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An isomorphic Dvoretzky's theorem for convex bodies

by

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Abstract. We prove that there exist constants $C > 0$ and $0 < \lambda < 1$ so that for all convex bodies K in $\mathbb{R}^n$ with non-empty interior and all integers k so that $1 \le k \le \lambda n/\ln(n+1)$, there exists a k-dimensional affine subspace Y of $\mathbb{R}^n$ satisfying

$$d(Y \cap K, B_2^k) \le C\left(1 + \sqrt{\frac{k}{\ln\left(\frac{n}{k\ln(n+1)}\right)}}\right).$$

This formulation of Dvoretzky's theorem for large dimensional sections is a generalization with a new proof of the result due to Milman and Schechtman for centrally symmetric convex bodies. A sharper estimate holds for the n-dimensional simplex.

1. Section of a convex body. By a *convex body*, we always mean a closed convex set with non-empty interior in the Euclidean space. Let K be an arbitrary convex body in $\mathbb{R}^n$ with the origin in its interior. The gauge functional of K is defined by $\mathrm{p}_K(x) = \inf\{t \ge 0 : x \in tK\}$ for all $x \in \mathbb{R}^n$. We define the distance between two convex bodies A and B included in $\mathbb{R}^n$ by

$$d(A,B) = \inf_{u\in\mathbb{R}^n,\, T\in Gl_n(\mathbb{R})} \{\lambda > 0 : B + u \subset T(A) \subset \lambda(B+u)\}.$$

This is the analogue to the Banach–Mazur distance between two Banach spaces.

Denote by $(e_i)_{1\le i\le k}$ the canonical basis of $\mathbb{R}^k$, ℓ_2^k the space $\mathbb{R}^k$ equipped with the Euclidean norm $|\cdot|_2$, and B_2^k the unit ball of this space.

By $(g_j)_{1\le j\le n}$ and $(g_{ij})_{1\le i\le k,\, 1\le j\le n}$ we always denote some independent, centered, normalized gaussian random variables. If $(t_p)_{1\le p\le N} \in \mathbb{R}^N$, we denote by $((t_p)_{p=1}^N)_q^*$ the qth coordinate of the decreasing rearrangement of

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$(t_p)_{1\le p\le N}$. When no confusion is possible, we shall denote it by t_q^*. The letters c, C will denote universal positive constants which can be different in each case.

We shall prove the following theorem, which extends to general convex bodies a result proved by Milman and Schechtman [MS1], [MS2] and which was reproved in a different way in [Gu].

THEOREM 1. *There exist $C>0$ and $0<\lambda<1$ such that for all convex bodies K in $\mathbb{R}^n$ and for all integers k so that $1\le k\le \lambda n/\ln n$, there exists a k-dimensional affine subspace Y of $\mathbb{R}^n$ satisfying*

$$d(Y\cap K, B_2^k)\le C\Bigl(1+\sqrt{\frac{k}{\ln(1+\frac{n}{k\ln n})}}\Bigr).$$

Proof. The proof consists of two steps. First, we use a result of Rudelson to obtain a new convex body which is near our original convex body but which has less contact points with its John ellipsoid and such that the John decomposition of the identity has all the coefficients c_p of the same order of magnitude. Secondly, we reduce the study to the case of a polytope with few contact points. In this part, we use the sharp generalization of the Slepian–Fernique inequality proved in [Go2] for the decreasing rearrangement of gaussian processes.

