Remark 4.3. It can be proved that if $\|\cdot\|$ is a Gateaux differentiable and locally uniformly convex norm on c_0, and B is an isomorphically precisely norming set for $(c_0, \|\cdot\|)$ then for any $A \subset c_0$ with $\{b(a) : a \in A\}$ bounded for every $b \in B$, the set A is bounded.

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Mathematical Sciences Building
University of Missouri-Columbia
Columbia, Missouri 65211
U.S.A.
E-mail: giorgis@math.missouri.edu

Received October 29, 1996
Revised version June 9, 1997 (3764)

STUDIA MATHEMATICA 127 (1) (1998)

The Abel equation and total solvability of linear functional equations

by

G. BELITSKII (Beer-Sheva) and Yu. LYUBICH (Haifa)

Abstract. We investigate the solvability in continuous functions of the Abel equation $\varphi(Fx) - \varphi(x) = 1$ where F is a given continuous mapping of a topological space X. This property depends on the dynamics generated by F. The solvability of all linear equations $P(x)\psi(Fx) + Q(x)\psi(x) = \gamma(x)$ follows from solvability of the Abel equation in case F is a homeomorphism. If F is noninvertible but X is locally compact then such a total solvability is determined by the same property of the cohomological equation $\varphi(Fx) - \varphi(x) = \gamma(x)$. The smooth situation can also be considered in this way.

1. Introduction. Results and applications. The *Abel equation* (A.e.) is a special kind of functional equation, namely,

$$\varphi(Fx) - \varphi(x) = 1 \quad (x \in X) \tag{1.1}$$

where $F : X \to X$ is a given continuous mapping of a given arbitrary topological space X, and $\varphi : X \to \mathbb{C}$ is an unknown function. N. H. Abel [1] (pp. 36–39) considered this equation on an interval $[0, a) \subset \mathbb{R}$.

We say that (1.1) is *solvable* if this equation has a continuous solution φ. Note that if the A.e. has a solution φ then the real part of φ is also a solution which is continuous since φ is. Therefore the solvability of the A.e. over $\mathbb{C}$ is equivalent to its solvability over $\mathbb{R}$.

Being written in the form

$$\varphi(Fx) = \varphi(x) + 1$$

equation (1.1) means that we have the commutative diagram

1991 *Mathematics Subject Classification*: Primary 39B22.

Key words and phrases: functional equation, Abel equation, cohomological equation, wandering set.

Research of the first author is supported by Guastello Foundation.

$$\begin{array}{ccc} X & \xrightarrow{F} & X \\ \varphi\downarrow & & \downarrow\varphi \\ \mathbb{C} & \xrightarrow{S} & \mathbb{C} \end{array}$$

with $Sz = z+1$, i.e. F and S are *semiconjugate* via φ. Thus, the solvability of the A.e. is connected with some deep intrinsic properties of the mapping F.

We consider the equation

$$P(x)\psi(Fx) + Q(x)\psi(x) = \gamma(x) \tag{1.2}$$

with an unknown vector-valued function $\psi : X \to \mathbb{C}^l$. The functions $P, Q : X \to \mathrm{Hom}(\mathbb{C}^l, \mathbb{C}^r)$ and $\gamma : X \to \mathbb{C}^r$ are given and they are supposed to be continuous. (1.2) can be viewed as a system of r scalar linear equations with l unknown scalar functions; then P and Q are $r \times l$-matrix-valued functions.

Equation (1.2) is called *totally solvable* if it has a continuous solution ψ for every continuous γ. From now on, *solution* means "continuous solution".

The following theorem shows a crucial role of the A.e. for the total solvability of the general equation (1.2). Call the latter equation *nondegenerate* if

$$\operatorname{rank} P(x) = \operatorname{rank} Q(x) = r \quad (x \in X).$$

THEOREM 1.1. *Let F be a homeomorphism. If the A.e. is solvable then any nondegenerate equation of the form* (1.2) *is totally solvable.*

In the case of an injective mapping F the conclusion of Theorem 1.1 is valid at least under some additional assumptions on the topology of X (see Corollary 1.6 below). If F is not injective then the solvability of the A.e. does not imply the total solvability even in the case of the so-called cohomological equation (see (1.8) and Example 1.7 below).

The point is that the solvability of the A.e. implies some special properties of the topological dynamical system (X, F). To explain this we note that if φ is a solution of the A.e. then

$$\varphi(F^n x) - \varphi(x) = n \tag{1.3}$$

where F^n is the nth iteration of F, $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$.

PROPOSITION 1.2. *If the space X is compact then the A.e. has no solution.*

Proof. Being continuous, any solution φ is bounded, which contradicts (1.3). ■

Thus, the A.e. is not solvable if there exists an F-invariant compact nonempty set $K \subset X$. A fortiori, if the A.e. is solvable then F has no periodic points. Certainly, this can be seen directly from (1.3).

Now let us call a closed set $A \subset X$ an *absorber* if its image $F(A)$ is contained in A (i.e. A is invariant) and for any point $x_0 \in X$ there exists a neighborhood $V_0 = V(x_0)$ of x_0 and a number $k_0 = k(x_0)$ such that $F^n(V_0) \subset A$ for $n \geq k_0$. Similarly, a closed set $N \subset X$ is said to be a *nozzle* if its preimage $F^{-1}(N)$ is contained in N (i.e. $X \setminus N$ is invariant) and for any $x_0 \in X$ there exists a neighborhood $W_0 = W(x_0)$ of x_0 and a number $k_0 = k(x_0)$ such that $F^{-n}(W_0) \subset N$ for $n \geq k_0$. Thus, if F is a homeomorphism then a nozzle for F is an absorber for the inverse mapping F^{-1}.

