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Norm attaining multilinear forms and polynomials on preduals of Lorentz sequence spaces

by

M. JIMÉNEZ SEVILLA (Madrid) and RAFAEL PAYÁ (Granada)

Abstract. For each natural number N, we give an example of a Banach space X such that the set of norm attaining N-linear forms is dense in the space of all continuous N-linear forms on X, but there are continuous $(N+1)$-linear forms on X which cannot be approximated by norm attaining $(N+1)$-linear forms. Actually, X is the canonical predual of a suitable Lorentz sequence space. We also get the analogous result for homogeneous polynomials.

1. Introduction. A classical result by E. Bishop and R. Phelps [6] asserts that the set of norm attaining linear functionals on a Banach space is dense in the dual space. Very recently, some attention has been paid to the question if the Bishop–Phelps Theorem still holds for multilinear forms or polynomials. To pose the question more precisely, given a real or complex Banach space X and a natural number N, let us denote by $\mathcal{L}^N(X)$ the space of all continuous N-linear forms on X and let us say that $\varphi \in \mathcal{L}^N(X)$ *attains its norm* if there are $y_1, \ldots, y_N \in B_X$ (the closed unit ball of X) such that

$$|\varphi(y_1, \ldots, y_N)| = \|\varphi\| := \sup\{|\varphi(x_1, \ldots, x_N)| : x_1, \ldots, x_N \in B_X\}.$$

We denote by $\mathcal{AL}^N(X)$ the set of norm attaining continuous N-linear forms on X. In the same way, if $\mathcal{P}^N(X)$ denotes the Banach space of continuous N-homogeneous polynomials on X, we say that $P \in \mathcal{P}^N(X)$ *attains its norm* if there is $x_0 \in B_X$ such that

$$|P(x_0)| = \|P\| := \sup\{|P(x)| : x \in B_X\},$$

and we denote by $\mathcal{AP}^N(X)$ the set of norm attaining continuous N-homogeneous polynomials. The question is whether or not $\mathcal{AL}^N(X)$ (resp. $\mathcal{AP}^N(X)$) is dense in $\mathcal{L}^N(X)$ (resp. $\mathcal{P}^N(X)$).

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As a matter of fact, the answer in general is negative. An example of a Banach space X such that the sets $\mathcal{AL}^2(X)$ and $\mathcal{AP}^2(X)$ are not dense was exhibited in [1]. Actually, the example was a predual $d_*(w,1)$ of a Lorentz sequence space $d(w,1)$ (see below for details) and the fact that these spaces are useful in problems related to norm attaining operators was first observed by W. Gowers [14].

Sufficient conditions for the denseness of the norm attaining multilinear forms were given by R. Aron, C. Finet and E. Werner in [5]. They proved that this denseness holds in spaces with either the Radon–Nikodým property or the so-called property α and deduced a quite general renorming result. Some other sufficient or necessary conditions for the denseness of the norm attaining multilinear forms or polynomials can be found in [3], [8] and [9].

The above results and counterexamples work for every $N \geq 2$, so they lead us to the following natural question: is the denseness of the norm attaining bilinear forms on a Banach space X sufficient to ensure the denseness of the norm attaining N-linear forms on X for all $N \geq 2$? Or, on the contrary, does the denseness of $\mathcal{AL}^N(X)$ depend heavily on the integer N? Analogous questions can be posed for polynomials. We will show in this paper that there is a dependence on N.

We start with the easy observation that $\mathcal{AL}^N(X)$ is dense whenever $\mathcal{AL}^{N+1}(X)$ is. Then our main result shows that $\mathcal{AL}^N(d_*(w,1))$ is dense if and only if $w \notin \ell_N$. The "only if" part is an extension of the results in [1] and ultimately depends on the lack of extreme points in the unit ball of the spaces $d_*(w,1)$, which makes it difficult for a multilinear form to attain its norm. The proof of the much more interesting "if" part actually shows that when $w \notin \ell_N$ there are "few" continuous N-linear forms on $d_*(w,1)$. For example, for $w \notin \ell_2$ we show that every bounded linear operator from $d_*(w,1)$ into its dual $d(w,1)$ is compact, or equivalently, every continuous bilinear form on $d_*(w,1)$ is weakly sequentially continuous. In general, we show that all continuous N-linear forms on $d_*(w,p)$ are weakly sequentially continuous if and only if a certain simple relation between N, w and $p \geq 1$ is satisfied. For $p = 1$ the relation is just $w \notin \ell_N$. Moreover, we give a new abstract sufficient condition for the denseness of the norm attaining multilinear forms or polynomials, based on upper p-estimates, which covers part of our results on Lorentz sequence spaces and applies to some other classical Banach spaces.

Concerning polynomials, we also prove that $\mathcal{AP}^N(d_*(w,1))$ is dense in $\mathcal{P}^N(d_*(w,1))$ if and only if $w \notin \ell_N$. The "if" part follows easily from the corresponding result for multilinear forms. Curiously enough, the proof of the "only if" part is very easy in the complex case but more delicate in the real case.

2. Norm attaining multilinear forms on $d_*(w,1)$. Let us start with the simple observation that perturbed optimization of multilinear forms becomes easier as the number of variables decreases.

