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Associated weights and spaces of holomorphic functions

by

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Abstract. When treating spaces of holomorphic functions with growth conditions, one is led to introduce associated weights. In our main theorem we characterize, in terms of the sequence of associated weights, several properties of weighted (LB)-spaces of holomorphic functions on an open subset $G \subset \mathbb{C}^N$ which play an important role in the projective description problem. A number of relevant examples are provided, and a "new projective description problem" is posed. The proof of our main result can also serve to characterize when the embedding of two weighted Banach spaces of holomorphic functions is compact. Our investigations on conditions when an associated weight coincides with the original one and our estimates of the associated weights in several cases (mainly for $G = \mathbb{C}$ or D) should be of independent interest.

Spaces of continuous functions with O- or o-growth conditions occur in approximation theory; the corresponding spaces of holomorphic functions arise in complex analysis, spectral theory, Fourier analysis, partial differential equations and convolution equations. (For concrete examples and references, see [11], Section 4.) In contrast to the case of spaces of continuous functions, however, not all continuous and strictly positive weights v are natural and intrinsically defined for spaces of holomorphic functions, as simple phenomena demonstrate (e.g., in connection with Liouville's theorem, see Section 1.A). Therefore, one is led to introduce *associated weights* $\widetilde{v}$ which contain information on the holomorphic functions estimated by $1/v$. (For the exact definition see 1.1 and the start of Section 1.B.)

In fact, associated weights have been part of the "folklore" of the subject; for instance, they were mentioned explicitly in Anderson–Duncan [2], beginning of Section 2. But, to the best of our knowledge, so far nobody has ever undertaken a *systematic study* of associated weights in reasonable generality. In particular, it was not clear which conditions on a weight v

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would imply that $v = \tilde{v}$, and how far $\tilde{v}$ could actually be different from v. The first part of Section 1 and parts of Section 3 of the present article can be considered as the start of such an investigation. (But the situation is rather complicated, and many questions remain open.)

Our main motivation for working with associated weights comes from *weighted (LB)-spaces* $\mathcal{V}H(G)$ *of holomorphic functions* (cf. [11], [9], [8], [15]). The "projective description problem", which is related to the notion of "analytically uniform spaces" due to Ehrenpreis, asks if the inductive limit topology of $\mathcal{V}H(G)$ can be described by weighted supseminorms with respect to a system $\overline{V}$ of weights depending on the decreasing sequence $\mathcal{V} = (v_n)_n$ in a natural way. In [11], it was proved that projective description holds, i.e., $\mathcal{V}H(G) = H\overline{V}(G)$ topologically, if $\mathcal{V}H(G)$ is a (DFS)-space or, more generally, just (semi-) Montel. But it is impossible to *characterize* these topological vector space properties of $\mathcal{V}H(G)$ in terms of the given sequence $\mathcal{V}$. Such characterizations have to involve the associated weights $\tilde{v}_n$, $n = 1, 2, \ldots$ (see Section 2), and hence it is sometimes very hard to evaluate them.

The first counterexamples to projective description (for spaces of holomorphic functions) were given in [15], but they were unnatural in the sense that they required enlarging the original domain of definition of the functions by adding another dimension in order to produce unexpected phenomena. The associated weights can serve to eliminate (at least part of) these constructions.

The article is organized as follows. Section 1 has two parts. In Part A, for a strictly positive continuous function w on an open subset G of $\mathbb{C}^N$, i.e., a *growth condition* on G, the definition of $\tilde{w}$ is given, together with a number of examples and remarks. Part B deals with *weighted Banach spaces* $Hv(G)$ *of holomorphic functions* and notes that, for $w = 1/v$ and $\tilde{v} = 1/\tilde{w}$, one has $Hv(G) = H\tilde{v}(G)$ isometrically. In our first theorem (Theorem 1.13), the canonical biduality $Hv(G) = Hv_0(G)''$ is characterized in terms of associated weights.

Section 2 is devoted to weighted (LB)-spaces $\mathcal{V}H(G)$ of holomorphic functions. After a short account of the projective description problem, we proceed to characterize the (DFS)-property and bounded retractivity for $\mathcal{V}H(G)$ as well as the semi-Montel property for $H\overline{V}(G)$ by inequalities involving the associated weights (Theorem 2.1). Several interesting examples follow. At the end of Section 2, we are led to define a new system $\tilde{V}$ of weights, depending on $\tilde{\mathcal{V}} = (\tilde{v}_n)_n$ in the same way as $\overline{V}$ depended on $\mathcal{V} = (v_n)_n$, and to pose the "new projective description problem" when $\mathcal{V}H(G) = H\tilde{V}(G)$ holds topologically. In an Appendix to Section 2, we show that the method of proof of Theorem 2.1 can also serve to characterize when two weighted topologies coincide on the unit ball of a weighted Banach space of holomorphic functions and, in particular, when the embedding of two such spaces is compact (Theorem 2.8).

Section 3 is of a more special character. It collects additional results, remarks and estimates of the associated weights, as well as further examples, mainly for $G = \mathbb{C}$ or D. For a more detailed introduction to the last section we refer to its beginning.

Notation. Our notation is standard: $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ = \{r \in \mathbb{R} : r \geq 0\}$, D denotes the open unit disk in $\mathbb{C}$. For an open set G in $\mathbb{C}^N$, $H(G)$ denotes the space of all holomorphic functions on G; it is usually endowed with the compact-open topology co. For notation on locally convex spaces and locally convex inductive limits, see Pérez-Carreras and Bonet [23] and the survey article [5].

1.A. Associated growth conditions, some examples. In the sequel, let G be an open subset of $\mathbb{C}^N$ ($N \geq 1$) and w a continuous (strictly) positive function, i.e., a *growth condition*, on G.

1.1. DEFINITION. Put $B_w := \{f \in H(G) : |f| \leq w \text{ on } G\}$. By $\tilde{w} : G \to \mathbb{R}_+$ we denote the function (*associated* with w) defined by

$$\tilde{w}(z) := \sup\{|f(z)| : f \in B_w\}, \quad z \in G.$$

Since w is bounded on each compact subset of G, Montel's theorem implies that B_w is compact in $(H(G), \mathrm{co})$. Using this fact and the continuity of point evaluations immediately yields that the sup in the definition of $\tilde{w}$ is a maximum. Moreover, Ascoli's theorem implies that B_w is equicontinuous, and thus $\tilde{w}$, in the pointwise closure of the equicontinuous set of all finite suprema of functions $|f|$, $f \in B_w$, must be continuous. Now $\tilde{w}$ is plurisubharmonic (p.s.h.) by Hörmander [19], 1.6.2 and 1.6.6. We have obtained:

1.2. PROPERTIES OF $\tilde{w}$. (i) $0 \leq \tilde{w} \leq w$,

(ii) *$\tilde{w}$ is continuous and p.s.h.*,

(iii) *for $f \in H(G)$, $|f| \leq w$ (i.e., $f \in B_w$) $\Leftrightarrow$ $|f| \leq \tilde{w}$ (i.e., $f \in B_{\tilde{w}}$),*

(iv) *for each $z \in G$ there is $f = f_z \in B_w$ with $|f(z)| = \tilde{w}(z)$ (or even $f(z) = \tilde{w}(z)$),*

(v) $(\tilde{w})^\sim = \tilde{w}$,

(vi) $(Cw)^\sim = C\tilde{w}$ *for arbitrary* $C > 0$,

(vii) $w_1 \leq w_2 \Rightarrow \tilde{w}_1 \leq \tilde{w}_2$,

(viii) $(\min(w_1, w_2))^\sim = (\min(\tilde{w}_1, \tilde{w}_2))^\sim$.

Note that, according to [19], Example after 2.6.1, even $\log \tilde{w}$ is p.s.h. (at least if $\tilde{w}$ is not identically zero on any connected component of G).

If G is bounded and w extends to a continuous function on $\overline{G}$ with $w|_{\partial G} \equiv 0$, then $\tilde{w} \equiv 0$ by the maximum principle. Thus, $\tilde{w}$ may have zeros.

The following two examples show that $\widetilde{w}$ may be strictly smaller than w for other "natural" reasons.

1.3. EXAMPLES. (a) If $G = \mathbb{C}$ and $w(z) = \max(1, |z|^{n+p})$, $z \in \mathbb{C}$, where $n \in \mathbb{N}_0$ and $0 < p < 1$, then $\widetilde{w}(z) = \max(1, |z|^n)$ for each $z \in \mathbb{C}$.

(b) (cf. [15]) Let G_1 be an open subset of $\mathbb{C}$ and w_1 a continuous positive function on G_1. Put $G := G_1 \times \mathbb{C}$, and let $w : G \to \mathbb{R}_+$ be continuous such that for all $(z_1, z_2) \in G$,

$$(*) \qquad 0 < w(z_1, z_2) \le w_1(z_1)(1 + |z_2|)^p, \quad 0 < p < 1.$$

Then $\widetilde{w}(z_1, z_2) \le \widetilde{w}_1(z_1)$ for all $(z_1, z_2) \in G$.

Proof. (a) As the constant 1 belongs to B_w, we already know that $\widetilde{w}(z) = 1 = w(z)$ for $|z| \le 1$. Moreover, as B_w contains the function f, $f(z) = z^n$, we have $\widetilde{w}(z) \ge |z|^n$ for $|z| \ge 1$. It remains to show $\widetilde{w}(z) \le |z|^n$ on $\mathbb{C} \setminus \overline{D}$, i.e., $|f(z)| \le |z|^n$ for $|z| > 1$, $f \in B_w$ arbitrary.

Put $u := 1/z$, $g(u) := u^n f(1/u)$, $z \neq 0$. Then g is holomorphic on $D \setminus \{0\}$. But $f \in B_w$ implies

$$|g(u)| = |u|^n \left| f\left(\frac{1}{u}\right) \right| = \frac{1}{|z|^n}|f(z)| \le |z|^p = \frac{1}{|u|^p} \quad \text{for } 0 < |u| \le 1$$

so that g must have a removable singularity at 0. Removing the singularity, we get $g \in H(D)$, $|g(u)| \le 1$ on ∂D, hence $|f(z)/z^n| = |g(u)| \le 1$ on D by the maximum principle. Thus $|f(z)| \le |z|^n$ for $|z| > 1$. (We thank A. Galbis for this argument.)

(b) Take an arbitrary $f \in B_w$ and fix $z_1 \in G$. In view of $(*)$, $|f(z_1, \cdot)| \le w_1(z_1)(1 + |\cdot|^p)$, and an application of (a general form of) Liouville's theorem yields that $f(z_1, \cdot)$ is constant, hence $|f(z_1, z_2)| = |f(z_1, 0)| \le w_1(z_1)$ for all $z_2 \in \mathbb{C}$. As z_1 varies in G_1, $f(\cdot, 0) \in H(G_1)$ so that by 1.2(iii),

$$|f(z_1, z_2)| = |f(z_1, 0)| \le \widetilde{w}_1(z_1), \quad (z_1, z_2) \in G.$$

Since $f \in B_w$ was arbitrary, the desired inequality follows. ∎

On the other hand, there are simple cases in which $\widetilde{w} = w$ holds.

1.4. EXAMPLE. If the continuous function $w : G \to \mathbb{R}_+ \setminus \{0\}$ happens to be of the form $w = \sup\{|g| : g \in \mathcal{G}\}$ for some family $\mathcal{G} \subset H(G)$, then $g \in B_w$ for each $g \in \mathcal{G}$, and hence $|g| \le \widetilde{w}$ by 1.2(iii). As a consequence, $w = \sup\{|g| : g \in \mathcal{G}\} \le \widetilde{w}$, i.e., $\widetilde{w} = w$.

Note that in 1.4, $\mathcal{G}$ cannot be an arbitrary subset of $H(G)$. In fact, $\mathcal{G}$ must be bounded pointwise on G, its sup must be continuous, and the functions $g \in \mathcal{G}$ are not allowed to have a common zero. Of course, this does not involve any loss of generality for finite families $\mathcal{G} \subset H(G)$ without common zeros. On the other hand, if $G = \mathbb{C}^N$ and $\mathcal{G} = \{g\}$, g cannot take the value 0 on G, any $f \in B_w$ can be divided by g, and another application of Liouville's theorem shows that f must be of the form λg, $\lambda \in \mathbb{C}$.

By 1.4, for $w(z) = \max(1, |z|^n)$ on $\mathbb{C}$, $\widetilde{w} = w$ holds. (Compare with 1.3(a).)

If $w = w_1 w_2$ on G, then $\widetilde{w}_1 \widetilde{w}_2 \le \widetilde{w}$ follows from the definition. If, in addition, $w_1 = |g|$ for some $g \in H(G)$ without zeros, we get equality here: $f \in B_w$ implies $f/g \in B_{w_2}$, hence $|f|/|g| \le \widetilde{w}_2$ by 1.2(iii). Thus, $|f| \le |g| \widetilde{w}_2 = w_1 \widetilde{w}_2 = \widetilde{w}_1 \widetilde{w}_2$ by 1.4, and we have shown $\widetilde{w} \le \widetilde{w}_1 \widetilde{w}_2$.