Let us recall the result due to Rudelson [R1] which allows us to find a new convex body with a "good" John decomposition of the identity operator and with few contact points. He improves this result in [R2] and obtains the following statement:

Let K be a convex body in $\mathbb{R}^n$ and let $0<\varepsilon<1$. Then there exists a convex body $K_1\subset\mathbb{R}^n$ such that $d(K_1,K)\le 1+\varepsilon$, B_2^n is the John ellipsoid of K_1 and if we denote by $y_1,\dots,y_m$ all the contact points of K_1 with B_2^n, then

$$\mathrm{Id}=\sum_{p=1}^m c_p y_p\otimes y_p\quad \textit{for some } c_1,\dots,c_m>0, \tag{1}$$

$$\sum_{p=1}^m c_p y_p = 0, \tag{2}$$

$$\frac{n}{m}(1-\varepsilon)\le c_p\le \frac{n}{m}(1+\varepsilon)\quad \textit{for all } 1\le p\le m, \tag{3}$$

$$m\le c(\varepsilon) n\ln n. \tag{4}$$

For the proof of Theorem 1, we may take $\varepsilon=1/2$ and suppose from now on that $K=K_1$. Now, note that equation (1) as stated implies the Dvoretzky–Rogers lemma [DR]:

There exists an orthonormal basis $u_1,\dots,u_n$ of $\mathbb{R}^n$ and contact points $y_{p_1},\dots,y_{p_n}\in\{y_1,\dots,y_m\}$ such that for all $1\le i\le n$,

$$\mathrm{span}\{y_{p_1},\dots,y_{p_i}\}=\mathrm{span}\{u_1,\dots,u_i\},\qquad |\langle y_{p_i},u_i\rangle|\ge\sqrt{1-\frac{i-1}{n}}.$$

To see this, take $y_{p_1}=y_1=u_1$ and proceeding by induction over $1\le k\le n$, suppose that the preceding relations hold for $1\le i\le k$. Let $A=\{p_1,\dots,p_k\}$. Then

$$\begin{aligned}k=\sum_{i=1}^k|u_i|_2^2&=\sum_{i=1}^k\sum_{p=1}^m c_p\langle y_p,u_i\rangle^2\\ &=\sum_{p\in A}\sum_{i=1}^k c_p\langle y_p,u_i\rangle^2+\sum_{p\notin A}\sum_{i=1}^k c_p\langle y_p,u_i\rangle^2\\ &\ge\sum_{p\in A}c_p+\Bigl(\sum_{p\notin A}c_p\Bigr)\inf_{p\notin A}\Bigl(\sum_{i=1}^k\langle y_p,u_i\rangle^2\Bigr).\end{aligned}$$

Since $\sum_{p=1}^m c_p=n$ we get

$$\inf_{p\notin A}\Bigl(\sum_{i=1}^k\langle y_p,u_i\rangle^2\Bigr)\le\frac{k-\sum_{p\in A}c_p}{n-\sum_{p\in A}c_p}\le\frac{k}{n}.$$

It follows that there exists $p_{k+1}\notin A$ such that $\sum_{i=1}^k\langle y_{p_{k+1}},u_i\rangle^2\le k/n$. Define a vector u_{k+1} orthogonal to $\{u_1,\dots,u_k\}$ such that $|u_{k+1}|_2=1$ and $\mathrm{span}\{y_{p_1},\dots,y_{p_{k+1}}\}=\mathrm{span}\{u_1,\dots,u_{k+1}\}$. Then

$$\langle y_{p_{k+1}},u_{k+1}\rangle^2=1-\sum_{i=1}^k\langle y_{p_{k+1}},u_i\rangle^2\ge 1-\frac{k}{n}.$$

Of course, one can suppose that $y_{p_i}=y_i$ for $1\le i\le n$ and observe that $|\langle y_i,y_j\rangle|\le\sqrt{(j-1)/n}$ for $i<j\le n$.

Let us recall the following inequalities [Go1, Th. 2.5]:

Let K be a convex body with the origin in its interior, $(v_j)_{1\le j\le N}$ be N points of $\mathbb{R}^n$ and G_ω be the gaussian operator defined by

$$G_\omega=\sum_{i=1}^k\sum_{j=1}^N g_{ij}(\omega)e_i\otimes v_j:\ell_2^k\to(\mathbb{R}^n,\mathrm{p}_K).$$

