Associated weights and spaces of holomorphic functions

by

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Abstract. When treating spaces of holomorphic functions with growth conditions, one is led to introduce associated weights. In our main theorem we characterize, in terms of the sequence of associated weights, several properties of weighted (LB)-spaces of holomorphic functions on an open subset $G \subset \mathbb{C}^n$ which play an important role in the projective description problem. A number of relevant examples are provided, and a "new projective description problem" is posed. The proof of our main result can also serve to characterize when the embedding of two weighted Banach spaces of holomorphic functions is compact. Our investigations on conditions when an associated weight coincides with the original one and our estimates of the associated weights in several cases (mainly for $G = \mathbb{C}$ or $D$) should be of independent interest.

Spaces of continuous functions with $O$- or $o$-growth conditions occur in approximation theory; the corresponding spaces of holomorphic functions arise in complex analysis, spectral theory, Fourier analysis, partial differential equations and convolution equations. (For concrete examples and references, see [11], Section 4.) In contrast to the case of spaces of continuous functions, however, not all continuous and strictly positive weights $\nu$ are natural and intrinsically defined for spaces of holomorphic functions, as simple phenomena demonstrate (e.g., in connection with Liouville's theorem, see Section 1.A). Therefore, one is led to introduce associated weights $\tilde{\nu}$ which contain information on the holomorphic functions estimated by $1/\nu$. (For the exact definition see 1.1 and the start of Section 1.B.)

In fact, associated weights have been part of the "folklore" of the subject; for instance, they were mentioned explicitly in Anderson–Duncan [2], beginning of Section 2. But, to the best of our knowledge, so far nobody has ever undertaken a systematic study of associated weights in reasonable generality. In particular, it was not clear which conditions on a weight $\nu$
would imply that \( v = \tilde{v} \), and how far \( \tilde{v} \) could actually be different from \( v \). The first part of Section 1 and parts of Section 3 of the present article can be considered as the start of such an investigation. (But the situation is rather complicated, and many questions remain open.)

Our main motivation for working with associated weights comes from weighted \((LB)\)-spaces \( \mathcal{V} \mathcal{H}(G) \) of holomorphic functions (cf. [11], [9], [8], [15]). The "projective description problem," which is related to the notion of "analytically uniform spaces" due to Ehrenpreis, asks if the inductive limit topology of \( \mathcal{V} \mathcal{H}(G) \) can be described by weighted supseminorms with respect to a system \( V \) of weights depending on the decreasing sequence \( \nu = (\nu_n)_n \) in a natural way. In [11], it was proved that projective description holds, i.e., \( \mathcal{V} \mathcal{H}(G) = \mathcal{V} \mathcal{H}(\tilde{G}) \) topologically, if \( \mathcal{V} \mathcal{H}(G) \) is a \((DFS)\)-space or, more generally, just (semi-)Montel. But it is impossible to characterize these topological vector space properties of \( \mathcal{V} \mathcal{H}(G) \) in terms of the given sequence \( \nu \). Such characterizations have to involve the associated weights \( \tilde{v} \), \( v = 1, 2, \ldots \) (see Section 2), and hence it is sometimes very hard to evaluate them.

The first counterexamples to projective description (for spaces of holomorphic functions) were given in [15], but they were unnatural in the sense that they required enlarging the original domain of definition of the functions by adding another dimension in order to produce unexpected phenomena. The associated weights can serve to eliminate (at least part of) these constructions.

The article is organized as follows. Section 1 has two parts. In Part A, for a strictly positive continuous function \( w \) on an open subset \( G \) of \( \mathcal{C}^N \), i.e., a growth condition on \( G \), the definition of \( \tilde{w} \) is given, together with a number of examples and remarks. Part B deals with weighted Banach spaces \( \mathcal{H} \mathcal{V}(G) \) of holomorphic functions and notes that, for \( w = 1/\nu \) and \( \tilde{v} = 1/\tilde{w} \), one has \( \mathcal{H} \mathcal{V}(G) = \mathcal{V} \mathcal{H}(G) \) isometrically. In our first theorem (Theorem 1.13), the canonical biduality \( \mathcal{H} \mathcal{V}(G) = \mathcal{H} \mathcal{V}(\tilde{G}) \) is characterized in terms of associated weights.

Section 2 is devoted to weighted \((LB)\)-spaces \( \mathcal{V} \mathcal{H}(G) \) of holomorphic functions. After a short account of the projective description problem, we proceed to characterize the \((DFS)\)-property and bounded reactivity for \( \mathcal{V} \mathcal{H}(G) \) as well as the semi-Montel property for \( \mathcal{V} \mathcal{H}(G) \) by inequalities involving the associated weights (Theorem 2.1). Several interesting examples follow. At the end of Section 2, we are led to define a new system \( \tilde{V} \) of weights, depending on \( \tilde{V} = (\tilde{v}_n)_n \) in the same way as \( \tilde{V} \) depended on \( \tilde{V} = (v_n)_n \), and to pose the "new projective description problem" when \( \mathcal{V} \mathcal{H}(G) = \mathcal{V} \mathcal{H}(\tilde{G}) \) holds topologically. In an Appendix to Section 2, we show that the method of proof of Theorem 2.1 can also serve to characterize when two weighted topologies coincide on the unit ball of a weighted Banach space of holomorphic functions and, in particular, when the embedding of two such spaces is compact (Theorem 2.8).

Section 3 is of a more special character. It collects additional results, remarks and estimates of the associated weights, as well as further examples, mainly for \( G = C \) or \( D \). For a more detailed introduction to the last section we refer to its beginning.

Notation. Our notation is standard: \( \mathcal{N} = \mathcal{N} \cup \{0\} \), \( \mathbb{R}_+ = \{ r \in \mathbb{R} : r \geq 0 \} \), \( D \) denotes the open unit disk in \( C \). For an open set \( G \) in \( C^N \), \( \mathcal{H}(G) \) denotes the space of all holomorphic functions on \( G \); it is usually endowed with the compact-open topology \( co \). For notation on locally convex spaces and locally convex inductive limits, see Pérez-Carreras and Bonet [23] and the survey article [5].

1.A. Associated growth conditions, some examples. In the sequel, let \( G \) be an open subset of \( C^N \) \((N \geq 1)\) and \( w \) a continuous (strictly) positive function, i.e., a growth condition, on \( G \).

1.1. Definition. Put \( B_w := \{ f \in \mathcal{H}(G) : |f| \leq w \text{ on } G \} \). By \( \tilde{w} : G \to \mathbb{R}_+ \), we denote the function \( \tilde{w} \) with \( w \) defined by

\[
\tilde{w}(z) := \sup \{|f(z)| : f \in B_w\}, \quad z \in G.
\]

Since \( w \) is bounded on each compact subset of \( G \), Montel's theorem implies that \( B_w \) is compact in \( (\mathcal{H}(G), co) \). Using this fact and the continuity of point evaluations immediately yields that the sup in the definition of \( \tilde{w} \) is a maximum. Moreover, Ascoli's theorem implies that \( B_w \) is equicontinuous, and thus \( \tilde{w} \), in the pointwise closure of the equicontinuous set of all finite suprema of functions \( |f|, f \in B_w \), must be continuous. Now \( \tilde{w} \) is plurisubharmonic (p.s.h.) by Hörmander [19], 1.6.2 and 1.6.6. We have obtained:

1.2. Properties of \( \tilde{w} \). (i) \( 0 \leq \tilde{w} \leq w \),

(ii) \( \tilde{w} \) is continuous and p.s.h.,

(iii) for \( f \in \mathcal{H}(G) \), \( |f| \leq w \) (i.e., \( f \in B_w \)) \( \Leftrightarrow |f| \leq \tilde{w} \) (i.e., \( f \in B_{\tilde{w}} \)),

(iv) for each \( z \in G \) there is \( f = f_z \in B_{\tilde{w}} \) with \( |f(z)| = \tilde{w}(z) \) (or even \( f(z) = \tilde{w}(z) \)),

(v) \( \tilde{w} \) is plurisubharmonic (p.s.h.),

(vi) \( (Cw)^\sim = C\tilde{w} \) for arbitrary \( C > 0 \),

(vii) \( u_1 \leq u_2 \Rightarrow \tilde{w}_1 \leq \tilde{w}_2 \),

(viii) \( \min(u_1, u_2)^\sim = \min(\tilde{w}_1, \tilde{w}_2)^\sim \).

Note that, according to [19], Example after 2.6.1, even \( \log \tilde{w} \) is p.s.h. (at least if \( \tilde{w} \) is not identically zero on any connected component of \( G \)).

If \( G \) is bounded and \( w \) extends to a continuous function on \( \mathbb{C} \) with \( w|_{\partial G} \equiv 0 \), then \( \tilde{w} \equiv 0 \) by the maximum principle. Thus, \( \tilde{w} \) may have zeros.
The following two examples show that \( \bar{w} \) may be strictly smaller than \( w \) for other “natural” reasons.

1.3. Examples. (a) If \( G = \mathbb{C} \) and \( w(x) = \max(1, |x|^{n+p}) \), \( z \in \mathbb{C} \), where \( n \in \mathbb{N}_0 \) and \( 0 < p < 1 \), then \( \bar{w}(z) = \max(1, |z|^n) \) for each \( z \in \mathbb{C} \).

(b) (cf. [10]) Let \( G_1 \) be an open subset of \( C \) and \( \varphi \) a continuous positive function on \( G_1 \). Put \( G := G_1 \times \mathbb{C} \), and let \( w : G \to \mathbb{R}_+ \) be continuous such that for all \( (z_1, z_2) \in G \),

\[
0 < w(z_1, z_2) \leq w_1(z_1)(1 + |z_2|^p), \quad 0 < p < 1.
\]

Then \( \bar{w}(z_1, z_2) \leq \bar{w}_1(z_1) \) for all \( (z_1, z_2) \in G \).

Proof. (a) As the constant 1 belongs to \( B_w \), we already know that \( \bar{w}(1) = 1 = w(1) \) for \( |z| \leq 1 \). Moreover, as \( B_w \) contains the function \( f \), \( f(z) = z^n \), we have \( \bar{w}(z) \geq |z|^n \) for \( |z| \geq 1 \). It remains to show \( \bar{w}(z) \leq |z|^n \) on \( C \setminus D \), i.e., \( |f(z)| \leq |z|^n \) for \( |z| > 1 \), \( f \in \mathcal{B}_w \) arbitrary.

Put \( u := 1/z, g(u) := u^n f(1/u) \), \( u \neq 0 \). Then \( g \) is holomorphic on \( D \setminus \{0\} \).

But \( f \in \mathcal{B}_w \) implies

\[
|g(u)| = |u|^n |f(1/u)| = |z|^n |f(z)| \leq |z|^n \quad |u| \leq 1
\]

so that \( g \) must have a removable singularity at 0. Removing the singularity, we get \( g \in H(D), |g(u)| \leq 1 \) on \( \partial D \), hence \( |f(z)/z^n| = |g(u)| \leq 1 \) on \( D \) by the maximum principle. Thus \( |f(z)| \leq |z|^n \) for \( |z| > 1 \). (We thank A. Galbis for this argument.)

(b) Take an arbitrary \( f \in \mathcal{B}_w \) and fix \( z_1 \in G \). In view of (a), \( |f(z_1, \cdot)| \leq w_1(z_1)(1 + |z_2|^p) \), and an application of a (general form of) Liouville’s theorem yields that \( f(z_1, \cdot) \) is constant, hence \( |f(z_1, z_2)| = |f(z_1, 0)| \leq w_1(z_1) \) for all \( z_2 \in \mathbb{C} \). As \( z_1 \) varies in \( G_1 \), \( f(\cdot, 0) \in H(G_1) \) so that by 1.2(iii),

\[
|f(z_1, z_2)| = |f(z_1, 0)| \leq \bar{w}_1(z_1), \quad (z_1, z_2) \in G.
\]

Since \( f \in \mathcal{B}_w \) was arbitrary, the desired inequality follows.

On the other hand, there are simple cases in which \( \bar{w} = w \) holds.

1.4. Example. If the continuous function \( w : G \to \mathbb{R}_+ \setminus \{0\} \) happens to be of the form \( w = \sup\{|g| : g \in \mathcal{G}\} \) for some family \( \mathcal{G} \subset H(G) \), then \( w \in \mathcal{B}_w \) for each \( g \in \mathcal{G} \), and hence \( |g| \leq \bar{w} \) by 1.2(iii). As a consequence, \( w = \sup\{|g| : g \in \mathcal{G}\} \leq \bar{w}_1, i.e., \bar{w} = w \).