The equality $\widetilde{w} = w$ can also be deduced in another important case. To do this, we must first introduce some notation. For a moment, let G denote a *balanced* open subset of $\mathbb{C}^N$ (i.e., $z \in G$, $\lambda \in \mathbb{C}$, $|\lambda| \le 1 \Rightarrow \lambda z \in G$), and let w be a continuous positive function on G which is, in addition, *radial* in the sense that $z \in G$, $\lambda \in \mathbb{C}$, $|\lambda| = 1 \Rightarrow w(\lambda z) = w(z)$. For $f \in H(G)$ and $z \in G$ put

$$M(f, z) := \max\{|f(\lambda z)| : |\lambda| = 1\}.$$

1.5. OBSERVATION. *For a continuous positive radial function w on a balanced open set $G \subset \mathbb{C}^N$, one has*

$$\widetilde{w}(z) = \sup\{M(f, z) : f \in B_w\}, \quad z \in G,$$

and the sup is again a maximum. Hence $\widetilde{w}$ is also radial.

Proof. Put $\overline{w}(z) := \sup\{M(f, z) : f \in B_w\}$, $z \in G$. For fixed $z \in G$, $\{\lambda z : |\lambda| = 1\}$ is compact in G, and thus $M(\cdot, z)$ yields a continuous seminorm on $(H(G), \mathrm{co})$. Now compactness of B_w implies that the sup in the definition of $\overline{w}$ must really be a maximum.

Since $|f(z)| \le M(f, z)$, trivially $\widetilde{w} \le \overline{w}$. Fix an arbitrary $z_0 \in G$. There are $f \in B_w$ and $\lambda \in \mathbb{C}$, $|\lambda| = 1$, with $\overline{w}(z_0) = M(f, z_0) = |f(\lambda z_0)|$. Put $g(z) := f(\lambda z)$, $z \in G$. Then g is a well-defined element of $H(G)$, and since $|g(z)| = |f(\lambda z)| \le w(\lambda z) = w(z)$ for all $z \in G$ due to the radiality of w, we have $g \in B_w$. Finally,

$$\widetilde{w}(z_0) \ge |g(z_0)| = |f(\lambda z_0)| = \overline{w}(z_0). \ ∎$$

1.6. COROLLARY. *If the continuous positive radial function w on the balanced open set $G \subset \mathbb{C}^N$ happens to be of the form $w(z) = M(f, z)$, $z \in G$, for some $f \in H(G)$, then $\widetilde{w} = w$. In case $N = 1$, this holds in particular if $w(z) = f(|z|)$, $z \in G$, for some $f \in H(G)$ whose Taylor series (at 0) has nonnegative coefficients.*

Note that for $f \in H(G)$ as in 1.6, $f(0)$ must be $\neq 0$ since $w(0)$ was required to be positive. On the other hand, if $N = 1$ and if $w(z) = \max(1, f(|z|))$, $z \in G$, for some $f \in H(G)$ with $f(0) = 0$ whose Taylor series has nonnegative coefficients, then a simple modification of the argument again yields $\widetilde{w} = w$. In case $N = 1$, if G is a balanced open set in $\mathbb{C}$

(i.e., G is an open disk centered at 0 or $G = \mathbb{C}$), and if f is an element of $H(G)$ and $r \in \mathbb{R}_+ \cap G$, it is customary to write

$$M(f,r) = \sup\{|f(z)| : |z| = r\}.$$

By the maximum principle, $r \to M(f,r)$ is always increasing. Moreover, it satisfies Hadamard's Three Circles Theorem. Hence, in view of 1.5, for any radial w on $G \subset \mathbb{C}$, the function $r \to \widetilde{w}(r)$ must be increasing and also logarithmically convex. (Cf. the discussion after Prop. 3.1.)

1.7. EXAMPLES. We have $\widetilde{w} = w$ for each of the following radial functions w on balanced domains G, where C and α are positive constants and n denotes a natural number:

(a) $G = \mathbb{C}$, $w(z) = \exp(C|z|^n)$,
(b) $G = D$, $w(z) = \exp(C/(1-|z|)^\alpha)$,
(c) $G = D$, $w(z) = 1/(1-|z|)^\alpha$,
(d) $G = D$, $w(z) = \max(1, -C\log(1-|z|))$.

Now we would like to show that certain interesting *nonradial* functions w on $\mathbb{C}$, related to the Fourier–Laplace transforms of distributions or ultradistributions of Beurling type, do also satisfy $\widetilde{w} = w$. We start by deriving a slightly more general fact. In the sequel, let w_1 denote a radial weight on $\mathbb{C}$ such that, for some $C > 0$ and $\alpha \geq 0$,

$$w_1(z) \leq C(1+|z|)^\alpha \widetilde{w}_1(z), \qquad z \in \mathbb{C}.$$

While this hypothesis is clearly satisfied with $\alpha = 0$ and $C = 1$ whenever $\widetilde{w}_1 = w_1$, we will show in Proposition 3.1 below that it also holds for many radial growth conditions w_1 on $\mathbb{C}$ with $\alpha = 1$, $C > 0$. Now take

$$w(z) := \exp(n|\mathrm{Im}\, z|)w_1(z), \qquad z \in \mathbb{C},$$

and note that $w_1 \leq w$ on $\mathbb{C}$.

1.8. OBSERVATION. *Under our assumptions,*

$$w(z) \leq C(1+|z|)^\alpha \widetilde{w}(z), \qquad z \in \mathbb{C}.$$

In particular, if $\widetilde{w}_1 = w_1$, then $\widetilde{w} = w$.

Proof. Fix $z_0 \in \mathbb{C}$. We treat three cases, starting with $\mathrm{Im}\, z_0 > 0$. According to 1.2(iv), there is $g \in H(G)$ with $|g| \leq w_1$ and $|g(z_0)| = \widetilde{w}_1(z_0)$. Put

$$f(z) := \exp(-inz)g(z), \qquad z \in \mathbb{C}.$$

Then $f \in B_w$ since

$$|f(z)| = \exp(n\,\mathrm{Im}\, z)|g(z)| \leq \exp(n|\mathrm{Im}\, z|)w_1(z) = w(z), \qquad z \in \mathbb{C}.$$

Thus, by hypothesis on w_1 and z_0, we can conclude:

$$\begin{aligned} C(1+|z|)^\alpha \widetilde{w}(z_0) &\geq C(1+|z|)^\alpha |f(z_0)| = C(1+|z|)^\alpha \widetilde{w}_1(z_0)\exp(n\,\mathrm{Im}\, z_0) \\ &\geq w_1(z_0)\exp(n|\mathrm{Im}\, z_0|) = w(z_0). \end{aligned}$$

The case $\mathrm{Im}\, z_0 < 0$ can be treated in exactly the same way, replacing $\exp(-inz)$ by $\exp(inz)$. The remaining case $\mathrm{Im}\, z_0 = 0$ is even easier: Choose g as before and put $f := g$. (Once more we thank A. Galbis for this argument, which is related to the one used after 1.4.) ■

Since $w_1(z) = (1+|z|)^n$, $z \in \mathbb{C}$, clearly satisfies $\widetilde{w}_1 = w_1$, the following example is a consequence of 1.8.

1.9. EXAMPLE. $\widetilde{w} = w$ holds for

$$w(z) = \exp(n|\mathrm{Im}\, z|)(1+|z|)^n, \qquad z \in \mathbb{C}\ (n \in \mathbb{N} \text{ arbitrary}).$$

Incidentally, on an arbitrary open set $G \subset \mathbb{C}^N$ and for any growth condition w on G, the property $\widetilde{w} = w$ is inherited by:

(a) positive scalar multiples,
(b) suprema, whenever the sup is continuous (and finite at each point),
(c) finite products,
(d) finite sums.

Here, (b) follows directly from 1.2(vii) (cf. also 1.4), and (c) and (d) are consequences of 1.2(iv). In fact, (d) generalizes to arbitrary sums $w = \sum_\alpha w_\alpha$ (with $\widetilde{w}_\alpha = w_\alpha$ for all α, which are continuous functions and) which converge pointwise on G.

It is also noteworthy that, for any G and $w = \widetilde{w}$, if G_0 is an open subset of $\mathbb{C}^{N_0}$ and $f : G_0 \to G$ is a holomorphic mapping, then $\widetilde{w \circ f} = w \circ f$. Similarly, if the radial extension of a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ to $\mathbb{C}$ satisfies $\widetilde{\varphi} = \varphi$, then $\widetilde{\varphi \circ w} = \varphi \circ w$.

Using 1.4 and the inheritance properties (a) and (d) (in its more general form), one realizes that the following (in general nonradial) growth conditions on an open set $G \subset \mathbb{C}$ with $G \neq \mathbb{C}$ have the property $\widetilde{w} = w$:

$$w(z) = \mathrm{dist}(z, \mathbb{C} \setminus G)^{-n},$$
$$w(z) = \exp(C\,\mathrm{dist}(z, \mathbb{C} \setminus G)^{-n}) \quad \text{for } n \in \mathbb{N} \text{ and } C > 0.$$

The same holds with $\alpha > 0$ instead of n if G is simply connected (a generalization of 1.7(b) and (c)).

From *one-dimensional* examples of functions w with $\widetilde{w} = w$, one obtains *N-dimensional* ones ($N \geq 2$) on product sets, by use of the following simple observation (which clearly holds in slightly greater generality, but then the notation would become more complicated).

1.10. OBSERVATION. *Let $G_k \subset \mathbb{C}$ be open, $k = 1, \ldots, N$, and $G = \prod_{k=1}^N G_k \subset \mathbb{C}^N$. If w_k is a continuous positive function on G_k with $\widetilde{w}_k =$*

w_k *for each* k, *then* $w = w_1 \otimes \ldots \otimes w_N$, $w(z) = w_1(z_1)\ldots w_N(z_N)$ *for* $z = (z_1, \ldots, z_N) \in G$, *again satisfies* $\widetilde{w} = w$.

1.10 (another consequence of 1.2(iv)) and 1.9 yield:

1.11. EXAMPLE. $\widetilde{w} = w$ also holds for

$$w(z) = \prod_{k=1}^{N} \exp(n|\mathrm{Im}\, z_k|)(1 + |z_k|)^n, \quad z = (z_1, \ldots, z_N) \in \mathbb{C}^N,$$

where $n \in \mathbb{N}$ is arbitrary and $N \geq 1$.

While it is good to know that $\widetilde{w} = w$ for many natural growth conditions, also those w for which $\widetilde{w} \leq w \leq C\widetilde{w}$ holds for some constant $C > 0$, i.e., for which $\widetilde{w}$ is equivalent to w, are very important; see Sections 3.A and 3.B for examples.

1.B. Weighted Banach spaces of holomorphic functions. In the rest of Section 1, we deal with the role of associated weights in weighted Banach spaces of holomorphic functions. Let G again be an open subset of $\mathbb{C}^N$, and let v denote a *weight* on G, that is, a continuous function from G into $\mathbb{R}_+ \setminus \{0\}$. The *weighted Banach spaces of holomorphic functions* (with O- resp. o-growth conditions with respect to $1/v$) are

$$Hv(G) := \{f \in H(G) : \|f\| = \|f\|_v := \sup_G v|f| < \infty\},$$

$$Hv_0(G) := \{f \in H(G) : vf \text{ vanishes at } \infty \text{ on } G, \text{ i.e.,}$$
$$\forall \varepsilon > 0\ \exists K \text{ compact} \subset G\ \forall z \in G \setminus K : v(z)|f(z)| < \varepsilon\},$$

endowed with the induced norm. Now, look at the *growth condition* $w := 1/v$. Then B_w as defined in 1.1 is nothing but the closed unit ball of $Hv(G)$. Hence, if $\widetilde{w}$ is the function from Definition 1.1, then $\widetilde{w}(z) = \|\delta_z\|_{Hv(G)'}$, where $\delta_z : f \to f(z)$ is the point evaluation at $z \in G$. The "*weight*" *associated with* v is defined by $\widetilde{v} := 1/\widetilde{w}$, where $1/0 = +\infty$. From 1.2(i), we get $v \leq \widetilde{v}$.

1.12. OBSERVATION. *$\widetilde{v}$ may take the value $+\infty$. But if $H\widetilde{v}(G)$ is defined as above (with $\widetilde{v}$ replacing v, and adopting the convention that $0(+\infty) = 0$), then $Hv(G) = H\widetilde{v}(G)$, and the norms $\|\cdot\|_v$ and $\|\cdot\|_{\widetilde{v}}$ coincide.*

Indeed, from $v \leq \widetilde{v}$, we already know $\|\cdot\|_v \leq \|\cdot\|_{\widetilde{v}}$ and $H\widetilde{v}(G) \subset Hv(G)$. Fix $f \in Hv(G)$ with $\|f\|_v = 1$. As remarked above, then $f \in B_w$. By 1.2(iii), $|f| \leq \widetilde{w}$, whence, clearly, $f \in H\widetilde{v}(G)$ and $\|f\|_{\widetilde{v}} \leq 1 = \|f\|_v$.

The associated weight $\widetilde{v}$ has the advantage of being defined *intrinsically*; it may also enjoy additional properties which v did not have (see 1.2(ii) and (iv)). If $\widetilde{v}$ is equivalent to v (in the sense that $v \leq \widetilde{v} \leq Cv$ for some $C > 0$), then $Hv(G)$ and $H\widetilde{v}(G)$ are still (topologically) isomorphic.

So far, we have only considered associated weights in the case of $Hv(G)$, but we can also do the same construction for $Hv_0(G)$. Namely, take again $w = 1/v$, but now put

$$\widetilde{w}_0(z) := \sup\{|f(z)| : f \in Hv_0(G),\ \|f\| \leq 1\}, \quad z \in G,$$

and $\widetilde{v}_0 := 1/\widetilde{w}_0$. Clearly, $0 \leq \widetilde{w}_0 \leq \widetilde{w} \leq w$, hence $v \leq \widetilde{v} \leq \widetilde{v}_0$.