THEOREM 1.3. *If the A.e. is solvable then there exist an absorber A and a nozzle N with empty intersection. Conversely, let F be a homeomorphism and suppose the space X normal. If there exist an absorber and a nozzle with empty intersection then the A.e. is solvable.*

Both Theorems 1.1 and 1.3 are proved in Section 3. We do it using a technique of extension of solutions developed in Section 2.

A *regular* construction of an absorber is the following. Take an open covering of X,

$$X = \bigcup_{\alpha \in I} U_\alpha,$$

and some function $q : I \to \mathbb{N}$. Then

$$A = \overline{\bigcup_{\alpha} \bigcup_{k \geq q(\alpha)} F^k(U_\alpha)} \tag{1.4}$$

is an absorber for F (the bar means closure). *Any absorber A contains a regular absorber*, namely, the following one:

$$\overline{\bigcup_{x \in X} \bigcup_{k \geq k(x)} F^k(V(x))}$$

where $k(x)$ and $V(x)$ come from the definition of an absorber.

Similarly, any nozzle contains a set of the form

$$N = \overline{\bigcup_{\alpha} \bigcup_{k \geq q(\alpha)} F^{-k}(U_\alpha)}. \tag{1.5}$$

It is a nozzle if F is a homeomorphism. The construction (1.5) without the bar has all properties of the nozzle except possibly for being closed, but then it may happen that $X \setminus N$ is not invariant. Indeed, consider $X = \mathbb{R} \cup \{i\} \subset \mathbb{C}$ and let $F(x) = x + 1$, $x \in \mathbb{R}$, $F(i) = 0 \in \mathbb{R}$. Let $U_0 = \{i\}$, $U_1 = (-\infty, 1)$, $U_\alpha = (\alpha - 2, \alpha + 2)$ $(\alpha \geq 2)$, $q(0) = 1$, $q(1) = 1$, $q(\alpha) = \alpha + 2$ $(\alpha \geq 2)$. Then

$$\bigcup_{\alpha} \bigcup_{k \geq q(\alpha)} F^{-k}(U_\alpha) = (-\infty, 0).$$

Its closure is $N = (-\infty, 0]$, and $X \setminus N = (0, \infty) \cup \{i\}$. The latter is not invariant since $F(i) = 0 \notin X \setminus N$.

Thus, *if the A.e. has a solution, then there exists a pair* A, N *of the form* (1.4), (1.5) *respectively with* $A \cap N = \emptyset$. To formulate this in more relevant dynamical terms, let us introduce the concept of a wandering set (cf. [4], §0.2).

We call an open nonempty set $U \subset X$ *wandering* if there exists a number $\nu > 0$ such that

$$\overline{F^n(U)} \cap \overline{F^m(U)} = \emptyset \tag{1.6}$$

for all $n, m \in \mathbb{N}$ such that $|n - m| \geq \nu$. Note that if U is wandering then (1.6) holds for *all* integers n, m with $|n - m| \geq \nu$. Indeed,

$$F^p(\overline{F^n(V_1)} \cap \overline{F^m(V_2)}) \subset \overline{F^{n+p}(V_1)} \cap \overline{F^{m+p}(V_2)} \tag{1.7}$$

for all $p \geq 0$ and arbitrary sets V_1 and V_2 in X. An arbitrary nonempty set $S \subset X$ is called *wandering* if it has a wandering neighborhood.

Obviously, any subset of a wandering set is also wandering. Every periodic point (in particular, every fixed point) is nonwandering (i.e. the corresponding one-point set is).

In Section 4 we show that all compact sets are wandering whenever the A.e. is solvable. Furthermore, we prove

THEOREM 1.4. *If the A.e. is solvable then there exists a nondecreasing sequence of open wandering sets which covers* X. *Conversely, assume that the space* X *is normal and there exists a sequence* $\{U_k\}_{k=1}^{\infty}$ *with the above properties. In addition, suppose there exists a sequence* $\nu_1 \leq \nu_2 \leq \ldots$ *of positive integers such that all the mappings*

$$F_k : \overline{\bigcup_{n \geq \nu_k} F^n(U_k)} \xrightarrow{F} \overline{\bigcup_{n \geq \nu_k} F^{n+1}(U_k)}$$

are homeomorphisms. Then any nondegenerate equation of the form (1.2) *is totally solvable.*

The A.e. is a special case of the *cohomological equation* (c.e.)

$$\varphi(Fx) - \varphi(x) = \gamma(x), \tag{1.8}$$

which, in turn, is a particular case of (1.2).

The solvability problem for the c.e. in $C(X)$ on a compact space X was investigated earlier (see [2] for details and references). Now we suppose that X is *locally compact and countable at infinity* (l.c.c.i.). The countability at infinity means that there exists a covering

$$X = \bigcup_{i=1}^{\infty} K_i \tag{1.9}$$

where K_i are compact. (Without loss of generality one can assume that $K_i \subset K_{i+1}$.)

For any $K \subset X$ we consider the mappings

$$\bigcup_{n \geq \nu} F^n(K) \xrightarrow{F} \bigcup_{n \geq \nu} F^{n+1}(K), \quad \nu \in \mathbb{N}. \tag{1.10}$$

Obviously, all these mappings are surjective.

THEOREM 1.5. *Assume that the space* X *is l.c.c.i. Then the following statements are equivalent.*

(a) *Any nondegenerate equation* (1.2) *is totally solvable.*

(b) (1.8) *is totally solvable.*

(c) *Every compact set* $K \subset X$ *is wandering and there exists a number* $\nu = \nu(K)$ *such that the mapping* (1.10) *is injective.*

(d) *Every compact set* $K \subset X$ *is wandering and there exists a number* $\nu = \nu(K)$ *such that the mapping* (1.10) *is a homeomorphism.*

We prove this theorem in Section 5. As a consequence of Theorems 1.4 and 1.5 we obtain

COROLLARY 1.6. *Assume that the space* X *is l.c.c.i and* F *is injective. Then the following statements are equivalent.*

(α) *Any nondegenerate equation* (1.2) *is totally solvable.*

(β) *The A.e. is solvable.*

(γ) *Every compact set in* X *is wandering.*

Note that the implication (β)⇒(α) may fail in the noninjective case.