PROPOSITION 2.1. *Let X be a Banach space and $N \in \mathbb{N}$ such that $\mathcal{AL}^{N+1}(X)$ is dense in $\mathcal{L}^{N+1}(X)$. Then $\mathcal{AL}^N(X)$ is dense in $\mathcal{L}^N(X)$.*

Proof. Given $\varphi \in \mathcal{L}^N(X)$ with $\|\varphi\| = 1$ and $0 < \varepsilon < 1$, pick any $x^* \in X^*$ with $\|x^*\| = 1$ and define $\psi \in \mathcal{AL}^{N+1}(X)$ by

$$\psi(x_1,\ldots,x_N,x_{N+1}) = \varphi(x_1,\ldots,x_N)x^*(x_{N+1}).$$

Our assumption provides $\overline{\psi} \in \mathcal{AL}^{N+1}(X)$ such that $\|\overline{\psi}\| = 1$ and $\|\psi - \overline{\psi}\| < \varepsilon/2$. If $a_1,\ldots,a_{N+1} \in B_X$ are such that $|\overline{\psi}(a_1,\ldots,a_{N+1})| = 1$, we clearly have

$$1 - \varepsilon/2 \leq |\psi(a_1,\ldots,a_{N+1})| \leq |x^*(a_{N+1})|.$$

We now go back to $\mathcal{L}^N(X)$ by fixing the $(N+1)$th variable at a suitable multiple of a_{N+1}, namely we define

$$\overline{\varphi}(x_1,\ldots,x_N) = \overline{\psi}(x_1,\ldots,x_N,a_{N+1})/x^*(a_{N+1}).$$

It is easy to check that $\overline{\varphi} \in \mathcal{AL}^N(X)$ and $\|\overline{\varphi} - \varphi\| < \varepsilon$. ∎

Let us recall the definition of Lorentz sequence spaces and their preduals, a family of classical Banach spaces that will play a crucial role in this paper. By an *admissible sequence* we shall mean a decreasing sequence $w = (w(n))$ of positive numbers such that $w(1) = 1$ and $w \in c_0 \setminus \ell_1$. For $1 \leq p < \infty$, the Banach space of all sequences of (real or complex) scalars $a = (a(n))$ for which

$$\|a\| := \sup_\pi \Big(\sum_{n=1}^\infty |a(\pi(n))|^p w(n)\Big)^{1/p} < \infty,$$

where π ranges over all permutations of the integers, is denoted by $d(w,p)$ and called a *Lorentz sequence space* [17]. For $p > 1$, $d(w,p)$ is reflexive, so we are mainly interested in the case $p = 1$. It is known ([12], [20]) that $d(w,1)$ has a predual $d_*(w,1)$ which is defined by

$$d_*(w,1) = \Big\{a \in c_0 : \lim_{n\to\infty} \frac{\sum_{k=1}^n \widetilde{a}(k)}{\sum_{k=1}^n w(k)} = 0\Big\},$$

where $(\widetilde{a}(n))$ is the decreasing rearrangement of $(|a(n)|)$. The norm on $d_*(w,1)$ is given by

$$\|a\| := \sup_n \frac{\sum_{k=1}^n \widetilde{a}(k)}{\sum_{k=1}^n w(k)}.$$

Thus, $\|a\| \le 1$ if and only if

$$\sum_{j\in J} |a(j)| \le \sum_{k=1}^{n} w(k),$$

for any $n \in \mathbb{N}$ and any set $J \subset \mathbb{N}$ with n elements.

Our plan is to characterize the denseness of norm attaining N-linear forms on $d_*(w,1)$ by a simple relation between w and N. Half of this characterization will follow from the next lemma, which shows that faces in the unit ball of $d_*(w,1)$ look very much like those of c_0 and this implies severe restrictions on norm attaining multilinear forms. For the sake of completeness we include a proof of this lemma, an easy generalization of the arguments given in [1], [2] and [14] for the special case $w = (1/n)$, $N = 2$.

LEMMA 2.2. *Let (e_n) be the unit vector basis of $d_*(w,1)$. Then we have:*

(i) *For every $x \in d_*(w,1)$ with $\|x\| \le 1$, there exist $n_0 \in \mathbb{N}$ and $\delta > 0$ such that $\|x + \lambda e_n\| \le 1$ whenever $|\lambda| \le \delta$ and $n \ge n_0$.*

(ii) *If $\varphi \in \mathcal{AL}^N(d_*(w,1))$, there exists $n_0 \in \mathbb{N}$ such that $\varphi(e_{n_1}, \ldots, e_{n_N}) = 0$ whenever $n_1, \ldots, n_N \ge n_0$.*

Proof. (i) Given $x \in d_*(w,1)$ with $\|x\| \le 1$, there exists $q \in \mathbb{N}$ such that

$$\sum_{k=1}^{m} \widetilde{x}(k) < \frac{1}{2}\sum_{k=1}^{m} w(k) \quad \text{for } m > q.$$