Note that in 1.4, \( G \) cannot be an arbitrary subset of \( H(G) \). In fact, \( G \) must be bounded pointwise on \( G \), its sup must be continuous, and the functions \( g \in \mathcal{G} \) are not allowed to have a common zero. Of course, this does not involve any loss of generality for finite families \( \mathcal{G} \subset H(G) \) without common zeros. On the other hand, if \( G = \mathbb{C}^N \) and \( \mathcal{G} = \{g\}, g \) cannot take

the value 0 on \( G \), any \( f \in \mathcal{B}_w \) can be divided by \( g \), and another application of Liouville’s theorem shows that \( f \) must be of the form \( \lambda g, \lambda \in \mathbb{C} \).

By 1.4, for \( w(z) = \max(1, |z|^n) \) on \( \mathbb{C} \), \( \bar{w} = w \) holds. (Compare with 1.3(a).)

If \( w = \lambda_1 w_2 \) on \( G \), then \( \lambda_1 \bar{w}_2 \leq \bar{w} \) follows from the definition. If, in addition, \( w_1 = |g| \) for some \( g \in H(G) \) without zeros, we get equality here: \( f \in \mathcal{B}_w \) implies \( f/g \in \mathcal{B}_{w_2} \), hence \( |f/g| \leq \bar{w}_2 \) by 1.2(iii). Thus, \( |f| \leq |g| \bar{w}_2 = w_1 \bar{w}_2 = \bar{w}_2 \) by 1.4, and we have shown \( \bar{w} = \bar{w}_1 \bar{w}_2 \).

The equality \( \bar{w} = w \) can also be deduced in another important case. To do this, we must first introduce some notation. For a moment, let \( G \) denote a balanced open subset of \( \mathbb{C}^N \) (i.e., \( z \in G, \lambda \in \mathbb{C}, |\lambda| = 1 \Rightarrow \lambda z \in G \)), and let \( w \) be a continuous positive function on \( G \) which is, in addition, radial in the sense that \( z \in G, \lambda \in \mathbb{C}, |\lambda| = 1 \Rightarrow w(\lambda z) = w(z) \). For \( f \in H(G) \) and \( z \in G \) put

\[
M(f, z) := \max\{|f(\lambda z)| : |\lambda| = 1\}.
\]

1.5. Observation. For a continuous positive radial function \( w \) on a balanced open set \( G \subset \mathbb{C}^N \), one has

\[
\bar{w}(z) = \sup\{M(f, z) : f \in \mathcal{B}_w\}, \quad z \in G,
\]

and the sup is again a maximum. Hence \( \bar{w} \) is also radial.

Proof. Put \( \bar{w}(z) := \sup\{M(f, z) : f \in \mathcal{B}_w\}, \quad z \in G \). For fixed \( z \in G \), \{\( \lambda z : |\lambda| = 1 \}\) is compact in \( G \), and thus \( M(\cdot, z) \) yields a continuous seminorm on \( (H(G), \mathcal{C}) \). Now compactness of \( \mathcal{B}_w \) implies that the sup in the definition of \( \bar{w} \) must really be a maximum.

Since \( |f(z)| \leq M(f, z) \), trivially \( \bar{w} \leq w \). Fix an arbitrary \( z_0 \in G \). There are \( f \in \mathcal{B}_w \) and \( \lambda \in \mathbb{C}, |\lambda| = 1 \), with \( w(z_0) = M(f, z_0) = |f(z_0)| \). Put \( g(z) := f(\lambda z), \quad z \in G \). Then \( g \) is a well-defined element of \( H(G) \), and since \( |g(z)| = |f(\lambda z)| = \bar{w}(\lambda z) = \bar{w}(z) \) for all \( z \in G \) due to the radiality of \( w \), we have \( g \in \mathcal{B}_w \).

Finally,

\[
\bar{w}(z_0) \geq |g(z_0)| = |f(\lambda z_0)| = \bar{w}(z_0).
\]

1.6. Corollary. If the continuous positive radial function \( w \) on the balanced open set \( G \subset \mathbb{C}^N \) happens to be of the form \( w(z) = M(f, z), \quad z \in G \), for some \( f \in H(G) \), then \( \bar{w} = w \). In case \( N = 1 \), this holds in particular if \( w(z) = f(|z|), \quad z \in G \), for some \( f \in H(G) \) whose Taylor series (at 0) has nonnegative coefficients.

Note that for \( f \in H(G) \) as in 1.6, \( f(0) \) must be \( \neq 0 \) since \( w(0) \) was required to be positive. On the other hand, if \( N = 1 \) and \( w(z) = \max(1, f(|z|)), \quad z \in G \), for some \( f \in H(G) \) with \( f(0) = 0 \) whose Taylor series has nonnegative coefficients, then a simple modification of the argument again yields \( \bar{w} = w \). In case \( N = 1 \), if \( G \) is a balanced open set in \( \mathbb{C} \),
Thus, by hypothesis on $w_1$ and $z_0$, we can conclude:
\[
C(1 + |z|)^a \tilde{w}(z_0) \geq C(1 + |z|)^a |f(z_0)| = C(1 + |z|)^a \tilde{w}_1(z_0) \exp(n \operatorname{Im} z_0) \\
\geq w_1(z_0) \exp(n \operatorname{Im} z_0) = w(z_0).
\]

The case $\operatorname{Im} z_0 < 0$ can be treated in exactly the same way, replacing $\exp(-inz)$ by $\exp(inz)$. The remaining case $\operatorname{Im} z_0 = 0$ is even easier: Choose $g$ as before and put $f := g$. (Once more we thank A. Galbis for this argument, which is related to the one used after 1.4.) \(\blacksquare\)

Since $w_1(z) = (1 + |z|)^n$, $z \in \mathbb{C}$, clearly satisfies $\tilde{w}_1 = w_1$, the following example is a consequence of 1.8.

1.9. Example. $\tilde{w} = w$ holds for

\[
w(z) = \exp(n \operatorname{Im} z)(1 + |z|)^n, \quad z \in \mathbb{C} \quad (n \in \mathbb{N} \text{ arbitrary}).
\]

Incidentally, on an arbitrary open set $G \subset \mathbb{C}^N$ and for any growth condition $w$ on $G$, the property $\tilde{w} = w$ is inherited by:

(a) positive scalar multiples,
(b) suprema, whenever the sup is continuous (and finite at each point),
(c) finite products,
(d) finite sums.

Here, (b) follows directly from 1.2(vii) (cf. also 1.4), and (c) and (d) are consequences of 1.2(iv). In fact, (d) generalizes to arbitrary sums $w = \sum \omega_\alpha$ (with $\omega_\alpha = \omega$ for all $\alpha$, which are continuous functions and) which converge pointwise on $G$.

It is also noteworthy that, for any $G$ and $w = \tilde{w}$, if $G_0$ is an open subset of $\mathbb{C}^{N_0}$ and $f : G_0 \rightarrow G$ is a holomorphic mapping, then $w \circ f = w \circ f$. Similarly, if the radial extension of a function $f : \mathbb{R}^+ \rightarrow \mathbb{R}_+$ to $\mathbb{C}$ satisfies $f = f$, then $f \circ w = f \circ w$.

Using 1.4 and the inheritance properties (a) and (d) (in its more general form), one realizes that the following (in general nonradial) growth conditions on an open set $G \subset \mathbb{C}$ with $G \neq \mathbb{C}$ have the property $\tilde{w} = w$:

\[
w(z) = \exp(C \operatorname{dist}(z, G))^{-n}, \quad w(z) = \exp(C \operatorname{dist}(z, G)^{-n}) \quad \text{for } n \in \mathbb{N} \text{ and } C > 0.
\]

The same holds with $\alpha > 0$ instead of $n$ if $G$ is simply connected (a generalization of 1.7(b) and (c)).

From one-dimensional examples of functions $w$ with $\tilde{w} = w$, one obtains $N$-dimensional ones ($N \geq 2$) on product sets, by use of the following simple observation (which clearly holds in slightly greater generality, but then the notation would become more complicated).

1.10. Observation. Let $G_k \subset \mathbb{C}$ be open, $k = 1, \ldots, N$, and $G = \prod_{k=1}^N G_k \subset \mathbb{C}^N$. If $w_k$ is a continuous positive function on $G_k$ with $w_k = \tilde{w}_k = \ldots$
w_k for each k, then w = w_1 \otimes \ldots \otimes w_N, w(z) = w_1(z_1) \ldots w_N(z_N) for z = (z_1, \ldots, z_N) \in G, again satisfies \tilde{w} = w.

1.10. Example (another consequence of 1.2(iv)) and 1.9 yield:

1.11. Example. \tilde{w} = w also holds for

\[ w(z) = \prod_{k=1}^{N} \exp(n |\text{Im } z_k|) (1 + |z_k|)^n, \quad z = (z_1, \ldots, z_N) \in \mathbb{C}^N, \]

where n \in \mathbb{N} is arbitrary and N \geq 1.

While it is good to know that \tilde{w} = w for many natural growth conditions, also those w for which \tilde{w} \leq w \leq C\tilde{w} holds for some constant C > 0, i.e., for which \tilde{w} is equivalent to w, are very important; see Sections 3.A and 3.B for examples.

1. B. Weighted Banach spaces of holomorphic functions. In the rest of Section 1, we deal with the role of associated weights in weighted Banach spaces of holomorphic functions. Let G again be an open subset of \mathbb{C}^N, and let v denote a weight on G, that is, a continuous function from G into \mathbb{R}_+ \setminus \{0\}. The weighted Banach spaces of holomorphic functions (with O- resp. o-growth conditions with respect to 1/v) are

\[ H_v(G) := \{ f \in H(G) : \|f\| = \|f\|_v := \sup_{G} |f(z)| < \infty \}, \]

\[ H_v(0)(G) := \{ f \in H(G) : \forall \varepsilon > 0 \exists K \text{ compact } \subset G \forall z \in G \setminus K : v(z) |f(z)| < \varepsilon \}, \]

endowed with the induced norm. Now, look at the growth condition \omega := 1/v. Then \omega = 1/v if \omega \omega = 1/v. Hence, H_v is defined as described in 1.1 is nothing but the closed unit ball of H_v(G).

1.12. Observation. \tilde{v} may take the value +\infty. But if \tilde{H}(G) is defined as above (with \tilde{v} replacing v, and adopting the convention that 0(+\infty) = 0), then H_v(G) = \tilde{H}_v(G), and the norms \| \cdot \|_v and \| \cdot \|_{\tilde{v}} coincide.

Indeed, from v \leq \tilde{v}, we already know \| \cdot \|_v \leq \| \cdot \|_{\tilde{v}} and \tilde{H}(G) \subset H_v(G). Fix f \in H_v(G) with \|f\|_v = 1. As remarked above, then f \in B_v. By 1.2(iii), \|f\| \leq \tilde{v} \omega, whence, clearly, f \in H_v(G) and \|f\|_v \leq \|f\| \leq \|f\|_v.

The associated weight \tilde{v} has the advantage of being defined intrinsically; it may also enjoy additional properties which v did not have (see 1.2(ii) and (iv)). If \tilde{v} is equivalent to v (in the sense that v \leq \tilde{v} \leq C_v for some C > 0), then H_v(G) and \tilde{H}_v(G) are still (topologically) isomorphic.

So far, we have only considered associated weights in the case of H_v(G), but we can also do the same construction for H_v(0). Namely, take \omega = 1/v, but now put

\[ \tilde{w}_0(z) := \sup \{|f(z)| : f \in H_v(0), \|f\| \leq 1\}, \quad z \in G, \]

and \tilde{v}_0 := \tilde{v}/\tilde{w}_0. Clearly, 0 \leq \tilde{w}_0 \leq \tilde{w} \leq w, hence \omega \leq \tilde{\omega} \leq \tilde{v}_0.

1.13. Theorem. The natural biduality \tilde{H}_v(G)^* = H_v(G) holds isometrically if and only if \tilde{v} = \tilde{v}_0.

Proof. 1.13 is an easy consequence of the abstract characterization in [12, Theorem 1.1 and Cor. 1.2], whereby H_v(G)^* = H_v(G) isometrically (in the canonical way) if and only if the closed unit ball B_{v_0} of H_v(G) is dense in B_v, the closed unit ball of H_v(G), with respect to \omega or, equivalently, with respect to the topology of pointwise convergence.

If this density holds, then the definition implies \tilde{w}_0(z) = \tilde{w}(z) for each z \in G, hence \tilde{v}_0 = \tilde{v}. On the other hand, if B_{v_0} is not pointwise dense in B_v, there must clearly exist z \in G and f \in B_v such that

\[ \tilde{w}_0(z) \geq |f(z)| > \sup \{|g(z)| : g \in H_v(0), \|g\| \leq 1\} = \tilde{w}(z), \]

and hence \tilde{v}_0(z) > \tilde{v}(z). (The first-named author thanks D. Vogt for conversations on the subject of Theorem 1.13.)

