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Added in proof (October 1997). In the recent article by J. Bonet, P. Domański and M. Lindström, *Essential norm and weak compactness of composition operators on weighted Banach spaces of analytic functions*, preprint, 1997, the authors show, among other things, that for a radial continuous weight v on D which is decreasing as a function of $r \in [0,1)$ and satisfies $\lim_{r\to 1} v(r) = 0$, v is equivalent to the associated weight $\tilde{v}$ if and only if $r \to 1/v(r)$ is equivalent to a log-convex function.

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The Weyl asymptotic formula by the method of Tulovskiĭ and Shubin

by

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Abstract. Let A be a pseudodifferential operator on $\mathbb{R}^N$ whose Weyl symbol a is a strictly positive smooth function on $W = \mathbb{R}^N \times \mathbb{R}^N$ such that $|\partial^\alpha a| \le C_\alpha a^{1-\varrho}$ for some $\varrho > 0$ and all $|\alpha| > 0$, $\partial^\alpha a$ is bounded for large $|\alpha|$, and $\lim_{w\to\infty} a(w) = \infty$. Such an operator A is essentially selfadjoint, bounded from below, and its spectrum is discrete. The remainder term in the Weyl asymptotic formula for the distribution of the eigenvalues of A is estimated. This is done by applying the method of approximate spectral projectors of Tulovskiĭ and Shubin.

Introduction. Let $A = a^w(x,D)$ be a pseudodifferential operator on $\mathbb{R}^N$ given by the Weyl formula

$$Af(x) = \iint e^{2\pi i(x-y)\xi} a\left(\frac{x+y}{2}, \xi\right) f(y)\, dy\, d\xi,$$

where a is a strictly positive smooth function with derivatives of polynomial growth. It is assumed that A enjoys certain "hypoelliptic" properties to be specified later which imply that A is selfadjoint and has a purely discrete spectrum $\lambda_n \nearrow \infty$. Let

$$Af = \int_0^\infty \lambda\, \mathcal{E}(d\lambda) f$$

be the spectral resolution for A.

Tulovskiĭ and Shubin [13] give estimates for the error term in the Weyl asymptotic formula

$$\mathcal{N}(\lambda) \approx \iint_{a\le\lambda} dx\, d\xi$$

for the number of eigenvalues of A smaller than or equal to λ. Their proof is based on a construction of a family E_λ of pseudodifferential operators that approximate the spectral projectors $\mathcal{E}_\lambda$ of A sufficiently well. This method

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of *approximate spectral projectors* has been subsequently substantially improved and extended by Hörmander [5] within the framework of his general Weyl calculus [6] (and [8]).

Our aim here is to pursue this idea in order to cope with a class of symbols which extends that of Tulovskiĭ–Shubin and is not covered by Hörmander's approach. Let us recall that Tulovskiĭ and Shubin demand that there exist $\varrho, \varepsilon > 0$ such that

$$\text{(A)} \qquad |\partial^\alpha a(w)| \le C_\alpha a(w)^{1-\varrho|\alpha|}, \quad |\alpha| > 0,$$

and

$$\text{(B)} \qquad a(w) \ge c\|w\|^\varepsilon, \quad w \in W,$$

whereas we only need

$$\text{(a}_1) \qquad |\partial^\alpha a(w)| \le C_\alpha a(w)^{1-\varrho}, \quad |\alpha| > 0,$$

$$\text{(a}_2) \qquad |\partial^\alpha a| \le C_\alpha, \quad \text{large } |\alpha|,$$

$$\text{(b)} \qquad \lim_{w\to\infty} a(w) = \infty.$$

Incidentally, (a_2) can be dropped, as has been shown by Czaja and Rzeszotnik [2].

Symbols of this kind arise in a natural way in the study of unitary representations of non-smooth infinitesimal generators of continuous semigroups of measures on the Heisenberg groups (see [4]). To explain this statement, let us start with the following example. Let P be the Schwartz distribution

$$\langle f, P\rangle = \int_{\|w\|\le 1} \frac{f(w) - f(0)}{\|w\|^{2n}}\, dw$$

on the phase space $W = \mathbb{R}^N \times \mathbb{R}^N$. A direct computation shows that

$$a(w) = \widehat{P}(w) \approx \log(1 + \|w\|)$$

violates both (A) and (B), while (a_1), (a_2), and (b) are satisfied with any $\varrho > 0$.

More generally, every real negative definite function a on W is a symbol of a selfadjoint operator $A = \pi(P)$, where $P \in \mathcal{S}^\star(W \times \mathbb{R})$ is a generating functional of a continuous semigroup of measures on the Heisenberg group $G = W \times \mathbb{R}$ with multiplication

$$(w,t)\circ(v,s) = \big(w + v, s + t + \tfrac{1}{2}\langle w, v\rangle\big).$$

Here $\langle\cdot,\cdot\rangle$ is a symplectic form on W, and π stands for the Schrödinger representation of G. The relationship between a and P is very simple, namely

$$a(w) = \widehat{P}(w, 1), \quad w \in W.$$

Although such symbols a are always continuous, they need not be differentiable. However, one can write

$$a = H \star a - \tau,$$

where H is the Gauss function and τ is positive definite so that the operator $T = \tau^w(x, D)$ is bounded on $L^2(\mathbb{R}^N)$. If, furthermore, a happens to be homogeneous and does not vanish away from the origin, $H \star a$ satisfies (a_1), (a_2) and (B), though not (A). Thus a fits the framework of our theory (cf. Corollary (3.4) below as well as [4]).

It is a future study of similar operators descending from nilpotent Lie groups *via* unitary representations that primarily motivates our interest in the Weyl formula as considered here.

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1. A symbolic calculus. Let V be an N-dimensional real vector space. Let $V^\star$ be the dual vector space and $x\xi$ the pairing between $x \in V$ and $\xi \in V^\star$. We fix a euclidean norm $\|\cdot\|$ in V and hence the dual norm in $V^\star$ and the product norm in the phase space $W = V \times V^\star$ denoted in the same way. Let $\{e_j\}_{j=1}^N$ be an orthonormal basis in V and $\{e_j\}_{j=N+1}^{2N}$ the dual basis in $V^\star$. For a multi-index $\alpha \in \mathbb{N}^{2N}$, let

$$\partial^\alpha f = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\ldots\partial_{2N}^{\alpha_{2N}} f,$$

where

$$\partial_j f(w) = \frac{1}{2\pi i}\frac{d}{dt}\bigg|_{t=0} f(w + te_j).$$

The *length* of a multi-index α is defined, as usual, by $|\alpha| = \sum_{j=1}^{2N}\alpha_j$.

There is a natural symplectic form on W:

$$\langle w, v\rangle = y\xi - x\eta \quad \text{for } w = (x,\xi),\ v = (y,\eta).$$

If $\Delta = \sum_{j=1}^{2N}\partial_j^2$ is the (positive) Laplace operator on W, then

$$\text{(1.1)} \qquad (\Delta_u + 1)e^{\langle u,v\rangle} = (1 + \|v\|^2)e^{\langle u,v\rangle} \quad \text{for } u, v \in W.$$

The Lebesgue measures $dx, d\xi$ on V and $V^\star$, respectively, are normalized so that the volume of the unit cube is equal to 1. Then the same is true of $dw = dx d\xi$. Let $\mathcal{S}(V)$ denote the Schwartz function space on the vector space V. The relationship between a function $f \in \mathcal{S}(V)$ and its Fourier transform $\widehat{f} \in \mathcal{S}(V^\star)$ is given by

$$\widehat{f}(\xi) = \int e^{-2\pi i x\xi} f(x)\, dx, \qquad f(x) = \int e^{2\pi i x\xi}\widehat{f}(\xi)\, d\xi.$$

We shall identify W with its dual by means of the bilinear form (1.1). The Fourier transform on $\mathcal{S}(W)$ takes the form

$$\widehat{f}(w)=\iint f(v)e^{2\pi i\langle v,w\rangle}\,dv.$$

Note that under this identification the Fourier transform turns out to be equal to its inverse.