Since $\lim_n x(n) = 0$ we now find $n_0 \in \mathbb{N}$ such that $|x(n)| \le w(q)/2$ for $n \ge n_0$. Given $y = x + \lambda e_n$, with $n \ge n_0$ and $|\lambda| \le \delta := w(q)/2$, we want to show that $\sum_{j\in J} |y(j)| \le \sum_{k=1}^{m} w(k)$ for any $m \in \mathbb{N}$ and any set $J \subset \mathbb{N}$ with m elements. We may clearly assume that $n \in J$. Then, for $m > q$, we have

$$\sum_{j\in J} |y(j)| \le \sum_{j\in J} |x(j)| + \delta < \frac{1}{2}\sum_{k=1}^{m} w(k) + \frac{1}{2} w(q) < \sum_{k=1}^{m} w(k),$$

while if $m \le q$,

$$\sum_{j\in J} |y(j)| \le \sum_{j\in J\setminus\{n\}} |x(j)| + |x(n)| + \delta \le \sum_{k=1}^{m-1} w(k) + w(q) \le \sum_{k=1}^{m} w(k).$$

(ii) Let $\mathbf{x} = (x_1, \ldots, x_N)$ be an N-tuple of elements in $B_{d_*(w,1)}$ such that $\|\varphi\| = |\varphi(\mathbf{x})|$. We use (i) to find $n_0 \in \mathbb{N}$ and $\delta > 0$ such that $\|x_k + \lambda e_n\| \le 1$ for $n \ge n_0$, $|\lambda| \le \delta$ and $k = 1, \ldots, N$. Then, for $n_k \ge n_0$, we have

$$\begin{aligned}|\varphi(\mathbf{x}) &\pm \delta\varphi(x_1, \ldots, x_{k-1}, e_{n_k}, x_{k+1} \ldots, x_N)| \\ &= |\varphi(x_1, \ldots, x_{k-1}, x_k \pm \delta e_{n_k}, x_{k+1}, \ldots, x_N)| \le \|\varphi\| = |\varphi(\mathbf{x})|,\end{aligned}$$

so $\varphi(x_1, \ldots, x_{k-1}, e_{n_k}, x_{k+1}, \ldots, x_N) = 0$. Now, for any $j, k = 1, \ldots N$ (say $j < k$) and $n_j, n_k \ge n_0$ we get

$$\begin{aligned}|\varphi(\mathbf{x}) &\pm \delta^2\varphi(x_1, \ldots, x_{j-1}, e_{n_j}, x_{j+1}, \ldots, x_{k-1}, e_{n_k}, x_{k+1} \ldots, x_N)| \\ &= |\varphi(x_1, \ldots, x_{j-1}, x_j \pm \delta e_{n_j}, x_{j+1}, \ldots, x_{k-1}, x_k + \delta e_{n_k}, x_{k+1}, \ldots, x_N)| \\ &\le \|\varphi\| = |\varphi(\mathbf{x})|,\end{aligned}$$

so $\varphi(x_1, \ldots, x_{j-1}, e_{n_j}, x_{j+1}, \ldots, x_{k-1}, e_{n_k}, x_{k+1}, \ldots, x_N) = 0$. The proof is concluded after N steps. ∎

The next lemma, on the existence of "diagonal" multilinear forms, is probably known. The result for bilinear forms can be traced back to [21] and a short proof can be found in [17, Proposition 1.c.8]. We omit the proof of the general case, which is rather similar.

LEMMA 2.3. *Let X be a Banach space with unconditional basis (e_n) and $\varphi \in \mathcal{L}^N(X)$. Then there exists $\psi \in \mathcal{L}^N(X)$ such that $\psi(e_{k_1}, \ldots, e_{k_N}) = 0$ unless $k_1 = \ldots = k_N$ and $\psi(e_n, \ldots, e_n) = \varphi(e_n, \ldots, e_n)$ for all n. If the unconditional constant of (e_n) is 1, then $\|\psi\| \le \|\varphi\|$.*

The next proposition is the key result in our discussion. Later we will see that the norm attaining N-linear forms on $d_*(w,1)$ are dense if and only if all continuous N-linear forms are weakly sequentially continuous (Lemma 2.2 already pointed in this direction). The latter property will now be characterized in terms of w and N. Actually, our proof also works for $p > 1$ (then $d_*(w,p)$ is nothing but the dual of $d(w,p)$). We denote by $\mathcal{L}^N_{\rm wsc}(X)$ the space of weakly sequentially continuous N-linear forms on a Banach space X.

PROPOSITION 2.4. *Given an admissible sequence w, $1 \le p < \infty$ and $N \ge 2$, the following statements are equivalent:*

(i) $\mathcal{L}^N(d_*(w,p)) = \mathcal{L}^N_{\rm wsc}(d_*(w,p))$,

(ii) $p < N^*$ *and* $w \notin \ell_\alpha$ *where* $1/\alpha + p/N^* = 1$ *and* $1/N + 1/N^* = 1$.

Proof. (i)⇒(ii). Suppose (ii) fails. If $p \ge N^*$ it is clear that the formal identity $i : \ell_{N^*} \to d(w,p)$ is bounded, and the same happens in case $p < N^*$ and $w \in \ell_\alpha$ (Hölder inequality). Then the adjoint operator $i^* : d_*(w,p) \to \ell_N$ (take the restriction to the predual if $p = 1$) is also bounded. Therefore, we can define $\psi \in \mathcal{L}^N(d_*(w,p))$ by

$$\psi(x_1, \ldots, x_N) = \sum_{j=1}^{\infty} x_1(j) \ldots x_N(j).$$

Since $\psi(e_n, \ldots, e_n) = 1$ for all $n \in \mathbb{N}$, ψ is not weakly sequentially continuous.