2. Weighted (LB)-spaces of holomorphic functions. In this section we first show that associated weights can serve to characterize certain properties of weighted (LB)-spaces of holomorphic functions which play an important role in the projective description problem.

Here \mathcal{V} = (\mathcal{V}_n : n \in \mathbb{N}) will always denote a decreasing sequence of weights \mathcal{V}_n on an open subset G of \mathbb{C}^N. The weighted (LB)-space of holomorphic functions associated with \mathcal{V} is the locally convex (LF) inductive limit

\[ \mathcal{V}H(G) := \bigcap_{n} H_v(G), \]

that is, it is the union of the weighted Banach spaces H_v(G) of holomorphic functions, endowed with the strongest topology given by a system of seminorms such that the injections H_v(G) \rightarrow \mathcal{V}H(G), n \in \mathbb{N}, become continuous. In the sequel, we shall put \mathcal{V}_n = 1/v_n and let B_n := B_{v_0} denote the closed unit ball of H_v(G), n = 1, 2, \ldots. Note that \mathcal{V}H(G) is always complete; in fact, it is the inductive dual of a Fréchet space Y (see [9], Theorem 6). (Hence, the inductive limit topology of \mathcal{V}H(G) exactly equals \beta(\mathcal{V}H(G), Y)).
In an effort to describe the continuous seminorms for the inductive limit topology, one introduces (cf. [11])
\[ \overline{V} = \overline{V}(V) := \{ \overline{v} \text{ weight on } G : \text{ for each } n \in \mathbb{N}, \overline{v}/v_n \text{ is bounded on } G \} \]
and the corresponding weighted space of holomorphic functions
\[ \mathcal{H}\overline{V}(G) := \{ f \in \overline{H}(G) : \text{ for each } \overline{v} \in \overline{V}, p_\varphi(f) = \sup_G \overline{v}|f| < \infty \}. \]

Endowed with the l.c. topology given by the system \((p_\varphi)_{\varphi \in \mathcal{V}}\) of weighted supseminorms, \(\mathcal{H}\overline{V}(G)\) is complete; it is called the projective hull of \(\mathcal{H}V(G)\).

Clearly \(\mathcal{V}H(G) \subset \mathcal{H}\overline{V}(G)\) with continuous injection, but it is easy to see that \(\mathcal{V}H(G) = \mathcal{H}\overline{V}(G)\) algebraically and that the two topologies yield the same bounded sets. We say that projective description holds if \(\mathcal{V}H(G) = \mathcal{H}\overline{V}(G)\) topologically, i.e., if \((p_\varphi)\) induces the inductive limit topology; and the projective description problem asks when this is the case. In [15], the authors constructed the first counterexamples to projective description for spaces of holomorphic functions; for counterexamples in the case of entire functions, see [14].

In [11] and some subsequent articles, the following conditions on \(V\) were considered:
\begin{align*}
(S) & \forall n \in \mathbb{N} \exists m > n : \frac{v_m}{v_n} \text{ vanishes at } \infty \text{ on } G, \\
(M) & \forall n \in \mathbb{N} \forall Y \text{ not relatively compact in } G \exists m = m(n, Y) > n : \\
& \inf_{Y} \frac{v_m}{v_n} = 0, \\
(RD) & \forall n \in \mathbb{N} \exists m \geq n \forall Y \subset G : \\
& \inf_{Y} \frac{v_m}{v_n} > 0 \Rightarrow \inf_{Y} \frac{v_k}{v_n} > 0, \quad k = m + 1, m + 2, \ldots
\end{align*}

Clearly, \((S)\) implies both \((M)\) and \((RD)\). It is easy to see that, whenever \((S)\) holds, \(\mathcal{V}H(G)\) is a (DFS)-space; if \(m > n\) is chosen as in \((S)\), then the canonical injection \(\mathcal{H}v_m(G) \rightarrow \mathcal{H}v_n(G)\) is compact. In [11], Theorem 1.6, it was proved that \((S)\) also implies \(\mathcal{V}H(G) = \mathcal{H}\overline{V}(G)\) topologically. More generally, using [6], Prop. 6, and applying the Baernstein Lemma (cf. [11], 0.4) to \(\mathcal{V}H(G) \rightarrow \mathcal{C}V(G)\) (the analog of \(\mathcal{V}H(G)\) for continuous functions), projective description holds whenever \(\mathcal{V}H(G)\) is semi-Montel (i.e., each bounded subset of this space is relatively compact). By [9], Prop. 7, \((M)\) implies that \(\mathcal{H}\overline{V}(G)\) is semi-Montel. However, [15] gave an example of a space \(\mathcal{V}H(G)\) such that \((M)\) holds, hence \(\mathcal{H}\overline{V}(G)\) is semi-Montel, but \(\mathcal{V}H(G) \neq \mathcal{H}\overline{V}(G)\) topologically. By [9], Theorem 2 and Theorem 6(2), \((RD)\) implies that \(\mathcal{V}H(G)\) is a boundedly reductive inductive limit, i.e., for each bounded set \(B \subset \mathcal{V}H(G)\) there is \(n \in \mathbb{N}\) such that \(B\) is contained and bounded in \(\mathcal{H}v_n(G)\) and that, in addition, the topologies induced by \(\mathcal{V}H(G)\) and \(\mathcal{H}v_n(G)\) coincide on \(B\). In this case, \(\mathcal{V}H(G)\) is the strong dual of a (quasi-normable) Fréchet space \(Y\). The method of construction of counterexamples that was used in [15] does not work if \(\mathcal{V}H(G)\) is boundedly retractive; it is still possible that projective description holds whenever \(\mathcal{V}H(G)\) has this property.

As we will see below, none of the three implications \((S) \Rightarrow \mathcal{V}H(G)\) (DFS), \((M) \Rightarrow \mathcal{H}\overline{V}(G)\) semi-Montel and \((RD) \Rightarrow \mathcal{V}H(G)\) boundedly retractive is an equivalence. Characterizations of the corresponding properties of \(\mathcal{V}H(G)\) resp. \(\mathcal{H}\overline{V}(G)\) can be given (and are even not hard to prove), but they involve the decreasing sequence \(\mathcal{V} = (v_n)\) of associated weights \(v_n\) (see Section 1.B). In fact, the characterizations also involve associated weights for some more complicated constructions with minima (which makes it hard to evaluate the characterizations).

In the sequel, for \(w_n = 1/v_n, n \in \mathbb{N}\), \(\tilde{w}_n\) has the same meaning as in Section 1.A, and \(\tilde{w}_n = 1/\tilde{w}_n, K_\infty(G)\) is the set of all nonnegative continuous functions on \(G\) with compact support. Note that, due to the conventions introduced in Section 1.B, \(\min(1/v_n, 1/\varphi)\) is always a positive continuous function on \(G\) for \(n \in \mathbb{N}\) and \(\varphi \in K_\infty(G)\).

2.1. Theorem. \((a)\) \(\mathcal{V}H(G)\) is a (DFS)-space if and only if
\begin{align*}
(S_H) & \forall n \in \mathbb{N} \exists m > n \forall \varphi \geq 0 \exists \varphi \in K_\infty(G) : \\
& \left( \frac{1}{v_n} \frac{1}{\varphi} \right)^{\sim} \leq \left( \frac{1}{v_m} \frac{1}{\varphi} \right)^{\sim} \text{ or, equivalently, } \left( \min \left( \frac{w_n}{v_n}, \frac{1}{\varphi} \right) \right)^{\sim} \leq \varepsilon w_m. \\
(M_H) & \forall n \in \mathbb{N} \forall \varphi \in \overline{V} \exists \varphi \in K_\infty(G) : \\
& \left( \frac{1}{v_n} \frac{1}{\varphi} \right)^{\sim} \leq \left( \frac{1}{v} \frac{1}{\varphi} \right)^{\sim} \text{ or } \left( \frac{1}{v_n} \frac{1}{\varphi} \right)^{\sim} \leq \frac{1}{v}. \\
(RD_H) & \forall n \in \mathbb{N} \exists m \geq n \forall \varphi \geq m \forall \varphi \geq 0 \exists \delta > 0 : \\
& \left( \frac{1}{v_n} \frac{\delta}{v_k} \right)^{\sim} \leq \left( \frac{1}{v_n} \frac{1}{\varphi} \right)^{\sim} \text{ or } \left( \min \left( w_n, \delta w_k \right) \right)^{\sim} \leq \varepsilon w_m.
\end{align*}

Proof. We will first reformulate the properties of \(\mathcal{V}H(G)\) resp. \(\mathcal{H}\overline{V}(G)\) in \((a)-(c)\) in terms of a condition \((*)\) which essentially involves an inclusion of sets with an intersection of two sets on the left side. From this point on, the three proofs are exactly the same, and we will only treat the case \((c)\) in detail.

\((a)\) \(\mathcal{V}H(G)\) is a (DFS)-space if and only if for each \(n \in \mathbb{N}\) there is \(m > n\) such that the canonical injection \(\mathcal{H}v_m(G) \rightarrow \mathcal{H}v_n(G)\) is compact. Since \(B_n\) is compact in \((H(G), co)\), this holds precisely if \(\mathcal{H}v_n(G)\) and \(\mathcal{H}v_m(G)\) coincide on \(B_n\).
same topology on $B_n$. But as two l.c. topologies coincide on an absolutely convex set if and only if they yield the same systems of 0-neighborhoods, we arrive at the following equivalence: $\mathcal{VH}(G)$ is a (DFS)-space if and only if

(*) $\forall n \in \mathbb{N} \exists m > 0 \forall \varepsilon > 0 \exists \varphi \in \mathcal{K}_+(G)$:

$B_n \cap \{f : \varphi[f] \leq 1 \text{ on } G\} \subset \varepsilon B_m$.

(b) Since $\mathcal{H}(G)$ has the same bounded sets as the regular inductive limit $\mathcal{VH}(G)$ and since each $B_n$ is compact in $(H(G), \mathcal{C})$, $\mathcal{H}(G)$ is semi-Montel precisely if for each $n \in \mathbb{N}$, $\mathcal{H}(G)$ and $\mathcal{C}(G)$ induce the same topology (at 0) on $B_n$. This is clearly equivalent to

(*) $\forall n \in \mathbb{N} \forall \varepsilon \in \mathcal{V} \exists \varphi \in \mathcal{K}_+(G)$:

$B_n \cap \{f : \varphi[f] \leq 1 \text{ on } G\} \subset \{f \in H(G) : p\varepsilon(f) \leq 1\}$.

(c) Since $\mathcal{VH}(G)$ is always regular, the bounded reactivity of $\mathcal{VH}(G)$ is equivalent to the fact that for each $n \in \mathbb{N}$ there is $m \geq n$ such that $H\nu_n(G)$ and $\mathcal{VH}(G)$ induce the same topology on $B_n$. Moreover, since for regular (LB)-spaces bounded reactivity is equivalent to bounded stability (cf. [5], Appendix to Section 3, Prop. 9), $\mathcal{VH}(G)$ is boundedly retractive if and only if for each $n \in \mathbb{N}$ there is $m \geq n$ such that for each $k \geq m$, $H\nu_m(G)$ and $H\nu_k(G)$ induce the same 0-neighbourhood system on $B_n$, i.e.,

(*) $\forall n \in \mathbb{N} \exists m \geq n \forall k \geq m \forall \varepsilon > 0 \exists \delta > 0$:

$B_n \cap \delta B_k \subset \varepsilon B_m$.

Now it suffices to show that in (a), (b), (c) the corresponding (*) is equivalent to $(\mathcal{S}_H)$, $(\mathcal{M}_H)$, $(\mathcal{RD}_H)$, respectively. Here is the proof for (c); (a) and (b) need only minor modifications.

(c) $(\mathcal{RD}_H) \Rightarrow (*)$. If $f \in B_n \cap \delta B_k$, then $f \in H(G)$ satisfies $|f| \leq \min(\omega_n, \delta \omega_k)$, hence by 1.2(iii) and $(\mathcal{RD}_H)$,

$|f| \leq (\min(\omega_n, \delta \omega_k))^{-1} \leq \varepsilon \omega_k \leq \varepsilon \omega_m$,

that is, $f \in \varepsilon B_m$.

(*)$\Rightarrow (\mathcal{RD}_H)$ (indirect). Once more, fix $n, k, \varepsilon, \delta$. If there is $z \in G$ such that $(\min(\omega_n, \delta \omega_k))^{-1} > \varepsilon \omega_m(z)$, then by 1.2(iv) there is $f \in H(G)$ with $|f| \leq \min(\omega_n, \delta \omega_k)$ on $G$, but $|f(z)| > \varepsilon \omega_m(z)$. Now if $f \in B_n \cap \delta B_k$, but $f$ cannot belong to $\varepsilon B_m$ because in that case we would have $|f| \leq \varepsilon \omega_m$, hence by 1.2(iii) even $|f| \leq \varepsilon \omega_m$, a contradiction. Thus, (a) cannot hold.