EXAMPLE 1.7. In the space $X = [1, \infty)$ let us take the following sequences:

$$a_n^{(l)} = l + \frac{1}{2n}, \quad b_n^{(l)} = \begin{cases} l + \frac{1}{2n-1} & (1 \leq l \leq n-1), \\ l + \frac{1}{2n} & (l \geq n). \end{cases}$$

Obviously,

$$l < a_n^{(l)} \leq b_n^{(l)} < a_{n-1}^{(l)} \leq b_{n-1}^{(l)} < \ldots < a_1^{(l)} = b_1^{(l)} < l + 1,$$

and $a_n^{(l)} \to l$ and $b_n^{(l)} \to l$ as n tends to infinity. Let F be a piecewise linear mapping such that $F(a_n^{(l)}) = a_n^{(l+1)}$ and $F(b_n^{(l)}) = b_n^{(l+1)}$. The continuous function

$$\varphi(x) = \begin{cases} l & (l \leq x \leq a_1^{(l)} = b_1^{(l)}), \\ 2x - (l+1) & (a_1^{(l)} \leq x \leq l+1, \ l \geq 1), \end{cases}$$

satisfies the A.e. On the other hand, $F^l(a_n^{(1)}) \neq F^l(b_n^{(1)})$ $(l \leq n-2)$ and $F^{n-1}(a_n^{(1)}) = F^{n-1}(b_n^{(1)})$. This means that condition (c) of Theorem 1.5 is violated. Hence, the c.e. (1.8) is not totally solvable.

Let us point out an interesting application of our general results.

THEOREM 1.8. *Let X be a l.c.c.i. topological group. For any element $h \in X$ the c.e.*

$$\varphi(hx) - \varphi(x) = \gamma(x) \tag{1.11}$$

is totally solvable if and only if h is aperiodic and the set $\{h^n \mid n \in \mathbb{N}\}$ of powers has no limit points.

Proof. Since the mapping $Fx = hx$ is a homeomorphism of X, one can apply criterion (γ) from Corollary 1.6.

Suppose a compact set is nonwandering. Then it has a neighborhood whose closure K is compact, and there exists a subsequence $\{n_i\} \subset \mathbb{N}$ such that $h^{n_i}K \cap K \neq \emptyset$, hence $h^{n_i} \in KK^{-1}$. Since KK^{-1} is also a compact set, the set $\{h^{n_i}\}$ is either finite (and then h is periodic) or this set has a limit point.

Conversely, suppose all compact sets are wandering. In particular, so is the set $\{e\}$ where e is the unit element. This means that $h^n \neq e$ for all n, i.e. h is aperiodic. Now suppose that the set $\{h^n \mid n \in \mathbb{N}\}$ has a limit point g. Let W be a central symmetric compact neighborhood of e. Then $h^{n_i} \in Wg$ for a subsequence $\{n_i\} \subset \mathbb{N}$, hence $g^{-1} \in h^{-n_i}W$, and so W is nonwandering. ∎

In particular, Theorem 1.8 is applicable to the c.e.

$$\varphi(x+h) - \varphi(x) = \gamma(x) \tag{1.12}$$

on $\mathbb{R}$, so that (1.12) is totally solvable. However, this case is rather simple itself (see [3]). In general, the problem of total solvability on $\mathbb{R}$ can be effectively treated by Theorem 1.5. We can restrict ourselves to the case of c.e.

THEOREM 1.9. *The c.e.*

$$\varphi(Fx) - \varphi(x) = \gamma(x) \tag{1.13}$$

on $\mathbb{R}$ is totally solvable if and only if F has no fixed points and F is injective on a ray of the same direction as $\sigma = \operatorname{sign}(Fx - x)$.

Proof. Suppose (1.13) is totally solvable. Then the corresponding A.e. is solvable, hence $Fx \neq x$ for all x. Let, for definiteness, $Fx > x$ for all x, so that $\sigma = 1$. Consider $K = [0, F0]$. By Theorem 1.5 there exists $\nu \geq 0$ such that F is injective on $\bigcup_{n \geq \nu} F^n(K)$. A fortiori, F is injective on the ray $[F^\nu 0, \infty)$.

Conversely, let, for definiteness, $Fx > x$ for all x and let F be injective on a ray $[\varepsilon, \infty)$. By Theorem 1.5 we only need to establish that every compact set K is wandering. It is sufficient to prove that every segment $[a, b]$ is wandering.

For any x the increasing sequence $\{F^p x\}_{p=0}^\infty$ tends to infinity, otherwise its limit is a fixed point. Thus, for any x we get $F^p x > \varepsilon$ with some exponent p. Let $p = p(x)$ be minimal possible. Then for any x there exists a neighborhood U such that $F^{p(x)}y > \varepsilon$, i.e. $p(y) \leq p(x)$ for $y \in U$. Hence, the function p is bounded above on every compact set.

Let $q = \sup_{a \leq x \leq b} p(x)$ and let $[\alpha, \beta] = F^q[a, b]$. Then $[\alpha, \beta] \subset [\varepsilon, \infty)$. Since F is nondecreasing on $[\alpha, \beta]$, we get

$$F^n[a, b] = F^{n-q}[\alpha, \beta] = [F^{n-q}\alpha, F^{m-q}\beta] \tag{1.14}$$

for all $n \geq q$. Note that there exists a segment $[c, d]$ containing all $F^m[a, b]$ with $0 \leq m \leq q$. In particular, $[c, d] \supset [\alpha, \beta]$. Let $\nu = \mu + q$ where μ satisfies $F^\mu \alpha > d$. Then (1.14) is valid for $m \geq 0$ and $n - m \geq \nu$. We prove that

$$F^n[a, b] \cap F^m[a, b] = \emptyset.$$

In the case $m > q$ we have

$$F^n[a, b] \cap F^m[a, b] = [F^{n-q}\alpha, F^{n-q}\beta] \cap [F^{m-q}\alpha, F^{m-q}\beta]$$

by (1.14). The last intersection is empty since $F^{n-m}\alpha \geq F^\mu \alpha > d \geq \beta$, so that $F^{n-q}\alpha > F^{m-q}\beta$.