(ii)⇒(i). Suppose that (i) fails and $p < N^*$ to prove that $w \in \ell_\alpha$. Since α decreases with N (for fixed p) we may assume that N is the minimal natural number such that (i) fails. We use the following fact, which is known and easy to check: If all continuous k-linear forms on a Banach space X are weakly sequentially continuous, then a continuous $(k+1)$-linear form on X is weakly sequentially continuous if and only if it is weakly sequentially continuous at zero. Thus, by the minimality of N, there is $\varphi \in \mathcal{L}^N(d_*(w,p))$ which is not weakly sequentially continuous at zero. Take weakly null sequences (u_n^h), $h = 1,\ldots,N$, such that

$$|\varphi(u_n^1,\ldots,u_n^N)| = 1 \quad \text{for all } n \in \mathbb{N}.$$

Our purpose is to modify φ so that it satisfies the above condition with the last sequence (u_n^N) replaced by (e_n), the unit vector basis of $d_*(w,p)$. Let us define a seminormalized sequence (u_n^*) in $d(w,p)$ by

$$u_n^* = \varphi(u_n^1,\ldots,u_n^{N-1},\cdot).$$

For any fixed $x^{**} \in d(w,p)^*$, the mapping $(x_1,\ldots,x_{N-1}) \to x^{**}(\varphi(x_1,\ldots \ldots,x_{N-1},\cdot))$ is a continuous $(N-1)$-linear form on $d_*(w,p)$. By the minimality of N this form is weakly sequentially continuous, so $x^{**}(u_n^*) \to 0$ and the sequence (u_n^*) is weakly null. By standard arguments we can find a subsequence of (u_n^*), denoted in the same way, which is equivalent to a seminormalized block sequence (v_n^*) of (e_n^*), the unit vector basis of $d(w,p)$. We can also suppose $\|v_n^* - u_n^*\| < 1/n$ for all n. The sequence (v_n^*) has the form

$$v_n^* = \sum_{k=q_n+1}^{q_{n+1}} \alpha_k e_k^*,$$

with suitable sequences of integers (q_n) and scalars (α_k). Two cases will be considered:

(a) *The sequence (α_k) does not converge to zero.* Then we can find increasing sequences of integers (n_j), (k_j) and $\eta > 0$ such that $q_{n_j} < k_j \le q_{n_j+1}$, $1/n_1 < \eta$ and $|\alpha_{k_j}| > 2\eta$ for all j. It follows that

$$|\varphi(u_{n_j}^1,\ldots,u_{n_j}^{N-1},e_{k_j})| = |u_{n_j}^*(e_{k_j})| \ge |v_{n_j}^*(e_{k_j})| - 1/n_j = |\alpha_{k_j}| - 1/n_j \ge \eta$$

for all $j \in \mathbb{N}$. Since the basis (e_n) is symmetric, we can use a suitable isomorphism in the last variable to get a new form $\widetilde{\varphi} \in \mathcal{L}^N(d_*(w,p))$ satisfying

$$\widetilde{\varphi}(u_{n_j}^1,\ldots,u_{n_j}^{N-1},e_j) = 1 \quad \text{for all } j \in \mathbb{N}.$$

(b) *The sequence (α_k) converges to zero.* In this case, we use a result due to Altshuler, Casazza and Lin [4], [7] (see [17, Proposition 4.e.3]) to get a subsequence $(v_{n_j}^*)$ which is equivalent to the unit vector basis of ℓ_p and such that its closed linear span $X = [v_{n_j}^* : j \in \mathbb{N}]$ is complemented in $d(w,p)$. We compose the projection from $d(w,p)$ onto X with the isomorphism which takes $(v_{n_j}^*)$ into the ℓ_p-basis and then the formal identity from ℓ_p into $d(w,p)$ to get a bounded linear operator T from $d(w,p)$ into itself which satisfies $T(v_{n_j}^*) = e_j^*$, $j \in \mathbb{N}$. Now consider $\widehat{\varphi} \in \mathcal{L}^N(d_*(w,p))$ given by

$$\widehat{\varphi}(x_1,\ldots,x_{N-1},\cdot) = T(\varphi(x_1,\ldots,x_{N-1},\cdot)).$$

This time we have

$$|\widehat{\varphi}(u_{n_j}^1,\ldots,u_{n_j}^{N-1},e_j)| = |Tu_{n_j}^*(e_j)| \ge |Tv_{n_j}^*(e_j)| - \frac{\|T\|}{n_j} = 1 - \frac{\|T\|}{n_j}.$$

Again by a suitable isomorphism in the last variable we get a new form $\widetilde{\varphi} \in \mathcal{L}^N(d_*(w,p))$ satisfying exactly the same condition as in case (a).

