fact that L_m maps trivial points to trivial points (for both of these facts the reader is referred to [8]). With this information, we are now able to obtain conditions on a class of functions that will ensure that the asymptotic multiplicative conditions hold.

THEOREM 17. *Let c be an inner function such that c has modulus one on all trivial parts. If $f \in L^\infty$ is constant on any part where c is nonconstant, then $m(f\bar{c}) = m(f)m(\bar{c})$ for all $m \in M(H^\infty + C)$. If $f \in H^\infty + C$ then $m(f\bar{c}) = m(f)m(\bar{c})$ for all $m \in M(H^\infty + C)$ if and only if f is constant on any part where c is not.*

Proof. Suppose that c is an inner function with modulus one on all trivial parts and let $m \in M(H^\infty + C)$. Let f be any function in L^∞ which is constant wherever c is not. By our remarks preceding the theorem, we may assume that m is a thin point. By [8], $L_m(M(L^\infty))$ is a subset of the trivial points. Let $x_t \in L_m(M(L^\infty))$. Then we know that x_t is trivial. Let $y \in M(L^\infty)$ be such that $x_t = L_m(y)$. Then

$$x_t(f\bar{c}) = f\bar{c}(L_m(y)).$$

Since c has modulus one on all the trivial points and $x_t = L_m(y)$ is a trivial point, we know from Lemma 8 that

$$x_t(f\bar{c}) = f\bar{c}(L_m(y)) = f(L_m(y))\bar{c}(L_m(y)) = x_t(f)x_t(\bar{c}). \tag{4.1}$$

By our assumption, either c or f is constant on $P(m)$. This means that the functions defined on D by $h = (f\bar{c}) \circ L_m$ and $g \in C(M)$ given by $g(z) = (f \circ L_m)(z)(\bar{c} \circ L_m)(z)$ are both harmonic functions on D and for any $x \in M(L^\infty)$ satisfy

$$\begin{aligned} h(x) = x((f\bar{c}) \circ L_m) &= \lim_{z_\alpha \to x} ((f\bar{c}) \circ L_m)(z_\alpha) \\ &= \lim_{z_\alpha \to x} (f\bar{c})(L_m(z_\alpha)) = (f\bar{c})(L_m(x)), \end{aligned}$$

where we used the fact that the extension of L_m is a homeomorphism on the whole maximal ideal space and $f\bar{c} \in C(M)$ for this last equality. Now x is trivial and therefore $L_m(x)$ is trivial, so by (4.1), it follows that $(f\bar{c})(L_m(x)) = f(L_m(x))\bar{c}(L_m(x))$ and hence

$$\begin{aligned} h(x) &= (f(L_m(x)))(\bar{c}(L_m(x))) \\ &= \lim_{z_\alpha \to x} (f(L_m(z_\alpha)))(\bar{c}(L_m(z_\alpha))) = \lim_{z_\alpha \to x} g(z_\alpha) = g(x). \end{aligned}$$

Since h and g are bounded harmonic functions which agree everywhere on $M(L^\infty)$, we have $h = g$ on D. Now $L_m(0) = m$ so $m(f\bar{c}) = m(f)m(\bar{c})$. Thus the asymptotic multiplicity condition holds for all thin points m and from our comments preceding the proof of the theorem, that means it holds for all $m \in M(H^\infty + C)$. This completes the proof of the first statement.

Suppose that f is constant wherever c is not. Then the asymptotic multiplicativity in the second statement follows from the first. Now suppose that $m(f\bar{c}) = m(f)m(\bar{c})$ for all $m \in M(H^\infty + C)$. Let $P(m)$ be a part where c is not constant. Then (1) of Theorem 14 does not hold. Since $f \in H^\infty + C$, (3) cannot hold unless $\beta = 0$. In this case, $\alpha \neq 0$ so $\bar{f} \circ L_m \in H^\infty$. Similarly, if (2) holds, $\bar{f} \circ L_m \in H^\infty$. Since $f \circ L_m$ and $\bar{f} \circ L_m$ are both analytic, $f \circ L_m$ is constant.