If $m \leq q$ then

$$F^n[a, b] \cap F^m[a, b] \subset [F^{n-q}\alpha, F^{n-q}\beta] \cap [c, d].$$

The last intersection is empty since $F^{n-q}\alpha \geq F^\mu \alpha > d$. ∎

COROLLARY 1.10 ([3], §3.2). *If F is a homeomorphism of $\mathbb{R}$ without fixed points then the c.e. (1.13) is totally solvable.*

As an application of Theorem 1.8 we consider the c.e. (1.11) on $X = \mathrm{GL}(n)$, the group of all invertible matrices over $\mathbb{C}$ or $\mathbb{R}$.

Let us say that a matrix h *is quasiunitary* if $h = t^{-1}ut$, where u is a unitary matrix and $t \in GL(n)$. A matrix h is quasiunitary if and only if $|\lambda| = 1$ for all eigenvalues λ and there are no Jordan blocks of order > 1 for every λ. Equivalently, a matrix h is quasiunitary if and only if the sequence $\{h^n\}_{n=-\infty}^\infty$ is bounded, and moreover, if and only if there exist two bounded subsequences $\{h^{n_i}\}$ and $\{h^{-m_i}\}$ with $n_i \to \infty$ and $m_i \to \infty$.

LEMMA 1.11. *Let $h \in \mathrm{GL}(n)$. Then the sequence $\{h^n\}_{n=0}^\infty$ has no ω-limit points in $\mathrm{GL}(n)$ if and only if h is not quasiunitary.*

Proof. Let h be quasiunitary. Then the sequence $\{h^n\}_{n=-\infty}^\infty$ has an ω-limit point $g \in M(n)$. The matrix g is invertible. Indeed, let $g = \lim_{i\to\infty} h^{n_i}$, $0 < n_1 < n_2 < \ldots$, and suppose $\det(g) = 0$. Then $\det(h^{n_i}) \to 0$, hence $\det(h^{-n_i}) \to \infty$, contrary to the boundedness of $\{h^n\}_{n=-\infty}^\infty$.

Conversely, suppose that $g = \lim_{i\to\infty} h^{n_i}$, where $n_i \to \infty$ and $g \in \mathrm{GL}(n)$. Then $g^{-1} = \lim_{i\to\infty} h^{-n_i}$. Hence, h is quasiunitary. ∎

COROLLARY 1.12. *The c.e.* (1.11) *with* $X = \mathrm{GL}(n)$ *is totally solvable if and only if the matrix* h *is not quasiunitary.*

2. Extension of local solutions. Let us say that a continuous vector-valued function $\psi_0(x)$, $x \in X$, is a *local solution* of equation (1.2) on a subset $M \subset X$ if ψ_0 satisfies (1.2) for $x \in M$.

In the case $M = X$ any local solution is a solution in the previous sense. We call it then a *global solution*. We say that a global solution ψ is an *extension* of ψ_0, a local solution on a subset M, if $\psi|M = \psi_0|M$. For example, if $\gamma|M = 0$ then the function $\psi_0(x) \equiv 0$ is a local solution (*local zero solution*) and any global solution ψ such that $\psi|M = 0$ is its extension. Note that if ψ_0 is a local solution on M then by the substitution $\psi = \varphi + \psi_0$, (1.2) is reduced to the equation

$$P(x)\varphi(Fx) + Q(x)\varphi(x) = \widetilde{\gamma}(x) \tag{2.1}$$

with

$$\widetilde{\gamma}(x) = \gamma(x) - P(x)\psi_0(Fx) - Q(x)\psi_0(x).$$

Obviously, $\widetilde{\gamma}|M = 0$ and the global solution ψ of equation (1.2) is an extension of the local solution ψ_0 if and only if φ is a global solution of (2.1) extending the local zero solution.

We see that the extendability of local solutions does not depend on γ. Thus, every local solution can be extended to a global one if it is so in the case of the local zero solution.

LEMMA 2.1. *Let A be an absorber and let N be a nozzle. If*

$$\operatorname{rank} Q(x) \equiv r, \tag{2.2}$$

then any local solution of equation (1.2) *on A can be extended to a global solution. If*

$$\operatorname{rank} P(x) \equiv r \tag{2.3}$$

and the mapping

$$\widetilde{F} : F^{-1}(X \setminus \operatorname{int} N) \xrightarrow{F} X \setminus \operatorname{int} N \tag{2.4}$$

is a homeomorphism then any local solution ψ_0 of equation (1.2) *on N can be extended to a global solution.*

Proof. Let $\gamma|A = 0$. We need a global solution of (1.2) vanishing on A.

It follows from (2.2) that the matrix $Q(x)Q^*(x)$ is invertible for all x, so $Q^*(x)(Q(x)Q^*(x))^{-1}$ is a right inverse to $Q(x)$. By substituting

$$\psi(x) = Q^*(x)(Q(x)Q^*(x))^{-1}\omega(x)$$

into (1.2) we get

$$\omega(x) = (T\omega)(x) + \gamma(x) \tag{2.5}$$

where T is a linear operator in the space of continous vector-valued functions,

$$(T\omega)(x) = -P(x)Q^*(Fx)(Q(Fx)Q^*(Fx))^{-1}\omega(Fx). \tag{2.6}$$

We need a solution ω of (2.5) vanishing on A. Formally,

$$\omega(x) = \sum_{n=0}^{\infty}(T^n\gamma)(x).$$

In fact, this is a solution provided the series locally uniformly converges. In our case we have even more: for any $x \in X$ this series reduces to a finite sum with a locally constant number of terms. Indeed, since A is an absorber, any $x_0 \in X$ has a neighborhood V_0 such that $F^n(V_0) \subset A$ for $n \geq k_0$. This yields $T^n\gamma|V_0 = 0$ for $n \geq k_0$, hence

$$\omega(x) = \sum_{n=0}^{k_0-1}(T^n\gamma)(x)$$

for $x \in V_0$. It remains to note that $\omega|A = 0$ since A is an invariant set. The first part of Lemma 2.1 is proved.