The procedure used to pass from φ to $\widetilde{\varphi}$ can now be iterated $N-1$ more times, working in the $N-1$ remaining variables, to end up with the fact that there exists a bounded N-linear form ψ on $d_*(w,p)$ satisfying $\psi(e_j,\ldots,e_j) = 1$ for all $j \in \mathbb{N}$. By Lemma 2.3 we can arrange that ψ is diagonal, i.e.

$$\psi(x_1,\ldots,x_N) = \sum_{k=1}^{\infty} x_1(k)x_2(k)\ldots x_N(k).$$

For fixed $n \in \mathbb{N}$ let us take $x = \sum_{k=1}^n w(k)^{\alpha/N} e_k$. Using the Hölder inequality, it is easy to check that $\|x\| \le (\sum_{k=1}^n w(k)^\alpha)^{1-1/p}$, hence

$$\sum_{k=1}^n w(k)^\alpha = \psi(x,\ldots,x) \le \|\psi\| \Big(\sum_{k=1}^n w(k)^\alpha\Big)^{N(1-1/p)}.$$

Since $N(1-1/p) < 1$, this implies $w \in \ell_\alpha$, as required.

Note that the last calculation is valid (and easier) when $p = 1$. Actually, the whole proof is shorter for $p = 1$ because the sequence (u_n^*) is weakly convergent to zero, so no subsequence can be equivalent to the ℓ_1-basis. ∎

An easy weak-compactness argument shows that every weakly sequentially continuous multilinear form on a reflexive Banach space attains its norm. Therefore, the above proposition gives

COROLLARY 2.5. *If N, w, and $p > 1$ satisfy statement* (ii) *in Proposition* 2.4, *then*

$$\mathcal{L}^N(d_*(w,p)) = \mathcal{AL}^N(d_*(w,p)).$$

However, we are mainly interested in the case $p = 1$. For this case we get:

THEOREM 2.6. *Given an admissible sequence w and $N \ge 2$, the following statements are equivalent*:

(i) *$\mathcal{AL}^N(d_*(w,1))$ is dense in $\mathcal{L}^N(d_*(w,1))$.*

(ii) *$w \notin \ell_N$.*

Proof. (i)⇒(ii). As in the proof of Proposition 2.4 (note that $\alpha = N$ for $p=1$), if $w \in \ell_N$ there is $\psi \in \mathcal{L}^N(d_*(w,1))$ such that $\psi(e_n,\ldots,e_n)=1$ for all n. By Lemma 2.2, $\|\psi-\varphi\| \geq 1$ for every $\varphi \in \mathcal{AL}^N(d_*(w,1))$.

(ii)⇒(i). For each $i_1,\ldots,i_N \in \mathbb{N}$, consider the continuous N-linear form (monomial) given by

$$B_{i_1,\ldots,i_N}(x_1,\ldots,x_N) = e_{i_1}^*(x_1)\ldots e_{i_N}^*(x_N),$$

where (e_i^*) is the unit vector basis of $d(w,1)$, and let $\mathcal{M}$ denote the linear span of all these monomials. Using the fact that (e_n) is a monotone basis, one can easily check that $\mathcal{M} \subset \mathcal{AL}^N(d_*(w,1))$. If (ii) holds, by Proposition2.4 we have

$$\mathcal{L}^N(d_*(w,1)) = \mathcal{L}^N_{\mathrm{wsc}}(d_*(w,1)). \tag{2.1}$$

Recall that, for any Banach space X, there is a canonical identification of $\mathcal{L}^N(X)$ with the space $\mathcal{L}^{N-1}(X,X^*)$ of X^*-valued continuous $(N-1)$-linear mappings on X. When X has a shrinking basis, it is not hard to check that under such identification $\mathcal{L}^N_{\mathrm{wsc}}(X)$ becomes the space $\mathcal{L}^{N-1}_{\mathrm{wsc}}(X,X^*)$ of weakly sequentially continuous $(N-1)$-linear mappings, i.e. those $(N-1)$-linear mappings taking weakly convergent into norm-convergent sequences. Since (e_n) is shrinking, we infer from (2.1) that $\mathcal{L}^{N-1}(d_*(w,1),d(w,1)) = \mathcal{L}^{N-1}_{\mathrm{wsc}}(d_*(w,1),d(w,1))$, so we can apply [11, Theorem 1] to deduce that $\mathcal{M}$ is dense in $\mathcal{L}^N(d_*(w,1))$ and (i) follows. ■

As an immediate consequence we get the following result, which was the main motivation for the research of this paper.

COROLLARY 2.7. *For each natural number N there is a Banach space X such that $\mathcal{AL}^N(X)$ is dense in $\mathcal{L}^N(X)$ but $\mathcal{AL}^{N+1}(X)$ is not dense in $\mathcal{L}^{N+1}(X)$.*

Proof. Just take $X = d_*(w,1)$ with $w \in \ell_{N+1} \setminus \ell_N$. ■

Concerning the proof of Theorem 2.6 we remark that, for a Banach space with shrinking basis like $X = d_*(w,1)$, the fact that all continuous N-linear forms on X are weakly sequentially continuous is equivalent to some other remarkable properties, for example that all X^*-valued continuous $(N-1)$-linear mappings on X are compact, that is, they take bounded sets into relatively norm-compact sets [11]. The case $N=2$ is especially appealing. From Proposition 2.4, Theorem 2.6 and the above observations we get:

COROLLARY 2.8. *Given an admissible sequence w, the following statements are equivalent:*

(i) *$\mathcal{AL}^2(d_*(w,1))$ is dense in $\mathcal{L}^2(d_*(w,1))$.*
(ii) *$w \notin \ell_2$.*
(iii) *Every bounded linear operator from $d_*(w,1)$ into $d(w,1)$ is compact.*

It is worth mentioning that every bounded linear operator from $d_*(w,1)$ into $d(w,1)$ is weakly compact, regardless of the admissible sequence w. This follows from the fact that $d_*(w,1)$ is an M-ideal in its bidual (see [15, Example III.1.4.(c) and Corollary III.3.7]).