A strictly positive continuous function $\mathbf{m}$ on W is called a *temperate weight* or simply a *weight* if it satisfies

$$\mathbf{m}(w+v)\le C\mathbf{m}(w)(1+\|v\|)^n \tag{1.2}$$

for all $w,v\in W$ and some constants $C,n>0$ (cf. Hörmander [7], 10.1). In particular, every weight $\mathbf{m}$ satisfies

$$C^{-1}(1+\|w\|)^{-n}\le\mathbf{m}(w)\le C(1+\|w\|)^n$$

for $w\in W$. Note that the weights form a group under multiplication. Moreover, if $\mathbf{m}$ is a weight, then $\log(1+\mathbf{m})$ is a weight, and for every real θ, $\mathbf{m}^\theta$ is also a weight.

For a given weight $\mathbf{m}$, let us denote by $S(\mathbf{m})$ the class of all $a\in C^\infty(W)$ such that

$$|a|_k=|a|_k^{\mathbf{m}}=\max_{1\le j\le k}\sup_W\mathbf{m}(w)^{-1}\|a^{(j)}(w)\|<\infty$$

for all positive integers k. Obviously, $S(\mathbf{m})$ is a Fréchet space if endowed with the family of norms $a\mapsto|a|_k$. It is convenient to extend this definition to symbols with values in finite-dimensional vector spaces so that, for instance, we can write $a^{(k)}\in S(\mathbf{m})$ instead of $\partial^\alpha a\in S(\mathbf{m})$ for all $|\alpha|=k$.

Every $a\in C^\infty(W)$ with derivatives of polynomial growth defines a continuous endomorphism $A=\mathrm{Op}(a):\mathcal{S}(V)\to\mathcal{S}(V)$ by the Weyl prescription

$$Af(x)=\iint e^{2\pi i(x-y)\xi}a\left(\frac{x+y}{2},\xi\right)f(y)\,dy\,d\xi.$$

The function a is called the *symbol* of A. Then

$$k(x,y)=\int e^{2\pi i(x-y)\xi}a\left(\frac{x+y}{2},\xi\right)d\xi$$

is the kernel of A, and the symbol can be retrieved by

$$a(x,\xi)=\int e^{-2\pi iy\xi}k(x+y/2,x-y/2)\,dy.$$

The weak version of the above definition

$$\langle Af,g\rangle=\int_W a(w)\psi_{f,g}(w)\,dw, \tag{1.3}$$

where

$$\psi_{f,g}(x,\xi)=\int_V e^{-2\pi iy\xi}f(x+y/2)\overline{g}(x-y/2)\,dy, \tag{1.4}$$

makes sense for every $a\in\mathcal{S}^\star(W)$. Thus the Weyl correspondence extends to symbols in $\mathcal{S}^\star(W)$ (cf. Hörmander [8]).

We shall denote by $\mathcal{L}(\mathbf{m})$ the linear space of all the operators $\mathrm{Op}(a)$ with $a\in S(\mathbf{m})$. Every $A\in\mathcal{L}(\mathbf{m})$ is closable when regarded as a densely defined operator on $L^2(V)$ with $\mathcal{S}(V)$ for its domain $\mathcal{D}(A)$.

If $A\in\mathcal{L}(\mathbf{m}_1),B\in\mathcal{L}(\mathbf{m}_2)$, then $AB\in\mathcal{L}(\mathbf{m}_1\mathbf{m}_2)$. Denote by $a\circ b$ the symbol of AB. The explicit formula is

$$a\circ b(w)=4^N\iint\mathcal{P}_u^{k_0}a(w+u)\mathcal{Q}_v^{k_0}b(w+v)e^{4\pi i\langle v,u\rangle}\,du\,dv, \tag{1.5}$$

where $\mathcal{P}_u=(1+16\|u\|^2)^{-1}(1+\Delta_u)$ and $\mathcal{Q}_v=(1+\Delta_v)(1+16\|v\|^2)^{-1}$ are differential operators acting in the u and v variable respectively, and k_0 is so large that the integral (1.5) is absolutely convergent. By the mean value theorem and (1.1),

$$\begin{aligned}r(a,b)(w)&=a\circ b(w)-a(w)b(w)\\&=2^{2N-1}\sum_{j=1}^{2N}\int_0^1 dt\iint\mathcal{P}_u^{k_0}(\partial_ja)(w+tu)\mathcal{Q}_v^{k_0}(\overline{\partial}_jb)(w+v)e^{4\pi i\langle v,u\rangle}\,du\,dv\end{aligned} \tag{1.6}$$

if k_0 is sufficiently large. Here $\overline{\partial}_j=\partial_{j+N}$ if $1\le j\le N$, and $\overline{\partial}_j=-\partial_{j-N}$ if $N+1\le j\le 2N$. Note the following Leibniz formulas:

$$\partial_j(a\circ b)=\partial_ja\circ b+a\circ\partial_jb,\qquad\partial_jr(a,b)=r(\partial_ja,b)+r(a,\partial_jb). \tag{1.7}$$

The composition is continuous, that is, for every j there exist integers k,l and a constant $C=C_{jkl}$ such that

$$|a\circ b|_j^{\mathbf{m}_1\mathbf{m}_2}\le C|a|_k^{\mathbf{m}_1}|b|_l^{\mathbf{m}_2}, \tag{1.8}$$

which follows from (1.5) and (1.2).

Suppose now that a and b are of polynomial growth and $a'\in S(\mathbf{m}_1)$, $b'\in S(\mathbf{m}_2)$. Then $r(a,b)\in S(\mathbf{m}_1\mathbf{m}_2)$ and for every j there exist integers k,l and a constant $C=C_{jkl}$ such that

$$|r(a,b)|_j^{\mathbf{m}_1\mathbf{m}_2}\le C\sum_{s=1}^{2N}|\partial_sa|_k^{\mathbf{m}_1}|\partial_sb|_l^{\mathbf{m}_2}. \tag{1.9}$$

We also have

$$\|\partial^\alpha r(a,b)\|_\infty\le C_{\alpha,k_0}\max_{0<|\beta|\le|\alpha|+2k_0+1}\|\mathbf{m}_1\partial^\beta b\|_\infty. \tag{1.10}$$

Another simple estimate is

$$\|\partial^\alpha r(a,b)\|_1\le C_{\alpha,k_0}\max_{0<|\beta|\le|\alpha|+2k_0+1}\|\partial^\beta a\|_1\max_{0<|\beta|\le|\alpha|+2k_0+1}\|\partial^\beta b\|_\infty. \tag{1.11}$$

Let $e,f\in S(\mathbf{1})$. Let $a\in S(\mathbf{m})$ and $a'\in S(\mathbf{n})$. If

$$e\circ a\circ f=eaf+r(e,a,f),$$

then there exists k_0 such that, for all k,

(1.12) $$|r(e,a,f)|_k^1 \le C_{k,e,f} \max_\beta (\|\partial^\beta e\cdot \mathbf{n}\|_\infty + \|\partial^\beta e \cdot a\|_\infty + \|\partial^\beta f\cdot \mathbf{n}\|_\infty),$$

where

$$C_{k,e,f} = C_k(|e|^1_{k_0+k} + |f|^1_{k_0+k})$$

and $0 < |\beta| \le k_0 + k$. All the formulas (1.9)–(1.12) are direct consequences of (1.6) and (1.2).

(1.13) LEMMA. *If $a : W \to \mathbb{C}$ is a continuous positive definite function, then $A = \mathrm{Op}(a)$ is a bounded operator on $L^p(V)$ for $1 \le p \le \infty$. The norm of A is less than or equal to $a(0)$.*

Proof. By Bochner's theorem, there exists a bounded measure μ on W such that $\mu = \widehat{a}$ and $\|\mu\| = a(0)$. Thus, by (1.4),

$$\langle Af, g\rangle = \int_W a(w)\psi_{f,g}(w)\,dw = \int_W \widehat{\psi}_{f,g}(w)\,\mu(dw) \quad \text{for } f,g\in \mathcal{S}(V).$$

Since

$$\widehat{\psi}_{f,g}(z,\eta) = \int_V e^{-2\pi i \eta x} f(x+z/2)\overline{g}(x-z/2)\,dx,$$

we have $|\widehat{\psi}_{f,g}| \le \|f\|_p\|g\|_q$, where $1/p + 1/q = 1$, so that

$$|\langle Af, g\rangle| \le a(0)\|f\|_p\|g\|_q,$$

which is our claim. ■

Our next lemma is Theorem 3.1.1 of Howe [9].