1.13. THEOREM. *The natural biduality $Hv_0(G)'' = Hv(G)$ holds isometrically if and only if $\widetilde{v} = \widetilde{v}_0$.*

Proof. 1.13 is an easy consequence of the abstract characterization in [12], Theorem 1.1 and Cor. 1.2, whereby $Hv_0(G)'' = Hv(G)$ isometrically (in the canonical way) if and only if the closed unit ball B_{w_0} of $Hv_0(G)$ is dense in B_w, the closed unit ball of $Hv(G)$, with respect to co or, equivalently, with respect to the topology of pointwise convergence.

If this density holds, then the definition implies $\widetilde{w}_0(z) = \widetilde{w}(z)$ for each $z \in G$, hence $\widetilde{v}_0 = \widetilde{v}$. On the other hand, if B_{w_0} is not pointwise dense in B_w, there must clearly exist $z \in G$ and $f \in B_w$ such that

$$\widetilde{w}(z) \geq |f(z)| > \sup\{|g(z)| : g \in Hv_0(G),\ \|g\| \leq 1\} = \widetilde{w}_0(z),$$

and hence $\widetilde{v}_0(z) > \widetilde{v}(z)$. (The first-named author thanks D. Vogt for conversations on the subject of Theorem 1.13.) ∎

For examples in which one has the biduality $Hv_0(G)'' = Hv(G)$, see [12]. In particular, by [8], Theorem 1.5(d), it holds if v is a radial weight on a balanced open set $G \subset \mathbb{C}^N$ such that $Hv_0(G)$ contains the polynomials.

2. Weighted (LB)-spaces of holomorphic functions. In this section we first show that associated weights can serve to characterize certain properties of weighted (LB)-spaces of holomorphic functions which play an important role in the projective description problem.

Here $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ will always denote a decreasing sequence of weights v_n on an open subset G of $\mathbb{C}^N$. The *weighted (LB)-space of holomorphic functions* associated with $\mathcal{V}$ is the locally convex (l.c.) inductive limit

$$\mathcal{V}H(G) := \mathrm{ind}_n\, Hv_n(G),$$

that is, it is the union of the weighted Banach spaces $Hv_n(G)$ of holomorphic functions, endowed with the strongest topology given by a system of seminorms such that all the injections $Hv_n(G) \to \mathcal{V}H(G)$, $n \in \mathbb{N}$, become continuous. In the sequel, we shall put $w_n = 1/v_n$ and let $B_n = B_{w_n}$ denote the closed unit ball of $Hv_n(G)$, $n = 1, 2, \ldots$ Note that $\mathcal{V}H(G)$ is always complete; in fact, it is the inductive dual of a Fréchet space Y (see [9], Theorem 6). (Hence, the inductive limit topology of $\mathcal{V}H(G)$ exactly equals $\beta(\mathcal{V}H(G), Y'')$.)

In an effort to describe the continuous seminorms for the inductive limit topology, one introduces (cf. [11])

$$\overline{V} = \overline{V}(\mathcal{V}) := \{\overline{v} \text{ weight on } G : \text{for each } n \in \mathbb{N},\ \overline{v}/v_n \text{ is bounded on } G\}$$

and the corresponding *weighted space of holomorphic functions*

$$H\overline{V}(G) := \{f \in H(G) : \text{for each } \overline{v} \in \overline{V},\ p_{\overline{v}}(f) = \sup_G \overline{v}|f| < \infty\}.$$

Endowed with the l.c. topology given by the system $(p_{\overline{v}})_{\overline{v}\in\overline{V}}$ of weighted supseminorms, $H\overline{V}(G)$ is complete; it is called the *projective hull* of $\mathcal{V}H(G)$. Clearly $\mathcal{V}H(G) \subset H\overline{V}(G)$ with continuous injection, but it is easy to see that $\mathcal{V}H(G) = H\overline{V}(G)$ algebraically and that the two topologies yield the same bounded sets. We say that *projective description holds* if $\mathcal{V}H(G) = H\overline{V}(G)$ topologically, i.e., if $(p_{\overline{v}})_{\overline{v}}$ induces the inductive limit topology; and the *projective description problem* asks when this is the case. In [15], the authors constructed the first counterexamples to projective description for spaces of holomorphic functions; for counterexamples in the case of entire functions, see [14].

In [11] and some subsequent articles, the following conditions on $\mathcal{V}$ were considered:

(S) $\quad\forall n \in \mathbb{N}\ \exists m > n : \quad \dfrac{v_m}{v_n}$ vanishes at ∞ on G,

(M) $\quad\forall n \in \mathbb{N}\ \forall Y$ not relatively compact in G $\exists m = m(n, Y) > n :$

$$\inf_Y \frac{v_m}{v_n} = 0,$$

(RD) $\quad\forall n \in \mathbb{N}\ \exists m \ge n\ \forall Y \subset G :$

$$\inf_Y \frac{v_m}{v_n} > 0 \Rightarrow \inf_Y \frac{v_k}{v_n} > 0,\ k = m+1, m+2, \ldots$$

Clearly, (S) implies both (M) and (RD). It is easy to see that, whenever (S) holds, $\mathcal{V}H(G)$ is a (DFS)-space; if $m > n$ is chosen as in (S), then the canonical injection $Hv_n(G) \to Hv_m(G)$ is compact. In [11], Theorem 1.6, it was proved that (S) also implies $\mathcal{V}H(G) = H\overline{V}(G)$ topologically. More generally, using [6], Prop. 6, and applying the Baernstein Lemma (cf. [11], 0.4) to $\mathcal{V}H(G) \to C\overline{V}(G)$ (the analog of $H\overline{V}(G)$ for continuous functions), projective description holds whenever $\mathcal{V}H(G)$ is *semi-Montel* (i.e., each bounded subset of this space is relatively compact). By [9], Prop. 7, (M) implies that $H\overline{V}(G)$ is semi-Montel. However, [15] gave an example of a space $\mathcal{V}H(G)$ such that (M) holds, hence $H\overline{V}(G)$ is semi-Montel, but $\mathcal{V}H(G) \ne H\overline{V}(G)$ topologically. By [9], Theorem 2 and Theorem 6(2), (RD) implies that $\mathcal{V}H(G)$ is a *boundedly retractive* inductive limit, i.e., for each bounded set $B \subset \mathcal{V}H(G)$ there is $n \in \mathbb{N}$ such that B is contained and bounded in $Hv_n(G)$ and that, in addition, the topologies induced by $\mathcal{V}H(G)$ and $Hv_n(G)$ coincide on B. In this case, $\mathcal{V}H(G)$ is the strong dual of a (quasinormable) Fréchet space Y. The method of construction of counterexamples that was used in [15] does not work if $\mathcal{V}H(G)$ is boundedly retractive; it is still possible that projective description holds whenever $\mathcal{V}H(G)$ has this property.

As we will see below, none of the three implications "$(S) \Rightarrow \mathcal{V}H(G)$ (DFS)", "$(M) \Rightarrow H\overline{V}(G)$ semi-Montel" and "$(RD) \Rightarrow \mathcal{V}H(G)$ boundedly retractive" is an equivalence. Characterizations of the corresponding properties of $\mathcal{V}H(G)$ resp. $H\overline{V}(G)$ can be given (and are even not hard to prove), but they must involve the decreasing sequence $\widetilde{\mathcal{V}} = (\widetilde{v}_n)_n$ of associated weights $\widetilde{v}_n$ (see Section 1.B). In fact, the characterizations also involve associated weights for some more complicated constructions with minima (which makes it hard to evaluate the characterizations).

In the sequel, for $w_n = 1/v_n$, $n \in \mathbb{N}$, $\widetilde{w}_n$ has the same meaning as in Section 1.A, and $\widetilde{v}_n = 1/\widetilde{w}_n$. $\mathcal{K}_+(G)$ is the set of all nonnegative continuous functions on G with compact support. Note that, due to the conventions introduced in Section 1.B, $\min(1/v_n, 1/\varphi)$ is always a positive continuous function on G for $n \in \mathbb{N}$ and $\varphi \in \mathcal{K}_+(G)$.

2.1. THEOREM. (a) *$\mathcal{V}H(G)$ is a (DFS)-space if and only if*

(S_H) $\quad\forall n \in \mathbb{N}\ \exists m > n\ \forall \varepsilon > 0\ \exists \varphi \in \mathcal{K}_+(G) :$

$$\left(\min\left(\frac{1}{v_n}, \frac{1}{\varphi}\right)\right)^{\sim} \le \left(\frac{\varepsilon}{v_m}\right)^{\sim} \quad \text{or, equivalently,} \quad \left(\min\left(w_n, \frac{1}{\varphi}\right)\right)^{\sim} \le \varepsilon w_m.$$

(b) *$H\overline{V}(G)$ is semi-Montel if and only if*

(M_H) $\quad\forall n \in \mathbb{N}\ \forall \overline{v} \in \overline{V}\ \exists \varphi \in \mathcal{K}_+(G) :$

$$\left(\min\left(\frac{1}{v_n}, \frac{1}{\varphi}\right)\right)^{\sim} \le \left(\frac{1}{\overline{v}}\right)^{\sim} \quad \text{or} \quad \left(\min\left(w_n, \frac{1}{\varphi}\right)\right)^{\sim} \le \frac{1}{\overline{v}}.$$

(c) *$\mathcal{V}H(G)$ is boundedly retractive if and only if*

(RD_H) $\quad\forall n \in \mathbb{N}\ \exists m \ge n\ \forall k \ge m\ \forall \varepsilon > 0\ \exists \delta > 0 :$

$$\left(\min\left(\frac{1}{v_n}, \frac{\delta}{v_k}\right)\right)^{\sim} \le \left(\frac{\varepsilon}{v_m}\right)^{\sim} \quad \text{or} \quad (\min(w_n, \delta w_k))^{\sim} \le \varepsilon w_m.$$

Proof. We will first reformulate the properties of $\mathcal{V}H(G)$ resp. $H\overline{V}(G)$ in (a)–(c) in terms of a condition $(*)$ which essentially involves an inclusion of sets with an intersection of two sets on the left side. From this point on, the three proofs are exactly the same, and we will only treat the case (c) in detail.

(a) $\mathcal{V}H(G)$ is a (DFS)-space if and only if for each $n \in \mathbb{N}$ there is $m > n$ such that the canonical injection $Hv_n(G) \to Hv_m(G)$ is compact. Since B_n is compact in $(H(G), \mathrm{co})$, this holds precisely if $Hv_m(G)$ and co induce the same topology on B_n. But as two l.c. topologies coincide on an absolutely convex set if and only if they yield the same systems of 0-neighborhoods, we arrive at the following equivalence: $\mathcal{V}H(G)$ is a (DFS)-space if and only if

$$(*)\qquad \forall n\in\mathbb{N}\ \exists m>0\ \forall\varepsilon>0\ \exists\varphi\in\mathcal{K}_+(G):\quad B_n\cap\{f:\varphi|f|\le 1 \text{ on } G\}\subset\varepsilon B_m.$$

(b) Since $H\overline{V}(G)$ has the same bounded sets as the regular inductive limit $\mathcal{V}H(G)$ and since each B_n is compact in $(H(G),\mathrm{co})$, $H\overline{V}(G)$ is semi-Montel precisely if for each $n\in\mathbb{N}$, $H\overline{V}(G)$ and co induce the same topology (at 0) on B_n. This is clearly equivalent to

$$(*)\qquad \forall n\in\mathbb{N}\ \forall\overline{v}\in\overline{V}\ \exists\varphi\in\mathcal{K}_+(G):\quad B_n\cap\{f:\varphi|f|\le 1 \text{ on } G\}\subset\{f\in H(G):p_{\overline{v}}(f)\le 1\}.$$

(c) Since $\mathcal{V}H(G)$ is always regular, the bounded retractivity of $\mathcal{V}H(G)$ is equivalent to the fact that for each $n\in\mathbb{N}$ there is $m\ge n$ such that $Hv_m(G)$ and $\mathcal{V}H(G)$ induce the same topology on B_n. Moreover, since for regular (LB)-spaces bounded retractivity is equivalent to *bounded stability* (cf. [5], Appendix to Section 3, Prop. 9), $\mathcal{V}H(G)$ is boundedly retractive if and only if for each $n\in\mathbb{N}$ there is $m\ge n$ such that for each $k\ge m$, $Hv_m(G)$ and $Hv_k(G)$ induce the same 0-neighborhood system on B_n, i.e.,

$$(*)\qquad \forall n\in\mathbb{N}\ \exists m\ge n\ \forall k\ge m\ \forall\varepsilon>0\ \exists\delta>0:\quad B_n\cap\delta B_k\subset\varepsilon B_m.$$

Now it suffices to show that in (a), (b), (c) the corresponding $(*)$ is equivalent to (S_H), (M_H), (RD_H), respectively. Here is the proof for (c); (a) and (b) need only minor modifications.

(c), $(RD_H)\Rightarrow(*)$. If $f\in B_n\cap\delta B_k$, then $f\in H(G)$ satisfies $|f|\le\min(w_n,\delta w_k)$, hence by 1.2(iii) and (RD_H),

$$|f|\le(\min(w_n,\delta w_k))^{\sim}\le\varepsilon\widetilde{w}_m\le\varepsilon w_m,$$

that is, $f\in\varepsilon B_m$.