Remark 2.9. In Theorem 2.6 we have shown the denseness of the norm attaining N-linear forms in many situations where no previously known sufficient condition for this denseness is satisfied. Let us mention the result by Choi and Kim [9, Theorem 2.2] that $\mathcal{AL}^N(X)$ is dense in $\mathcal{L}^N(X)$ for every N, provided that X has the Dunford–Pettis property and a shrinking monotone basis. Since the unit vector bases of $d_*(w,1)$ and $d(w,1)$ are weakly null, $d_*(w,1)$ fails the Dunford–Pettis property. Nevertheless, by taking $w(n) = (1+\log n)^{-1}$ we get an admissible sequence w such that $\mathcal{AL}^N(d_*(w,1))$ is dense in $\mathcal{L}^N(d_*(w,1))$ for every N.

We conclude this section by generalizing some of the arguments we have been using for Lorentz sequence spaces, in order to get a new sufficient condition for the denseness of the norm attaining multilinear forms, under more abstract assumptions on the Banach space. The proof will also work for polynomials. Recall that a Banach X has *property S_p* $(1<p<\infty)$ if every seminormalized weakly null sequence (x_n) has a subsequence (y_n) with an upper p-estimate, i.e.

$$\Big\|\sum_{i=1}^n a_i y_i\Big\| \leq M\Big(\sum_{i=1}^n |a_i|^p\Big)^{1/p}$$

for some constant $M>0$ and every choice of scalars $\{a_i\}_{i=1}^n$, $n \in \mathbb{N}$.

PROPOSITION 2.10. *Let X be a Banach space with a shrinking monotone basis (e_n) such that X has property S_p for some $p > N \geq 2$. Then $\mathcal{AL}^N(X)$ is dense in $\mathcal{L}^N(X)$ and $\mathcal{AP}^N(X)$ is dense in $\mathcal{P}^N(X)$.*

Proof. Since X has property S_p and $N<p$, every continuous N-linear form on X is weakly sequentially continuous (see [13], for example). Also, X has a shrinking monotone basis, so we can use exactly the same argument as in the proof of Theorem 2.6 to get the denseness of $\mathcal{AL}^N(X)$ in $\mathcal{L}^N(X)$. The proof for polynomials is analogous, since $\mathcal{AP}^N(X)$ contains the linear span of all polynomials of the form $e_{i_1}^*\ldots e_{i_N}^*$, which is dense in $\mathcal{P}^N(X)$. ■

EXAMPLES 2.11. (i) Recall that, given $p>0$, an admissible sequence w is said to be *p-regular* [19] if

$$w(n)^p \sim \frac{1}{n}\sum_{i=1}^n w(i)^p, \qquad n \in \mathbb{N}.$$

In case w is p-regular, with $p > N \geq 2$ (which in particular implies that $w \notin \ell_N$) we obtain a shorter proof of the fact that $\mathcal{AL}^N(d_*(w,1))$ is dense in

$\mathcal{L}^N(d_*(w,1))$. Indeed, we first apply a result of Reisner [19] to deduce that $d(w,1)$ is p^*-concave and thus $d_*(w,1)$ is p-convex [18]. This implies that every seminormalized block sequence of the unit vector basis has an upper p-estimate, hence $d_*(w,1)$ has property S_p and Proposition 2.10 applies.

(ii) Fix $N \geq 2$ and let M be an Orlicz function with Boyd index $\alpha_M > N$ (cf. [17], [18]). Then the Orlicz sequence space associated with M, defined by

$$h_M := \Big\{(x(n)) : \sum_{n=1}^{\infty} M(|x(n)|/\varrho) < \infty \text{ for every } \varrho > 0\Big\},$$

satisfies the assumptions of Proposition 2.10 since h_M has property S_p for any $p < \alpha_M$ (see [16], for example). It can be seen that h_M fails the Dunford–Pettis property and if $\beta_M = \infty$, then h_M also fails the Radon–Nikodým property.

If we consider for instance

$$M(t) = e^{-1/t^2} \qquad (\text{with } t > 0),$$

we have $\alpha_M = \beta_M = \infty$, so Proposition 2.10 tells us that $\mathcal{AL}^N(h_M)$ (resp. $\mathcal{AP}^N(h_M)$) is dense in $\mathcal{L}^N(h_M)$ (resp. $\mathcal{P}^N(h_M)$), for every $N \in \mathbb{N}$.

3. Norm attaining polynomials on $d_*(w,1)$. The next lemma shows that norm attaining N-homogeneous polynomials on complex preduals of Lorentz sequence spaces "behave" like norm attaining N-linear forms.