$(\mathcal{S}_H)$ and $(\mathcal{M}_H)$ have the drawback that in their formulation not only the associated weights, but also functions $\varphi \in \mathcal{K}_+(G)$ appear. (We could have worked with characteristic functions of compact sets, but then the minima in those conditions would not have been continuous in general, violating the general hypotheses of Section 1.) In this respect, condition $(\mathcal{RD}_H)$ is better.

By what we have said at the beginning of Section 2, $(\mathcal{S})$ implies $\mathcal{VH}(G)$ being (DFS), and hence $(\mathcal{S}) \Rightarrow (\mathcal{S}_H)$. A direct proof of this implication is easy. Later on, we will give an example of a (DFS)-space $\mathcal{VH}(G)$ for which $\mathcal{V}$ does not satisfy $(\mathcal{S})$, and hence $(\mathcal{S}_H)$ does not imply $(\mathcal{S})$. Clearly, if all $\bar{\omega}_n$ are weights (i.e., they do not take the value $\infty$), and if $\mathcal{V}$ satisfies $(\mathcal{S})$, then in view of 1.12,

$\mathcal{VH}(G) = \lim_n H\nu_n(G) = \lim_n H\bar{\omega}_n(G) = \mathcal{VH}(G)$,

and thus $\mathcal{VH}(G)$ must also be a (DFS)-space. To deduce directly that if $\mathcal{V}$ satisfies $(\mathcal{S})$, then $(\mathcal{S}_H)$ must also hold, it is useful to observe 1.2(v) and 1.2(viii). It is trivial to exhibit an example of a sequence $\mathcal{V} = (\nu_n)$ with (S) such that $\mathcal{V} = (\bar{\omega}_n)$, is indeed a sequence of weights, but does not satisfy $(\mathcal{S})$: Just take $G = C$, fix $n_0 \in \mathbb{N}$ and let $\mathcal{V} = (\nu_n)$ be defined by

$\nu_n(z) := (\max(1, |z|^{-n_0+pn}))^{-1}$ for $z \in C$, $n = 1, 2, \ldots$,

where $(p_n)_n$ is an increasing sequence of numbers in $(0, 1)$. Note that each $\bar{\omega}_n(z)$ is of the form $(\max(1, |z|^{-n}))^{-1}$ by 1.3(a) and that all $\bar{\omega}_n$ are radial. Compare with Proposition 3.5 below and with the note after the proof of 3.5.

Similarly, $(\mathcal{M})$ implies $\mathcal{H}(G)$ Montel, and hence $(\mathcal{M}) \Rightarrow (\mathcal{M}_H)$; using [10], Prop. 5.2(3), (M) also can give a direct proof. A later example will serve to show that $(\mathcal{M}_H)$ does not imply (M). Finally, $(\mathcal{RD})$ implies $\mathcal{VH}(G)$ boundedly retractive, and hence one has $(\mathcal{RD}) \Rightarrow (\mathcal{RD}_H)$. This time, it is not so easy to give a direct proof, but it follows from [7], Theorem 1.1(a), 2(v). Later on, there will also be an example of a boundedly retractive space $\mathcal{VH}(G)$ for which $\mathcal{V}$ does not satisfy $(\mathcal{RD})$, and hence $(\mathcal{RD}_H)$ does not imply $(\mathcal{RD})$. Once more, if all $\bar{\omega}_n$ are weights and if $\mathcal{V}$ satisfies $(\mathcal{RD})$, then $\mathcal{VH}(G)$ (or $\mathcal{VH}(G)$) must have $(\mathcal{RD}_H)$; using [7], one can deduce this directly.

At this point, we turn to examples, some of which were promised before. We start with a simple construction, based on ideas from [15].

2.2. LEMMA. Let $G_1$ be an open subset of $C$, and put $G := G_1 \times C$. Let $T = (\nu_n)_n$ and $U = (\nu_n)_n$ be decreasing sequences of weights on $G_1$ and $C$, respectively. Let $0 < p < 1$ and assume that for each $n \in \mathbb{N}$ and each $z \in C$, $(1 + |z|)^{-p} \leq \nu_n(x) \leq 1$. Put $\mathcal{V} := (\nu_n, \nu_n(z_1, z_2) := \nu_n(z_1)\nu_n(z_2)$ for $(z_1, z_2) \in G$, $n \in \mathbb{N}$. Then:

(a) Every $f \in \mathcal{VH}(G)$ is constant in the second coordinate $z_2$.

(b) $\mathcal{VH}(G)$ and $\mathcal{VH}(G_1)$ are canonically isomorphic.

(a) follows from Liouville’s theorem (cf. 1.3(b)), and then a short glance at the map $f \mapsto f(\cdot, 0)$ suffices to see (b).
2.3. Example. Let $G_1 \subset C$ be open, and let the decreasing sequence $T = (\ell_n)_n$ on $G_1$ satisfy $(S)$. Then $TH(G_1)$ is a (DFS)-space. If we take $\ell_n \equiv 1$ on $C$ for each $n \in \mathbb{N}$ and if $V = (\ell_n)_n$ on $G = G_1 \times C$ is defined as in 2.2, then $V$ cannot satisfy $(S)$ on $G$, but by Lemma 2.2(b), $VH(G) \cong TH(G_1)$ is a (DFS)-space, and hence $VH(G) = H\overline{V}(G)$ topologically.

We note that each $f \in HV_n(G)$ in 2.3 is constant in the second coordinate by 2.2(a), and thus also each $\overline{v}_n$ must be constant in $s_2$, $n = 1, 2, \ldots$. Now it is easy to arrange situations as above in which all $\overline{v}_n$ are weights, but the sequence $V = (\overline{v}_n)_n$ does not satisfy $(S)$. The preceding example also serves to show that $(M_H)$ does not imply $(M)$. In fact, $V$ as in 2.3 does not have $(M)$ on $G$, and neither does $\overline{V}$.

Before passing to our next example, which will be constructed along the same lines as 2.3, but will in some sense be even “worst possible”, let us recall two other conditions on $V$ which arose in the corresponding projective description problem for spaces of continuous functions.

Thus, for a moment, let $X$ denote a locally compact and $\sigma$-compact space and $V = (\ell_n)_n$ a decreasing sequence of positive continuous functions on $X$. Following [10], $V$ is said to satisfy condition $(D)$ if there exists an increasing sequence $J = (X_m)_{m \in \mathbb{N}}$ of (closed) subsets of $X$ such that the following two properties hold:

\begin{align*}
(N, J) & \quad \forall m \in \mathbb{N} \exists n \geq m : \inf_{X_m} \ell_k > 0, \quad k = n_{m+1}, n_{m+2}, \ldots, \\
(M, J) & \quad \forall n \in \mathbb{N} \forall Y \subset X \text{ with } Y \cap (X \setminus X_n) = \emptyset \text{ for all } m \in \mathbb{N} \\
& \exists n' = n'(n, Y) > n : \inf_{Y} \frac{\ell_{n'}}{\ell_{n}} = 0.
\end{align*}

And $V$ satisfies $(ND)$ if there exists a decreasing sequence $J = (J_k)_k$ of subsets of $X$ such that

$$
\exists n_0 \in \mathbb{N} \forall k \geq n_0 \exists k > k : \inf_{J_k} \frac{\ell_k}{\ell_{n_0}} > 0, \quad \text{but} \quad \inf_{J_k} \frac{\ell_k}{\ell_{n_0}} = 0.
$$

It is easy to see that (even in the generalized setting above) each of the conditions $(M)$ and $(RD)$ implies $(D)$; thus $(D)$ is the weakest condition on $V$ considered so far. And by [3], Lemma 1, $(ND)$ means precisely that condition $(D)$ is not satisfied.

Now let the spaces $Co_n(X)$, $VC(X) = \text{ind}_X Co_n(X)$ and $CV(X)$ be the exact analogs of $HV_n(G)$, $VH(G) = \text{ind}_X HV_n(G)$ and $H\overline{V}(G)$ for $X$ replacing $G$ and continuous functions taking the role of the holomorphic ones. Then $VC(X) = CV(X)$ topologically if and only if $V$ satisfies $(D)$ (cf. [6], Theorem 11).

2.4. Example. Here, let simply $G_1 = C$, and take a decreasing sequence $T = (\ell_n)_n$ of weights $\ell_n$ on $C$ with $\ell_n(0) = 1$ for all $n$ such that $T$ satisfies $(S)$. Hence $TH(C)$ is a (DFS)-space. For any decreasing sequence $U = (\ell_n)_n$ on $C$ which satisfies the assumption of 2.2, if $V = (\ell_n)_n$ on $C_1 = C \times C$ is defined as in 2.2, then Lemma 2.2(b) yields $VH(C^2) \cong TH(C)$ (DFS), and projective description holds. By Theorem 2.1, $V$ satisfies $(S_H)$, hence a fortiori $(M_H)$ and $(RD_H)$.

We will now construct $U = (\ell_n)_n$ as above in such a way that $V = (\ell_n)_n$ does not satisfy $(D)$. Then, by what was said before, $VC(G) \neq CV(G)$ topologically, and $V$ cannot satisfy $(RD)$.

For $n \in \mathbb{N}$ and arbitrary $k \in \mathbb{N}$, let first $\ell_n : \mathbb{R} \to \mathbb{R}$ be given on $[2k, 2k + 1]$ by

$$
\ell_n(r) = \begin{cases} 1 & \text{for } r \in [2k, 2k + 2^{-n}) \\
1 + r^{-(n-1)/(2n)} & \text{for } r \in [2k + 1, 2k + 2), \\
(1 + r)^{-1/(2n + 4k + 1)} & \text{for } r \in [2k + 2^{-n-1}, 2k + 1 - 2^{-n-1}],
\end{cases}
$$

with $\ell_n$ affine on $[2k + 2^{-n}, 2k + 3 \cdot 2^{-n-1}]$ and $[2k + 1 - 2^{-n}, 2k + 1]$. Distinguishing several cases, it is elementary to verify that the sequence $(\ell_n)_n$ is decreasing. Now extend $\ell_n$ radially, $\ell_n(x) = \ell_n(|x|)$ for $x \in C$, $n = 1, 2, \ldots$, to obtain a decreasing sequence $(\ell_n)_n$ on $C$ for which the assumption of 2.2 holds with $p = 1/2$.

Now we wish to show that $V$ satisfies $(ND)$. For arbitrary $n \in \mathbb{N}$, take $J_n := \{(0, 2k+2^{-n}) : k, m \geq n\}$, given $n \geq n_0 := 1$, for any $(0, 2k+2^{-n}) \in J_n$ clearly $\ell_n(0, 2k+2^{-n}) = 1$, but also

$$
\ell_n(0, 2k+2^{-n}) = \ell_n(0) \ell_n(2k+2^{-n}) = 1
$$

since $n \geq n_0$; hence $\inf_{J_n} \ell_n / \ell_n > 0$. But for $l_n := n + 1$, $k \geq n$, $\ell_{n+1}(0, 2k+2^{-n}) = \ell_{n+1}(2k+2^{-n}) = (1 + 2k+2^{-n})^{-n/(2n+2)}$, and

$$
\inf_{J_n} \ell_n / \ell_n = 1 + 2k+2^{-n} 
$$

which proves that $\ell_n$ satisfies $(ND)$.

In the rest of this section, we sketch a new approach to projective description which directly takes the associated weights $\overline{v}_n$ into account.

Since the sequence $\overline{V} = (\overline{v}_n)_n$ is intrinsically defined, and hence “more natural” than the original sequence $V = (v_n)_n$ on some open $G \subset C^N$, one is led to introduce—instead of $\overline{V} = \overline{V}(\overline{V})$—its analog $\overline{V} := \overline{V}(\overline{V})$, i.e.,

$$
\overline{V} = \{ \overline{v} \text{ weight on } G : \text{for each } n \in \mathbb{N}, \overline{v} / \overline{v}_n \text{ is bounded on } G \}.
$$

Since $\overline{v}_n \leq \overline{v}_n$ for each $n \in \mathbb{N}$, it is clear that $\overline{V} \subset \overline{V}$. For arbitrary $\overline{v} \in \overline{V}$, put $p_G(f) := \sup_G |f| / f$ for $f \in VH(G)$.

2.5. Lemma. (a) For each $\overline{v} \in \overline{V}$, $p_G$ defines a continuous seminorm on $\overline{V}$.}
We finish this section with an example in which the topologies of $H\bar{V}(G)$ and $HV(G)$ are different. Note that in this example the sequence $\mathcal{V}=(v_n)_n$ satisfies condition $(M)$.