Putting this together with Theorem 15 we get

THEOREM 18. *Let c be an inner function such that c has modulus one on all trivial parts and let $f \in H^\infty + C$. Then $m(f\bar{c}) = m(f)m(\bar{c})$ for all $m \in M(H^\infty + C)$ if and only if*

$$H^\infty(D)[f, \bar{c}] \cap H^\infty(D)[\bar{f}, c] \cap \bigcap_{|\alpha|+|\beta|>0} H^\infty(D)[\alpha f + \beta \bar{c}, \overline{\alpha f - \beta \bar{c}}] \subseteq AOP.$$

Note that the proof of Theorem 17 is local in the sense that we only need either f or c constant on the part $P(m)$. Thus one half of the following theorem is an easy consequence of the proof above.

THEOREM 19. *Let c be a finite product of interpolating Blaschke products. Then c has modulus less than one on some trivial part if and only if there exists a nontrivial part $P(m)$ and a function $f \in L^\infty$ such that c is constant on $P(m)$ but $x(fc) \neq x(f)x(c)$ for some $x \in P(m)$.*

Proof. If c has modulus one on all trivial points then, since every part contains a trivial point in its closure, c must have modulus one on any part on which it is constant. Therefore, $x(f\bar{c}) = x(f)x(\bar{c})$ for all $x \in P(m)$ and all $f \in L^\infty$.

If, on the other hand, c has modulus less than one on a trivial part, we know from [17] that c must be a constant λ of modulus less than one on a nontrivial part. By Lemma 16 there is an interpolating sequence such that c is identically λ on any part outside D containing a cluster point of the sequence. Passing to a subsequence, we may assume that the sequence is thin. We conclude that c is identically λ on a thin part $P(m)$. By [16], we know that we can factor $c = c_1 c_2$ so that neither c_1 nor c_2 is constant on $P(m)$. Now let $f = c_1$. Then for $x \in P(m)$ we have $x(c\bar{f}) = x(c_2)$ while $x(c)x(\bar{f}) = x(c_1)x(c_2)\overline{x(c_1)} = |x(c_1)|^2 x(c_2)$. Since c is not identically zero on the part (as it is interpolating) these two can only be equal on the whole part if $|x(c_1)| = 1$. This in turn implies that c_1 is a constant of modulus one on the support of x, which would mean that c_1 is constant on the Gleason part of x, i.e. $P(m)$, a contradiction. So there must exist a point $x \in P(m)$ with $x(c\bar{f}) \neq x(c)x(\bar{f})$ and this completes the proof.

5. Concluding remarks. We remark that many of the proofs given in earlier sections have operator-theoretic formulations. These appear in the work of [5], [18] and [31], and will only be mentioned here. The reader is referred to those papers for proofs.

Let P be the projection of L^2 onto H^2. For $f \in L^\infty$, the *Toeplitz operator* T_f on H^2 is defined by $T_f h = P(fh)$ and the *Hankel operator* from H^2 to $L^2 \ominus H^2$ is defined by $H_f h = (1-P)(fh)$. For $z \in D$, let k_z denote the normalized reproducing kernel $(1-|z|^2)^{1/2}/(1-\bar{z}w)$ of the Hardy space, and let ϕ_z denote the Möbius map on the unit disc, i.e. $\phi_z(w) = (z-w)/(1-\bar{z}w)$.

Although the Böttcher–Silbermann condition does not hold, we have shown that the following is true. The proof is almost exactly the same as that in [18] and therefore will not be repeated here.

THEOREM 20. *Let $f, g \in L^\infty$. Then the following are equivalent:*

(1) *For each $m \in M(H^\infty + C)$ one of the following holds:*

(1a) *$f|\operatorname{supp} m$ and $g|\operatorname{supp} m$ are in $H^\infty|\operatorname{supp} m$.*

(1b) *$\bar{f}|\operatorname{supp} m$ and $\bar{g}|\operatorname{supp} m$ are in $H^\infty|\operatorname{supp} m$.*

(1c) *There exist constants α, β, not both zero, such that both $(\alpha f + \beta)g|\operatorname{supp} m$ and $(\overline{\alpha f - g})|\operatorname{supp} m$ are in $H^\infty|\operatorname{supp} m$.*

(2) *We have*

$$H^\infty[f,g] \cap H^\infty[\bar{f},\bar{g}] \cap \bigcap_{|\alpha|+|\beta|>0} H^\infty[\alpha f + \beta g, \overline{\alpha f - \beta g}] \subseteq H^\infty + C.$$

(3) *$H^*_{\bar{f}} H_g + H^*_{\bar{g}} H_f$ is compact on the Hardy space.*

When we look at algebras generated as subalgebras of $L^\infty(D)$ we obtain operator-theoretic conditions equivalent to those in Theorem 15.