(1.14) LEMMA. *Let $a : W \to \mathbb{C}$ be a bounded function whose Fourier transform $\widehat{a}$ has compact support. Then $A = \mathrm{Op}(a)$ is a bounded operator on $L^2(V)$.* ■

For the proofs of the following three classical results the reader is referred to, e.g., Shubin [12] or Folland [3].

(1.15) PROPOSITION. *Let $a \in C^{2N+1}(W)$. If $\partial^\alpha a$ are bounded for $|\alpha| \le 2N+1$, then $\mathrm{Op}(a)$ has a unique extension to an operator $A \in \mathcal{L}(L^2(V))$ whose norm is estimated by*

$$\|A\| \le C \max_{|\alpha|\le 2N+1} \|\partial^\alpha a\|_\infty.$$

If, in addition, $\lim_{\|w\|\to\infty} a(w) = 0$, then A is compact. ■

Let

$$H(w) = e^{-2\pi\|w\|^2}$$

denote the square of the Gauss function. For every weight $\mathbf{m}$, $H \star \mathbf{m}$ is a weight equivalent to $\mathbf{m}$, that is,

$$C^{-1}\mathbf{m} \le H\star \mathbf{m} \le C\mathbf{m}.$$

(1.16) PROPOSITION. *Let $\mathbf{m}$ be a weight and $a \in S(\mathbf{m})$. If $a \ge 0$, then $A = \mathrm{Op}(H\star a)$ is positive.* ■

(1.17) COROLLARY. *There exists a constant $L > 0$ such that if $a \ge 0$ satisfies*

$$\max_{0<|\alpha|\le 2N+1} |\partial^\alpha a| \le C,$$

then the operator $\mathrm{Op}(a) + LC$ is positive.

Proof. We have $a = H\star a + (\delta - H)\star a$, where, by Proposition (1.16), $\mathrm{Op}(H\star a)$ is positive, and

(1.18) $$(\delta - H)\star a(w) = \int (a(w) - a(w-v))H(v)\,dv = \int_0^1 dt \int a'(w-tv)vH(v)\,dv,$$

so that for $|\alpha| \le 2N+1$,

$$|\partial^\alpha(\delta - H)\star a(w)| \le \sum_{s=1}^{2N}\int_0^1 dt \int |\partial^\alpha \partial_s a(w-tv)|\cdot|v_s|H(v)\,dv \le C\sum_s \int |v_s| H(v)\,dv,$$

and our claim follows by Proposition (1.15). ■

Recall that $A = \mathrm{Op}(a)$ is a Hilbert–Schmidt operator on $L^2(V)$ if and only if $a \in L^2(W)$, and then its Hilbert–Schmidt norm is

(1.19) $$\|A\|_{\mathrm{HS}} = \Big(\int |a(w)|^2\,dw\Big)^{1/2}.$$

(1.20) PROPOSITION. *Let $a \in C^{2N+1}(W)$. If $\partial^\alpha a \in L^1(W)$ for $|\alpha| \le 2N+1$, then $\mathrm{Op}(a)$ has a unique extension to a trace class operator A on $L^2(V)$ whose trace norm is estimated by*

$$\|A\|_{\mathrm{Tr}} \le C \max_{|\alpha|\le 2N+1}\|\partial^\alpha a\|_1.$$

In addition,

$$\mathrm{Tr}\,A = \int_W a(w)\,dw. \ ■$$

Let $\mathbf{n}$ be a weight and $\varrho > 0$. We say that a symbol $a \in C^\infty(W)$ such that

(1.21) $$\lim_{\|w\|\to\infty} |a(w)| = \infty$$

is $(\mathbf{n},\varrho)$-*hypoelliptic* if $\mathbf{n} \le C(1+|a|)^{1-\varrho}$, $a' \in S(\mathbf{n})$, and $a^{(k)} \in S(\mathbf{1})$ for some $k \in \mathbb{N}$. Let us remark that if a symbol $a > 0$ is $(\mathbf{n},\varrho)$-hypoelliptic, then

$$\mathbf{m}(w) = a(w) \tag{1.22}$$

is a temperate weight, which follows immediately by the Taylor expansion formula.

The proof of our next lemma is an adaptation of those of Hörmander [5], Lemmas 3.1 and 3.2, and Manchon [10], Proposition II.2.6.

(1.23) LEMMA. *Let $0 < a \in C^\infty(W)$ be $(\mathbf{n},\varrho)$-hypoelliptic. Let $\mathbf{m} = a$. There exist real symbols $b \in S(\mathbf{m}^{-1})$ and $q \in S(\mathbf{m}^{1/2})$ such that*

$$b \circ a - 1 \in S(\mathbf{m}^{-1}), \qquad q \circ q - a \in S(\mathbf{m}^{-1}).$$

Proof. Let $b_0 = 1/a$, $r_0 = 1 - b_0 \circ a$, and

$$r_n = r_0^{\circ 2^n}, \qquad b_{n+1} = (1 + r_{n+1}) \circ b_n.$$

Then the symbolic calculus gives $b_n \in S(\mathbf{m}^{-1})$, $r_n \in S(\mathbf{m}^{-2^{n+1}\varrho})$, and

$$b_n \circ a = 1 - r_n$$

so that $b = b_n$ has the required property if n is sufficiently large.

The other part of the assertion is proved in a similar way. In fact, let $q_0 = a^{1/2}$ and

$$r_n = q_n \circ q_n - a, \qquad q_{n+1} = q_n - \frac{r_n}{2q_n}.$$

The symbolic calculus gives $q_n \in S(\mathbf{m}^{1/2})$, $q_n' \in S(\mathbf{m}^{1/2-\varrho})$, and $r_n \in S(\mathbf{m}^{1-(n+1)\varrho})$ so that $q = q_n$ has the desired property if n is large enough. ∎

Recall that in the Weyl calculus an operator $A \in \mathcal{L}(\mathbf{m})$ is symmetric if and only if its symbol a is real. Let $A \in \mathcal{L}(\mathbf{m})$ be symmetric. The formula

$$\langle f, \widetilde{A}g\rangle = \langle Af, g\rangle, \qquad f \in \mathcal{S}(V),\ g \in \mathcal{S}^\star(V),$$

defines a continuous extension of A to $\mathcal{S}^\star(V)$.

We say that a selfadjoint operator A on a Hilbert space $\mathbf{H}$ has *discrete spectrum* if the spectrum of A consists of a discrete sequence $\{\lambda_k\}$ of eigenvalues of finite multiplicity.

(1.24) PROPOSITION. *If $a > 0$ is an $(\mathbf{n},\varrho)$-hypoelliptic symbol, then*

$$A = \mathrm{Op}(a) : \mathcal{S}(V) \to L^2(V)$$

is essentially selfadjoint, bounded from below, and its spectrum is discrete.

Proof. Let $u \in \mathcal{D}(A^\star)$. This implies that $v = \widetilde{A}u \in L^2(V)$. Let $\mathbf{m} = a$. By Lemma (1.23), there exist $B, R \in \mathcal{L}(\mathbf{m}^{-1})$ such that

$$u = Bv - Ru.$$

Once we prove that $Cv \in \mathcal{D}(\overline{A})$ for $C \in \mathcal{L}(\mathbf{m}^{-1})$ and $v \in L^2(V)$, we shall have $u \in \mathcal{D}(\overline{A})$, and consequently $A^\star = \overline{A}$.

Let v_n be a sequence of Schwartz functions converging to v in $L^2(V)$. The operators $C \in \mathcal{L}(\mathbf{m}^{-1})$ and $AC \in \mathcal{L}(\mathbf{1})$ are bounded so

$$Cv_n \to Cv, \qquad A(Cv_n) \to (AC)v.$$

Since A is closable, this implies $Cv \in \mathcal{D}(\overline{A})$.

By Lemma (1.23) and Proposition (1.15), there exist compact operators $B, S \in \mathcal{L}(\mathbf{m}^{-1})$ such that $BA = I - S$. Consequently, A has discrete spectrum.