LEMMA 3.1. *Consider the complex space $d_*(w,1)$. If $P \in \mathcal{AP}^N(d_*(w,1))$, then $P(e_n) = 0$ for n large enough.*

Proof. Let $x_0 \in B_{d_*(w,1)}$ be such that $\|P\| = |P(x_0)|$, and find $n_0 \in \mathbb{N}$, $\delta > 0$ such that $\|x_0 + \lambda e_n\| \leq 1$ for $|\lambda| \leq \delta$ and $n \geq n_0$. Then, for each $n \geq n_0$, the modulus of the complex polynomial $\lambda \to P(x_0 + \lambda e_n)$ attains a local maximum at the origin. By the maximum modulus principle this polynomial is constant, so $P(e_n) = 0$. ∎

The real case is different. Just take $w \in \ell_2$ with $\|w\|_2 = 2$, and $P \in \mathcal{P}^2(d_*(w,1))$ defined by

$$P(x) = 4x(1)^2 - \sum_{n\geq 2} x(n)^2. \tag{3.1}$$

It is easy to check that $\|P\| = 4 = P(e_1)$ and $P(e_n) = -1$ for $n \geq 2$. Nevertheless, we have the following analog of Theorem 2.6 for polynomials. We denote by $\mathcal{P}^N_{\mathrm{wsc}}(X)$ the space of weakly sequentially continuous N-homogeneous polynomials on a Banach space X.

THEOREM 3.2. *Given an admissible sequence w and $N \geq 2$, the following statements are equivalent:*

(i) *$\mathcal{AP}^N(d_*(w,1))$ is dense in $\mathcal{P}^N(d_*(w,1))$.*
(ii) *$w \notin \ell_N$.*
(iii) *$\mathcal{P}^N(d_*(w,1)) = \mathcal{P}^N_{\mathrm{wsc}}(d_*(w,1))$.*

Proof. (ii)⇒(iii). This follows directly from Proposition 2.4.

(iii)⇒(i). By [10, Proposition 10], (iii) implies that $\mathcal{P}^N(d_*(w,1))$ is the closed linear span of polynomials of the form $e^*_{i_1}\dots e^*_{i_N}$ with $i_1,\dots,i_N \in \mathbb{N}$, where (e^*_n) is the unit vector basis of $d(w,1)$. The fact that the unit vector basis of $d_*(w,1)$ is monotone easily implies that any finite linear combination of these polynomials attains its norm, and (i) follows.

(i)⇒(ii). We assume that $w \in \ell_N$ and find a polynomial $Q \in \mathcal{P}^N(d_*(w,1))$ which cannot be approximated by norm attaining polynomials. In the complex case this is easy: we just take

$$Q(x) = \sum_{k=1}^{\infty} x(k)^N.$$

Since $Q(e_n) = 1$ for all n, Lemma 3.1 tells us that $\|Q - P\| \geq 1$ for every $P \in \mathcal{AP}^N(d_*(w,1))$.

In the real case consider the minimal natural number $M \leq N$ such that $w \in \ell_M$. For $P \in \mathcal{AP}^N(d_*(w,1))$, let $x_0 \in B_{d_*(w,1)}$ be such that $\|P\| = |P(x_0)|$, assume for the moment that $P(x_0) > 0$ and use Lemma 2.2 to find $n_0 \in \mathbb{N}$ and $\delta > 0$ such that $\|x_0 + \lambda e_n\| \leq 1$ for $n \geq n_0$ and $|\lambda| \leq \delta$. Then, if we denote by φ the symmetric N-linear form associated with P, we have

$$\begin{aligned} P(x_0 + \lambda e_n) &= P(x_0) + N\lambda\varphi(x_0,\dots,x_0,e_n) + \dots \\ &\quad + \tbinom{N}{M}\lambda^M \varphi(x_0,\dots,x_0,e_n,\overset{(M)}{\dots},e_n) + \dots + \lambda^N P(e_n) \\ &\leq P(x_0), \end{aligned} \tag{3.2}$$

for $n \geq n_0$ and $|\lambda| \leq \delta$. By Proposition 2.4 every continuous k-linear form on $d_*(w,1)$ is weakly sequentially continuous for $k < M$, so $\lim_n \varphi(x_0,\dots,x_0,e_n,\overset{(k)}{\dots},e_n) = 0$ for $k < M$. Therefore, by letting $n \to \infty$ in (3.2) and dividing by λ^M we get

$$\begin{aligned} \limsup_n \Big(\tbinom{N}{M}\varphi(x_0,\dots,x_0,e_n,\overset{(M)}{\dots},e_n) \\ + \lambda\tbinom{N}{M+1}\varphi(x_0,\dots,x_0,e_n,\overset{(M+1)}{\dots},e_n) + \dots + \lambda^{N-M}P(e_n)\Big) \leq 0 \end{aligned}$$

for $0 < \lambda \leq \delta$. It follows that

$$\limsup_n \varphi(x_0,\dots,x_0,e_n,\overset{(M)}{\dots},e_n) \leq 0.$$

In case $P(x_0) < 0$, the above argument applies to $-P$ and we get

$$\liminf_n \varphi(x_0, \ldots, x_0, e_n, \overset{(M)}{\ldots}, e_n) \geq 0.$$

Now define $Q \in \mathcal{P}^N(d_*(w,1))$ by

$$Q(x) = x(1)^{N-M} \sum_{k=1}^{\infty} (-1)^k x(k)^M.$$

If ψ denotes the symmetric N-linear form associated with Q, it is easy to check that

$$\binom{N}{M} \psi(x, \ldots, x, e_n, \overset{(M)}{\ldots}, e_n) = (-1)^n x(1)^{N-M}$$

for every $x \in d_*(w,1)$ and $n \geq 2$, so

$$\binom{N}{M} \limsup_n \psi(x, \ldots, x, e_n, \overset{(M)}{\ldots}, e_n) = |x(1)|^{N-M} = -\binom{N}{M} \liminf_n \psi(x, \ldots, x, e_n, \overset{(M)}{\ldots}, e_n).$$