Let dA denote the usual normalized area measure on D. The *Bergman space* L^2_{a} is the Hilbert space of analytic functions $g : D \to C$ with inner product given by

$$\langle f, g\rangle = \int_D f(z)\bar{g}(z)\, dA(z).$$

Let P denote the orthogonal projection of $L^2(D, dA)$ onto L^2_{a}. For $f \in L^\infty(D)$, we still use H_f to denote the Hankel operator $H_f : L^2_{\mathrm{a}} \to L^2$ which is defined by $H_f(h) = (1-P)(fh)$.

THEOREM 21. *Let f and g be bounded harmonic functions on the unit disc. The following conditions are equivalent:*

(a) *For each nontrivial part $P(m)$ in $M(H^\infty + C)$, one of the following conditions holds:*

(a1) *$f \circ L_m$ and $g \circ L_m$ are in H^∞.*

(a2) *$\bar{f} \circ L_m$ and $\bar{g} \circ L_m$ are in H^∞.*

(a3) *There exist constants α, β, not both zero, such that both $(\alpha f + \beta g) \circ L_m$ and $\overline{\alpha f - \beta g} \circ L_m$ are in H^∞*

(b) *$H^*_{\bar{f}} H_g + H^*_{\bar{g}} H_f$ is compact on the Bergman space.*

(c)

$$H^\infty(D)[f,g] \cap H^\infty(D)[\bar{f},\bar{g}] \cap \bigcap_{|\alpha|+|\beta|>0} H^\infty(D)[\alpha f + \beta g, \overline{\alpha f - \beta g}] \subseteq AOP.$$

Proof. The equivalence of (a) and (b) is shown in [5]. The rest follows from Theorem 15.

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Existence and uniqueness results for solutions of nonlinear equations with right hand side in L^1

by

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Abstract. We prove an existence and uniqueness theorem for the elliptic Dirichlet problem for the equation $\operatorname{div} a(x, \nabla u) = f$ in a planar domain Ω. Here $f \in L^1(\Omega)$ and the solution belongs to the so-called *grand Sobolev space* $W_0^{1,2)}(\Omega)$. This is the proper space when the right hand side is assumed to be only L^1-integrable. In particular, we obtain the exponential integrability of the solution, which in the linear case was previously proved by Brezis–Merle and Chanillo–Li.

1. Introduction. We consider the Dirichlet problem on a bounded open set $\Omega \subset \mathbb{R}^2$ with C^1 boundary,

$$(1.1)\qquad \begin{cases} Au = f & \text{in } \Omega \subset \mathbb{R}^2, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f \in L^1(\Omega)$ and A is a differential operator defined by

$$(1.2)\qquad Au = \operatorname{div} a(x, \nabla u).$$

Here $a : \Omega \times \mathbb{R}^2 \to \mathbb{R}^2$ is a mapping such that

$$(1.3)\qquad \begin{cases} x \to a(x, \xi) & \text{is measurable for all } \xi \in \mathbb{R}^2, \\ \xi \to a(x, \xi) & \text{is continuous for almost every } x \in \Omega. \end{cases}$$

Furthermore, we assume that there exists $m \geq 1$ such that for almost every $x \in \Omega$ we have

$$(1.4)\qquad \begin{aligned} &\text{(i) } |a(x,\xi) - a(x,\eta)| \leq m|\xi - \eta| && \text{(Lipschitz continuity)}, \\ &\text{(ii) } \frac{1}{m}|\xi - \eta|^2 \leq \langle a(x,\xi) - a(x,\eta), \xi - \eta\rangle && \text{(strong monotonicity)}, \\ &\text{(iii) } a(x, 0) = 0, \end{aligned}$$

where ξ, η are arbitrary vectors in $\mathbb{R}^2$ ([LL]).

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