We have the following inequalities:

$$\mathbb{E}\Bigl(\mathrm{p}_K\Bigl(\sum_{j=1}^N g_jv_j\Bigr)\Bigr)-a_k\varepsilon_2(\{v_j\}_{1\le j\le N})\le\mathbb{E}\inf_{|x|_2=1}\mathrm{p}_K(G_\omega(x)) \tag{5}$$

$$\le\mathbb{E}\sup_{|x|_2=1}\mathrm{p}_K(G_\omega(x))\le\mathbb{E}\Bigl(\mathrm{p}_K\Bigl(\sum_{j=1}^N g_jv_j\Bigr)\Bigr)+a_k\varepsilon_2(\{v_j\}_{1\le j\le N}) \tag{6}$$

where $\{g_{ij}\}$ *and* $\{g_i\}$ *denote sequences of orthonormal standard gaussian variables, and*

$$a_k = \sqrt{2}\,\frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})} \le \sqrt{k}, \qquad \varepsilon_2(\{v_j\}_{1\le j\le N}) = \sup_{|t|_2=1} \mathrm{p}_K\Big(\sum_{j=1}^{N}\langle t, e_j\rangle v_j\Big).$$

In our case, set again

$$G_\omega = \sum_{i=1}^{k}\sum_{j=1}^{n} g_{ij}(\omega) e_i \otimes e_j : \ell_2^k \to (\mathbb{R}^n, \mathrm{p}_K).$$

Since $B_2^n \subset K$ and $(e_1,\dots,e_n)$ is an orthonormal basis of $\mathbb{R}^n$, it is easy to see that $\varepsilon_2(\{e_j\}_{1\le j\le n}) = 1$. We shall consider two different cases:

1. *Dvoretzky's theorem case*: $\mathbb{E}(\mathrm{p}_K(\sum_{j=1}^n g_j e_j)) \ge 2a_k$. In this case, we deduce from (5) and (6) that

$$\frac{\mathbb{E}\sup_{|x|_2=1}\mathrm{p}_K(G_\omega(x))}{\mathbb{E}\inf_{|x|_2=1}\mathrm{p}_K(G_\omega(x))} \le \Big\{1 + \frac{a_k\varepsilon_2(\{e_j\}_{1\le j\le n})}{\mathbb{E}(\mathrm{p}_K(\sum_{j=1}^n g_j e_j))}\Big\}\Big/\Big\{1 - \frac{a_k\varepsilon_2(\{e_j\}_{1\le j\le n})}{\mathbb{E}(\mathrm{p}_K(\sum_{j=1}^n g_j e_j))}\Big\} \le 3.$$

So, there exists ω_0 such that $\dim(\operatorname{Im} G_{\omega_0}) = k$ and

$$\frac{\sup_{|x|_2=1}\mathrm{p}_K(G_{\omega_0}(x))}{\inf_{|x|_2=1}\mathrm{p}_K(G_{\omega_0}(x))} \le 3.$$

Let $Y = \operatorname{Im} G_{\omega_0}$. Then $\dim Y = k$ and $d(Y\cap K, B_2^k) \le 3$.

2. *Second case*: $\mathbb{E}(\mathrm{p}_K(\sum_{j=1}^n g_j e_j)) \le 2a_k$. By (6), we see that

$$\mathbb{E}\sup_{|x|_2=1}\mathrm{p}_K(G_\omega(x)) \le 3a_k.$$

To find a lower bound of $\mathbb{E}\inf_{|x|_2=1}\mathrm{p}_K(G_\omega(x))$, we follow the same idea as in [Gué, Lemma 4]. We apply (5) to new gauges p_l on $\mathbb{R}^n$, defined for $l \le m-1$ and for $t\in\mathbb{R}^n$ by

$$\mathrm{p}_l(t) = \frac{1}{l}\sum_{q=1}^{l}((c_p\langle t, y_p\rangle)_{p=1}^m)_q^*.$$