We assume that Q can be approximated by norm attaining polynomials to get a contradiction. Fix $\varepsilon > 0$ and let $P \in \mathcal{AP}^N(d_*(w,1))$ be such that $\|Q - P\| < \varepsilon N!/N^N$, so that the corresponding symmetric N-linear form φ satisfies $\|\psi - \varphi\| < \varepsilon$. Now let $x_0 \in B_{d_*(w,1)}$ be such that $\|P\| = |P(x_0)|$. If $P(x_0) > 0$ we have

$$\begin{aligned}\binom{N}{M}^{-1} |x_0(1)|^{N-M} &= \limsup_n \psi(x_0, \ldots, x_0, e_n, \overset{(M)}{\ldots}, e_n) \\ &\leq \limsup_n \varphi(x_0, \ldots, x_0, e_n, \overset{(M)}{\ldots}, e_n) + \varepsilon \leq \varepsilon,\end{aligned}$$

while if $P(x_0) < 0$ we get the same conclusion:

$$\begin{aligned}-\binom{N}{M}^{-1} |x_0(1)|^{N-M} &= \liminf_n \psi(x_0, \ldots, x_0, e_n, \overset{(M)}{\ldots}, e_n) \\ &\geq \liminf_n \varphi(x_0, \ldots, x_0, e_n, \overset{(M)}{\ldots}, e_n) - \varepsilon \geq -\varepsilon.\end{aligned}$$

It follows that

$$\begin{aligned}\|Q\| &\leq \|P\| + \varepsilon = |P(x_0)| + \varepsilon \leq |Q(x_0)| + 2\varepsilon \\ &= |x_0(1)|^{N-M} \sum_{k=1}^{\infty} |x_0(k)|^M + 2\varepsilon \leq \varepsilon \Big(\binom{N}{M} \sum_{k=1}^{\infty} w(k)^M + 2 \Big).\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we have the desired contradiction. ∎

Remark 3.3. Note that in the real case, if $w \in \ell_N \setminus \ell_{N-1}$ with N odd, the above proof shows that $\lim_n P(e_n) = 0$ for every $P \in \mathcal{AP}^N(d_*(w,1))$. For N even we cannot expect the same conclusion, as shown by the example given after Lemma 3.1.

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Departamento de Análisis Matemático
Facultad de Ciencias Matemáticas
Universidad Complutense de Madrid
28040 Madrid, Spain
E-mail: marjim@sunam1.mat.ucm.es

Departamento de Análisis Matemático
Facultad de Ciencias
Universidad de Granada
18071 Granada, Spain
E-mail: rpaya@goliat.ugr.es

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The Berezin transform on the Toeplitz algebra

by

SHELDON AXLER (San Francisco, Calif.) and
DECHAO ZHENG (Nashville, Tenn.)

Abstract. This paper studies the boundary behavior of the Berezin transform on the C^*-algebra generated by the analytic Toeplitz operators on the Bergman space.

1. Introduction. Let dA denote Lebesgue area measure on the unit disk D, normalized so that the measure of D equals 1. The *Bergman space* L^2_{a} is the Hilbert space consisting of the analytic functions on D that are also in $L^2(D, dA)$. For $z \in D$, the *Bergman reproducing kernel* is the function $K_z \in L^2_{\mathrm{a}}$ such that

$$f(z) = \langle f, K_z \rangle$$

for every $f \in L^2_{\mathrm{a}}$. The *normalized Bergman reproducing kernel* k_z is the function $K_z/\|K_z\|_2$. Here, as elsewhere in this paper, the norm $\| \ \|_2$ and the inner product $\langle\, , \rangle$ are taken in the space $L^2(D, dA)$. The set of bounded operators on L^2_{a} is denoted by $\mathcal{B}(L^2_{\mathrm{a}})$.

For $S \in \mathcal{B}(L^2_{\mathrm{a}})$, the *Berezin transform* of S is the function $\widetilde{S}$ on D defined by

$$\widetilde{S}(z) = \langle Sk_z, k_z \rangle.$$

Often the behavior of the Berezin transform of an operator provides important information about the operator.

For $u \in L^\infty(D, dA)$, the *Toeplitz operator* T_u with symbol u is the operator on L^2_{a} defined by $T_u f = P(uf)$; here P is the orthogonal projection from $L^2(D, dA)$ onto L^2_{a}. Note that if $g \in H^\infty$ (the set of bounded analytic functions on D), then T_g is just the operator of multiplication by g on L^2_{a}.

The Berezin transform $\widetilde{u}$ of a function $u \in L^\infty(D, dA)$ is defined to be the Berezin transform of the Toeplitz operator T_u. In other words, $\widetilde{u} = \widetilde{T_u}$.

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