$(*)\Rightarrow(RD_H)$ (indirect). Once more, fix n, m, k, ε, δ. If there is $z\in G$ such that $(\min(w_n,\delta w_k))^{\sim}(z)>\varepsilon\widetilde{w}_m(z)$, then by 1.2(iv) there is $f\in H(G)$ with $|f|\le\min(w_n,\delta w_k)$ on G, but $|f(z)|>\varepsilon\widetilde{w}_m(z)$. Now $f\in B_n\cap\delta B_k$, but f cannot belong to εB_m because in that case we would have $|f|\le\varepsilon w_m$, hence by 1.2(iii) even $|f|\le\varepsilon\widetilde{w}_m$, a contradiction. Thus, $(*)$ cannot hold. ■

(S_H) and (M_H) have the drawback that in their formulation not only the associated weights, but also functions $\varphi\in\mathcal{K}_+(G)$ appear. (We could have worked as well with characteristic functions of compact sets, but then the minima in those conditions would not have been continuous in general, violating the general hypotheses of Section 1.) In this respect, condition (RD_H) is better.

By what we have said at the beginning of Section 2, (S) implies $\mathcal{V}H(G)$ being (DFS), and hence $(S)\Rightarrow(S_H)$. A direct proof of this implication is easy. Later on, we will give an example of a (DFS)-space $\mathcal{V}H(G)$ for which $\mathcal{V}$ does not satisfy (S), and hence (S_H) does not imply (S). Clearly, if all $\widetilde{v}_n$ are *weights* (i.e., they do not take the value ∞), and if $\widetilde{\mathcal{V}}$ satisfies (S), then in view of 1.12,

$$\mathcal{V}H(G)=\mathrm{ind}_n\,Hv_n(G)=\mathrm{ind}_n\,H\widetilde{v}_n(G)=\widetilde{\mathcal{V}}H(G),$$

and thus $\mathcal{V}H(G)$ must also be a (DFS)-space. To deduce directly that if $\widetilde{\mathcal{V}}$ satisfies (S), then (S_H) must also hold, it is useful to observe 1.2(v) and 1.2(viii). It is trivial to exhibit an example of a sequence $\mathcal{V}=(v_n)_n$ with (S) such that $\widetilde{\mathcal{V}}=(\widetilde{v}_n)_n$ is indeed a sequence of weights, but does not satisfy (S): Just take $G=\mathbb{C}$, fix $n_0\in\mathbb{N}$ and let $\mathcal{V}=(v_n)_n$ be defined by

$$v_n(z):=(\max(1,|z|^{n_0+p_n}))^{-1}\quad\text{for } z\in\mathbb{C},\ n=1,2,\ldots,$$

where $(p_n)_n$ is an increasing sequence of numbers in $(0,1)$. Note that each $\widetilde{v}_n(z)$ is of the form $(\max(1,|z|^{n_0}))^{-1}$ by 1.3(a) and that all v_n are radial. Compare with Proposition 3.5 below and with the note after the proof of 3.5.

Similarly, (M) implies $H\overline{V}(G)$ Montel, and hence $(M)\Rightarrow(M_H)$; using [10], Prop. 5.2(3), one can also give a direct proof. A later example will serve to show that (M_H) does not imply (M). Finally, (RD) implies $\mathcal{V}H(G)$ boundedly retractive, and hence one has $(RD)\Rightarrow(RD_H)$. This time, it is not so easy to give a direct proof, but it follows from [7], Theorem 1.1(a), 2(v). Later on, there will also be an example of a boundedly retractive space $\mathcal{V}H(G)$ for which $\mathcal{V}$ does not satisfy (RD), and hence (RD_H) does not imply (RD). Once more, if all $\widetilde{v}_n$ are weights and if $\widetilde{\mathcal{V}}$ satisfies (RD), then $\mathcal{V}H(G)$ $(=\widetilde{\mathcal{V}}H(G))$ must have (RD_H); using [7], one can deduce this directly.

At this point, we turn to examples, some of which were promised before. We start with a simple construction, based on ideas from [15].

2.2. Lemma. *Let G_1 be an open subset of $\mathbb{C}$, and put $G:=G_1\times\mathbb{C}$. Let $\mathcal{T}=(t_n)_n$ and $\mathcal{U}=(u_n)_n$ be decreasing sequences of weights on G_1 and $\mathbb{C}$, respectively. Let $0<p<1$, and assume that for each $n\in\mathbb{N}$ and each $z\in\mathbb{C}$, $(1+|z|)^{-p}\le u_n(z)\le 1$. Put $\mathcal{V}:=(v_n)_n$, $v_n(z_1,z_2):=t_n(z_1)u_n(z_2)$ for $(z_1,z_2)\in G$, $n\in\mathbb{N}$. Then:*

(a) *Every $f\in\mathcal{V}H(G)$ is constant in the second coordinate z_2.*

(b) *$\mathcal{V}H(G)$ and $\mathcal{T}H(G_1)$ are canonically isomorphic.*

(a) follows from Liouville's theorem (cf. 1.3(b)), and then a short glance at the map $f\to f(\cdot,0)$ suffices to see (b).

2.3. EXAMPLE. Let $G_1 \subset \mathbb{C}$ be open, and let the decreasing sequence $\mathcal{T} = (t_n)_n$ on G_1 satisfy (S). Then $\mathcal{T}H(G_1)$ is a (DFS)-space. If we take $u_n \equiv 1$ on $\mathbb{C}$ for each $n \in \mathbb{N}$ and if $\mathcal{V} = (v_n)_n$ on $G = G_1 \times \mathbb{C}$ is defined as in 2.2, then $\mathcal{V}$ cannot satisfy (S) on G, but by Lemma 2.2(b), $\mathcal{V}H(G) \cong \mathcal{T}H(G_1)$ is a (DFS)-space, and hence $\mathcal{V}H(G) = H\overline{V}(G)$ topologically.

We note that each $f \in Hv_n(G)$ in 2.3 is constant in the second coordinate by 2.2(a), and thus also each $\widetilde{v}_n$ must be constant in z_2, $n = 1, 2, \ldots$ Now it is easy to arrange situations as above in which all $\widetilde{v}_n$ are *weights*, but the sequence $\widetilde{\mathcal{V}} = (\widetilde{v}_n)_n$ does not satisfy (S). The preceding example also serves to show that (M_H) does not imply (M). In fact, $\mathcal{V}$ as in 2.3 does not have (M) on G, and neither does $\widetilde{\mathcal{V}}$.

Before passing to our next example, which will be constructed along the same lines as 2.3, but will in some sense be even "worst possible", let us recall two other conditions on $\mathcal{V}$ which arose in the corresponding projective description problem for spaces of *continuous* functions.

Thus, for a moment, let X denote a locally compact and σ-compact space and $\mathcal{V} = (v_n)_n$ a decreasing sequence of positive continuous functions on X. Following [10], $\mathcal{V}$ is said to satisfy *condition* (D) if there exists an increasing sequence $J = (X_m)_{m\in\mathbb{N}}$ of (closed) subsets of X such that the following two properties hold:

$$(N, J) \quad \forall m \in \mathbb{N}\ \exists n_m \geq m: \quad \inf_{X_m} \frac{v_k}{v_{n_m}} > 0, \quad k = n_{m+1}, n_{m+2}, \ldots,$$

$$(M, J) \quad \forall n \in \mathbb{N}\ \forall Y \subset X \text{ with } Y \cap (X \setminus X_m) \neq \emptyset \text{ for all } m \in \mathbb{N}$$
$$\exists n' = n'(n, Y) > n: \quad \inf_{Y} \frac{v_{n'}}{v_n} = 0.$$

And $\mathcal{V}$ satisfies (ND) if there exists a decreasing sequence $J = (J_k)_k$ of subsets of X such that

$$\exists n_0 \in \mathbb{N}\ \forall k \geq n_0\ \exists l_k > k: \quad \inf_{J_k} \frac{v_k}{v_{n_0}} > 0, \quad \text{but} \quad \inf_{J_k} \frac{v_{l_k}}{v_{n_0}} = 0.$$

It is easy to see that (even in the generalized setting above) each of the conditions (M) and (RD) implies (D); thus (D) is the *weakest* condition on $\mathcal{V}$ considered so far. And by [3], Lemma 1, (ND) means precisely that condition (D) is *not* satisfied.

Now let the spaces $Cv_n(X)$, $\mathcal{V}C(X) = \operatorname{ind}_n Cv_n(X)$ and $C\overline{V}(X)$ be the exact analogs of $Hv_n(G)$, $\mathcal{V}H(G) = \operatorname{ind}_n Hv_n(G)$ and $H\overline{V}(G)$ for X replacing G and continuous functions taking the role of the holomorphic ones. Then $\mathcal{V}C(X) = C\overline{V}(X)$ topologically if and only if $\mathcal{V}$ satisfies (D) (cf. [6], Theorem 11).

2.4. EXAMPLE. Here, let simply $G_1 = \mathbb{C}$, and take a decreasing sequence $\mathcal{T} = (t_n)_n$ of weights t_n on $\mathbb{C}$ with $t_n(0) = 1$ for all n such that $\mathcal{T}$ satisfies

(S). Hence $\mathcal{T}H(\mathbb{C})$ is a (DFS)-space. For any decreasing sequence $\mathcal{U} = (u_n)_n$ on $\mathbb{C}$ which satisfies the assumption of 2.2, if $\mathcal{V} = (v_n)_n$ on $\mathbb{C}^2 = \mathbb{C} \times \mathbb{C}$ is defined as in 2.2, then Lemma 2.2(b) yields $\mathcal{V}H(\mathbb{C}^2) \cong \mathcal{T}H(\mathbb{C})$ (DFS), and projective description holds. By Theorem 2.1, $\mathcal{V}$ satisfies (S_H), hence a fortiori (M_H) and (RD_H).

We will now construct $\mathcal{U} = (u_n)_n$ *as above in such a way that* $\mathcal{V} = (v_n)_n$ *does not satisfy* (D). *Then, by what was said before,* $\mathcal{V}C(G) \neq C\overline{V}(G)$ *topologically, and* $\mathcal{V}$ *cannot satisfy* (RD).

For $n \in \mathbb{N}$ and arbitrary $k \in \mathbb{N}_0$, let first $u_n : \mathbb{R}_+ \to \mathbb{R}_+$ be given on $[2k, 2k+2]$ by

$$u_n(r) := \begin{cases} 1 & \text{for } r \in [2k, 2k+2^{-n}] \text{ and} \\ & \phantom{\text{for }} r \in [2k+1, 2k+2], \\ (1+r)^{-(n-1)/(2n)} & \text{for } r \in [2k+3\cdot 2^{-(n+1)}, 2k+1-2^{-(n+1)}], \end{cases}$$

with u_n affine on $[2k+2^{-n}, 2k+3\cdot 2^{-(n+1)}]$ and $[2k+1-2^{-(n+1)}, 2k+1]$. Distinguishing several cases, it is elementary to verify that the sequence $(u_n)_n$ is decreasing. Now extend u_n radially, $u_n(z) = u_n(|z|)$ for $z \in \mathbb{C}$, $n = 1, 2, \ldots$, to obtain a decreasing sequence $(u_n)_n$ on $\mathbb{C}$ for which the assumption of 2.2 holds with $p = 1/2$.

Now, it suffices to show that $\mathcal{V}$ satisfies (ND). For arbitrary $n \in \mathbb{N}$, take $J_n := \{(0, 2k+2^{-m}) : k, m \geq n\}$. Given $n \geq n_0 := 1$, for any $(0, 2k+2^{-m}) \in J_n$ clearly $v_{n_0}(0, 2k+2^{-m}) = 1$, but also

$$v_n(0, 2k+2^{-m}) = t_n(0)u_n(2k+2^{-m}) = 1$$

since $m \geq n$; hence $\inf_{J_n} v_n/v_{n_0} > 0$. But for $l_n := n+1$, $k \geq n$,

$$v_{n+1}(0, 2k+2^{-n}) = u_{n+1}(2k+2^{-n}) = (1+2k+2^{-n})^{-n/(2n+2)},$$
$$\inf_{J_n} \frac{v_{l_n}}{v_{n_0}} \leq (1+2k+2^{-n})^{-n/(2n+2)} \to 0 \quad (k \to \infty),$$

which proves that $\mathcal{V}$ satisfies (ND).

In the rest of this section, we sketch a new approach to projective description which directly takes the associated weights $\widetilde{v}_n$ into account.

Since the sequence $\widetilde{\mathcal{V}} = (\widetilde{v}_n)_n$ is intrinsically defined, and hence "more natural" than the original sequence $\mathcal{V} = (v_n)_n$ (on some open $G \subset \mathbb{C}^N$), one is led to introduce—instead of $\overline{V} = \overline{V}(\mathcal{V})$—its analog $\widetilde{V} := \overline{V}(\widetilde{\mathcal{V}})$, i.e.,

$$\widetilde{V} = \{\widetilde{v} \text{ weight on } G : \text{for each } n \in \mathbb{N},\ \widetilde{v}/\widetilde{v}_n \text{ is bounded on } G\}.$$

Since $v_n \leq \widetilde{v}_n$ for each $n \in \mathbb{N}$, it is clear that $\overline{V} \subset \widetilde{V}$. For arbitrary $\widetilde{v} \in \widetilde{V}$, put $p_{\widetilde{v}}(f) := \sup_G \widetilde{v}|f|$ for $f \in \mathcal{V}H(G)$.