The p_l are gauges because by (2), $\sum_{p=1}^m c_p y_p = 0$. Moreover, K is contained in the polytope $\{x\in\mathbb{R}^n : \max_{1\le p\le m}\langle x, y_p\rangle \le 1\}$, and by (3), for all $1\le p\le m$, $c_p \le 3n/(2m)$; it follows that for all $t\in\mathbb{R}^n$, we have

$$0 \le \mathrm{p}_l(t) \le \max_{1\le p\le m} c_p\langle t, y_p\rangle \le \frac{3n}{2m}\max_{1\le p\le m}\langle t, y_p\rangle \le \frac{3n}{2m}\mathrm{p}_K(t). \tag{7}$$

We now apply (5) to p_l:

$$\mathbb{E}\inf_{|x|_2=1}\mathrm{p}_l(G_\omega(x)) \ge \mathbb{E}\Big(\mathrm{p}_l\Big(\sum_{j=1}^n g_j e_j\Big)\Big) - a_k\max_{|t|_2=1}\mathrm{p}_l(t). \tag{8}$$

First we note that

(a) $$\max_{|t|_2=1}\mathrm{p}_l(t) \le \sqrt{\frac{3n}{2lm}}.$$

Indeed, fix a $t\in\ell_2^n$. Then for all $1\le q\le l$, there exists $p_q\in\{1,\dots,m\}$ such that

$$((c_p\langle t, y_p\rangle)_{p=1}^m)_q^* = c_{p_q}\langle t, y_{p_q}\rangle.$$

By (1) and (3), we deduce that

$$\mathrm{p}_l(t) = \frac{1}{l}\Big(\sum_{q=1}^{l} c_{p_q}\langle t, y_{p_q}\rangle\Big) \le \frac{1}{l}\Big(\sum_{q=1}^{l} c_{p_q}(\langle t, y_{p_q}\rangle)^2\Big)^{1/2}\Big(\sum_{q=1}^{l} c_{p_q}\Big)^{1/2} \le \sqrt{\frac{3n}{2lm}}\,|t|_2.$$

Next we prove that

(b) $$\mathbb{E}\Big(\mathrm{p}_l\Big(\sum_{j=1}^n g_j e_j\Big)\Big) \ge C\frac{n}{m}\sqrt{\ln\Big(1+\frac{n}{l}\Big)} \quad \text{for } l\le[n/2],$$

Indeed, we have

$$\mathrm{p}_l\Big(\sum_{j=1}^n g_j e_j\Big) = \frac{1}{l}\sum_{q=1}^{l}\Big(\Big(\sum_{j=1}^n g_j c_p\langle e_j, y_p\rangle\Big)_{p=1}^m\Big)_q^* \ge \frac{1}{l}\sum_{q=1}^{l}\Big(\Big(\sum_{j=1}^n g_j c_p\langle e_j, y_p\rangle\Big)_{p=1}^{[n/2]}\Big)_q^* \tag{9}$$

because this is the decreasing rearrangement of the same sequence with fewer terms. Let now $X_p(\omega)$ be the random variables defined for $1\le p\le[n/2]$ by

$$X_p(\omega) = c_p\sum_{j=1}^n g_j(\omega)\langle e_j, y_p\rangle.$$

By the choice of the $y_1,\dots,y_{[n/2]}$ in the Dvoretzky–Rogers lemma, and by (3), we deduce that for all $p\ne q$,

$$\mathbb{E}|X_p - X_q|^2 = |c_p y_p - c_q y_q|^2 \ge c_p^2 + c_q^2 - \sqrt{2}\,c_p c_q \ge \frac{n^2}{8m^2}.$$

Let $h_1, \ldots, h_{[n/2]}$ be $[n/2]$ centered, independent gaussian variables with variance $\sigma = n/(4m)$. We then have for all $1 \le p, q \le [n/2]$,

$$\mathbb{E}|X_p - X_q|^2 \ge \mathbb{E}|h_p - h_q|^2 = \frac{n^2}{8m^2}.$$

By a generalized Slepian–Fernique inequality [Go2, Theorem 1.3], for all $l = 1, \ldots, [n/2]$ we get

$$\mathbb{E}\frac{1}{l}\sum_{p=1}^{l} X_p^* \ge \mathbb{E}\frac{1}{l}\sum_{p=1}^{l} h_p^*.$$