Again, by Lemma (1.23) and Proposition (1.15), $A = Q^2 + R$, where $Q : \mathcal{S}(V) \to L^2(V)$ is symmetric and R is bounded. Thus A is bounded from below. ∎

For general information concerning pseudodifferential operators, as they are presented in this paper, the reader is referred to Folland [3], Hörmander [8], and Shubin [12].

2. The Weyl formula. Let $A : \mathcal{D}_A \to \mathbf{H}$ be a selfadjoint operator bounded from below on a Hilbert space with a discrete spectrum $\lambda_n \nearrow \infty$. Let

$$\mathcal{N}(\lambda) = \mathcal{N}_A(\lambda) = \#\{n \in \mathbb{N} : \lambda_n \le \lambda\}$$

be the *spectral function* of A. The following version of the Tulovskiĭ–Shubin Lemma on approximate spectral projections is due to Hörmander (see [5], Lemma 2.1).

(2.1) LEMMA. *Let E be a selfadjoint operator of trace class such that AE is bounded. Let $\lambda, K \ge 0$. If $E(\lambda - A)E \ge -K$, then*

$$\mathcal{N}_A(\lambda + 4K) \ge \operatorname{Tr} E - 2\|E - E^2\|_{\mathrm{Tr}}.$$

If $(I-E)(A-\lambda)(I-E) \ge -K$, then

$$\mathcal{N}_A(\lambda - 4K) \le \operatorname{Tr} E + 2\|E - E^2\|_{\mathrm{Tr}}. \ ∎$$

For a positive $a \in C(W)$, let

$$V(\lambda) = V_a(\lambda) = \int_{a(w)\le\lambda} dw$$

be the *volume function* associated with a. For a function $\mu : [\lambda_0, \infty) \to \mathbb{R}^+$, let $\mu_0(\lambda) = \mu(\lambda_0)$ and

$$\mu_{n+1}(\lambda) = \begin{cases} \mu(\lambda - \mu_n(\lambda)) & \text{if } \lambda - \mu_n(\lambda) \ge \lambda_0, \\ \mu(\lambda_0) & \text{if } \lambda - \mu_n(\lambda) < \lambda_0. \end{cases} \tag{2.2}$$

The following is the main result of this paper.

(2.3) THEOREM. *Let a be a strictly positive $(\mathbf{n},\varrho)$-hypoelliptic symbol. Let $\lambda_0 = \inf a(w)$, and let $\mu : [\lambda_0,\infty) \to \mathbb{R}^+$ be a monotonic function such that*

$$C^{-1}\mathbf{n}(w) \le \mu(a(w)) \le Ca(w)^{1-\varrho}, \quad w \in W.$$

Assume that either μ is increasing and

(2.4) $$\frac{\mu(\lambda)}{\lambda} \le C_1\frac{\mu(t)}{t}, \quad \lambda \ge t \ge \lambda_0,$$

or μ is decreasing and there exists $n \in \mathbb{N}$ such that, for every $r > 0$,

(2.5) $$(r\mu)_n(\lambda) \le C_{n,r}\mu(\lambda), \quad \lambda \ge \lambda_0.$$

Then $A = \mathrm{Op}(a)$ is essentially selfadjoint and bounded from below, its spectrum is discrete, and there exists a constant $R > 0$ such that, for large λ,

$$\left|\frac{\mathcal{N}(\lambda)}{V(\lambda)} - 1\right| \le R\frac{V(\lambda + R\mu) - V(\lambda - R\mu)}{V(\lambda)}.$$

Before starting the proof of Theorem (2.3), let us make some comments on our hypotheses. If a is an $(\mathbf{n},\varrho)$-hypoelliptic symbol, there always exists a function μ such that all the remaining assumptions are satisfied for we can let, for instance,

$$\mu(\lambda) = \lambda^{1-\varrho}, \quad \lambda \ge \lambda_0.$$

Sometimes, however, one can do much better: see Example (3.8) and Proposition (3.11) below.

If μ is decreasing, then, by definition,

(2.6) $$\mu \le \mu_{n+1} \le \mu_n \le \mu_1$$

for every n. Note also that for every $C_1 \in \mathbb{R}$, there exists $C_2 > 0$ such that

(2.7) $$\mu(\lambda + C_1\mu(\lambda)) \le C_2\mu(\lambda), \quad \lambda \ge \lambda_0,$$

which is a direct consequence of (2.4) when μ is increasing, or of (2.5), (2.6) when μ is decreasing. Finally, observe that (2.4) is trivially satisfied also in the case when μ is decreasing.

Proof of Theorem (2.3). The proof is divided into a sequence of propositions. We are going to define a family of selfadjoint trace class pseudodifferential operators E_λ and show that they uniformly fulfil the hypotheses of Lemma (2.1) with $K = R\mu(\lambda)$.

To this end, let $\phi \in C^\infty(\mathbb{R})$ be a positive function such that $\phi(t) = 1$ for $t \le 0$ and $\phi(t) = 0$ for $t \ge 1$. For a given $\lambda \ge \lambda_0$, let

$$\phi_\lambda(t) = \phi(\mu(\lambda)^{-1}(t-\lambda))$$

so that

$$\left|\frac{d^k}{dt^k}\phi_\lambda\right| \le C_k\mu(\lambda)^{-k}$$

for $k \in \mathbb{N}$. Define

$$e_\lambda(w) = \phi_\lambda(a(w)), \quad E_\lambda = \mathrm{Op}(e_\lambda)$$

for $w \in W$ and $\lambda \ge \lambda_0$. If a satisfies the hypothesis of Theorem (2.3), then

(2.8) $$|\partial^\alpha e_\lambda(w)| \le C_\alpha$$

for all α, which, by Proposition (1.15), implies that E_λ are uniformly bounded. Since e_λ are smooth with compact support, it is clear that E_λ are of trace class (Proposition (1.20)) and selfadjoint.

(2.9) Remark. *For every $\lambda \ge \lambda_0$, AE_λ is bounded on $L^2(V)$.*

Proof. In fact, $a \circ e_\lambda = ae_\lambda + r(a, e_\lambda) = a_\lambda + r_\lambda$, where $|\partial^\alpha a_\lambda| \le C_\alpha\lambda$, and, by (1.10), $|\partial^\alpha r_\lambda| \le C_\alpha\mu(\lambda)$. Thus our claim follows from Proposition (1.15). ■

(2.10) LEMMA. *We have*

(2.11) $$\|E_\lambda - E_\lambda^2\|_{\mathrm{Tr}} \le C(V(\lambda+\mu) - V(\lambda))$$

for all $\lambda \ge \lambda_0$.

Proof. The symbol of $E_\lambda - E_\lambda^2$ decomposes as

(2.12) $$(1 - e_\lambda) \circ e_\lambda = p_\lambda + r_\lambda,$$

where $p_\lambda = (1 - e_\lambda)e_\lambda$ and $r_\lambda = -r(e_\lambda, e_\lambda)$. Since p_λ is supported where $\lambda \le a \le \lambda + \mu$ and, by (2.8), all its derivatives are bounded, we have

$$\|\partial^\alpha p_\lambda\|_1 \le C_\alpha(V(\lambda+\mu) - V(\lambda)).$$

Similarly, combining (2.8) with (1.11), we get

$$\|\partial^\alpha r_\lambda\|_1 \le C_\alpha(V(\lambda+\mu) - V(\lambda)).$$

By Proposition (1.20), these two estimates imply the desired conclusion. ■

(2.13) LEMMA. *For all $\lambda \ge \lambda_0$,*

$$V(\lambda) \le \mathrm{Tr}\, E_\lambda \le V(\lambda+\mu).$$

Proof. By Proposition (1.20), $\mathrm{Tr}\, E_\lambda = \int e_\lambda\, dw$, so our claim is a consequence of the definition of e_λ. ■