For the second part, we consider the space E of all continuous vector-valued functions ω such that $\omega|N = 0$.

It follows from (2.3) that the matrix $P(x)P^*(x)$ is invertible for all $x \in X$, so $P^*(x)(P(x)P^*(x))^{-1}$ is a right inverse to $P(x)$. For any $\omega \in E$ we define the function

$$h_\omega(y) = \begin{cases} P^*(\widetilde{F}^{-1}y)(P(\widetilde{F}^{-1}y)P^*(\widetilde{F}^{-1}y))^{-1}\omega(\widetilde{F}^{-1}y) & (y \in X \setminus \operatorname{int} N), \\ 0 & (y \in \operatorname{int} N). \end{cases}$$

If $y \in N \setminus \operatorname{int} N$, then $\widetilde{F}^{-1}y \in N$ by definition of a nozzle, and therefore $h_\omega(y) = 0$ since $\omega|N = 0$. The function h_ω turns out to be continuous. Indeed, $h_\omega|N = 0$ and $\overline{\operatorname{int} N} \subset N$ since N is closed. Thus, $h_\omega \in E$. It is easy to see that $P(x)h_\omega(Fx) = \omega(x)$, i.e. $P(x)(L\omega)(Fx) = \omega(x)$, $x \in X$, where $L : E \to E$ is the linear operator defined by $L\omega = h_\omega$.

Let $\gamma \in E$. By the substitution $\psi(x) = (L\omega)(x)$, $\omega \in E$, equation (1.2) takes the form

$$\omega(x) = (S\omega)(x) + \gamma(x) \tag{2.7}$$

where $S : E \to E$ is the linear operator given by $(S\omega)(x) = -Q(x)(L\omega)(x)$. Since N is a nozzle and $\gamma \in E$, the series

$$\omega(x) = \sum_{n=0}^{\infty}(S^n\gamma)(x)$$

defines a solution $\omega \in E$ of equation (2.7) as before in the case of absorber. ■

COROLLARY 2.2. *Under the conditions of Lemma* 2.1 *equation* (1.2) *is globally solvable if it is locally solvable on an absorber or on a nozzle.*

COROLLARY 2.3. *Suppose the space X is normal. Let A be an absorber and N be a nozzle. Suppose that all conditions of Lemma* 2.1 *are satisfied. Then any local solution ψ_0 on the intersection $A \cap N$ can be extended to a global solution.*

Proof. The function

$$\widetilde{\gamma}(x) = \begin{cases} \gamma(x) & (x \in A), \\ P(x)\psi_0(Fx) + Q(x)\psi_0(x) & (x \in N), \end{cases}$$

is continuous on the closed set $A \cup N$. It follows from normality that there exists a continuous extension γ_1 of $\widetilde{\gamma}$ to the whole X. By Lemma 2.1 there exists a solution ψ_1 of (1.2) with γ_1 instead of γ such that $\psi_1|N = \psi_0|N$. Similarly, there exists a solution ψ_2 of (1.2) with $\gamma_2 = \gamma - \gamma_1$ instead of γ satisfying $\psi_2|A = 0$. (Note that $\gamma_2|A = 0$.) Then the sum $\psi = \psi_1 + \psi_2$ is a solution of (1.2) which coincides with ψ_0 on $A \cap N$. ∎

3. Proof of Theorems 1.1 and 1.3. We start with the following simple

LEMMA 3.1. *Let φ be a real-valued solution of the A.e. Then the Lebesgue sets*

$$A(\varepsilon) = \{x \mid \varphi(x) \geq \varepsilon\}, \quad N(\delta) = \{x \mid \varphi(x) \leq \delta\}$$

are an absorber and a nozzle for F respectively.

Proof. For instance, consider $A(\varepsilon)$. It is closed since φ is continuous. It is invariant by (1.3). Now we choose a neighborhood V_0 with $a \equiv \inf\{\varphi(x) \mid x \in V_0\} > -\infty$. By (1.3), $F^n(V_0) \subset A(\varepsilon)$ if $n \geq \varepsilon - a$. ∎

Passing to the proof of Theorem 1.1, fix $\varepsilon > 0$ and take a continuous function $\theta : \mathbb{R} \to \mathbb{R}$ such that $\theta(t) = 0$ for $t \geq \varepsilon$ and $\theta(t) = 1$ for $t \leq -\varepsilon$. Let φ be a real-valued solution of the A.e. Then the composition $\tau = \theta \circ \varphi$ is a continuous function on X such that $\tau(x) = 0$ for $x \in A(\varepsilon)$ and $\tau(x) = 1$ for $x \in N(-\varepsilon)$.

Turning to equation (1.2) we consider two equations,

$$P(x)\psi_1(Fx) + Q(x)\psi_1(x) = \tau(x)\gamma(x) \tag{3.1}$$

and

$$P(x)\psi_2(Fx) + Q(x)\psi_2(x) = (1 - \tau(x))\gamma(x). \tag{3.2}$$

Equation (3.1) has the local zero solution on $A(\varepsilon)$. By Lemma 2.1 it has a global solution ψ_1. Similarly, equation (3.3) has a global solution ψ_2. The sum $\psi = \psi_1 + \psi_2$ is a solution of (1.2). ∎

Now let us prove Theorem 1.3.