Applying classical estimates to the gaussian vector $(h_1, \ldots, h_{[n/2]})$ with law $\mathcal{N}(0, \sigma\,\mathrm{Id})$, we have

$$\mathbb{E}\frac{1}{l}\sum_{p=1}^{l} h_p^* \ge C\sigma\sqrt{\ln\left(1+\frac{n}{l}\right)} \quad \text{for } l = 1, \ldots, [n/2].$$

It follows from (9) that

$$\mathbb{E}\Big(\mathrm{p}_l\Big(\sum_{j=1}^{n} g_j e_j\Big)\Big) \ge C\frac{n}{m}\sqrt{\ln\left(1+\frac{n}{l}\right)}.$$

This concludes the proof of (b).

By (8), (a) and (b), we obtain

$$\mathbb{E}\inf_{|x|_2=1} \mathrm{p}_l(G_\omega(x)) \ge C\frac{n}{m}\sqrt{\ln\left(1+\frac{n}{l}\right)} - a_k\sqrt{\frac{3n}{2lm}}.$$

One can now choose a universal constant $0 < \lambda < 1/2$ so that if $l = [km/n] \le \lambda n$ then

$$\mathbb{E}\inf_{|x|_2=1} \mathrm{p}_l(G_\omega(x)) \ge \frac{C}{2}\frac{n}{m}\sqrt{\ln\left(1+\frac{n}{l}\right)}.$$

Observe that this imposes $k \le \lambda n^2/m$, that is, $k \le \lambda cn/\ln n$, since $m \le cn\ln n$. Then we deduce from (7) that

$$\mathbb{E}\inf_{|x|_2=1} \mathrm{p}_K(G_\omega(x)) \ge \frac{2m}{3n}\mathbb{E}\inf_{|x|_2=1} \mathrm{p}_l(G_\omega(x)) \ge C\sqrt{\ln\left(1+\frac{n^2}{km}\right)}.$$

Using the hypothesis of the second case, we find ω_0 such that $\dim(\operatorname{Im} G_{\omega_0}) = k$ and

$$\frac{\sup_{|x|_2=1} \mathrm{p}_K(G_{\omega_0}(x))}{\inf_{|x|_2=1} \mathrm{p}_K(G_{\omega_0}(x))} \le c\sqrt{\frac{k}{\ln(1+\frac{n^2}{km})}} \le c\left(1+\sqrt{\frac{k}{\ln(1+\frac{n}{k\ln n})}}\right).$$

As in case 1, this concludes the proof. ■

2. The case of the simplex. It is well known and easy to see that the distance of an n-dimensional simplex S_n to the Euclidean ball is n. Here we consider k-dimensional sections of S_n. We note below that S_n has an $[(n+1)/2]$-dimensional section which is precisely a parallelepiped, hence this section through the center of mass is symmetric. A question then arises whether there exist sections of S_n of dimension greater than κn, for $1 > \kappa > 1/2$, whose distance to the Euclidean ball is asymptotically $c(\kappa)\sqrt{n}$. When $\kappa > 1/2$ sections of S_n of dimension κn cannot be centrally symmetric. The following theorem states these results precisely.

THEOREM 2. *Denote by S_n an n-dimensional simplex, with center of mass at the origin.*

(A) *If $k \le n/2$ then there is a subspace E of dimension k such that*

$$d(S_n \cap E, B_2^k) \le C\sqrt{\frac{k}{\ln(n/k)}}.$$

where C is a universal constant.

(B) *If $n/2 < p = \kappa n < n$ then there is a subspace E of dimension p such that*

$$d(S_n \cap E, B_2^p) \le \sqrt{\frac{n}{1-\kappa}}.$$

Proof. (A) The simplex has a section of dimension $[(n+1)/2]$ which is a parallelepiped. This is shown as follows.