(2.14) LEMMA. *For large λ,*

$$\|E_\lambda(\lambda - A)(I - E_\lambda)\| \le C\mu(\lambda).$$

Proof. We have

$$e_\lambda \circ (\lambda - a) \circ (1 - e_\lambda) = a_\lambda + r_\lambda,$$

where $a_\lambda = e_\lambda(\lambda - a)(1 - e_\lambda)$ and $r_\lambda = r(e_\lambda, \lambda - a, 1 - e_\lambda)$. Note that $|a_\lambda| \le \mu$, and, by (2.8),

$$|\partial^\alpha a_\lambda| \le C_\alpha\mu(\lambda).$$

In a similar way, by (1.12) and (2.7),

$$|\partial^\alpha r_\lambda| \le C_\alpha \mu(\lambda+\mu) \le C'_\alpha \mu(\lambda)$$

for all α, which, by Proposition (1.15), proves the required estimate. ■

(2.15) PROPOSITION. *If μ is increasing, then there exists a constant $C > 0$ such that*

$$E_\lambda(\lambda - A)E_\lambda \ge -C\mu(\lambda) \quad \text{for large } \lambda.$$

Proof. We have

$$e_\lambda \circ (\lambda - a) \circ e_\lambda = a_\lambda + r_\lambda,$$

where $a_\lambda = e_\lambda^2(\lambda - a) \ge -\mu$ and $r_\lambda = r(e_\lambda, \lambda - a, e_\lambda)$. Since, by (2.7),

$$|\partial^\alpha a_\lambda| \le C_\alpha \mu(\lambda+\mu) \le C'_\alpha \mu(\lambda)$$

for $|\alpha| > 0$, it follows, by Corollary (1.17), that

$$\mathrm{Op}(a_\lambda) \ge -C\mu(\lambda).$$

Moreover, by (1.12), $|\partial^\alpha r_\lambda| \le C_\alpha \mu(\lambda)$ for all α so that, by Proposition (1.15),

$$\|\mathrm{Op}(r_\lambda)\| \le C\mu(\lambda).$$

Thus, our claim follows by Corollary (1.17). ■

(2.16) LEMMA. *Let μ be decreasing. Let $K : \mathbb{R}^+ \to \mathbb{R}^+$ be decreasing and $\mu_K^*(\lambda) = \mu(\lambda - K)$. Then, for large λ,*

$$(E_\lambda - E_\kappa)(\lambda - A)(2E_\lambda - E_\kappa) \ge -C\mu_K^*(\lambda),$$

where $\lambda - K \le \kappa \le \lambda$.

Proof. In fact,

$$(e_\lambda - e_\kappa) \circ (\lambda - a) \circ (2e_\lambda - e_\kappa) = a_\lambda + r_\lambda,$$

where $a_\lambda = (e_\lambda - e_\kappa)(\lambda - a)(2e_\lambda - e_\kappa)$ and $r_\lambda = -r(e_\lambda - e_\kappa, \lambda - a, 2e_\lambda - e_\kappa)$. We have $a_\lambda \ge -\mu(\lambda) \ge -\mu_K^*(\lambda)$, and $|\partial^\alpha a_\lambda| \le C_\alpha \mu(\kappa) \le C'_\alpha \mu_K^*(\lambda)$ for $|\alpha| > 0$, hence, by Corollary (1.17),

$$\mathrm{Op}(a_\lambda) \ge -C\mu_K^*(\lambda).$$

At the same time, by (1.12), $|\partial^\alpha r_\lambda| \le C_\alpha \mu(\kappa) \le C'_\alpha \mu_K^*(\lambda)$ for all α, which, by Proposition (1.15), proves that

$$\|\mathrm{Op}(r_\lambda)\| \le C\mu_K^*(\lambda).$$

To complete the proof, it is sufficient to invoke Corollary (1.17). ■

(2.17) LEMMA. *Let μ be decreasing. Let K be as in Lemma (2.16). If*

$$E_\lambda(\lambda - A)E_\lambda \ge -K(\lambda)$$

for large λ, then, for λ still larger,

$$E_\lambda(\lambda - A)E_\lambda \ge -C\mu_K^*(\lambda).$$

Proof. By hypothesis,

$$E_\lambda^2(\lambda - A)E_\lambda^2 \ge -KE_\lambda^2.$$

Therefore, for large λ and $\kappa = \lambda - K$,

$$\begin{aligned} E_\lambda(\lambda - A)E_\lambda &= E_\kappa(\lambda - A)E_\kappa + T_{\lambda,\kappa} = E_\kappa^2(\kappa - A)E_\kappa^2 + KE_\kappa^2 + T_{\lambda,\kappa} + T_\kappa \\ &\ge T_{\lambda,\kappa} - \|T_\kappa\| \ge -C\mu_K^*(\lambda), \end{aligned}$$

where

$$T_\kappa = E_\kappa(I - E_\kappa)(\kappa - A)E_\kappa(2I - E_\kappa)$$

with $\|T_\kappa\| \le C\mu_K^*(\lambda)$ estimated by Lemma (2.14), and

$$T_{\lambda,\kappa} = (E_\lambda - E_\kappa)(\lambda - A)(2E_\lambda - E_\kappa) \ge -C\mu_K^*(\lambda),$$

by Lemma (2.16). ■

(2.18) PROPOSITION. *Let μ be decreasing. There exists a constant $C > 0$ such that*

$$E_\lambda(\lambda - A)E_\lambda \ge -C\mu(\lambda) \quad \text{for large } \lambda.$$

Proof. In fact,

$$E_\lambda(\lambda - A)E_\lambda = E_\lambda(\lambda - A) - E_\lambda(\lambda - A)(I - E_\lambda).$$

To estimate the first term, write

$$e_\lambda \circ (\lambda - a) = a_\lambda + r_\lambda,$$

where $a_\lambda = e_\lambda(\lambda - a) \ge -\mu(\lambda)$ and $r_\lambda = -r(e_\lambda, a)$. Since, by (2.8), $|\partial^\alpha a_\lambda| \le C_\alpha \mu(\lambda_0)$ for $|\alpha| > 0$, it follows, by Corollary (1.17), that $\mathrm{Op}(a_\lambda) \ge -K$ for a constant $K \ge 0$.

In a similar way, by (1.10) and (2.7), $|\partial^\alpha r_\lambda| \le C_\alpha \mu(\lambda)$ for all α so that, by Proposition (1.15), $\|\mathrm{Op}(r_\lambda)\| \le C\mu(\lambda)$. The other term is estimated by Lemma (2.14) so that

$$E_\lambda(\lambda - A)E_\lambda \ge -K \tag{2.19}$$

for large λ and a constant $K \ge 0$.

Now, by repeated use of Lemma (2.17), we arrive at (2.19), where

$$K = K(\lambda) = (r\mu)_n(\lambda),$$

for r sufficiently large, so that, finally, our conclusion follows by (2.5). ■

Propositions (2.15) and (2.18) provide the first estimate required by Lemma (2.1). Now we turn to the other one. This time, however, we start with the case when μ is decreasing, which is much simpler.

(2.20) PROPOSITION. *Let μ be decreasing. Then there exists a constant $C>0$ such that, for large λ,*
$$(I-E_\lambda)(A-\lambda)(I-E_\lambda)\ge -C\mu(\lambda).$$

Proof. We have
$$(I-E_\lambda)(A-\lambda)(I-E_\lambda)=(A-\lambda)(I-E_\lambda)+E_\lambda(\lambda-A)(I-E_\lambda).$$
To estimate the first term, write
$$(a-\lambda)\circ(1-e_\lambda)=a_\lambda+r_\lambda,$$
where $a_\lambda=(a-\lambda)(1-e_\lambda)\ge 0$ and $r_\lambda=-r(a,e_\lambda)$. Since μ is decreasing, we have $|\partial^\alpha a_\lambda|\le C_\alpha\mu(\lambda)$ for $|\alpha|>0$, and $|\partial^\alpha r_\lambda|\le C_\alpha\mu(\lambda)$ for all α so that, by Corollary (1.17) and Proposition (1.15), $\mathrm{Op}(a_\lambda+r_\lambda)\ge -C\mu(\lambda)$. The other term is estimated by Lemma (2.14). ■

Note that from the formal point of view the argument which follows covers the general case of monotone μ.

Recall from Section 1 that $H(w)=e^{-2\pi\|w\|^2}$, $w\in W$.