2.5. LEMMA. (a) *For each* $\widetilde{v} \in \widetilde{V}$, $p_{\widetilde{v}}$ *defines a continuous seminorm on* $\mathcal{V}H(G)$.

(b) *Hence, for* $H\widetilde{V}(G) := \{f \in H(G) : \text{for each } \widetilde{v} \in \widetilde{V},\ p_{\widetilde{v}}(f) < \infty\}$, *endowed with the l.c. topology given by the system* $(p_{\widetilde{v}})_{\widetilde{v}\in\widetilde{V}}$ *of seminorms, we have*

$$\mathcal{V}H(G) \subset H\widetilde{V}(G) \subset H\overline{V}(G)$$

with continuous inclusions, and the three spaces coincide algebraically and have the same bounded sets.

Proof. By what is known about $\mathcal{V}H(G) \subset H\overline{V}(G)$, it suffices to show (a). And for this, it is enough to verify that $p_{\widetilde{v}}|_{Hv_n(G)}$ is a continuous seminorm on $Hv_n(G)$ for every $n \in \mathbb{N}$. But for $f \in Hv_n(G)$, 1.12 implies

$$p_{\widetilde{v}}(f) = \sup_G \widetilde{v}|f| \le \sup_G \left(\frac{\widetilde{v}}{\widetilde{v}_n}\widetilde{v}_n|f|\right) \le \left(\sup_G \frac{\widetilde{v}}{\widetilde{v}_n}\right)\left(\sup_G \widetilde{v}_n|f|\right) = \left(\sup_G \frac{\widetilde{v}}{\widetilde{v}_n}\right)\left(\sup_G v_n|f|\right). \ \blacksquare$$

In the light of Lemma 2.5, the "correct" projective hull of $\mathcal{V}H(G)$ is rather $H\widetilde{V}(G)$ (even though it may be harder to compute the seminorms $p_{\widetilde{v}}$, $\widetilde{v} \in \widetilde{V}$, than the corresponding ones $p_{\overline{v}}$, $\overline{v} \in \overline{V}$). And there is even a better chance that $\mathcal{V}H(G) = H\widetilde{V}(G)$ holds topologically. *Thus, the projective description problem should rather ask when this equality holds.*

Clearly, if $\mathcal{V}H(G)$ is boundedly retractive, the three spaces $\mathcal{V}H(G)$, $H\widetilde{V}(G)$ and $H\overline{V}(G)$, which always have the same bounded sets, also induce the same topology on these bounded sets. With the method of Theorem 2.1, the following characterizations are easy:

2.6. Proposition. (a) $H\widetilde{V}(G) = H\overline{V}(G)$ *holds topologically if and only if*

$$(N) \qquad \forall \widetilde{v} \in \widetilde{V}\ \exists \overline{v} \in \overline{V}\ \forall n \in \mathbb{N}: \quad (\min(w_n, 1/\overline{v}))^\sim \le 1/\widetilde{v}.$$

(b) $H\widetilde{V}(G)$ *is semi-Montel if and only if*

$$(\widetilde{M_H}) \qquad \forall n \in \mathbb{N}\ \forall \widetilde{v} \in \widetilde{V}\ \exists \varphi \in \mathcal{K}_+(G): \quad (\min(w_n, 1/\varphi))^\sim \le 1/\widetilde{v}.$$

Proof. (a) $\widetilde{V}$ and $\overline{V}$ yield the same topology if and only if for each $\widetilde{v} \in \widetilde{V}$ there is $\overline{v} \in \overline{V}$ such that $p_{\widetilde{v}}(f) \le p_{\overline{v}}(f)$ for each $f \in \mathcal{V}H(G)$ or, equivalently, for each $f \in B_n$, n arbitrary. Thus, $H\widetilde{V}(G) = H\overline{V}(G)$ topologically if and only if

$(*)$ $\quad \forall \widetilde{v} \in \widetilde{V}\ \exists \overline{v} \in \overline{V}\ \forall n \in \mathbb{N}:$

$$B_n \cap \{f : \overline{v}|f| \le 1\} \subset \{f \in H(G) : p_{\widetilde{v}}(f) \le 1\}.$$

Now, the proof of Theorem 2.1 immediately shows that $(*)$ is equivalent to (N). Finally, (b) is nothing but Theorem 2.1(b) with $\widetilde{V}$ replacing $\overline{V}$. ∎

We finish this section with an example in which the topologies of $H\widetilde{V}(G)$ and $H\overline{V}(G)$ are *different*. Note that in this example the sequence $\mathcal{V} = (v_n)_n$ satisfies condition (M).

2.7. Example. In the sequel, we stick to the notation of [15]. In Section 2 of that article, the authors constructed a decreasing sequence $\mathcal{W} = (w_n)_n$ of weights on the open set $G_1 = \{z \in \mathbb{C} : 1/2 < |z| < 1,\ 0 < \arg z < \pi\}$ such that $H\overline{W}(G_1)$ contains a complemented subspace isomorphic to a non-bornological sequence space K_∞ (i.e., to the strong dual of some non-distinguished Köthe echelon space λ_1); hence $\mathcal{W}H(G_1)$ and $H\overline{W}(G_1)$ have different topologies.

Then, in [15], Section 3, $\mathcal{V} = (v_n)_n$, $v_n(z_1, z_2) = w_n(z_1)u_n(z_1, |z_2|)$, is constructed on $G = G_1 \times \mathbb{C}$ in such a way (similar to the method of our Lemma 2.2) that $\mathcal{V}$ satisfies (M), but $\mathcal{V}H(G) \ne H\overline{V}(G)$ topologically. Each $f \in \mathcal{V}H(G)$ is constant in the second variable; the map $A : H\overline{V}(G) \to H\overline{W}(G_1)$, $Af(z_1) = f(z_1, 0)$ for $z_1 \in G_1$, $f \in H\overline{V}(G)$, is a linear bijection with A^{-1} continuous, but A is *not* continuous.

Similarly to our Example 1.3(b), one can now verify that the corresponding associated weights satisfy

$$\widetilde{w}_n(z_1) \le C_n \widetilde{v}_n(z_1, z_2) \quad \text{for } (z_1, z_2) \in G,$$

where C_n is some positive constant, $n = 1, 2, \ldots$ *Now, we shall prove that* $A : H\widetilde{V}(G) \to H\widetilde{W}(G_1)$ *is continuous, whence so is* $A : H\widetilde{V}(G) \to H\overline{W}(G_1)$ *and thus* $H\widetilde{V}(G) \ne H\overline{V}(G)$ *topologically.*

Fix an arbitrary $\widetilde{w} \in \widetilde{W}$ and define $\widetilde{v}(z_1, z_2) := \widetilde{w}(z_1)$, $(z_1, z_2) \in G$. For each $n \in \mathbb{N}$, putting $\alpha_n := \sup_G \widetilde{w}/\widetilde{w}_n$, we get

$$\widetilde{v}(z_1, z_2) = \widetilde{w}(z_1) \le \alpha_n \widetilde{w}_n(z_1) \le \alpha_n C_n \widetilde{v}_n(z_1, z_2), \qquad (z_1, z_2) \in G.$$

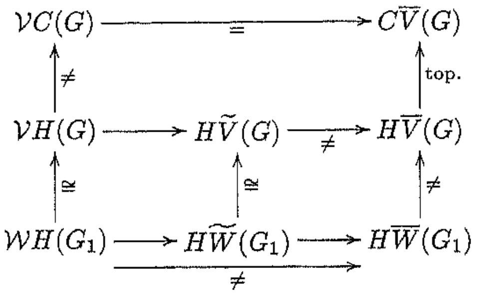
Hence $\widetilde{v}$ is an element of $\widetilde{V}$, and for any $f \in H\widetilde{V}(G)$,

$$p_{\widetilde{w}}(Af) = \sup_{G_1} \widetilde{w}|Af| = \sup_{z_1 \in G_1} \widetilde{w}(z_1)|f(z_1, 0)| \le \sup_{(z_1,z_2)\in G} \widetilde{v}(z_1, z_2)|f(z_1, z_2)| = p_{\widetilde{v}}(f),$$

which implies the desired continuity.

To complete the picture (as far as possible), let us note that $A^{-1} : H\widetilde{W}(G_1) \to H\widetilde{V}(G)$ *is continuous as well so that* $H\widetilde{W}(G_1)$ *and* $H\widetilde{V}(G)$ *are isomorphic.* Indeed, Definition 1.1 implies that each $\widetilde{v}_k$ must also be constant in the second coordinate. And from $0 \le u_k \le 1$, it follows not only that $v_k(z_1, z_2) \le w_k(z_1)$, but in view of 1.2(vii), it is also clear that $\widetilde{v}_k(z_1, z_2) \le \widetilde{w}_k(z_1)$ for $(z_1, z_2) \in G$, $k = 1, 2, \ldots$ Then, proceeding along the lines of the corresponding part of the proof of [15], Prop. 4, the continuity of $A^{-1} : H\widetilde{W}(G_1) \to H\widetilde{V}(G)$ becomes obvious.

Thus, we have the following situation:



It remains open whether the topologies of $\mathcal{W}H(G_1)$ *and* $H\widetilde{W}(G_1)$, *and hence also those of* $\mathcal{V}H(G)$ *and* $H\widetilde{V}(G)$, *are different.* However, Bonet and Vogt have observed that there are examples of sequences $\mathcal{V} = (v_n)_n$ of weights on $G = D$ such that $v_n = \widetilde{v}_n$ for each n, hence $\overline{V} = \widetilde{V}$, but still $\mathcal{V}H(G) \neq H\widetilde{V}(G)$ topologically.

In case $\mathcal{V}C(G) = C\overline{V}(G)$ holds topologically, the "old projective description problem" (concerning $\mathcal{V}H(G) = H\overline{V}(G)$ topologically) was equivalent to asking if $\mathcal{V}H(G)$ is a topological subspace of $\mathcal{V}C(G)$. The disadvantage of the "new projective description problem" (concerning $\mathcal{V}H(G) = H\widetilde{V}(G)$ topologically) is that it is not directly linked to $\mathcal{V}C(G)$ any more. (But, of course, if $\mathcal{V}H(G)$ is a topological subspace of $\mathcal{V}C(G)$ and if $\mathcal{V}$ satisfies (D), then clearly $\mathcal{V}H(G) = H\widetilde{V}(G) = H\overline{V}(G)$ topologically.)

Appendix. In this appendix, we explain a result on compact embeddings of weighted spaces of holomorphic functions which, in some sense, generalizes (the main point behind) Theorem 2.1(a).

A general *weighted space* $HV(G)$ *of holomorphic functions* is defined exactly as $H\overline{V}(G)$, but with $\overline{V}$ replaced by a general system V of *nonnegative upper semicontinuous* functions on G, which (can and) will always be assumed to be a *Nachbin family*, i.e., for $v_1, v_2 \in V$ and any $\lambda > 0$, there is $v_3 \in V$ with $\lambda v_1, \lambda v_2 \leq v_3$, and for each $z \in G$ there exists $v \in V$ such that $v(z) > 0$. The (Hausdorff) topology of $HV(G)$ is given by the directed system $(p_v)_{v\in V}$ of seminorms, $p_v(f) = \sup_G v|f|$ for $f \in HV(G)$, and $(B_v)_{v\in V}$ yields a basis of 0-neighborhoods in $HV(G)$, where $B_v := \{f \in HV(G) : p_v(f) \leq 1\}$ for any $v \in V$.

In the sequel, let v_1 denote a weight, V_2 a Nachbin family of *weights* and V_3 a Nachbin family of nonnegative *continuous* functions on G. We assume that for each $v_2 \in V_2$ there is $C > 0$ with $v_2 \leq Cv_1$ and that $V_3 \leq V_2$ (in the sense that for each $v_3 \in V_3$ there is $v_2 \in V_2$ with $v_3 \leq v_2$). Then the canonical injections

$$Hv_1(G) \to HV_2(G) \to HV_3(G)$$

are continuous.

2.8. THEOREM. (a) *Under our assumptions,* $HV_3(G)$ *induces the topology of* $HV_2(G)$ *on the unit ball* $B_1 := B_{v_1}$ *of* $Hv_1(G)$ *if and only if*

$$\forall v_2 \in V_2\ \exists v_3 \in V_3 : \quad \left(\min\left(\frac{1}{v_1}, \frac{1}{v_3}\right)\right)^{\sim} \leq \frac{1}{v_2}.$$

(b) *The embedding* $Hv_1(G) \to HV_2(G)$ *is compact if and only if*

$$\forall v_2 \in V_2\ \exists \varphi \in \mathcal{K}_+(G) : \quad \left(\min\left(\frac{1}{v_1}, \frac{1}{\varphi}\right)\right)^{\sim} \leq \frac{1}{v_2}.$$

(c) *Let* v_1, v_2 *denote weights on* G *with* $v_2 \leq v_1$, *and put* $w_k = 1/v_k$, $k = 1, 2$. *Then the canonical injection* $Hv_1(G) \to Hv_2(G)$ *is compact if and only if*

$$\forall \varepsilon > 0\ \exists \varphi \in \mathcal{K}_+(G) : \quad \left(\min\left(w_1, \frac{1}{\varphi}\right)\right)^{\sim} \leq \varepsilon w_2.$$

Proof. (a) Since the topology of $HV_3(G)$ is weaker than the one of $HV_2(G)$, the two topologies coincide (at 0) on B_1 if and only if, for each $v_2 \in V_2$, there is $v_3 \in V_3$ with $B_1 \cap B_{v_3} \subset B_{v_2}$ or, equivalently (since $B_1 \subset HV_2(G) \subset HV_3(G)$),

$$B_1 \cap \{f : v_3|f| \leq 1\} \subset \{f : v_2|f| \leq 1\}.$$

But the proof of Theorem 2.1 shows that this inclusion is equivalent to

$$\left(\min\left(\frac{1}{v_1}, \frac{1}{v_3}\right)\right)^{\sim} \leq \frac{1}{v_2}.$$

(b) Since V_2 is a system of *continuous positive* functions, the topology of $HV_2(G)$ must be stronger than co. Hence the embedding $Hv_1(G) \to HV_2(G)$ is compact if and only if co induces the topology of $HV_2(G)$ on B_1. But for the Nachbin family $V_3 = \mathcal{K}_+(G)$, we have $HV_3(G) = (H(G), \mathrm{co})$, and thus it suffices to apply (a). Finally, (c) follows directly from (b) by taking $V_2 = \{\lambda v_2 : \lambda > 0\}$. ∎

Associated weights can also serve to characterize (the boundedness and) the compactness of *composition operators* between weighted Banach spaces of holomorphic functions (see [13]).