2.7. Example. In the sequel, we stick to the notation of [15]. In Section 2 of that article, the authors constructed a decreasing sequence $\mathcal{W}=(w_n)_n$ of weights on the open set $G_1=\{z\in C: 1/2 < |z| < 1, 0 < \arg z < \pi\}$ such that $H\bar{W}(G_1)$ contains a complemented subspace isomorphic to a non-borelological sequence space $K_\infty$ (i.e., to the strong dual of some non-distinguished Köthe echelon space $\lambda_1$); hence $HV(G_1)$ and $H\bar{W}(G_1)$ have different topologies.

Then, in [15], Section 3, $\mathcal{V}=(v_n)_n$, $v_n(x_1, x_2) = w_n(x_1)u_{(x_2)}$, is constructed on $G = G_1 \times C$ in such a way (similar to the method of our Lemma 2.2) that $\mathcal{V}$ satisfies $(M)$, but $H\bar{W}(G) \neq H\bar{V}(G)$ topologically. Each $f \in HV(G)$ is constant in the second variable; the map $A: HV(G) \to H\bar{W}(G)$, $Af(x_1) = f(x_1, 0)$ for $x_1 \in G_1$, $f \in HV(G)$, is a linear bijection with $A^{-1}$ continuous, but $A$ is not continuous.

Similarly to our Example 1.3(b), one can now verify that the corresponding associated weights satisfy

$$\bar{w}_n(x_1) \leq \alpha_n \bar{u}_n(x_1, x_2) \quad \text{for } (x_1, x_2) \in G,$$

where $\alpha_n$ is some positive constant, $n = 1, 2, \ldots$ Now, we shall prove that $A: HV(G) \to H\bar{W}(G)$ is continuous, whence so is $A: H\bar{W}(G) \to H\bar{W}(G_1)$ and thus $HV(G) \neq H\bar{V}(G)$ topologically.

Fix an arbitrary $\bar{w} \in \bar{W}$ and define $\bar{v}(x_1, x_2) := \bar{w}(x_1), (x_1, x_2) \in G$. For each $n \in \mathbb{N}$, putting $\alpha_n := \sup_{G_1} \bar{u}_n(x_1, x_2)$, we get

$$\bar{w}(x_1, x_2) = \bar{w}(x_1) \leq \alpha_n \bar{u}_n(x_1) \leq \alpha_n C_n \bar{u}_n(x_1, x_2), \quad (x_1, x_2) \in G.$$

Hence $\bar{v}$ is an element of $\bar{V}$, and for any $f \in H\bar{V}(G)$,

$$p_{\bar{V}}(Af) = \sup_{G_1} \bar{w}(Af) = \sup_{G_1} \bar{w}(x_1) \cdot |f(x_1, 0)|,$$

which implies the desired continuity.

To complete the picture (as far as possible), let us note that $A^{-1} : H\bar{W}(G_1) \to H\bar{V}(G)$ is continuous as well so that $H\bar{W}(G_1)$ and $H\bar{V}(G)$ are isomorphic. Indeed, Definition 1.1 implies that each $\bar{v}_k$ must also be constant in the second coordinate. And from $0 \leq w_k \leq 1$, it follows not only that $w_k(x_1, x_2) \leq w_k(x_1)$, but in view of 1.2(vii), it is also clear that $\bar{w}_k(x_1, x_2) \leq \bar{w}_k(x_1)$ for $(x_1, x_2) \in G$, $k = 1, 2, \ldots$. Then, proceeding along the lines of the corresponding part of the proof of [15], Prop. 4, the continuity of $A^{-1} : H\bar{W}(G_1) \to H\bar{V}(G)$ becomes obvious.
Thus, we have the following situation:
\[
\begin{align*}
\mathcal{V}(G) & \xrightarrow{\#} \mathcal{V}(G) \\
\mathcal{V}(H(G)) & \xrightarrow{\#} \mathcal{V}(G) \\
\mathcal{V}(H(G)) & \xrightarrow{\#} \mathcal{V}(G) \\
\mathcal{W}(G) & \xrightarrow{\#} \mathcal{W}(G) \\
\mathcal{W}(G) & \xrightarrow{\#} \mathcal{W}(G) \\
\mathcal{W}(G_1) & \xrightarrow{\#} \mathcal{W}(G_1) \\
\mathcal{W}(G_1) & \xrightarrow{\#} \mathcal{W}(G_1) \\
\mathcal{W}(G_1) & \xrightarrow{\#} \mathcal{W}(G_1) \\
\end{align*}
\]

It remains open whether the topologies of $\mathcal{W}(G_1)$ and $\mathcal{W}(G_1)$, and hence also those of $\mathcal{V}(G)$ and $\mathcal{V}(G)$, are different. However, Bonet and Vogt have observed that there are examples of sequences $\mathcal{V} = (v_n)_n$ of weights on $G = D$ such that $v_n = v_n$ for each $n$, hence $\mathcal{V} = \mathcal{V}$, but still $\mathcal{V}(G) \neq \mathcal{V}(G)$ topologically.

In case $\mathcal{V}(G) = \mathcal{V}(G)$ holds topologically, the “old projective description problem” (concerning $\mathcal{V}(G) = \mathcal{V}(G)$ topologically) was equivalent to asking if $\mathcal{V}(G)$ is a topological subspace of $\mathcal{V}(G)$. The disadvantage of the “new projective description problem” (concerning $\mathcal{V}(G) = \mathcal{V}(G)$ topologically) is that it is not directly linked to $\mathcal{V}(G)$ any more. (But, of course, if $\mathcal{V}(G)$ is a topological subspace of $\mathcal{V}(G)$ and if $\mathcal{V}$ satisfies (D), then clearly $\mathcal{V}(G) = \mathcal{V}(G)$ topologically.)

Appendix. In this appendix, we explain a result on compact embeddings of weighted spaces of holomorphic functions which, in some sense, generalizes (the main point behind) Theorem 2.1(a).

A general weighted space $\mathcal{V}(G)$ of holomorphic functions is defined exactly as $\mathcal{V}(G)$, but with $\mathcal{V}$ replaced by a general system $\mathcal{V}$ of non-negative upper semi-continuous functions on $G$, which (can and) will always be assumed to be a Nachbin family, i.e., for $v_1, v_2 \in V$ and any $\lambda > 0$, there is $v_3 \in V$ with $v_1, v_2 \leq v_3$ and for each $x \in G$ there exists $v \in V$ such that $v(x) > 0$. The (Hausdorff) topology of $\mathcal{V}(G)$ is given by the directed system $(\mathcal{V}(G)_v \subset \mathcal{V}(G))$ of seminorms, $p_v(f) = \sup_{x \in G} v(f)$ for $f \in \mathcal{V}(G)$, and $B_v \subset \mathcal{V}(G)$ yields a basis of $\mathcal{V}(G)$ neighborhoods in $\mathcal{V}(G)$, where $B_v := \{ f \in \mathcal{V}(G) : p_v(f) \leq 1 \}$ for any $v \in V$.

In the sequel, let $v_1$ denote a weight, $v_2$ a Nachbin family of weights and $v_3$ a Nachbin family of non-negative continuous functions on $G$. We assume that for each $v_2 \in V$ there is $C > 0$ with $v_2 \leq C v_1$ and that $v_3 \leq v_2$ (in the sense that for each $v_2 \in V$ there is $v_2 \in V$ with $v_2 \leq v_2$). Then the canonical injections

$\mathcal{V}(G) \rightarrow \mathcal{V}(G) \rightarrow \mathcal{V}(G)$

are continuous.

2.8. Theorem. (a) Under our assumptions, $\mathcal{V}(G)$ induces the topology of $\mathcal{V}(G)$ on the unit ball $B_1 := B_{v_1}$ of $\mathcal{V}(G)$ if and only if

\[
\forall v_2 \in V_2 \exists v_3 \in V_3 : \left( \min \left( \frac{1}{v_2} ; \frac{1}{v_3} \right) \right) \leq \frac{1}{v_2}.
\]

(b) The embedding $\mathcal{V}(G) \rightarrow \mathcal{V}(G)$ is compact if and only if

\[
\forall v_2 \in V_2 \exists \varphi \in \mathcal{K}(G) : \left( \min \left( \frac{1}{v_2} , \frac{1}{\varphi} \right) \right) \leq \frac{1}{v_2}.
\]

(c) Let $v_1, v_2$ denote weights on $G$ with $v_2 \leq v_1$, and put $v_k = 1/v_k$, $k = 1, 2$. Then the canonical injection $\mathcal{V}(G) \rightarrow \mathcal{V}(G)$ is compact if and only if

\[
\forall \varepsilon > 0 \exists \varphi \in \mathcal{K}(G) : \left( \min \left( \frac{1}{v_1} ; \frac{1}{\varphi} \right) \right) \leq \varepsilon v_2.
\]

Proof. (a) Since the topology of $\mathcal{V}(G)$ is weaker than the one of $\mathcal{V}(G)$, the two topologies coincide (at 0) on $B_1$, and only if, for each $v_2 \in V_2$, there is $v_3 \in V_3$ with $B_1 \cap B_{v_3} \subset B_{v_2}$ or, equivalently (since $B_1 \subset \mathcal{V}(G) \subset \mathcal{V}(G)$),

\[
B_1 \cap \{ f : v_3 |f| \leq 1 \} \subset \{ f : v_2 |f| \leq 1 \}.
\]

But the proof of Theorem 2.1 shows that this inclusion is equivalent to

\[
\left( \min \left( \frac{1}{v_1} ; \frac{1}{v_3} \right) \right) \leq \frac{1}{v_2}.
\]

(b) Since $V_2$ is a system of continuous positive functions, the topology of $\mathcal{V}(G)$ must be stronger than $\mathcal{V}(G)$. Hence the embedding $\mathcal{V}(G) \rightarrow \mathcal{V}(G)$ is compact if and only if it induces the topology of $\mathcal{V}(G)$ on $B_1$. But for the Nachbin family $V_3 = \mathcal{K}(G)$, we have $\mathcal{V}(G) = (H(G), \mathcal{V}(G))$, and thus it suffices to apply (a). Finally, (c) follows directly from (b) by taking $V_2 = \{ \lambda v_2 : \lambda > 0 \}$.

Associated weights can also serve to characterize (the boundedness and) the compactness of composition operators between weighted Banach spaces of holomorphic functions (see [13]).

3. Additional results and remarks. This section is divided into three subsections. Part A is devoted to estimates for $\mathcal{V}$ when $w$ is a radial growth condition on $C$ satisfying certain natural assumptions (Prop. 3.1).
In Part B, it is first pointed out (Prop. 3.4) that for radial weights \( v \) with condition \( (U) \) on the unit disk, Shields and Williams [26] already showed that \( v \) and the associated weight \( \overline{v} \) are equivalent. Then we prove (Prop. 3.5) that for a sequence \( \mathcal{V} = (v_n) \) of radial weights on \( D \) for which \( r \to v_n(r) \) is strictly decreasing, \( n = 1, 2, \ldots \), the space \( VH(D) \) is \( (DFS) \) if and only if the sequence \( \mathcal{V} = (\overline{v}_n) \) of associated weights satisfies condition \( (S) \). This is remarkable because a simple example after Theorem 2.1 demonstrated that such a result cannot hold for \( C \) instead of \( D \). Finally, there is a way to calculate \( \overline{w} \), up to a constant, if \( w = 1/v \) is a growth condition on \( D \) which has a radial limit function \( w^* \) on \( \partial D \) with certain properties: In this situation, \( \overline{w} = \frac{1}{1/\overline{v}} \) is equivalent to the modulus of the corresponding outer function \( Q_{w^*} \) (Prop. 3.6).

Part C sketches applications of Hörmander’s \( \overline{\partial} \)-technique and of Phragmén–Lindelöf type theorems to estimates for \( \overline{w} \) on \( C^N \) (Props. 3.8, 3.9). We finish the article by constructing an example of a space \( VH(G) \) with some “strange” properties, this time on a set \( G \) which is the union of a sequence of pairwise disjoint disks in \( C \).

In contrast to the previous sections, not all the arguments here are given in (full) detail, and some (easy) calculations are omitted. But we provide the reader with sufficient information and relevant references.

3. A. Estimates for \( \overline{w} \) on \( C \). Let \( w \) be a radial growth condition on \( C \). If \( r \to w(r) \) is increasing and logarithmically convex for \( r \in [1, \infty) \), then \( w \) satisfies

\[
(*) \quad w(r) = w(1) \exp \left( \frac{\omega(t)}{t} \right) dt, \quad r \geq 1,
\]

for some positive increasing function \( \omega \) (see Clunie–Kővári [17], Theorem 4). Classical results on Taylor series and growth conditions of entire functions lead to estimates of \( \overline{w} \), as follows.