(2.21) LEMMA. *Let $a_\lambda=(a-\lambda)(1-e_\lambda)$. Let $k\in\mathbb{N}$. Then*
$$(\delta-H)^{\star k}\star a_\lambda=b_{\lambda,k}+c_{\lambda,k},$$
where

(2.22) $$|b_{\lambda,k}|\le C(\lambda^{-\varrho}a_\lambda+\mu(\lambda)),$$

(2.23) $$|\partial^\alpha c_{\lambda,k}|\le C_\alpha\mu(\lambda),$$

for all α, and

(2.24) $$|\partial^\alpha b_{\lambda,k}|\le C_\alpha$$

for $|\alpha|$ sufficiently large.

Proof. By (1.18), the mean value theorem, and the Leibniz rule,
$$\begin{aligned}&(\delta-H)^{\star k}\star a_\lambda(w)\\
&=\int_{[0,1]^k}\int_{W^k} a_\lambda^{(k)}\Big(w+\sum_{j=1}^k t_jv_j\Big)(v_1,\ldots,v_k)H(v_1)\ldots H(v_k)\,dv\,dt\\
&=\int_{[0,1]^k}\int_{W^k}(1-e_\lambda)a^{(k)}\Big(w+\sum_{j=1}^k t_jv_j\Big)\cdot vH(v_1)\ldots H(v_k)\,dv\,dt\\
&\quad-\sum_{0<j<k}\binom{k}{j}\int_{[0,1]^k}\int_{W^k}e_\lambda^{(j)}\otimes a^{(k-j)}\Big(w+\sum_{j=1}^k t_jv_j\Big)\cdot vH(v_1)\ldots H(v_k)\,dv\,dt\\
&\quad-\int_{[0,1]^k}\int_{W^k}(a-\lambda)e_\lambda^{(k)}\Big(w+\sum_{j=1}^k t_jv_j\Big)\cdot vH(v_1)\ldots H(v_k)\,dv\,dt\\
&=b_{\lambda,k}+c_{\lambda,k},\end{aligned}$$

where $b_{\lambda,k}$ is equal to the first term on the right-hand side and $c_{\lambda,k}$ is the sum of the remaining terms. Since
$$|\partial^{\beta-\gamma}a\partial^\gamma(1-e_\lambda)|\le C_{\beta,\gamma}\mathbf{1}_{\{\lambda\le a\le\lambda+\mu(\lambda)\}}\mathbf{n}\le C'_{\beta,\gamma}\mu(\lambda)$$
for $0<\gamma<\beta$, and
$$|(a-\lambda)\partial^\beta(1-e_\lambda)|\le C_\beta\mu(\lambda),$$
it is obvious that $c_{\lambda,k}$ satisfies (2.23). Recall from (1.22) that a is a weight. At the same time μ satisfies (2.4) so
$$|(1-e_\lambda)\partial^\beta a|\le C_\beta\mathbf{1}_{\{a\ge\lambda\}}\mathbf{n}\le C'_\beta\frac{\mu(\lambda)}{\lambda}a,$$
whence
$$|b_{\lambda,k}|\le C_{\alpha,k}\frac{\mu(\lambda)}{\lambda}a.$$
To end the proof of (2.22), observe that
$$a\le a_\lambda+\lambda+\mu(\lambda).$$
Finally, by hypothesis of Theorem (2.3), all derivatives of large order $|\alpha|$ are bounded, which implies (2.24). ■

(2.25) PROPOSITION. *For large λ,*
$$(I-E_\lambda)(A-\lambda)(I-E_\lambda)\ge -C\mu(\lambda).$$

Proof. We have
$$(I-E_\lambda)(A-\lambda)(I-E_\lambda)=(A-\lambda)(I-E_\lambda)+E_\lambda(\lambda-A)(I-E_\lambda).$$
To estimate the first term, write
$$(a-\lambda)\circ(1-e_\lambda)=a_\lambda+r_\lambda,$$
where $a_\lambda=(a-\lambda)(1-e_\lambda)\ge 0$ and $r_\lambda=-r(a,e_\lambda)$. By (1.10), $|\partial^\alpha r_\lambda|\le C_\alpha\mu(\lambda)$ so that $\|\mathrm{Op}(r_\lambda)\|\le C\mu(\lambda)$. To handle a_λ we decompose it as
$$a_\lambda=H\star\sum_{k=0}^{m-1}(\delta-H)^{\star k}\star a_\lambda+(\delta-H)^{\star m}\star a_\lambda=H\star p_\lambda+q_\lambda,$$
where
$$p_\lambda=a_\lambda+\sum_{k=1}^{m-1}b_{\lambda,k},\qquad q_\lambda=H\star\sum_{k=1}^{m}c_{\lambda,k}+b_{\lambda,m},$$
and $b_{\lambda,k}$, $c_{\lambda,k}$ are as in Lemma (2.21). Now, by Lemma (2.21), $p_\lambda\ge -C_m\mu(\lambda)$ for large λ, and $|\partial^\alpha q_\lambda|\le C_\alpha$ so that, by Propositions (1.15) and (1.16), $\mathrm{Op}(a_\lambda)\ge -C\mu(\lambda)$. The other term is estimated by Lemma (2.14). ■

Conclusion of the proof of Theorem (2.3). By Proposition (1.24), A is essentially selfadjoint and bounded from below. Moreover, its spectrum is discrete. Propositions (2.15), (2.18), (2.20), (2.25), and Remark (2.9) show that Lemma (2.1) applies. We get

$$\mathcal{N}(\lambda - C\mu) \le \operatorname{Tr} E_\lambda + 2\|E_\lambda - E_\lambda^2\|_{\mathrm{Tr}}$$

and

$$\mathcal{N}(\lambda + C\mu) \ge \operatorname{Tr} E_\lambda - 2\|E_\lambda - E_\lambda^2\|_{\mathrm{Tr}}$$

so that

$$\mathcal{N}(\lambda) \le \operatorname{Tr} E_{\lambda+C\mu} + 2\|E_{\lambda+C\mu} - E_{\lambda+C\mu}^2\|_{\mathrm{Tr}}$$

and

$$\mathcal{N}(\lambda) \ge \operatorname{Tr} E_{\lambda-C\mu} - 2\|E_{\lambda-C\mu} - E_{\lambda-C\mu}^2\|_{\mathrm{Tr}}.$$

In view of Lemmas (2.10), (2.13), and inequality (2.7), this completes the proof. ∎

3. Applications. This section contains some corollaries to our main theorem and examples.

(3.1) Corollary. *Let $A = \operatorname{Op}(a)$, where a satisfies the hypothesis of Theorem (2.3) with μ increasing. Let $T = T^\star$ be a bounded operator on $L^2(V)$ with a symbol τ which is a continuous function such that $|\tau(w)| \le C_0\mu(a(w))$. Let $B = A + T$ and $b = a + \tau$. Then, for large λ,*

$$\left|\frac{\mathcal{N}_B(\lambda)}{V_b(\lambda)} - 1\right| \le R\frac{V_b(\lambda + R\mu) - V_b(\lambda - R\mu))}{V_b(\lambda)}.$$

Proof. Let E_λ be the family of approximate projections for A, as constructed in the course of the proof of Theorem (2.3). Since T is bounded,

$$E_\lambda(\lambda - B)E_\lambda \ge -C_1\mu - K, \quad (I - E_\lambda)(B - \lambda)(I - E_\lambda) \ge -C_1\mu - K$$

so that, for λ large enough,

$$E_\lambda(\lambda - B)E_\lambda \ge -C_2\mu, \quad (I - E_\lambda)(B - \lambda)(I - E_\lambda) \ge -C_2\mu.$$

Consequently, by Lemma (2.1) and the properties of V_a, we get

$$\left|\frac{\mathcal{N}_B(\lambda)}{V_a(\lambda)} - 1\right| \le C\frac{V_a(\lambda + C\mu) - V_a(\lambda - C\mu)}{V_a(\lambda)}.$$