3. Additional results and remarks. This section is divided into three subsections. Part A is devoted to estimates for $\widetilde{w}$ when w is a radial growth condition on $\mathbb{C}$ satisfying certain natural assumptions (Prop. 3.1).

In Part B, it is first pointed out (Prop. 3.4) that for radial weights v with condition (U) on the unit disk, Shields and Williams [26] already showed that v and the associated weight $\widetilde{v}$ are equivalent. Then we prove (Prop. 3.5) that for a sequence $\mathcal{V} = (v_n)_n$ of radial weights on D for which $r \to v_n(r)$ is strictly decreasing, $n = 1, 2, \ldots$, the space $\mathcal{V}H(D)$ is (DFS) if and only if the sequence $\widetilde{\mathcal{V}} = (\widetilde{v}_n)_n$ of associated weights satisfies condition (S). This is remarkable because a simple example after Theorem 2.1 demonstrated that such a result *cannot* hold for $\mathbb{C}$ instead of D. Finally, there is a way to calculate $\widetilde{w}$, up to a constant, if $w = 1/v$ is a growth condition on D which has a radial limit function w^* on ∂D with certain properties: In this situation, $\widetilde{w} = 1/\widetilde{v}$ is equivalent to the modulus of the corresponding outer function Q_{w^*} (Prop. 3.6).

Part C sketches applications of Hörmander's $\overline{\partial}$-technique and of Phragmén–Lindelöf type theorems to estimates for $\widetilde{w}$ on $\mathbb{C}^N$ (Props. 3.8, 3.9). We finish the article by constructing an example of a space $\mathcal{V}H(G)$ with some "strange" properties, this time on a set G which is the union of a sequence of pairwise disjoint disks in $\mathbb{C}$.

In contrast to the previous sections, not all the arguments here are given in (full) detail, and some (easy) calculations are omitted. But we provide the reader with sufficient information and relevant references.

3.A. Estimates for $\widetilde{w}$ on $\mathbb{C}$. Let w be a radial growth condition on $\mathbb{C}$. If $r \to w(r)$ is *increasing and logarithmically convex* for $r \in [1, \infty)$, then w satisfies

$$(*) \qquad w(r) = w(1) \exp \int_1^r \frac{\omega(\varrho)}{\varrho}\, d\varrho, \qquad r \geq 1,$$

for some positive increasing function ω (see Clunie–Kővári [17], Theorem 4). Classical results on Taylor series and growth conditions of entire functions lead to estimates of $\widetilde{w}$, as follows.

3.1. Proposition. *Let w be a radial growth condition on $\mathbb{C}$.*

(a) *If (even) $r \to w(r)/\sqrt{r}$ is increasing and logarithmically convex for $r \geq 1$, and if $1/w$ is rapidly decreasing (i.e., for every $n \in \mathbb{N}$, $w(r)/r^n \to \infty$ as $r \to \infty$), then there is $C > 0$ with*

$$\widetilde{w}(r) \leq w(r) \leq C(1 + r)\widetilde{w}(r), \qquad r \geq 0.$$

(b) *If $r \to w(r)$ is increasing and logarithmically convex for $r \geq 1$, and if, for ω as in $(*)$, there is $c > 1$ with $\omega(cr) - \omega(r) \geq 1$ for all $r \geq 1$, then there exists $C > 1$ such that*

$$\widetilde{w}(r) \leq w(r) \leq C\widetilde{w}(r), \quad r \geq 0.$$

Proof. (a) By Clunie and Kővári [17], Theorem 2, applied to $r \to w(r)/\sqrt{r}$, there is an entire function f (whose Taylor series has positive coefficients) such that

$$(\times) \qquad 1 \leq \frac{w(r)}{M(f, r)} \leq 9r, \qquad r \geq \frac{9}{5}.$$

Put $w_1(z) := M(f, |z|)$, $z \in \mathbb{C}$. Then w_1 is a radial growth condition on $\mathbb{C}$ which satisfies $w_1(r) \leq w(r) \leq 9rw_1(r)$ for $r \geq 2$. Hence, by 1.2(vii), $\widetilde{w}_1(r) \leq \widetilde{w}(r)$ for $r \geq 2$, and by 1.6, $\widetilde{w}_1 = w_1$ holds. It follows that for $r \geq 2$,

$$\widetilde{w}(r) \leq w(r) \leq 9rw_1(r) = 9r\widetilde{w}_1(r) \leq 9r\widetilde{w}(r).$$

The proof of (b) is similar: By [17], Theorem 4, applied to $r \to w(r)$, $(\times)$ can now be replaced by

$$1 \leq \frac{w(r)}{M(f, r)} \leq C, \qquad r \geq 1. \ \blacksquare$$

The conditions of Proposition 3.1 are quite natural: E.g., $v = 1/w$ rapidly decreasing means exactly that $Hv(\mathbb{C})$ (or $Hv_0(\mathbb{C})$) contains the polynomials. And if the radial growth condition on $\mathbb{C}$ is taken in the form $w(z) = \exp \omega(|z|)$, $z \in \mathbb{C}$, for a continuous *increasing* function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ with, say, $\omega(0) = \omega(1) = 0$, then supposing that $\log w$ is *subharmonic* is equivalent to assuming that the function φ, $\varphi(t) := \omega(e^t)$ for $t \geq 1$, is *convex* (see [4], 4.4.19 and 4.4.26), i.e., $r \to w(r)$ is *logarithmically convex* (cf. the notes after 1.2 and 1.6). In this setting, $1/w$ is rapidly decreasing if and only if $\log r = o(\omega(r))$ as $r \to \infty$.

Let us now prove directly (without recourse to [17]) that, in the setting which we have just described, $w(r) \leq r\widetilde{w}(r)$ for $r \geq 1$.

We will make use of the *Young conjugate* $\varphi^* : \mathbb{R}_+ \to \mathbb{R}_+$ of φ; it is defined by the formula $\varphi^*(y) := \sup\{xy - \varphi(x) : x \geq 0\}$. φ^* is again increasing, convex and satisfies $\varphi^*(0) = 0$, $y = o(\varphi^*(y))$ as $y \to \infty$; moreover, $\varphi^{**} = \varphi$ (cf. e.g. [16]).

First note that for $n \in \mathbb{N}_0$,

$$\begin{aligned}\inf_{r>0} \frac{\exp \omega(r)}{r^n} &= \inf_{r\geq 1} \frac{\exp \omega(r)}{r^n} = \inf_{r \geq 1} \exp(\omega(r) - n \log r) \\ &= \exp(-\sup_{r\geq 1}(n \log r - \varphi(\log r))) = \exp(-\varphi^*(n)),\end{aligned}$$

hence

$$(*) \qquad 0 < \exp(-\varphi^*(n)) \leq \frac{\exp \omega(r)}{r^n} \qquad \text{for each } r > 0.$$

Next, to prove the desired estimate, we fix $r_0 \geq 1$, let $0 < \varepsilon < 1$ and choose $n_0 \in \mathbb{N}$ with

$$r_0^{n_0} \exp(-\varphi^*(n_0)) > (1-\varepsilon) \sup_{n \in \mathbb{N}_0} (r_0^n \exp(-\varphi^*(n)).$$

Then $g(z) := \exp(-\varphi^*(n_0)) z^{n_0}$ for $z \in \mathbb{C}$ defines an entire function g such that, in view of $(*)$,

$$|g(z)| = \exp(-\varphi^*(n_0))|z|^{n_0} \leq \exp \omega(|z|) = w(z)$$

for each $z \in \mathbb{C}$, while

$$w(r_0) = \exp \varphi^{**}(\log r_0) = \sup_{t \geq 0}(r_0^t \exp(-\varphi^*(t)))$$
$$\leq \sup_{n \in \mathbb{N}_0} (r_0^{n+1} \exp(-\varphi^*(n))) \leq \frac{r_0}{1-\varepsilon} g(r_0)$$

by our choice of n_0. So far, we have proved that for each $\varepsilon \in (0,1)$ there is $g = g_\varepsilon \in B_w$ (see Def. 1.1) with $g(r_0) > (1-\varepsilon) w(r_0)/r_0$. Since $\varepsilon > 0$ is arbitrary and B_w is compact in the topology of pointwise convergence, it follows that for each $r_0 \geq 1$ there exists $f \in B_w$ with

$$|f(r_0)| \geq \frac{1}{r_0} w(r_0), \quad \text{whence} \quad \widetilde{w}(r_0) \geq \frac{1}{r_0} w(r_0). \ \blacksquare$$

It is interesting to note that if, in addition, $(\varphi^*)'$ exists and is strictly increasing, then there is an increasing sequence $(\varrho_n)_{n \in \mathbb{N}} \subset \mathbb{R}_+$, $\varrho_n \to \infty$, such that $\widetilde{w}(\varrho_n) = w(\varrho_n)$ for each $n \in \mathbb{N}$.

In fact, put $r_n := (\varphi^*)'(n)$ and $\varrho_n := e^{r_n}$, $n = 1, 2, \ldots$; since $y = o(\varphi^*(y))$, clearly $\lim_{n \to \infty} \varrho_n = \infty$. Now, for arbitrary $n \in \mathbb{N}$, a glance at the derivative yields

$$w(\varrho_n) = \exp \omega(\varrho_n) = \exp \varphi(r_n) = \exp(\max_{y \geq 0}(y r_n - \varphi^*(y)))$$
$$= \exp(n r_n - \varphi^*(n)) = \varrho_n^n \exp(-\varphi^*(n)).$$

On the other hand, $g(z) := \exp(-\varphi^*(n)) z^n$ for $z \in \mathbb{C}$ defines an entire function g which, in view of $(*)$, satisfies $|g| \leq w$ on $\mathbb{C}$ and $g(\varrho_n) = w(\varrho_n)$, i.e., $\widetilde{w}(\varrho_n) = w(\varrho_n)$. $\blacksquare$

Now we modify an idea of [17], Theorem 3, to show that, even under the conditions of Proposition 3.1(a), $\widetilde{w}$ can really be *essentially* smaller than w.

For this purpose, let us fix a sequence $(r_n)_{n \in \mathbb{N}_0} \subset \mathbb{R}_+$ with $r_0 = r_1 = 1$, $r_n < r_{n+1}$ for $n \geq 1$, $\lim_{n \to \infty} r_n = \infty$, and set $A_n := (r_1 \ldots r_n)^{-1}$, $n \in \mathbb{N}$. Next, put $w|_{[0,1]} \equiv 1$ and $w(r) := A_n r^{n+1/2}$ for $r \in [r_n, r_{n+1})$, $n = 1, 2, \ldots$ Then $w : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, increasing and satisfies $\lim_{r \to \infty} r^n / w(r) = 0$ for each $n \in \mathbb{N}$; moreover, the corresponding function φ, $\varphi(t) = \log(w(e^t)/\sqrt{e^t})$ for $t \in \mathbb{R}_+$, is convex. Finally, extend w to a radial function on $\mathbb{C}$ by taking $w(z) := w(|z|)$ for each $z \in \mathbb{C}$.