Let φ be a real-valued solution of the A.e. By Lemma 3.1 the set $A(\varepsilon)$ is an absorber and $N(\delta)$ is a nozzle. Obviously, $A(\varepsilon) \cap N(\delta) = \emptyset$ if $\delta < \varepsilon$.

Conversely, let $F : X \to X$ be a homeomorphism of a normal space X. Given an absorber A and a nozzle N such that $A \cap N = \emptyset$, one can construct a continuous function $\tau : X \to \mathbb{R}$ such that $\tau(x) = 0$ for $x \in A$ and $\tau(x) = 1$ for $x \in N$. Then both of the equations

$$\varphi_1(Fx) - \varphi_1(x) = \tau(x), \quad \varphi_2(Fx) - \varphi_2(x) = 1 - \tau(x)$$

have some global solutions φ_1 and φ_2 (which are extensions of the corresponding local zero solutions). The sum $\varphi = \varphi_1 + \varphi_2$ is a solution of the A.e. ∎

4. Proof of Theorem 1.4

LEMMA 4.1. *Let φ be a solution of the A.e. and let $U \subset X$ be a subset such that*

$$c \equiv \sup_{x \in U} |\varphi(x)| < \infty.$$

Then U is wandering.

Proof. Suppose that

$$x \in \overline{F^n(U)} \cap \overline{F^m(U)} \tag{4.1}$$

for some $n, m \in \mathbb{N}$. Fix $\varepsilon > 0$ and choose a neighborhood $W \ni x$ such that $|\varphi(z) - \varphi(x)| < \varepsilon$ for $z \in W$. It follows from (4.1) that $W \cap F^n(U) \neq \emptyset$ and $W \cap F^m(U) \neq \emptyset$. Let $F^n u_1 \in W$ and $F^m u_2 \in W$ where $u_1 \in U$ and $u_2 \in U$. By (1.3), $\varphi(F^n u_1) - \varphi(u_1) = n$ and $\varphi(F^m u_2) - \varphi(u_2) = m$, hence

$$|n - m| \leq 2c + |\varphi(F^n u_1) - \varphi(F^m u_2)| \leq 2c + 2\varepsilon.$$

Therefore

$$\overline{F^n(U)} \cap \overline{F^m(U)} = \emptyset \quad \text{if } |n - m| > 2c. \ ∎$$

Since any continuous function is bounded in a neighborhood of a compact set we obtain

COROLLARY 4.2. *If the A.e. is solvable then every compact set $K \subset X$ is wandering.*

Now we can prove Theorem 1.4.

The nondecreasing sequence of open sets $U_k = \varphi^{-1}(-k, k)$ $(k = 1, 2, \ldots)$ covers X and the sets are wandering by Lemma 4.1.

Conversely, let U_k be open, $U_k \subset U_{k+1}$ $(k = 1, 2, \ldots)$, $X = \bigcup_k U_k$. For every k there exists $\nu_k > 0$ such that the mapping

$$F_k : \overline{\bigcup_{n \geq \nu_k} F^n(U_k)} \xrightarrow{F} \overline{\bigcup_{n \geq \nu_k} F^{n+1}(U_k)} \tag{4.2}$$

is a homeomorphism and

(4.3) $$\overline{F^n(U_k)} \cap \overline{F^m(U_k)} = \emptyset \quad (|n-m| \geq \nu_k).$$

Let $V_k = \overline{F^{\nu_k}(U_k)}$ and let

(4.4) $$X_k = \bigcup_{n=-\infty}^{\infty} F^n(V_k).$$

Note that all $F^n(V_k)$ are closed via the homeomorphisms F_k for $n \geq 0$ and F itself for $n < 0$.

We prove that X_k is also closed.

Let $x \notin X_k$. We have to find a neighborhood $W \ni x$ such that $W \cap X_k = \emptyset$. We start with a neighborhood U_s containing x. Let $\nu = \max(\nu_k, \nu_s)$. The finite union

$$R = \bigcup_{|n+\nu_k|<\nu} F^n(V_k)$$

is closed. Since $x \notin R$, there exists a neighborhood $\widetilde{W} \ni x$ such that $\widetilde{W} \cap R = \emptyset$. If we know that $F^n(V_k) \cap U_s = \emptyset$ for $|n+\nu_k| \geq \nu$ then $W = \widetilde{W} \cap U_s$ is a neighborhood we need. In fact, we can prove more, namely,

$$F^n(V_k) \cap \overline{F^m(U_s)} = \emptyset$$

for $|n-m+\nu_k| \geq \nu$. This follows from (4.3). Indeed, by (1.7) one can suppose that $n \geq 0$ and $m \geq 0$. Let $s \leq k$, so $U_s \subset U_k$. Then

$$F^n(V_k) \cap \overline{F^m(U_s)} \subset F^n(V_k) \cap \overline{F^m(U_k)} \subset \overline{F^{n+\nu_k}(U_k)} \cap \overline{F^m(U_k)} = \emptyset$$

by (4.3). Now if $s > k$, so that $U_k \subset U_s$, then

$$F^n(V_k) \cap \overline{F^m(U_s)} \subset \overline{F^{n+\nu_k}(U_s)} \cap \overline{F^m(U_s)} = \emptyset$$

by (4.3) with s in place of k.

Obviously, all the sets X_k are invariant. Furthermore,

$$V_k = \overline{F^{\nu_k}(U_k)} \subset F^{-(\nu_{k+1}-\nu_k)}\overline{F^{\nu_{k+1}}(U_{k+1})} = F^{-(\nu_{k+1}-\nu_k)}(V_{k+1}) \subset X_{k+1},$$

hence $X_k \subset X_{k+1}$. Finally, $U_k \subset X_k$ since $U_k \subset F^{-\nu_k}(V_k)$. Hence $X = \bigcup X_k$, and in this covering all the X_k are closed and invariant.