However, since $|\tau| \le C_0\mu(a)$,

$$V_a(\lambda) \le V_b(\lambda + C_0\mu), \quad V_b(\lambda) \le V_a(\lambda + C_0\mu),$$

which implies our assertion. ∎

(3.2) Remark. *Let a satisfy the hypothesis of Theorem (2.3). Then*

$$\left|\frac{\mathcal{N}_A(\lambda)}{V(\lambda)} - 1\right| \le \exp\left\{\int_{\lambda-R\mu}^{\lambda+R\mu} \frac{V'(t)}{V(t)}\,dt\right\} \cdot \int_{\lambda-R\mu}^{\lambda+R\mu} \frac{V'(t)}{V(t)}\,dt.$$

Proof. Note that V is increasing, hence differentiable almost everywhere on $\mathbb{R}^+$. For $m \ge 0$,

$$\frac{V(\lambda+m) - V(\lambda-m)}{V(\lambda)} \le \exp\left\{\int_{\lambda-m}^{\lambda+m} \frac{V'(t)}{V(t)}\right\} dt - 1 \tag{3.3}$$

$$\le \exp\left\{\int_{\lambda-m}^{\lambda+m} \frac{V'(t)}{V(t)}\,dt\right\} \cdot \int_{\lambda-m}^{\lambda+m} \frac{V'(t)}{V(t)}\,dt$$

so our claim follows by Theorem (2.3). ∎

Let D be a non-degenerate semisimple linear transformation of W. Let $d_j > 0$ be the eigenvalues of D. The number

$$Q = \operatorname{Tr} D = \sum_{j=1}^{2N} d_j$$

is called the *D-homogeneous dimension* of W. Recall that there exists a Borel measure σ_D on $\Sigma = \{w \in W : \|w\| = 1\}$ such that for every $g \in C_c(W)$,

$$\int g(w)\,dw = \int_0^\infty r^{Q-1} \int_\Sigma g(\delta_r \overline{w})\,\sigma_D(d\overline{w})\,dr,$$

where $\delta_t = t^D$ is the *family of dilations* generated by D. As a matter of fact, the measure σ_D has a smooth density relative to the spherical Lebesgue measure on Σ.

We say that a function $f : W \to \mathbb{C}$ is *D-homogeneous* of degree θ if

$$f(\delta_t w) = t^\theta f(w) \quad \text{for } t > 0 \text{ and } w \in W.$$

Recall that a symmetric function $F : W \to \mathbb{C}$ is called *negative definite* if for every $t > 0$ the function e^{-tF} is positive definite. This is equivalent to saying that $F(0) \ge 0$ and

$$\sum F(w_i - w_j)c_i\overline{c}_j \le 0$$

for every finite collection of vectors w_j and complex numbers c_j with $\sum c_j = 0$. Every negative definite function is continuous and its real part is positive. If F is a negative definite function and $F(0) > 0$, then F^{-z} is positive definite for every $\operatorname{re} z > 0$ (cf. Berg–Forst [1], Corollary 7.9).

(3.4) Corollary. *Let $\{u_j^\star\}_{j=1}^{2N}$ be linearly independent linear functionals on W. Let*

$$b(w) = \sum_{j=1}^{2N} |u_j^\star(w)|^{r_j},$$

where $r_j > 0$. Then $B = \operatorname{Op}(b) : \mathcal{S}(V) \to L^2(V)$ is bounded from below and essentially selfadjoint. Its spectrum is discrete, and there exists a constant

$C > 0$ such that

$$\mathcal{N}_B(\lambda) = C\lambda^Q(1 + O(\lambda^{-\varrho})),$$

where $Q = \sum_{j=1}^{2N} 1/r_j$ and $\varrho = \min(1, \{r_j\})$.

Proof. Let

$$Dw = \left(\frac{1}{r_1}u_1^\star(w), \frac{1}{r_2}u_2^\star(w), \ldots, \frac{1}{r_{2N}}u_{2N}^\star(w)\right)$$

so that b becomes D-homogeneous of degree $\theta = 1$. Then

$$V_b(\lambda) = \int_\Sigma \frac{\sigma_D(d\overline{w})}{b(\overline{w})} \cdot \lambda^Q,$$

and

$$\frac{V_b'(\lambda)}{V_b(\lambda)} = \frac{Q}{\lambda}. \tag{3.5}$$

Let

$$a(w) = \sum_{j=1}^{2N}(|u_j^\star(w)|^2 + h(u_j^\star(w)))^{r_j/2},$$

where $h \in C_c^\infty(\mathbb{R})$ is a fixed positive function such that $h(0) = 1$. Then $\tau = b - a$ is a bounded function.

Now, a satisfies the hypothesis of Theorem (2.3) with $\varrho = \min(1, \{r_j\})$ and $\mu(\lambda) = \lambda^{1-\varrho}$. Let $A = \mathrm{Op}(a)$. Assume for the moment that

(3.6) $T = B - A = \mathrm{Op}(\tau)$ is bounded on $L^2(V)$.

Then, by Corollary (3.1), Remark (3.2), and (3.5), we get

$$\left|\frac{\mathcal{N}_B(\lambda)}{V_b(\lambda)} - 1\right| \le C \int_{\lambda - C\lambda^{1-\varrho}}^{\lambda + C\lambda^{1-\varrho}} \frac{dt}{t} \le C\lambda^{-\varrho}$$

for large λ, which is our claim.

To prove that (3.6) holds true, let

$$R = \max\{r_j\}_{j=1}^{2N}, \qquad r = \min\{r_j\}_{j=1}^{2N}.$$

For $1/R \le \mathrm{re}\, z \le 4(N+1)/r$ let

$$a_z(w) = \sum_{j=1}^{2N}(|w_j|^2 + h(w_j))^{r_j z/2}, \qquad b_z(w) = \sum_{j=1}^{2N}|w_j|^{r_j z}$$

so that

$$a(w) = a_1 \circ U(w), \qquad b(w) = b_1 \circ U(w),$$

where U is a non-singular linear transformation of W such that $u_j^\star(Uw) = w_j$ for $1 \le j \le 2N$.

Let $\phi \in \mathcal{S}(W)$ be such that $\widehat{\phi}$ is compactly supported and equal to 1 in a neighbourhood of the origin. Let $c_z = b_z - a_z$ and

$$\Phi_z = \mathrm{Op}(\phi \star (c_z \circ U)).$$

The symbols $\phi \star (c_z \circ U)$ are bounded and their Fourier transforms have compact support (depending on ϕ) so, by Lemma (1.14), the operators Φ_z are bounded with

$$\|\Phi_z\| \le C_{\phi,z}.$$

Now, for $\mathrm{re}\, z = 4(N+1)/r$, $c_z \in C^{2N+1}(W)$ and

$$|\partial^\alpha c_z(w)| \le C_1$$

so the same is true for $c \circ U$, and, by Proposition (1.15),

$$\|\Phi_z\| \le C_1, \qquad \mathrm{re}\, z = 4(N+1)/r.$$

On the other hand, if $\mathrm{re}\, z = 1/R$, then

$$c_z = b_z - a_z = (b_z - d_z) + (d_z - a_z),$$

where

$$d_z(w) = \sum_{j=1}^{2N}(|w_j|^2 + 1)^{r_j/2}.$$

Note that, by the mean value theorem,

$$d_z - b_z = \frac{r_j z}{2}\int_0^1 (|w_j|^2 + t)^{r_j z/2 - 1}\, dt,$$

which, by the remarks preceding this corollary, implies that $d_z - b_z$, and consequently $(d_z - b_z) \circ U$, is positive definite. It is also clear that $d_z - a_z$, and consequently $(d_z - a_z) \circ U$, is bounded along with all its derivatives so that, by Lemma (1.13) and Proposition (1.15),

$$\|\Phi_z\| \le C_0, \qquad \mathrm{re}\, z = 1/R.$$

Now we are in a position to apply the Stein interpolation theorem (cf. Simon–Reed [11], Theorem IX.21) which yields $\|\Phi_1\| \le C$, where C does not depend on ϕ. Thus $T = B - A$ is bounded. ∎

(3.7) Remark. If U is symplectic, then $T = \mathrm{Op}(\tau)$ is unitarily equivalent to $T^U = \mathrm{Op}(\tau \circ U)$ (cf. Hörmander [8], Theorem 18.5.9) and T^U is evidently bounded since the variables in $\tau \circ U$ are separated.