3.2. Lemma. *Let f be an entire function with $|f| \leq w$ on $\mathbb{C}$ such that the Taylor series of f, $f(z) = \sum_{k=0}^\infty a_k z^k$, has nonnegative coefficients a_k, $k \in \mathbb{N}_0$. Then*

$$0 \leq f(r) \leq \left(\left(\frac{r_n}{r} \right)^{1/2} + \left(\frac{r}{r_{n+1}} \right)^{1/2} \right) w(r), \quad r \in [r_n, r_{n+1}), \ n = 1, 2, \ldots$$

Proof. Fix $n \in \mathbb{N}$. Clearly, $|f| \leq w$ implies

$$\sum_{k=0}^\infty a_k r^{k-n-1/2} \leq A_n, \quad r \in [r_n, r_{n+1}).$$

Hence, for all such r,

$$\sum_{k=0}^n a_k r^{k-n-1/2} = \left(\frac{r_n}{r}\right)^{1/2} \sum_{k=0}^n a_k r_n^{k-n-1/2} \left(\frac{r_n}{r}\right)^{n-k} \leq A_n \left(\frac{r_n}{r}\right)^{1/2},$$
$$\sum_{k=n+1}^\infty a_k r^{k-n-1/2} = \left(\frac{r}{r_{n+1}}\right)^{1/2} \sum_{k=n+1}^\infty a_k r_{n+1}^{k-n-1/2} \left(\frac{r}{r_{n+1}}\right)^{k-n-1}$$
$$\leq A_n \left(\frac{r}{r_{n+1}}\right)^{1/2}.$$

Summing up, we get

$$\sum_{k=0}^\infty a_k r^{k-n-1/2} \leq A_n \left(\left(\frac{r_n}{r}\right)^{1/2} + \left(\frac{r}{r_{n+1}}\right)^{1/2} \right),$$

and multiplication by $r^{n+1/2}$ yields the desired inequality. $\blacksquare$

3.3. Example. For w as above, for $n \in \mathbb{N}$ and $r \in [r_n, r_{n+1})$, the corresponding $\widetilde{w}$ satisfies

$$\widetilde{w}(r) \leq 18 \left(\left(\frac{r_n}{r}\right)^{1/2} + \left(\frac{r}{r_{n+1}}\right)^{1/2} \right) w(r).$$

Proof. Fix $g \in B_w$. By a result of Erdős–Kővári [18], there is a function $f \in H(\mathbb{C})$, $f(z) = \sum_{k=0}^\infty a_k z^k$ with $a_k > 0$ for all $k \in \mathbb{N}_0$, such that

$$(*) \qquad \frac{1}{6} f(r) \leq M(g, r) \leq 3 f(r) \quad \text{for each } r > 0.$$

Since w is radial and $|g| \leq w$, it follows from $(*)$ that $f(r) \leq 6w(r)$, $r > 0$. Now Lemma 3.2 yields, for $n \in \mathbb{N}$ and each $r \in [r_n, r_{n+1})$,

$$f(r) \leq 6 \left(\left(\frac{r_n}{r}\right)^{1/2} + \left(\frac{r}{r_{n+1}}\right)^{1/2} \right) w(r),$$

hence by another application of $(*)$,

$$M(g,r)\le 18\Big(\Big(\frac{r_n}{r}\Big)^{1/2}+\Big(\frac{r}{r_{n+1}}\Big)^{1/2}\Big)w(r).$$

Finally, invoking Observation 1.5, one realizes that the last estimate also holds for $\widetilde{w}(r)$ instead of $M(g,r)$. ■

Note that, taking $r_{n+1}=2^n r_n$ for $n=1,2,\ldots$, one gets by 3.3,

$$\widetilde{w}(2^{n/2}r_n)\le 36\Big(\frac{1}{2^n}\Big)^{1/4}w(2^{n/2}r_n).$$

3.B. Functions on the unit disk. Let $v:[0,1]\to\mathbb{R}_+$ be a continuous decreasing function with $v(0)=1$, $v(1)=0$ and $v(r)>0$ for $r\in(0,1)$. Then v becomes a radial weight on D by taking $v(z):=v(|z|)$, $z\in D$. The auxiliary function $\psi:[1,\infty)\to\mathbb{R}_+\setminus\{0\}$ is defined by $\psi(t):=1/v(1-1/t)$, $t\ge 1$; note that $1/v(r)=\psi(1/(1-r))$, $r\in[0,1)$. According to Shields–Williams [26], ψ is said to satisfy *condition* (U) if there are $a,c>0$ such that

$$1\le x<y\Rightarrow \frac{\psi(y)}{y^a}\le c\frac{\psi(x)}{x^a};$$

in this case, we will also say that v satisfies condition (U). As communicated to the authors by W. Lusky, (U) is equivalent to his condition $(*)$ (on v) in [21]. For examples of weights v which satisfy (U), we refer to [21] or [25]. Now Proposition 3.4 follows directly from Shields–Williams [26], Lemma 1(iv).

3.4. PROPOSITION. *For a radial weight v on D as above (which, in particular, satisfies (U)), there is a constant $C>0$ such that*

$$v(r)\le\widetilde{v}(r)\le Cv(r)\quad \textit{for each } r\in[0,1),$$

i.e., v and $\widetilde{v}$ are equivalent weights.

If the radial weight v on D decreases very rapidly (e.g., exponentially) as $r\to 1_-$, (U) is not satisfied. Then the corresponding growth condition $w=1/v$ increases very rapidly. The method of Proposition 3.1(b) can still be used to prove equivalence of w and $\widetilde{w}$ (under certain assumptions).

We suppose that the radial growth condition w on D satisfies

$$w(r)=w(0)\exp\int_0^r\frac{\tau(\varrho)}{1-\varrho}\,d\varrho,\quad 0\le r<1,$$

where τ is a positive increasing function on $[0,1)$ such that, for some $c>1$ and all $\varrho\in[0,1)$,

$$(*)\qquad \tau(1-(1-\varrho)/c)-\tau(\varrho)\ge 1.$$

Then there is $C>1$ with $\widetilde{w}\le w\le C\widetilde{w}$ on D.

In fact, starting from τ, one defines a positive increasing function ψ on $[1,\infty)$ by $\psi(\varrho):=\tau(1-1/\varrho)$ and puts

$$\Phi(r):=w(0)\exp\int_1^r\frac{\psi(\varrho)}{\varrho}\,d\varrho,\quad r\ge 1.$$

Since $\Phi(1/(1-r))=w(r)$ for all $r\in[0,1)$ and $(*)$ holds, Φ satisfies the assumptions of [17], Theorem 4. Hence there is an entire function f whose Taylor series has positive coefficients such that, for some $C_0>0$,

$$C_0^{-1}\le\frac{\Phi(r)}{M(f,r)}=\frac{\Phi(r)}{f(r)}\le C_0,\quad r\ge 1.$$

Define $g(z):=f(1/(1-z))$ and $w_1(z):=g(|z|)$ for $z\in D$. Then g is an analytic function on D whose Taylor series at 0 also has positive coefficients, and hence the growth condition w_1 on D satisfies $\widetilde{w}_1=w_1$ by 1.6. Moreover,

$$w(r)=\Phi\Big(\frac{1}{1-r}\Big)\le C_0f\Big(\frac{1}{1-r}\Big)=C_0w_1(r),\quad 0\le r<1,$$

and $w_1(r)\le C_0w(r)$ follows in the same way. Now 1.2(vii) yields

$$w(r)\le C_0w_1(r)=C_0\widetilde{w}_1(r)\le C_0^2\widetilde{w}(r),\quad 0\le r<1. ■$$

Next we turn to the characterization of the (DFS)-property of the space $\mathcal{V}H(D)=\mathrm{ind}_n Hv_n(D)$ in terms of condition (S) (cf. Section 2) on the sequence $\widetilde{\mathcal{V}}=(\widetilde{v}_n)_n$ of associated weights $\widetilde{v}_n$, when $\mathcal{V}=(v_n)_n$ is a decreasing sequence of "nice" radial weights v_n on D.

3.5. PROPOSITION. *Let $\mathcal{V}=(v_n)_n$ denote a decreasing sequence of positive continuous radial weights such that each $r\to v_n(r)$ is strictly decreasing on $[0,1)$, $n=1,2,\ldots$ Then $\mathcal{V}H(D)$ is a (DFS)-space if and only if $\widetilde{\mathcal{V}}=(\widetilde{v}_n)_n$ satisfies (S).*

Proof. In a remark after the proof of Theorem 2.1, it was already pointed out that $\mathcal{V}H(D)=\widetilde{\mathcal{V}}H(D)$ is a (DFS)-space if $\widetilde{\mathcal{V}}$ satisfies (S) (and this holds for any sequence $\mathcal{V}$ on an arbitrary open set $G\subset\mathbb{C}^N$).

Conversely, assume that $\mathcal{V}H(D)$ is a (DFS)-space and again write $w_n=1/v_n$, $n=1,2,\ldots$ Exactly as in the proof of Theorem 2.1(1), one shows that

(*) for each $n\in\mathbb{N}$ there is $m>n$ with the property that for each $\varepsilon>0$ one can find $\delta>0$ (without loss of generality, $\delta<w_n(0)$) and $r\in(0,1)$ such that each $f\in H(D)$ with $|f|\le w_n$ on D and $|f|\le\delta$ on $\overline{D}_r$, the closed disk of radius r around 0, must satisfy $|f|\le\varepsilon\widetilde{w}_m$ on D.

To show that $\widetilde{\mathcal{V}}=(\widetilde{v}_n)_n$ satisfies (S), fix $n\in\mathbb{N}$ and choose $m>n$ as in $(*)$; for arbitrary $\varepsilon>0$, take $\delta>0$ and $r\in(0,1)$ as provided by $(*)$. We *claim* that there exists $r_0\in[r,1)$ such that $\widetilde{v}_m(z)\le 2\varepsilon\widetilde{v}_n(z)$ for all $z\in D\setminus\overline{D}_{r_0}$.

This claim will follow after the introduction of an auxiliary function φ. Let us first put $a := \delta(1-r)/(w_n(r)-\delta) > 0$; clearly $a/(1+a-r) = \delta/w_n(r)$. Now define $\varphi \in H(D)$ by $\varphi(z) := a/(1+a-z)$, $z \in D$; it is easy to check that

(i) $|\varphi| \le 1$ on D and
(ii) $|\varphi| \le \delta/w_n(r)$ on $\overline{D}_r$.

Since $\varphi(x)$ is real for $x \in [0,1)$ and $\lim_{x\to 1}\varphi(x) = 1$, there is $r_0 \in [r,1)$ such that $\varphi(s) > 1/2$ for $s \in [r_0, 1)$.

To prove the claim, we fix $z \in D \setminus \overline{D}_{r_0}$ and let $s := |z|$, whence $s > r_0$. By Observation 1.5, $\widetilde{w}_n$ and $\widetilde{w}_m$ are radial, i.e., $\widetilde{w}_n(z) = \widetilde{w}_n(s)$ and $\widetilde{w}_m(z) = \widetilde{w}_m(s)$. By 1.2(iv), there is $g \in H(D)$ with $|g| \le w_n$ on D such that $|g(s)| = \widetilde{w}_n(s)$. Now, define $f := \varphi g \in H(D)$. By (i), we have $|f| \le w_n$ on D; on the other hand, for $\zeta \in \overline{D}_r$, we get from (ii),

$$|f(\zeta)| = |g(\zeta)| \cdot |\varphi(\zeta)| \le w_n(\zeta)\frac{\delta}{w_n(r)} = w_n(|\zeta|)\frac{\delta}{w_n(r)} \le \delta.$$

$(*)$ implies that $|f| \le \varepsilon\widetilde{w}_m$ on D; in particular, $|f(s)| \le \varepsilon\widetilde{w}_m(s)$. Since $\varphi(s) > 1/2$, we finally obtain

$$\widetilde{w}_n(z) = \widetilde{w}_n(s) = |g(s)| = \frac{|f(s)|}{\varphi(s)} \le 2\varepsilon\widetilde{w}_m(s) = 2\varepsilon\widetilde{w}_m(z),$$

which is equivalent to the claim. ∎

Recently, D. Vogt extended Proposition 3.5 to general open sets G if the weights v_n satisfy the condition of 1.13, i.e., he proved the following theorem:

Let G be an open subset of $\mathbb{C}^N$ and $\mathcal{V} = (v_n)_n$ a decreasing sequence of weights on G. If, for each $n \in \mathbb{N}$, $\widetilde{v}_n$ is a weight with $\widetilde{v}_n = \widetilde{v}_{n0}$ (in the notation introduced before 1.13), *then the following assertions are equivalent*:

(1) *$\mathcal{V}H(G)$ is a (DFS)-space,*
(2) *$\mathcal{V}_0H(G) = \mathrm{ind}_n\, H(v_n)_0(G)$ is a (DFS)-space,*
(3) *$\widetilde{\mathcal{V}} = (\widetilde{v}_n)_n$ satisfies (S).*

So far, our results in Section 3 have dealt with radial weights. But *non-radial* weights on the unit disk are also important: They come up naturally e.g. when functions on a half plane H (with growth conditions on the boundary) are considered and the conformal map $H \to D$ is used. The following remarks apply to certain nonradial weights v on D.

For a function $f : D \to \mathbb{C}$ and $0 < r < 1$, let $f_r : \partial D \to \mathbb{C}$ be defined by $f_r(e^{it}) := f(re^{it})$, $t \in [-\pi, \pi]$. Now let v be a weight on D which has an extension $v : \overline{D} \to [0,\infty]$ with $v|_{\partial D} = \lim_{r\to 1-} v_r$ a.e. on ∂D such that, for $w := 1/v$ and for some $1 \le p \le \infty$, $\sup_{r\in[0,1)} \|w_r\|_{L^p(\partial D)} < \infty$, $w|_{\partial D} =: w^* \in L^p(\partial D)$ and $\log w^* \in L^1(\partial D)$. In this case, $Hv(D)$ is contained in the *Hardy space* H^p (cf. Rudin [24], 17.7); in particular, each $f \in Hv(D)$ has a *radial limit function* $f^* : \partial D \to \mathbb{C}$, defined at almost all points of ∂D. Also recall (see [24], 17.14) that for any $g \in L^p(\partial D)$ with $g \ge 0$ and $\log g \in L^1(\partial D)$, the *outer function* $Q_g \in H(D)$ is defined by

$$Q_g(z) := \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}\log g(e^{it})\,dt\right), \quad z \in D.$$

3.6. Proposition. *Let v be a weight as above and $w = 1/v$. If there is a constant $C > 0$ with $C|Q_{w^*}| \le w$ on D, then $\widetilde{w}$ is equivalent to $|Q_{w^*}|$, i.e., $C|Q_{w^*}| \le \widetilde{w} \le |Q_{w^*}|$ on D.*

Proof. The first inequality is clear from 1.2(iii). To see the second estimate take an arbitrary $f \in H(D)$, $f \not\equiv 0$, with $|f| \le w$ on D. Then $f \in Hv(D) \subset H^p$ and $|f^*| \le w^*$ a.e. on D. Now, it follows from [24], Theorems 17.16 and 17.17, that

$$|f| \le |Q_{|f^*|}| \le |Q_{w^*}|.$$

Thus, we get $\widetilde{w} \le |Q_{w^*}|$ on D. ∎

Now take a function $g \in L^1(\partial D)$ with $g > 0$ a.e. and $\log g \in L^1(\partial D)$. Then the *Poisson integral* of g (see [24], 11.7),

$$w := P[g] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{Re}\left(\frac{e^{it}+z}{e^{it}-z}\right)g(e^{it})\,dt,$$

is harmonic and positive on D; it satisfies all the assumptions on w needed for 3.6 (with $p = 1$ and $w^* = g$; cf. [24], Corollary to 11.10).