If S_n is a regular simplex in $\mathbb{R}^n$, denote by S_n^0 its polar body with respect to the center of mass of S_n and let $p = [(n+1)/2]$. If we prove that there exists an orthogonal projection P onto a p-dimensional subspace E of $\mathbb{R}^n$ such that $P(S_n^0)$ is a symmetric convex hull of p points, then $E \cap S_n$ will be a parallelepiped. We distinguish between the two cases (i) $n = 2p-1$, and (ii) $n = 2p$.

In case (i) let $S_n^0 = \operatorname{conv}\{x_1, \ldots, x_p, y_1, \ldots, y_p\}$ and $F = \operatorname{span}\{x_1 + y_1, \ldots, x_p + y_p\}$. Then since $\sum_{i=1}^{p}(x_i + y_i) = 0$, F is $(p-1)$-dimensional and $E = F^\perp = \operatorname{span}\{x_1 - y_1, \ldots, x_p - y_p\}$ is p-dimensional. Let P be the orthogonal projection onto E. Then $\ker P = F$, thus $Px_i = -Py_i$ for $i = 1, \ldots, p$, so that $P(S_n^0) = \operatorname{conv}\{Px_i, -Px_i\}_{i=1}^p$. Moreover, $E = \operatorname{span}\{Px_1, \ldots, Px_p\}$ so $P(S_n^0)$ is a section of the simplex which is an ℓ_1^p ball.

In case (ii) let $S_n^0 = \operatorname{conv}\{x_1, \ldots, x_p, y_1, \ldots, y_p, z\}$, and let $F = \operatorname{span}\{x_1 + y_1, \ldots, x_p + y_p, z\}$. Then F is p-dimensional and we conclude as above.

Now apply the theorem of Milman–Schechtman [MS1], [MS2] (see also [Gué]) to the space $\ell_\infty^{[(n+1)/2]}$, which yields the result.

(B) Let S_n be an n-dimensional regular simplex with the origin as center of mass. Let $k = n - p$, $r + 1 = [n/k]$, and $m = k(r+1)$. We describe the vertices of S_n as follows:

$$S_n = \operatorname{conv}((e_1^1, \ldots, e_{r+1}^1), \ldots, (e_1^k, \ldots, e_{r+1}^k), e_{n+1}) \quad \text{if } m = n,$$

and if $m \le n-1$ and $l = n - m$,

$$S_n = \operatorname{conv}((e_1^1, \ldots, e_{r+1}^1, e_{m+1}), \ldots, (e_1^l, \ldots, e_{r+1}^l, e_n), (e_1^{l+1}, \ldots, e_{r+1}^{l+1}), \ldots, (e_1^k, \ldots, e_{r+1}^k), e_{n+1}).$$

Let

$$F = \operatorname{span}\Big(e_{m+i} + \sum_{j=1}^{r+1} e_j^i, \sum_{j=1}^{r+1} e_j^q : 1 \le i \le l,\ l+1 \le q \le k\Big).$$

Then F is a k-dimensional subspace of $\mathbb{R}^n$; denote by P the orthogonal projection onto $E = F^\perp$. For $1 \le i \le l$ and $l+1 \le q \le k$, we have $Pe_{m+i} + \sum_{j=1}^{r+1} Pe_j^i = 0$ and $\sum_{j=1}^{r+1} Pe_j^q = 0$. For $1 \le i \le l$ and $l+1 \le q \le k$, if $E_i = \operatorname{span}(Pe_1^i, \ldots, Pe_{r+1}^i)$ and $F_q = \operatorname{span}(Pe_1^q, \ldots, Pe_r^q)$, then

$$\Delta^i = \operatorname{conv}(Pe_1^i, \ldots, Pe_{r+1}^i, Pe_{m+i}) \quad \text{and} \quad S^q = \operatorname{conv}(Pe_1^q, \ldots, Pe_{r+1}^q)$$

are simplices in E_i and F_q.