For the mappings $X_k \xrightarrow{F} X_k$ the sets

$$A_k = \bigcup_{n \geq \nu_k} F^N(V_k), \quad N_k = \bigcup_{n \geq 0} F^{-m}(V_k)$$

are an absorber and a nozzle respectively. First of all, they are closed like X_k. Obviously, $F(A_k) \subset A_k$ and $F^{-1}(N_k) \subset N_k$. Finally, for any point $x_0 \in X_k$ and $U_l \ni x_0$ the set $W = X_k \cap U_l$ is a neighborhood of x_0 in X_k. Then

$$W = \bigcup_{|j+\nu_k|<\mu} (F^j(V_k) \cap U_l)$$

where $\mu = \max(\nu_k, \nu_l)$ because $F^j(V_k) \cap U_l = \emptyset$ for $|j+\nu_k| \geq \mu$. Hence, for all integers p we have

$$F^p(W) \subset \bigcup_{|j+\nu_k|<\mu} F^{j+p}(V_k).$$

The last union is contained in A_k if p is positive and large enough. The union is contained in N_k if p is negative and $|p|$ is large enough.

The intersection of A_k and N_k is empty since if $n \geq \nu_k$ and $m \geq 0$ then

$$F^n(V_k) \cap F^{-m}(V_k) \subset \overline{F^{n+\nu_k}(U_k)} \cap \overline{F^{-m+\nu_k}(U_k)} = \emptyset$$

by (4.3).

Now we are going to construct a solution of equation (1.2) by induction on k. Namely, let $\psi_{k-1} \in C(X_{k-1}, \mathbb{C}^l)$ be a local solution on X_{k-1}. In order to obtain a local solution ψ_k on X_k which extends ψ_{k-1} we note that the union $A'_k = A_k \cup X_{k-1}$ is an absorber and $N'_k = N_k \cup X_{k-1}$ is a nozzle for F_k. Obviously, $A'_k \cap N'_k = X_{k-1}$. Since

$$X_k \setminus N_k \subset \bigcup_{n\geq 1} F^n(V_k) \subset \overline{\bigcup_{n \geq \nu_k} F^{n+1}(U_k)},$$

the mapping $\widetilde{F}_k : F^{-1}(X_k \setminus \operatorname{int} N_k) \xrightarrow{F} X_k \setminus \operatorname{int} N_k$ is a homeomorphism. By Corollary 2.3 the required local solution ψ_k does exist. In particular, ψ_1 is a local solution on X_1 obtained as before with $X_0 = \emptyset$ and with no ψ_0. ■

5. Proof of Theorem 1.5. We start with a topological statement.

LEMMA 5.1. *Suppose the space X is locally compact and all compact subsets of X are wandering. Then for any compact subset $K \subset X$ and any integer ν the set $S = \bigcup_{n\geq\nu} F^n(K)$ is closed. If, moreover, the mapping (1.10) is injective for some ν, then it is a homeomorphism.*

Proof. Let $x \notin S$ and let $M \ni x$ be a compact neighborhood. Then there exists ν_1 such that $F^n(K) \cap M = \emptyset$ for $n \geq \nu_1$ because $K \cup M$ is wandering. Since all the $F^n(K)$ $(n \geq 0)$ are compact, the set $S_1 = \bigcup_{\nu\leq n\leq\nu_1} F^n(K)$ is closed. Take a neighborhood $U \ni x$ such that $\overline{U} \cap S_1 = \emptyset$. Then $V = M \cap \overline{U}$ is a compact neighborhood of x such that $V \cap S = \emptyset$. The first statement is proved.

Now we suppose that the mapping (1.10) is injective; then, actually, it is bijective. It is sufficient to prove that $F(V)$ is closed for any closed $V \subset S$. However,

$$V = \bigcup_{n\geq\nu} V_n, \quad V_n = V \cap F^n(K),$$

so every V_n is compact and

$$F(V) = \bigcup_{n\geq\nu} F(V_n).$$

This set is closed by the same argument as before. ∎

LEMMA 5.2. *Suppose the space X is locally compact. If the c.e. is totally solvable then for any compact subset $K \subset X$ there exists $\nu_0 = \nu_0(K) \geq 0$ such that all the mappings $F^n(K) \xrightarrow{F} F^{n+1}(K)$ with $n \geq \nu_0$ are injective.*

Proof. Suppose that the statement is not true for a compact $K \subset X$. Then there exist some sequences $\{n_j\} \subset \mathbb{N}$ and $\{u_j\}$, $\{v_j\} \subset K$ such that

$$(5.1)\qquad F^{n_j}u_j \neq F^{n_j}v_j, \quad F^{n_j+1}u_j = F^{n_j+1}v_j.$$

The A.e. is solvable as a particular case of (1.8). Therefore all compact sets are wandering by Corollary 4.2. Hence, there exists ν_1 such that

$$(5.2)\qquad F^n(K) \cap F^m(K) = \emptyset, \quad |n-m| \geq \nu_1.$$

A fortiori, F has no periodic points, hence $F^l u_j \neq F^s u_j$ for $l \neq s$ and $F^l u_j \neq F^s v_j$ for $l, s \leq n_j$.