3.7. Corollary. *For $w = P[g]$ in the present situation, one actually has $\widetilde{w} = |Q_g|$.*

Proof. From [24], proof of 17.16 (using Jensen's inequality), $|Q_g| \le P[g] = w$ follows, hence by 1.2(iii), $|Q_g| \le \widetilde{w}$. The second inequality is clear from 3.6. ∎

Note that, using 3.7, one can give *simple examples* of functions $w = P[g]$ such that $\widetilde{w}(r) < w(r)$ for each $r \in [0,1)$. (E.g., take $g(e^{it}) = m$ for $t \in [-\pi, 0]$ and $g(e^{it}) = M$ for $t \in (0,\pi]$, where $m \ne M$.) And it follows from techniques of [22] that for other interesting weights on D the conditions of 3.6 are satisfied.

3.C. Final remarks. Let p denote a nonnegative continuous p.s.h. function on $\mathbb{C}^N$ and let q be defined by $q(z) := \sup\{p(\zeta) : \|\zeta - z\|_\infty \le 1\}$, $z \in \mathbb{C}^N$. Let $|\cdot|$ denote the Euclidean norm. We consider the continuous positive function w on $\mathbb{C}^N$,

$$w(z) := \exp(q(z) + 3\log(1 + |z|^2)) \quad \text{for } z \in \mathbb{C}^N.$$

The $\overline{\partial}$-techniques of Hörmander can be applied to estimate $\widetilde{w}$, as follows.

3.8. Proposition. *For w as above, there is $M > 0$ such that $\exp(p(z)) \le M\widetilde{w}(z)$ for each $z \in \mathbb{C}^N$.*

Proof. By a result of Abanin ([1], Lemma 4), there is $M > 0$ such that, for each $z_0 \in \mathbb{C}^N$, there is $f \in H(\mathbb{C}^N)$ with $f(z_0) = \exp(p(z_0))$ and $|f| \le Mw$ on $\mathbb{C}^N$. Now

$$\frac{1}{M}\exp p(z_0) \le \sup\{|g(z_0)| : g \in B_w\} = \widetilde{w}(z_0). \ \blacksquare$$

Phragmén–Lindelöf theorems can also be applied to estimate associated weights; here is an example:

3.9. Proposition. *Let w be a positive continuous function on $\mathbb{C}^N$ such that the following estimates hold for some $B > 0$ and some $n \in \mathbb{N}$:*

(i) *$w(z) \le \exp(B|z|)$ for each $z \in \mathbb{C}^N$ and*
(ii) *$w(x) \le (1+|x|)^n$ for each $x \in \mathbb{R}^N$.*

Then $\widetilde{w}(z) \le (1+|z|)^n \exp(B|\mathrm{Im}\, z|)$ for each $z \in \mathbb{C}^N$.

Proof. Let $f \in H(\mathbb{C}^N)$ satisfy $|f| \le w$ on $\mathbb{C}^N$; then the estimates (i) and (ii) also hold for $|f|$. Applying the Phragmén–Lindelöf principle as in Hörmander [20], proof of 16.3.10, we conclude

$$|f(z)| \le (1+|z|)^n \exp(B|\mathrm{Im}\, z|) \quad \text{for each } z \in \mathbb{C}^N,$$

whence the desired estimate. ■

We note that it is possible to replace the term $(1+|x|)^n$ in 3.9 by $\exp(C\omega(x))$, $C > 0$, where ω is a nonquasianalytic weight as in [16]; for example, $\omega(x) = |x|^p$, $0 < p < 1$.

For our final example, we adopt the following setting:

Let v, w be continuous real-valued functions on $[0,1)$, $v(0) = w(0) = 1$, v strictly increasing with $\lim_{r\to 1_-} v(r) = \infty$ and w strictly decreasing with $\lim_{r\to 1_-} w(r) = 0$. Both are extended to D radially.

3.10. Lemma. *Given $0 < \varepsilon < 1$, let r_0 be the unique number in $(0,1)$ with $v(r_0) = \varepsilon^{-1}$. Put $b := v(r_0)/w(r_0) > 1$, define $u := \min(v, bw)$ on $[0,1)$ and then $u(z) := u(|z|)$ for $z \in D$. Now, if $f \in H(D)$ satisfies $u|f| \le 1$, then $w|f| \le \varepsilon$ on D.*

Proof. Fix $f \in H(D)$ with $u|f| \le 1$ on D and take a point z_0 with $|z_0| = r$ such that $|f(z_0)| = \max_{|z|=r_0}|f(z)| = \max_{|z|\le r_0}|f(z)|$. Since $v(0) - bw(0) = 1 - b < 0$, $v(r_0) - bw(r_0) = 0$ and $v(r) - bw(r) \to \infty$ as $r \to 1_-$, it follows that $u(r)$ equals $v(r)$ for $r \le r_0$ while $u(r) = bw(r)$ for $r \ge r_0$. Now,

$$\begin{aligned}\sup_{|z|\le r_0} w(z)|f(z)| &\le w(0)\sup_{|z|\le r_0}|f(z)| = |f(z_0)| \\ &\le u(r_0)^{-1}\sup_{|z|=r_0} u(z)|f(z)| \le u(r_0)^{-1} = v(r_0)^{-1} = \varepsilon, \\ \sup_{r_0\le|z|<1} w(z)|f(z)| &= b^{-1}\sup_{r_0\le|z|<1} bw(z)|f(z)| \le b^{-1}\sup_{z\in D} u(z)|f(z)| \\ &\le b^{-1} < b^{-1}w(r_0)^{-1} = v(r_0)^{-1} = \varepsilon. \ \blacksquare\end{aligned}$$

Turning to our example, put

$$D_m := 4m + D, \quad m = 1, 2, \ldots, \quad \text{and} \quad G := \bigcup_{m\in\mathbb{N}} D_m.$$

The decreasing sequence $\mathcal{V} = (v_n)_n$ of weights on G is defined by

$$v_n(z) := \begin{cases} w(z-4m), & z \in D_m,\ m < n, \\ v(z-4m), & z \in D_m,\ m \ge n. \end{cases}$$

3.11. Example. $\mathcal{V}$ satisfies (ND) (so that the topologies of $\mathcal{V}C(G)$ and $C\overline{V}(G)$ are different, cf. Section 2), but $\mathcal{V}H(G) = H\overline{V}(G)$ topologically, and this space is isomorphic to $\bigoplus_{\mathbb{N}} Hw(D)$.

Proof. Let $n_0 = 1$ and put

$$J_k := \{4m + 1 - 1/s : m \ge k,\ s \in \mathbb{N}\}, \quad k = 1, 2, \ldots$$

Clearly, $(J_k)_k$ is decreasing, and if $z \in J_n$, then $z \in D_m$ for $m \ge n$, hence $v_n(z) = v(z-4m) = v_1(z)$, i.e., $\inf_{z\in J_n} v_n(z)/v_1(z) = 1$. But $4n+1-1/s \in J_n \cap D_n$ for arbitrary $s \in \mathbb{N}$, and we have $v_1(4n+1-1/s) = v(1-1/s)$ while $v_{n+1}(4n+1-1/s) = w(1-1/s)$; thus,

$$\lim_{s\to\infty}\frac{v_{n+1}(4n+1-1/s)}{v_1(4n+1-1/s)} = \lim_{s\to\infty}\frac{w(1-1/s)}{v(1-1/s)} = 0,$$

i.e., we have verified that $\mathcal{V}$ has (ND).

Next, note that, clearly, $Hv(D) = \{0\}$. Therefore,

$$\psi : \mathcal{V}H(G) \to \bigoplus_{\mathbb{N}} Hw(D), \quad \psi(f) := (f(4m + \cdot))_{m\in\mathbb{N}},$$

is well-defined, linear and, in fact, bijective and continuous, hence an isomorphism by the open mapping theorem for (LB)-spaces.

It now suffices to show that $\psi : H\overline{V}(G) \to \bigoplus_{\mathbb{N}} Hw(D)$ is also continuous. To see this, take an arbitrary 0-neighborhood in $\bigoplus_{\mathbb{N}} Hw(D)$; it contains a set of the form

$$V = \bigoplus_{m\in\mathbb{N}} \{f \in Hw(D) : w|f| \le \varepsilon_m \text{ on } D\},$$

where $(\varepsilon_m)_m \subset (0,1)$ is decreasing. For each $m \in \mathbb{N}$, let $r_m \in (0,1)$ satisfy $v(r_m) = \varepsilon_m^{-1}$, and put $b_m := v(r_m)/w(r_m) > 1$; then $(r_m)_m$ and $(b_m)_m$ are increasing. By Lemma 3.10, if $u_m := \min(v, b_m w)$ and $f \in H(D)$, then $u_m|f| \le 1$ implies $w|f| \le \varepsilon_m$ on D.

We let $\alpha_1 := 1$, $\alpha_n := b_{n-1}$ for $n \ge 2$ and take $\overline{v} \in \overline{V}$ with $\inf_n \alpha_n v_n \le \overline{v}$ (cf. [11], 0.2, Prop.). Since $(\alpha_n)_n$ is increasing, we have

$$(\inf_n \alpha_n v_n)(z) = \min(v(z-4m), b_m w(z-4m))$$

for $z \in D_m$, $m = 1, 2, \ldots$

Now suppose that $f \in H\overline{V}(G)$ satisfies $\overline{v}|f| \le 1$ on G and let $\psi(f) = (g_m)_m$. Then, for each $m \in \mathbb{N}$,

$$\sup_{z\in D_m} \min(v(z-4m), b_m w(z-4m))|f(z)| \le \overline{v}(z)|f(z)| \le 1,$$

hence $u_m|g_m| \le 1$ and thus $w|g_m| \le \varepsilon_m$ on D. It follows that

$$\psi(\{f \in H\overline{V}(G) : \overline{v}|f| \le 1 \text{ on } G\}) \subset V,$$

that is, ψ is continuous on $H\overline{V}(G)$. ■

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Added in proof (October 1997). In the recent article by J. Bonet, P. Domański and M. Lindström, *Essential norm and weak compactness of composition operators on weighted Banach spaces of analytic functions*, preprint, 1997, the authors show, among other things, that for a radial continuous weight v on D which is decreasing as a function of $r \in [0,1)$ and satisfies $\lim_{r\to 1} v(r) = 0$, v is equivalent to the associated weight $\widetilde{v}$ if and only if $r \to 1/v(r)$ is equivalent to a log-convex function.

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The Weyl asymptotic formula by the method of Tulovskiĭ and Shubin

by

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Abstract. Let A be a pseudodifferential operator on $\mathbb{R}^N$ whose Weyl symbol a is a strictly positive smooth function on $W = \mathbb{R}^N \times \mathbb{R}^N$ such that $|\partial^\alpha a| \le C_\alpha a^{1-\varrho}$ for some $\varrho > 0$ and all $|\alpha| > 0$, $\partial^\alpha a$ is bounded for large $|\alpha|$, and $\lim_{w\to\infty} a(w) = \infty$. Such an operator A is essentially selfadjoint, bounded from below, and its spectrum is discrete. The remainder term in the Weyl asymptotic formula for the distribution of the eigenvalues of A is estimated. This is done by applying the method of approximate spectral projectors of Tulovskiĭ and Shubin.

Introduction. Let $A = a^w(x, D)$ be a pseudodifferential operator on $\mathbb{R}^N$ given by the Weyl formula

$$Af(x) = \iint e^{2\pi i(x-y)\xi} a\left(\frac{x+y}{2}, \xi\right) f(y)\, dy\, d\xi,$$

where a is a strictly positive smooth function with derivatives of polynomial growth. It is assumed that A enjoys certain "hypoelliptic" properties to be specified later which imply that A is selfadjoint and has a purely discrete spectrum $\lambda_n \nearrow \infty$. Let

$$Af = \int_0^\infty \lambda\, \mathcal{E}(d\lambda) f$$

be the spectral resolution for A.

Tulovskiĭ and Shubin [13] give estimates for the error term in the Weyl asymptotic formula

$$\mathcal{N}(\lambda) \approx \iint_{a\le\lambda} dx\, d\xi$$

for the number of eigenvalues of A smaller than or equal to λ. Their proof is based on a construction of a family E_λ of pseudodifferential operators that approximate the spectral projectors $\mathcal{E}_\lambda$ of A sufficiently well. This method

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