3.1. PROPOSITION. Let \( w \) be a radial growth condition on \( C \).

(a) If (even) \( r \to w(r) / \sqrt{r} \) is increasing and logarithmically convex for \( r \geq 1 \), and if \( 1/w \) is rapidly decreasing (i.e., for every \( n \in \mathbb{N} \), \( w(r)/r^n \to \infty \) as \( r \to \infty \)), then there is \( C > 0 \) with

\[
\overline{w}(r) \leq w(r) \leq C(1 + r)\overline{w}(r), \quad r \geq 0.
\]

(b) If \( r \to w(r) \) is increasing and logarithmically convex for \( r \geq 1 \), and if, for \( \omega \) as in \( (*) \), there is \( c > 1 \) with \( \omega(c) - \omega(r) \geq 1 \) for all \( r \geq 1 \), then there exists \( C > 1 \) such that

\[
\overline{w}(r) \leq w(r) \leq C\overline{w}(r), \quad r \geq 0.
\]

Proof. (a) By Clunie and Kővári [17], Theorem 2, applied to \( r \to w(r)/\sqrt{r} \), there is an entire function \( f \) (whose Taylor series has positive coefficients) such that

\[
(*) \quad 1 \leq \frac{w(r)}{M(f, r)} \leq 9r, \quad r \geq \frac{9}{5}.
\]

Put \( w_1(z) := M(f, |z|), \quad z \in C \). Then \( w_1 \) is a radial growth condition on \( C \) which satisfies \( w_1(r) \leq w(r) \leq 9w_1(r) \) for \( r \geq 2 \). Hence, by 1.2(vii), \( \overline{w}_1(r) \leq \overline{w}(r) \) for \( r \geq 2 \), and by 1.6, \( \overline{w}_1 = \overline{w} \) holds. It follows that for \( r \geq 2 \),

\[
\overline{w}(r) \leq w(r) \leq 9w_1(r) \leq 9\overline{w}_1(r) \leq 9\overline{w}(r).
\]

The proof of (b) is similar: By [17], Theorem 4, applied to \( r \to w(r) \), \( (*) \) can now be replaced by

\[
1 \leq \frac{w(r)}{M(f, r)} \leq C, \quad r \geq 1.
\]

The conditions of Proposition 3.1 are quite natural: E.g., \( v = 1/w \) rapidly decreasing means exactly that \( H_v(C) \) (or \( H_{uv}(C) \)) contains the polynomials. And if the radial growth condition on \( C \) is taken in the form \( w(z) = \exp(\omega(|z|)) \), \( z \in C \), for a continuous increasing function \( \omega : R_+ \to R_+ \), say, \( \omega(0) = \omega(1) = 0 \), then supposing that log \( w \) is subharmonic is equivalent to assuming that the function \( \psi : \psi(t) := \omega(e^t) \) for \( t \geq 1 \), is convex (see [4], 4.4.19 and 4.4.26), i.e., \( r \to w(r) \) is logarithmically convex (cf. the notes after 1.2 and 1.6). In this setting, \( 1/w \) is rapidly decreasing if and only if \( \log r = o(\omega(r)) \) as \( r \to \infty \).

Let us now prove directly (without recourse to [17]) that, in the setting which we have just described, \( w(r) \leq w_1(r) \) for \( r \geq 1 \).

We will make use of the Young conjugate \( \varphi^* : R_+ \to R_+ \) of \( \varphi \); it is defined by the formula \( \varphi^*(y) := \sup\{xy - \varphi(x) : x \geq 0\} \). \( \varphi^* \) is again increasing, convex and satisfies \( \varphi^*(0) = 0 \), \( y = o(\varphi^*(y)) \) as \( y \to \infty \); moreover, \( \varphi^* = \varphi \) (cf. e.g. [16]).

First note that for \( n \in \mathbb{N} \),

\[
\inf_{r > 0} \frac{\exp(\omega(r))}{r^n} = \inf_{r > 1} \frac{\exp(\omega(r))}{r^n} = \inf_{r > 1} \exp\left(\varphi^*(n)\right) = \exp(\varphi^*(n)) = \exp(\varphi^*(n)),
\]

hence

\[
(*) \quad 0 < \exp(-\varphi^*(n)) \leq \frac{\exp(\omega(r))}{r^n} \quad \text{for each } r > 0.
\]
Next, to prove the desired estimate, we fix \( r_0 \geq 1 \), let \( 0 < \varepsilon < 1 \) and choose \( n_0 \in \mathbb{N} \) with
\[
\varepsilon n_0 \exp(-\varepsilon^* (n_0)) > (1 - \varepsilon) \sup_{n \in \mathbb{N}} (\varepsilon n_0 \exp(-\varepsilon^* (n))).
\]
Then \( g(z) := \exp(-\varepsilon^* (n_0))\) for \( z \in \mathbb{C} \) defines an entire function \( g \) such that, in view of (*),
\[
|g(z)| = \exp(-\varepsilon^* (n_0)) |z|^{n_0} \leq \exp(\omega(|z|)) = w(z)
\]
for each \( z \in \mathbb{C} \), while
\[
w(r_0) = \exp \varphi^* (\log r_0) = \sup_{i \geq 0} (r_0^i \exp(-\varepsilon^* (i)))
\]
\[
\leq \sup_{n \in \mathbb{N}} (r_0^{n+1} \exp(-\varepsilon^* (n))) \leq \frac{r_0}{1 - \varepsilon} g(r_0)
\]
by our choice of \( n_0 \). So far, we have proved that for each \( \varepsilon \in (0,1) \) there is \( g = g_\varepsilon \in B_w \) (see Def. 1.1) with \( g(r_0) > (1 - \varepsilon)w(r_0)/r_0 \). Since \( \varepsilon > 0 \) is arbitrary and \( B_w \) is compact in the topology of pointwise convergence, it follows that for each \( r_0 \geq 1 \) there exists \( f \in B_w \) with
\[
|f(r_0)| \geq \frac{1}{r_0} w(r_0), \quad \text{whence} \quad \overline{w}(r_0) \geq \frac{1}{r_0} w(r_0).
\]

It is interesting to note that if, in addition, \( \varphi' \) exists and is strictly increasing, then there is an increasing sequence \( \{\varphi_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}_+^* \), \( \varphi_n \to \infty \), such that \( \overline{w}(\varphi_n) = w(\varphi_n) \) for each \( n \in \mathbb{N} \).

In fact, put \( \varphi_n := (\varphi')^{-1}(n) \) and \( \varphi_n := e^{\varphi_n} \), \( n = 1, 2, \ldots \); since \( y = o(\varphi(y)) \), clearly \( \lim_{n \to \infty} \varphi_n = \infty \). Now, for arbitrary \( n \in \mathbb{N} \), a glance at the derivative yields
\[
w(\varphi_n) = \exp \omega(\varphi_n) = \exp \varphi(r_n) = \exp(\max(y r_n - \varphi^*(y)))
\]
\[
= \exp(nr_n - \varphi^*(n)) = r_n \exp(-\varphi^*(n)).
\]
On the other hand, \( g(z) := \exp(-\varepsilon^* (n))z^{n_0} \) for \( z \in \mathbb{C} \) defines an entire function \( g \) which, in view of (*), satisfies \( |g| \leq w \) on \( \mathbb{C} \) and \( g(\varphi_n) = w(\varphi_n) \), i.e., \( \overline{w}(\varphi_n) = w(\varphi_n) \).

Now we modify an idea of [17], Theorem 3, to show that, even under the conditions of Proposition 3.1(a), \( \overline{w} \) can really be essentially smaller than \( w \).

For this purpose, let us fix a sequence \( \{r_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}_+^* \) with \( r_0 = r_1 = 1, \ r_n < r_{n+1} \) for \( n \geq 1 \), \( \lim_{n \to \infty} r_n = \infty \), and set \( A_n := (r_1 \ldots r_n)^{-1} \), \( n \in \mathbb{N} \). Next, put \( w_{[0,1]} \equiv 1 \) and \( w(r) := A_n r^{n+1}/2 \) for \( r \in [r_n, r_{n+1}) \), \( n = 1, 2, \ldots \) Then \( w : \mathbb{R}_+^* \to \mathbb{R}_+^* \) is continuous, increasing and satisfies \( \lim_{n \to \infty} r^n/w(r) = 0 \) for each \( n \in \mathbb{N} \); moreover, the corresponding function \( \varphi, \varphi(t) = \log(w(e^t))/\sqrt{e^t} \) for \( t \in \mathbb{R}_+^* \), is convex. Finally, extend \( w \) to a radial function on \( \mathbb{C} \) by taking \( w(z) := w(|z|) \) for each \( z \in \mathbb{C} \).

3.2. Lemma. Let \( f \) be an entire function with \( |f| \leq w \) on \( \mathbb{C} \) such that the Taylor series of \( f, f(z) = \sum_{k=0}^{\infty} a_k z^k \), has nonnegative coefficients \( a_k \), \( k \in \mathbb{N}_0 \). Then
\[
0 \leq f(r) \leq \left( \frac{r_n}{r} \right)^{\frac{1}{2}} + \left( \frac{r}{r_{n+1}} \right)^{\frac{1}{2}} w(r), \quad r \in [r_n, r_{n+1}), \ n = 1, 2, \ldots
\]

**Proof.** Fix \( n \in \mathbb{N} \). Clearly, \( |f| \leq w \) implies
\[
\sum_{k=0}^{\infty} a_k r^{k-n-1/2} \leq A_n \left( \frac{r_n}{r} \right)^{\frac{1}{2}}, \quad r \in [r_n, r_{n+1}).
\]
Hence, for all such \( r \),
\[
\sum_{k=0}^{n} a_k r^{k-n-1/2} = \left( \frac{r_n}{r} \right)^{1/2} \sum_{k=0}^{n} a_k r_n^{k-n-1/2} \left( \frac{r_n}{r} \right)^{n-k} \leq A_n \left( \frac{r_n}{r} \right)^{1/2},
\]
\[
\sum_{k=n+1}^{\infty} a_k r^{k-n-1/2} = \left( \frac{r_{n+1}}{r_{n+1}} \right)^{1/2} \sum_{k=n+1}^{\infty} a_k r_{n+1}^{k-n-1/2} \left( \frac{r_{n+1}}{r_{n+1}} \right)^{k-n-1} \leq A_n \left( \frac{r_{n+1}}{r_{n+1}} \right)^{1/2}.
\]
Summing up, we get
\[
\sum_{k=0}^{\infty} a_k r^{k-n-1/2} \leq A_n \left( \frac{r_n}{r} \right)^{1/2} + \left( \frac{r}{r_{n+1}} \right)^{1/2},
\]
and multiplication by \( r_n^{n+1}/2 \) yields the desired inequality.

3.3. Example. For \( w \) as above, for \( n \in \mathbb{N} \) and \( r \in [r_n, r_{n+1}) \), the corresponding \( \overline{w} \) satisfies
\[
\overline{w}(r) \leq \frac{1}{18} \left( \frac{r_n}{r} \right)^{1/2} + \left( \frac{r}{r_{n+1}} \right)^{1/2} w(r),
\]

**Proof.** Fix \( g \in B_w \). By a result of Erdös–Kővári [18], there is a function \( f \in H(\mathbb{C}), f(z) = \sum_{k=0}^{\infty} a_k z^k \) with \( a_k > 0 \) for all \( k \in \mathbb{N}_0 \), such that
\[
\frac{1}{18} f(r) \leq M(g, r) \leq 3 f(r) \quad \text{for each} \ r > 0.
\]
Since \( w \) is radial and \( |g| \leq w \), it follows from (*) that \( f(r) \leq 6 w(r) \), \( r > 0 \). Now Lemma 3.2 yields, for \( n \in \mathbb{N} \) and each \( r \in [r_n, r_{n+1}) \),
\[
f(r) \leq 6 \left( \frac{r_n}{r} \right)^{1/2} + \left( \frac{r}{r_{n+1}} \right)^{1/2} w(r),
\]
hence by another application of (\*),

\[ M(g, r) \leq 18 \left( \left( \frac{r_n}{r} \right)^{1/2} + \left( \frac{r}{r_{n+1}} \right)^{1/2} \right) w(r). \]

Finally, invoking Observation 1.5, one realizes that the last estimate also holds for \( \tilde{w}(r) \) instead of \( M(g, r) \).