Let us recall a lemma of Tulovskiĭ and Shubin (see Shubin [12], Proposition 28.3).

(3.8) Lemma. *Let $0 < a \in C^\infty(W)$ fulfil*

$$|a'(w) \cdot w| \ge ca(w)$$

for large $\|w\|$ *and some* $c>0$. *Then there exists* $C>0$ *such that*

$$\frac{V'(\lambda)}{V(\lambda)}\le C\lambda^{-1}\quad \text{for large } \lambda. \blacksquare$$

Let us denote (2.5) by $(2.5)_n$. Let

$$\langle x\rangle=(1+\|x\|^2)^{1/2},\qquad x\in V.$$

(3.9) EXAMPLE. Let $A=\mathrm{Op}(a)$, where

$$a(x,\xi)=\langle x\rangle+\langle\xi\rangle.$$

Then a is $(\mathbf{1},1)$-hypoelliptic and satisfies hypothesis $(2.5)_1$ of Theorem (1.15) with $\mu(\lambda)=1$. Moreover,

$$\lim_{\|w\|\to\infty}\frac{a'(w)\cdot w}{a(w)}=1.$$

Therefore, by Theorem (1.15), Remark (3.2), and Lemma (3.8),

$$\mathcal{N}_A(\lambda)=V(\lambda)(1+O(\lambda^{-1})),$$

where

(3.10) $$V(\lambda)=\iint_{\langle x\rangle+\langle\xi\rangle\le\lambda}dx\,d\xi. \blacksquare$$

For a function $\mu:\mathbb{R}^+\to\mathbb{R}^+$, let

$$\widetilde{\mu}(\lambda)=e^{-\lambda}\mu(e^\lambda),\qquad \lambda\ge 0.$$

Recall that the functions μ_n have been defined by (2.2).

(3.11) LEMMA. *Let* $\mu:\mathbb{R}^+\to\mathbb{R}^+$ *be decreasing. Then, for every* $n\in\mathbb{N}$, *there exists* $r=r_n>0$ *such that*

(3.12) $$\widetilde{\mu}_{n+1}(\lambda)\le((r\mu)^\sim)_n(\lambda).$$

Moreover, if μ *satisfies* $(2.5)_n$, *then* $\widetilde{\mu}$ *satisfies* $(2.5)_{n+1}$.

Proof. Observe that (3.12) is trivial for $n=0$. Suppose it holds true for some $n\ge 0$. Then

$$\begin{aligned}\widetilde{\mu}_{n+2}(\lambda)&=\widetilde{\mu}(\lambda-\widetilde{\mu}_{n+1}(\lambda))=e^{(\mu_{n+1})^\sim}e^{-\lambda}\mu(e^{\lambda-(\mu_{n+1})^\sim})\\&\le Ce^{-\lambda}\mu(e^\lambda-e^\lambda((r\mu)^\sim)_n(\lambda))\le e^{-\lambda}(R\mu)_{n+1}(e^\lambda)\\&=((R\mu)^\sim)_{n+1}(\lambda),\end{aligned}$$

which completes the proof of (3.12). The remaining part of the lemma follows immediately from (3.12). ∎

(3.13) PROPOSITION. *Let* a *be an* $(\mathbf{n},1)$-*hypoelliptic symbol satisfying the hypothesis of Theorem* (2.3) *with some* μ. *Let* $0<b\in C^\infty(W)$ *and*

$$b(w)=\log a(w)$$

for large $\|w\|$. *Then* b *is* $(\mathbf{n}/a,1)$-*hypoelliptic and satisfies the hypothesis of Theorem* (2.3) *with* $\widetilde{\mu}$. *Moreover, for large* λ,

$$\mathcal{N}_B(\lambda)=V_b(\lambda)(1+O(\Delta_R(e^\lambda))),$$

where $B=\mathrm{Op}(b)$, *and*

$$\Delta_R(\lambda)=\int_{\lambda-R\mu}^{\lambda+R\mu}\frac{V_a'(t)}{V_a(t)}\,dt.$$

Proof. By hypothesis, μ satisfies $(2.5)_n$ for some n. Thus, by Lemma (3.11), $\widetilde{\mu}$ satisfies $(2.5)_{n+1}$. That b and $\widetilde{\mu}$ fulfil the remaining assumptions of Theorem (2.3) is quite obvious. We also have

$$V_b(\lambda)=V_a(e^\lambda),\qquad \frac{V_b'(\lambda)}{V_b(\lambda)}=e^\lambda\frac{V_a'(e^\lambda)}{V_a(e^\lambda)}$$

for large λ outside a set of measure zero. Therefore,

$$\begin{aligned}\left|\frac{\mathcal{N}_B(\lambda)}{V_b(\lambda)}-1\right|&\le C\int_{\lambda-R\widetilde{\mu}}^{\lambda+R\widetilde{\mu}}\frac{V_a'(e^t)}{V_a(e^t)}e^t\,dt=C\int_{\exp(\lambda-R\widetilde{\mu})}^{\exp(\lambda+R\widetilde{\mu})}\frac{V_a'(s)}{V_a(s)}\,ds\\&\le C\int_{e^\lambda-R_1\mu(e^\lambda)}^{e^\lambda+R_1\mu(e^\lambda)}\frac{V_a'(s)}{V_a(s)}\,ds=C\Delta_{R_1}(e^\lambda),\end{aligned}$$

which completes the proof. ∎

Let $\log^n$ and $\exp^n$ denote the n-fold iteration of the corresponding function.

(3.14) EXAMPLE. Let $n\in\mathbb{N}$ and let $0<a_n\in C^\infty(W)$ be such that

$$a_n(x,\xi)=\log^n(\langle x\rangle+\langle\xi\rangle)$$

for large $\|x\|+\|\xi\|$. By induction, starting with Example (3.9) and using Proposition (3.13), we get

$$\mathcal{N}_{A_n}(\lambda)=V(\exp^n(\lambda))(1+O(1/\exp^n(\lambda))),$$

where $A_n=\mathrm{Op}(a_n)$ and V is as in (3.10). ∎

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An isomorphic Dvoretzky's theorem for convex bodies

by

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Abstract. We prove that there exist constants $C > 0$ and $0 < \lambda < 1$ so that for all convex bodies K in $\mathbb{R}^n$ with non-empty interior and all integers k so that $1 \le k \le \lambda n/\ln(n+1)$, there exists a k-dimensional affine subspace Y of $\mathbb{R}^n$ satisfying

$$d(Y \cap K, B_2^k) \le C\left(1 + \sqrt{\frac{k}{\ln(\frac{n}{k\ln(n+1)})}}\right).$$

This formulation of Dvoretzky's theorem for large dimensional sections is a generalization with a new proof of the result due to Milman and Schechtman for centrally symmetric convex bodies. A sharper estimate holds for the n-dimensional simplex.

1. Section of a convex body. By a *convex body*, we always mean a closed convex set with non-empty interior in the Euclidean space. Let K be an arbitrary convex body in $\mathbb{R}^n$ with the origin in its interior. The gauge functional of K is defined by $p_K(x) = \inf\{t \ge 0 : x \in tK\}$ for all $x \in \mathbb{R}^n$. We define the distance between two convex bodies A and B included in $\mathbb{R}^n$ by

$$d(A,B) = \inf_{u\in\mathbb{R}^n,\, T\in Gl_n(\mathbb{R})} \{\lambda > 0 : B + u \subset T(A) \subset \lambda(B+u)\}.$$

This is the analogue to the Banach–Mazur distance between two Banach spaces.

Denote by $(e_i)_{1\le i\le k}$ the canonical basis of $\mathbb{R}^k$, ℓ_2^k the space $\mathbb{R}^k$ equipped with the Euclidean norm $|\cdot|_2$, and B_2^k the unit ball of this space.

By $(g_j)_{1\le j\le n}$ and $(g_{ij})_{1\le i\le k,\,1\le j\le n}$ we always denote some independent, centered, normalized gaussian random variables. If $(t_p)_{1\le p\le N} \in \mathbb{R}^N$, we denote by $((t_p)_{p=1}^N)_q^*$ the qth coordinate of the decreasing rearrangement of

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