Moreover, $\dim E_i = r+1$, $\dim F_q = r$,

$$E = \bigoplus_{i=1}^{l} E_i \oplus \bigoplus_{q=l+1}^{k} F_q,$$

$\dim E = p$ and

$$Pe_{n+1} = -\sum_{j=1}^{n} Pe_j = 0\ .$$

It follows that $P(S_n) = \operatorname{conv}(\Delta^1, \ldots, \Delta^l, S^{l+1}, \ldots, S^k)$, and thus

$$d(P(S_n), B_2^p) \le \sqrt{k} \max\big(\max_{1\le i\le l} d(\Delta^i, B_2^{r+1}), \max_{l+1\le q\le k} d(S^q, B_2^r)\big).$$

We conclude that

$$d(P(S_n), B_2^p) \le (r+1)\sqrt{k}.$$

By definition of r and k, the projection P is of rank $p = n - k$ and satisfies

$$d(P(S_n), B_2^p) \le \frac{\sqrt{n}}{\sqrt{1 - p/n}}.$$

By self-duality of the simplex, this gives part (B) of the theorem. ■

Remark 1. We now prove that this estimate is sharp. Indeed, suppose that 0 is in the interior of S_n and let Y be a k-dimensional subspace of $\mathbb{R}^n$. Let $\mathcal{E}$ be any ellipsoid contained in $Y \cap S_n$; by the inverse Santaló inequality

[BM], for some $c > 0$, we have

$$\left(\frac{\operatorname{vol}(Y \cap S_n)\operatorname{vol}((Y\cap S_n)^0)}{\operatorname{vol}(\mathcal{E})\operatorname{vol}(\mathcal{E}^0)}\right)^{1/k} \ge c,$$

where the polar is taken with respect to the center of $\mathcal{E}$.

If P_Y denotes the orthogonal projection onto Y, we have

$$(Y \cap S_n)^0 = P_Y S_n^0 \subset \mathcal{E}^0.$$

Thus if $S_n = \operatorname{conv}\{y_j\}_{j=1}^{n+1}$, we get $(Y\cap S_n)^0 = \operatorname{conv}(P_Y y_1, \ldots, P_Y y_{n+1})$. Since $(Y \cap S_n)^0 \subset \mathcal{E}^0$, we deduce from a classical result due independently to many authors (Maurey, Bárány and Füredy, Carl and Pajor, Gluskin, see for instance the general version due to [GMP]) that for some $c' > 0$,

$$\left(\frac{\operatorname{vol}((Y\cap S_n)^0)}{\operatorname{vol}(\mathcal{E}^0)}\right)^{1/k} = \left(\frac{\operatorname{vol}\operatorname{conv}(P_Y y_1, \ldots, P_Y y_{n+1})}{\operatorname{vol}(\mathcal{E}^0)}\right)^{1/k} \le c'\sqrt{\frac{\ln(1+n/k)}{k}},$$

so that for some constant $C > 0$ we have

$$\left(\frac{\operatorname{vol}(Y\cap S_n)}{\operatorname{vol}(\mathcal{E})}\right)^{1/k} \ge C\sqrt{\frac{k}{\ln(1+n/k)}}.$$

Applying this inequality to an ellipsoid $\mathcal{E}$ such that

$$\mathcal{E} \subset Y \cap S_n \subset d(Y\cap S_n, B_2^k)\mathcal{E},$$

we get

$$d(Y\cap S_n, B_2^k) \ge \left(\frac{\operatorname{vol}(Y\cap S_n)}{\operatorname{vol}(\mathcal{E})}\right)^{1/k} \ge C\sqrt{\frac{k}{\ln(1+n/k)}}.$$

Remark 2. Part (A) of Theorem 2 can also be proved directly using the same method as in Theorem 1, but adapted to the case of the simplex, and without using Rudelson's theorem. As the proof of Theorem 1 uses random methods, it produces a set of large probability (see e.g. [Go2]) such that the k-dimensional subspaces $\operatorname{Im} G_\omega$ satisfy the conclusion of the theorem.

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