Note that, taking \( r_{n+1} = 2^nr_n \) for \( n = 1, 2, \ldots \), one gets by 3.3,

\[ \tilde{w}(2^{n/2}r_n) \leq 36 \left( \frac{1}{2^n} \right)^{1/6} w(2^{n/2}r_n). \]

3.B. Functions on the unit disk. Let \( \psi : [0, 1] \to \mathbb{R}_+ \) be a continuous decreasing function with \( \psi(0) = 1, \psi(1) = 0 \) and \( \psi(r) > 0 \) for \( r \in (0, 1) \). Then \( \psi \) becomes a radial weight on \( D \) by taking \( \psi(z) := \psi(|z|) \), \( z \in D \). The auxiliary function \( \psi : [1, \infty) \to \mathbb{R}_+ \setminus \{ 0 \} \) is defined by \( \psi(t) := 1/\psi(1-t) \), \( t \geq 1 \); note that \( 1/\psi(r) = \psi(1/(1-r)) \), \( r \in [0, 1] \). According to Shields–Williams [26], \( \psi \) is said to satisfy condition \((U)\) if there are \( a, c > 0 \) such that

\[ 1 \leq x < y \Rightarrow \frac{\psi(y)}{y^a} \leq c \frac{\psi(x)}{x^a}; \]

in this case, we will also say that \( \psi \) satisfies condition \((U)\). As communicated to the authors by W. Luxky, \((U)\) is equivalent to his condition (\*) (on \( \psi \)) in [21]. For examples of weights \( \psi \) which satisfy \((U)\), we refer to [21] or [26]. Now Proposition 3.4 follows directly from Shields–Williams [26], Lemma 1(iv).

3.4. Proposition. For a radial weight \( \psi \) on \( D \) as above (which, in particular, satisfies \((U)\)), there is a constant \( C > 0 \) such that

\[ \psi(r) \leq \tilde{\psi}(r) \leq Cr(r) \quad \text{for each } r \in [0, 1], \]

i.e., \( \psi \) and \( \tilde{\psi} \) are equivalent weights.

If the radial weight \( \psi \) on \( D \) decreases very rapidly (e.g., exponentially) as \( r \to 1^- \), \((U)\) is not satisfied. Then the corresponding growth condition \( \tilde{\psi} = 1/\psi \) increases very rapidly. The method of Proposition 3.1(b) can still be used to prove equivalence of \( \psi \) and \( \tilde{\psi} \) (under certain assumptions).

We suppose that the radial growth condition \( \psi \) on \( D \) satisfies

\[ \psi(r) = \omega(0) \exp \int_0^r \frac{\tau(\theta)}{1 - \theta} d\theta, \quad 0 \leq r < 1, \]

where \( \tau \) is a positive increasing function on \( [0, 1] \) such that, for some \( c > 1 \) and all \( \theta \in [0, 1], \)

\[ \tau(1 - (1 - \theta)/c) - \tau(\theta) \geq 1. \]

Then there is \( C > 1 \) with \( \tilde{\psi} \leq \tilde{w} \leq C\tilde{w} \) on \( D \).

In fact, starting from \( \tau \), one defines a positive increasing function \( \psi \) on \([1, \infty)\) by \( \psi(\theta) := \tau(1 - 1/\theta) \) and puts

\[ \Phi(r) := \omega(0) \exp \int_0^r \frac{\psi(\theta)}{\theta} d\theta, \quad r \geq 1. \]

Since \( \Phi(1/(1-r)) = \psi(r) \) for all \( r \in [0, 1] \) and (\*) holds, \( \Phi \) satisfies the assumptions of [17], Theorem 4. Hence there is an entire function \( f \) whose Taylor series has positive coefficients such that, for some \( C_0 > 0, \)

\[ C_0^{-1} \leq \frac{\Phi(r)}{M(f, r)} = \tilde{\Phi}(r) \leq C_0, \quad r \geq 1. \]

Define \( g(z) := f(1/(1-z)) \) and \( w_1(z) := g(|z|) \) for \( z \in D \). Then \( g \) is an analytic function on \( D \) whose Taylor series at 0 also has positive coefficients, and hence the growth condition \( \tilde{w}_1 \) on \( D \) satisfies \( \tilde{w}_1 = \omega_1 \) by 1.6. Moreover,

\[ \psi(r) = \frac{\Phi(1/(1-r))}{M(f, r)} \leq C_0 \psi \left( \frac{1}{1-r} \right) = C_0 w_1(r), \quad 0 \leq r < 1, \]

and \( w_1(r) \leq C_1 \tilde{w}(r) \) follows in the same way. Now 1.2(vii) yields

\[ w(r) \leq C_0 w_1(r) = C_0 \tilde{w}_1(r) \leq C_1 \tilde{w}(r), \quad 0 \leq r < 1. \]

Next we turn to the characterization of the (DFS)-property of the space \( V(H(D) = \text{ind}_w H_n(D) \) in terms of condition \((S)\) (cf. Section 2) on the sequence \( \tilde{w} = (w_n)_n \) of associated weights \( w_n \), when \( V = (w_n)_n \) is a decreasing sequence of “nice” radial weights \( w_n \) on \( D \).

3.5. Proposition. Let \( V = (w_n)_n \) denote a decreasing sequence of positive continuous radial weights such that each \( r \to w_n(r) \) is strictly decreasing on \([0, 1], n = 1, 2, \ldots \). Then \( V(H(D) \) is a (DFS)-space if and only if \( \tilde{w} = (w_n)_n \) satisfies \((S)\).

Proof. In a remark after the proof of Theorem 2.1, it was already pointed out that \( V(H(D) = \tilde{V}(H(D)) \) is a (DFS)-space if \( \tilde{w} \) satisfies \((S)\) (and this holds for any sequence \( V \) on an arbitrary open set \( G \subset C^N \)).

Conversely, assume that \( V(H(D) \) is a (DFS)-space and again write \( w_n = 1/w_n \), \( n = 1, 2, \ldots \). Exactly as in the proof of Theorem 2.1, it is shown that

\[ \text{for each } n \in \mathbb{N} \text{ there is } m > n \text{ with the property that for each } \epsilon > 0 \text{ one can find } \delta > 0 \text{ without loss of generality, } \delta < w_n(0) \text{ and } r \in (0, 1) \text{ such that each } f \in H(D) \text{ with } |f| \leq w_n \text{ on } D \text{ and } |f| \leq \delta \text{ on } \tilde{D}, \text{ the closed disk of radius } r \text{ around } 0, \text{ must satisfy } |f| \leq C \tilde{w} \text{ on } D. \]

To show that \( \tilde{w} = (w_n)_n \) satisfies \((S)\), fix \( n \in \mathbb{N} \) and choose \( m > n \) as in (\*); for arbitrary \( \epsilon > 0 \), take \( \delta > 0 \) and \( r \in (0, 1) \) as provided by (\*). We claim that there exists \( r_0 \in [r, 1) \) such that \( \tilde{w}_m(z) \leq 2 \tilde{w}_n(z) \) for all \( z \in D \setminus \tilde{D}_m \).
This claim will follow after the introduction of an auxiliary function \( \varphi \). Let us first put \( a := \delta(1-r)/(w_n(r) - \delta) > 0 \); clearly \( a/((1+a-r)) = \delta/w_n(r) \).

Now define \( \varphi \in H(D) \) by \( \varphi(z) := a/(1+a-z) \), \( z \in D \); it is easy to check that

1. \( |\varphi| \leq 1 \) on \( D \) and
2. \( |\varphi| \leq \delta/w_n(r) \) on \( \partial D \).

Since \( \varphi(z) \) is real for \( z \in [0,1] \) and \( \lim_{z \to 1} \varphi(z) = 1 \), there is \( r_0 \in (r,1) \) such that \( \varphi(z) > 1/2 \) for \( z \in [r_0,1] \).

To prove the claim, we fix \( z \in D \setminus \partial D \) and let \( s := |\zeta| \), whence \( s > r_0 \). By Observation 1.5, \( \omega_n \) and \( \omega_m \) are radial, i.e., \( \omega_n(z) = \omega_n(s) \) and \( \omega_m(z) = \omega_m(s) \). By 1.2(2), there is \( g \in H(D) \) with \( |g| \leq w_n \) on \( D \) such that \( g(s) = \omega_n(s) \). Now, define \( f := \varphi g \in H(D) \). By (i), we have \( |f| \leq w_n \) on \( D \); on the other hand, for \( \zeta \in \partial D \), we get from (ii),

\[
|f(\zeta)| = |g(\zeta)| \cdot |\varphi(\zeta)| \leq w_n(\zeta) \frac{\delta}{w_n(r)} = w_n(|\zeta|) = \frac{\delta}{w_n(r)} \leq \delta.
\]

(* ) implies that \( |f| \leq \varepsilon \omega_m \) on \( D \); in particular, \( |f(s)| \leq \varepsilon \omega_m(s) \). Since \( \varphi(s) > 1/2 \), we finally obtain

\[
\omega_n(z) = \omega_n(s) = |g(s)| = |f(s)| = \frac{|f(s)|}{\varphi(s)} \leq 2 \varepsilon \omega_m(s) = 2 \varepsilon \omega_m(z),
\]

which is equivalent to the claim. \( \square \)

Recently, D. Vogt extended Proposition 3.5 to general open sets \( G \) if the weights \( u_n \) satisfy the condition of 1.13, i.e., he proved the following theorem:

Let \( G \) be an open subset of \( \mathbb{C}^N \) and \( V = (u_n)_n \) a decreasing sequence of weights on \( G \). If, for each \( n \in \mathbb{N} \), \( \bar{v}_n \) is a weight with \( \bar{v}_n = \bar{v}_0 \) (in the notation introduced before 1.13), then the following assertions are equivalent:

1. \( V \) is a (DFS)-space,
2. \( \bar{V} := \left\{ (\bar{v}_n)_{n} \right\} \) is a (DFS)-space,
3. \( \bar{V} = (\bar{v}_n)_{n} \) satisfies (S).

So far, our results in Section 3 have dealt with radial weights. But non-radial weights on the unit disk are also important: They come up naturally e.g. when functions on a half plane \( H \) (with growth conditions on the boundary) are considered and the conformal map \( H \to D \) is used. The following remarks apply to certain non-radial weights \( v \) on \( D \).

For a function \( f : D \to \mathbb{C} \) and \( 0 < r < 1 \), let \( f_r : \partial D \to \mathbb{C} \) be defined by \( f_r(e^{it}) := f(re^{it}), \) \( t \in [-\pi, \pi] \). Now let \( v \) be a weight on \( D \) which has an extension \( v : D \to [0, \infty] \) with \( v|_{\partial D} = \lim_{r \to 1} v_r \), a.e. on \( \partial D \) such that, for \( v := 1/v \) and for some \( 1 \leq p < \infty \), \( \sup_{r \in [0,1]} \|v_r\|_{L^p(\partial D)} < \infty \), \( w|_{\partial D} = w^{*} \in L^p(\partial D) \) and \( \log w^* \in L^1(\partial D) \). In this case, \( H_v(D) \) is contained in the Hardy space \( H^p \) (cf. Rudin [24], 17.7); in particular, each \( f \in H_v(D) \) has a radial limit function \( f^* : \partial D \to \mathbb{C} \), defined at almost all points of \( \partial D \). Also, recall (see [24], 17.14) that for any \( g \in L^p(\partial D) \) with \( g \geq 0 \) and \( \log g \in L^1(\partial D) \), the outer function \( Q_g \in H(D) \) is defined by

\[
Q_g(z) := \exp \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} g(e^{it}) dt \right), \quad z \in D.
\]

3.6. Proposition. Let \( v \) be a weight as above and \( w = 1/v \). If there is a constant \( C > 0 \) with \( C|Q_w| \leq w \) on \( D \), then \( \tilde{w} \) is equivalent to \( |Q_w| \), i.e., \( C|Q_w| \leq \tilde{w} \leq |Q_w| \) on \( D \).

Proof. The first inequality is clear from 1.2(iii). To see the second estimate take an arbitrary \( f \in H(D) \), \( f \neq 0 \), with \( |f| \leq w \) on \( D \). Then \( f \in H_v(D) \subseteq H^p \) and \( |f^*| \leq w^* \) a.e. on \( D \). Now, it follows from [24], Theorems 17.16 and 17.17, that

\[
|f| \leq |Q_f| \leq |Q_w|.
\]

Thus, we get \( \tilde{w} \leq |Q_w| \) on \( D \).

Now take a function \( g \in L^1(\partial D) \) with \( g > 0 \) a.e. and \( \log g \in L^1(\partial D) \). Then the Poisson integral of \( g \) (see [24], 11.7),

\[
w := P[g] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{Re} \left( \frac{e^{it} + z}{e^{it} - z} \right) g(e^{it}) dt,
\]

is harmonic and positive on \( D \); it satisfies all the assumptions on \( w \) needed for 3.6 (with \( p = 1 \) and \( w^* = g \); cf. [24], Corollary to 11.10).

3.7. Corollary. For \( w = P[g] \) in the present situation, one actually has \( \tilde{w} = |Q_g| \).

Proof. From [24], proof of 17.16 (using Jensen’s inequality), \( |Q_g| \leq P[g] = w \) follows, hence by 1.2(iii), \( |Q_g| \leq \tilde{w} \). The second inequality is clear from 3.6. \( \square \)

Note that, using 3.7, one can give simple examples of functions \( w = P[g] \) such that \( \tilde{w}(r) < w(r) \) for each \( r \in [0,1) \). (E.g., let \( g(e^{it}) = m \) for \( t \in [-\pi,0] \) and \( g(e^{it}) = M \) for \( t \in (0,\pi] \), where \( m \neq M \).) And it follows from techniques of [22] that for other interesting weights on \( D \) the conditions of 3.6 are satisfied.

3.C. Final remarks. Let \( p \) denote a nonnegative continuous p.a.e. function on \( \mathbb{N} \) and let \( q \) be defined by \( q(z) := \sup_{r \in [0,1]} |z - r| \leq 1 \), \( z \in \mathbb{C}^N \). Let \( |\cdot| \) denote the Euclidean norm. We consider the continuous
positive function \( w \) on \( \mathbb{C}^N \),
\[
w(z) := \exp(q(z) + 3 \log(1 + |z|^2)) \quad \text{for} \quad z \in \mathbb{C}^N.
\]
The \( \tilde{\omega} \)-techniques of Hörmander can be applied to estimate \( \tilde{\omega} \), as follows.

3.8. Proposition. For \( w \) as above, there is \( M > 0 \) such that \( \exp(p(z)) \leq M \tilde{\omega}(z) \) for each \( z \in \mathbb{C}^N \).

Proof. By a result of Abanin ([1], Lemma 4), there is \( M > 0 \) such that, for each \( z_0 \in \mathbb{C}^N \), there is \( f \in H(\mathbb{C}^N) \) with \( f(z_0) = \exp(p(z_0)) \) and \( |f| \leq Mw \) on \( \mathbb{C}^N \). Now
\[
\frac{1}{M} \exp(p(z_0)) \leq \sup \{ |g(z_0)| : g \in B_w \} = \tilde{\omega}(z_0).
\]

Phragmén–Lindelöf theorems can also be applied to estimate associated weights; here is an example:

3.9. Proposition. Let \( w \) be a positive continuous function on \( \mathbb{C}^N \) such that the following estimates hold for some \( B > 0 \) and some \( n \in \mathbb{N} \):

(i) \( w(z) \leq \exp(B|z|) \) for each \( z \in \mathbb{C}^N \) and
(ii) \( w(z) \leq (1 + |z|)^n \) for each \( z \in \mathbb{R}^N \).

Then \( \tilde{\omega}(z) \leq (1 + |z|)^n \exp(B|\text{Im} z|) \) for each \( z \in \mathbb{C}^N \).

Proof. Let \( f \in H(\mathbb{C}^N) \) satisfy \( |f| \leq w \) on \( \mathbb{C}^N \); then the estimates (i) and (ii) also hold for \( |f| \). Applying the Phragmén–Lindelöf principle as in Hörmander [20], proof of 16.3.10, we conclude
\[
|f(z)| \leq (1 + |z|)^n \exp(B|\text{Im} z|) \quad \text{for} \quad z \in \mathbb{C}^N,
\]
whence the desired estimate.

We note that it is possible to replace the term \( (1 + |z|)^n \) in 3.9 by \( \exp(Cw(z)) \); \( C > 0 \), where \( w \) is a nonquasianalytic weight as in [10]; for example, \( w(z) = |z|^p \), \( 0 < p < 1 \).

For our final example, we adopt the following setting:

Let \( \nu, w \) be continuous real-valued functions on \( [0, 1] \), \( \nu(0) = w(0) = 1 \), \( \nu \) strictly increasing with \( \lim_{s \to 1-} \nu(s) = \infty \) and \( w \) strictly decreasing with \( \lim_{s \to 0} \nu(s) = 0 \). Both are extended to \( D \) radially.

3.10. Lemma. Given \( 0 < \varepsilon < 1 \), let \( r_0 \) be the unique number in \( (0, 1) \) with \( \nu(r_0) = \varepsilon^{-1} \). Put \( b := \nu(r_0)/w(r_0) > 1 \), define \( u := \min(\nu, bw) \) on \( [0, 1] \) and then \( u(z) := u(|z|) \) for \( z \in D \). Now, if \( f \in H(D) \) satisfies \( |f| \leq 1 \), then \( |u(f)| \leq \varepsilon \) on \( D \).

Proof. Fix \( f \in H(D) \) with \( |f| \leq 1 \) on \( D \) and take a point \( z_0 \) with \( |z_0| = r \) such that \( |f(z_0)| = \max_{|z| = r_0} |f(z)| = \max_{|z| \leq r_0} |f(z)| \). Since \( \nu(0) - bw(0) = 1 - b < 0 \), \( \nu(r_0) - bw(r_0) = 0 \) and \( \nu(r) - bw(r) \to \infty \) as \( r \to 1- \), it follows that \( u(r) \) equals \( \nu(r) \) for \( r \leq r_0 \) while \( u(r) = bw(r) \) for \( r \geq r_0 \). Now,
\[
sup_{|z| \leq r_0} w(z)|f(z)| \leq \sup_{|z| \leq r_0} |f(z)| = |f(z_0)|
\]
\[
\leq u(r_0)^{-1} \sup_{|z| \leq r_0} u(z)|f(z)| \leq u(r_0)^{-1} = \varepsilon.
\]
\[
sup_{r_0 \leq |z| < 1} \sup_{z \in D} w(z)|f(z)| = b^{-1} \sup_{z \in D} bw(z)|f(z)| \leq b^{-1} \sup_{r_0 \leq |z| < 1} u(z)|f(z)|
\]
\[
\leq b^{-1} < b^{-1}w(r_0)^{-1} = \nu(r_0)^{-1} = \varepsilon.
\]

Turning to our example, put
\[
D_m := 4m + D, \quad m = 1, 2, \ldots, \quad \text{and} \quad G := \bigcup_{m \in \mathbb{N}} D_m.
\]
The decreasing sequence \( \nu = (\nu_n) \) of weights on \( G \) is defined by
\[
\nu_n(z) := \begin{cases} 
\nu(z - 4m), & z \in D_m, \quad m < n, \\
\nu(z - 4m), & z \in D_m, \quad m \geq n.
\end{cases}
\]

3.11. Example. \( \nu \) satisfies \( (ND) \) (so that the topologies of \( VC(G) \) and \( CV(G) \) are different, cf. Section 2), but \( \mathcal{V}H(G) = \mathcal{H}V(G) \) topologically, and this space is isomorphic to \( \mathcal{O}_N Hw(D) \).

Proof. Let \( n_0 = 1 \) and put
\[
J_k := \{ 4m + 1 - 1/s : m \geq k, \ s \in \mathbb{N} \}, \quad k = 1, 2, \ldots
\]
Clearly, \( (J_k) \) is decreasing, and if \( z \in J_k \), then \( z \in D_m \) for \( m \geq n \), hence \( \nu_n(z) = \nu(z - 4m) = \nu_1(z) \), i.e., \( \inf_{z \in J_k} \nu_n(z) / \nu_1(z) = 1 \). But \( 4n + 1 - 1/s \in J_n \cap D_n \) for arbitrary \( s \in \mathbb{N} \), and we have \( \nu_1(4n + 1 - 1/s) = \nu(1 - 1/s) \) while \( \nu_{n+1}(4n + 1 - 1/s) = \nu(1 - 1/s) \); thus,
\[
\lim_{s \to \infty} \frac{\nu_{n+1}(4n + 1 - 1/s)}{\nu_1(4n + 1 - 1/s)} = \lim_{s \to \infty} \frac{\nu(1 - 1/s)}{\nu(1 - 1/s)} = 0,
\]
i.e., we have verified that \( \nu \) has \( (ND) \).

Next, note that, clearly, \( \mathcal{H}u(D) = \{ 0 \} \). Therefore,
\[
\psi : \mathcal{V}H(G) \to \bigoplus_{N} \mathcal{H}w(D), \quad (\psi(f) := (f(4m + 1)))_{m \in \mathbb{N}},
\]
is well-defined, linear and, in fact, bijective and continuous, hence an isomorphism by the open mapping theorem for \( (L^B) \)-spaces.
It now suffices to show that \( \psi : H\bar{V}(G) \to \bigoplus_{m} Hw(D) \) is also continuous. To see this, take an arbitrary 0-neighborhood in \( \bigoplus_{m} Hw(D) \); it contains a set of the form

\[
V = \bigoplus \{ f \in Hw(D) : w|f| \leq \varepsilon_{m} \text{ on } D \},
\]

where \( (\varepsilon_{m})_{m} \subset (0,1) \) is decreasing. For each \( m \in \mathbb{N} \), let \( r_{m} \in (0,1) \) satisfy \( v(r_{m}) = \varepsilon_{m}^{-1} \), and put \( b_{m} := v(r_{m})/w(r_{m}) > 1 \); then \( (r_{m})_{m} \) and \( (b_{m})_{m} \) are increasing. By Lemma 3.10, if \( u_{m} := \min(v,b_{m}w) \) and \( f \in H(D) \), then \( u_{m}|f| \leq 1 \) implies \( w|f| \leq \varepsilon_{m} \) on \( D \).

We let \( \alpha_{1} := 1, \alpha_{m} := b_{m-1} \) for \( n \geq 2 \) and take \( \psi \in V \) with \( \inf \alpha_{n} v_{n} \leq \psi \) (cf. [11], 0.2, Prop.). Since \( (\alpha_{n}) \) is increasing, we have

\[
(\inf \alpha_{n} v_{n})(z) = \min(v(z-4m), b_{m}w(z-4m))
\]

for \( z \in D, m = 1, 2, \ldots \).

Now suppose that \( f \in H\bar{V}(G) \) satisfies \( w|f| \leq 1 \) on \( G \) and let \( \psi(f) = (g_{m})_{m} \). Then, for each \( m \in \mathbb{N} \),

\[
\sup_{z \in D} \min(v(z-4m), b_{m}w(z-4m)) |f(z)| \leq \psi(z)|f(z)| \leq 1,
\]

hence \( u_{m}|g_{m}| \leq 1 \) and thus \( w|g_{m}| \leq \varepsilon_{m} \) on \( D \). It follows that

\[
\psi(\{ f \in H\bar{V}(G) : w|f| \leq 1 \text{ on } G \}) \subset V,
\]

that is, \( \psi \) is continuous on \( H\bar{V}(G) \).

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References

The Weyl asymptotic formula by the method of Tulovskii and Slubin

by

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Abstract. Let $A$ be a pseudodifferential operator on $\mathbb{R}^n$ whose Weyl symbol $a$ is a strictly positive smooth function on $W = \mathbb{R}^n \times \mathbb{R}^n$ such that $|\partial^\alpha a| \leq C_\alpha |\xi|^{-\delta}$ for some $\delta > 0$ and all $|\alpha| > 0$, $\partial^\alpha a$ is bounded for large $|\alpha|$, and $\lim_{|\xi| \to \infty} a(\xi) = \infty$. Such an operator $A$ is essentially selfadjoint, bounded from below, and its spectrum is discrete. The remainder term in the Weyl asymptotic formula for the distribution of the eigenvalues of $A$ is estimated. This is done by applying the method of approximate spectral projectors of Tulovskii and Slubin.

Introduction. Let $A = a^w(x, D)$ be a pseudodifferential operator on $\mathbb{R}^n$ given by the Weyl formula

$$Af(x) = \int \left| e^{2\pi i (x-y) \cdot \xi} a\left(\frac{x+y}{2}, \xi\right) f(y) dy d\xi,$$

where $a$ is a strictly positive smooth function with derivatives of polynomial growth. It is assumed that $A$ enjoys certain "hypoelliptic" properties to be specified later which imply that $A$ is selfadjoint and has a purely discrete spectrum $\lambda_n \nearrow \infty$. Let

$$Af = \int_0^\infty \lambda E(d\lambda) f$$

be the spectral resolution for $A$.

Tulovskii and Slubin [13] give estimates for the error term in the Weyl asymptotic formula

$$N(\lambda) \approx \int_{\alpha \leq \lambda} dx d\xi$$

for the number of eigenvalues of $A$ smaller than or equal to $\lambda$. Their proof is based on a construction of a family $E_\lambda$ of pseudodifferential operators that approximate the spectral projectors $E_\lambda$ of $A$ sufficiently well. This method