One can assume that $\ln(n_j+1) \geq n_{j-1}^2 + \ln(n_{j-1}+\nu_1)$. Consider the subset W which consists of the points $F^l u_j, F^s v_j$ where l and s satisfy $n_j \geq l$, $s \geq n_{j-1}+\nu_1$. Now one can construct a bounded continuous function γ on X which equals 0 at every point $F^s v_j \in W$ and $1/l$ at every $F^l u_j \in W$. We show that equation (1.8) with such a γ is not solvable. Indeed, let φ be a solution. Then

$$(5.3)\qquad \varphi(F^n x) - \varphi(x) = \sum_{l=0}^{n-1} \gamma(F^l x).$$

In particular,

$$\varphi(F^{n_j+1}u_j) - \varphi(u_j) = \sum_{l=0}^{n_j} \gamma(F^l u_j),$$

$$\varphi(F^{n_j+1}v_j) - \varphi(v_j) = \sum_{l=0}^{n_j} \gamma(F^l v_j).$$

Then (5.1) yields

$$|\varphi(u_j) - \varphi(v_j)| = \Big|\sum_{l=0}^{n_j} [\gamma(F^l u_j) - \gamma(F^l v_j)]\Big| \geq \sum_{l=n_{j-1}+\nu_1}^{n_j} \frac{1}{l} - 2M(n_{j-1}+\nu_1)$$

where $M = \sup_{x\in X} |\gamma(x)|$. Thus,

$$|\varphi(u_j) - \varphi(v_j)| \geq \ln\frac{n_j+1}{n_{j-1}+\nu_1} - 2M(n_{j-1}+\nu_1) \geq n_{j-1}^2 - 2M(n_{j-1}+\nu_1).$$

This contradicts the boundedness of $\varphi|K$. ∎

Now let us prove Theorem 1.5.

(a)⇒(b) is trivial.

(b)⇒(c). By Corollary 4.2 every compact set K is wandering. Hence, (5.2) holds for some $\nu_1 = \nu_1(K)$. It remains to prove that the mapping (1.10) is injective for some $\nu = \nu(K)$. Assume that the latter is false. Then there exist some sequences $\{n_j\}, \{m_j\} \subset \mathbb{N}$ and $\{u_j\}, \{v_j\} \subset K$ such that $F^{n_j}u_j \neq F^{m_j}v_j$ but $F^{n_j+1}u_j = F^{m_j+1}v_j$. It follows from (5.2) that $|n_j - m_j| < \nu_1$. Let, for definiteness, $0 \leq m_j - n_j < \nu_1$. Then for $x_j = F^{n_j}u_j$ and $y_j = F^{n_j}v_j'$ with $v_j' = F^{m_j-n_j}v_j$ we obtain

$$(5.4)\qquad x_j \neq y_j, \quad Fx_j = Fy_j, \quad x_j, y_j \in F^{n_j}(K_1),$$

where $K_1 = \bigcup_{l\leq\nu_1} F^l(K)$. By Lemma 5.2, the mapping

$$F : F^m K_1 \to F^{m+1} K_1$$

is injective for $m \geq \nu_0(K_1)$. Hence, $Fx_j \neq Fy_j$ if $m_j \geq \nu_0(K_1)$. This contradicts (5.4).

(c)⇒(d) follows from Lemma 5.1.

(d)⇒(a). One can choose an open covering $X = \bigcup U_i$, $U_i \subset U_{i+1}$, such that each $K_i \equiv \overline{U_i}$ is compact. Then it follows from Lemma 5.1 that

$$\overline{\bigcup_{n\geq\nu_i} F^n(U_i)} \subset \bigcup_{n\geq\nu_i} F^n(K_i)$$

where $\nu_i = \nu(K_i)$. Since the mappings (1.10) are homeomorphisms for $K = K_i$ and $\nu = \nu_i$, one can apply Theorem 1.4.

Remark 5.4. Only (d)⇒(a) needs the countability at infinity. The chain (a)⇒(b)⇒(c)⇒(d) is true for any locally compact space X.

6. Appendix. Smooth solutions. Let X be a C^∞-manifold and let $F : X \to X$ and $P, Q : X \to \mathrm{Hom}(\mathbb{C}^l, \mathbb{C}^r)$ be C^k-mappings, $0 \leq k \leq \infty$. Then one can consider equation (1.2) in the class of C^k-vector-valued functions. The C^k-*total solvability* is defined as in the case $k = 0$.

THEOREM 6.1. *Let F be a C^k-diffeomorphism. If the A.e. is C^0-solvable then any nondegenerate equation (1.2) is C^k-totally solvable.*

Proof. By Theorem 1.3 there exist an absorber A and a nozzle N with $A \cap N = \emptyset$. By a Whitney theorem (see [5], Appendix III, §1), there exists a C^∞-function $\tau : X \to \mathbb{R}$ such that $\tau(x) = 0$ for $x \in A$ and $\tau(x) = 1$ for

$x \in N$. With this function τ one can repeat our proof of Theorem 1.1 and Lemma 2.1 for the class C^k. ∎

In Theorem 6.1 the class of C^k-diffeomorphisms cannot be extended to C^k-homeomorphisms.

EXAMPLE 6.2. Let $0 < \alpha < 1$ and

$$F_\alpha x = x + 1 - \alpha \sin(\alpha^{-1} x), \quad x \in \mathbb{R}.$$

The mapping F_α is a real-analytic homeomorphism without fixed points. By Corollary 1.11 *the corresponding c.e. is totally solvable*. However, F_α is not a diffeomorphism since it has critical points,

$$c(F_\alpha) \equiv \{x \mid F'_\alpha x = 0\} = \{x_s\}_{s=-\infty}^{\infty}$$

where $x_s = 2s\pi\alpha$. We show that *if*

$$F_\alpha^n(c(F_\alpha)) \cap c(F_\alpha) = \emptyset \quad (n > 0) \tag{6.1}$$

then the corresponding c.e. is not C^1-totally solvable. Since by induction $F_\alpha^n(x_s) = x_s + F_\alpha^n 0$ $(s, n \in \mathbb{Z})$, condition (6.1) means that $F_\alpha^n 0 \neq 2s\pi\alpha$ for all s and $n > 0$.

Indeed, let

$$\varphi(F_\alpha x) - \varphi(x) = \gamma(x), \quad \varphi \in C^1(\mathbb{R}). \tag{6.2}$$

By differentiation it follows from (5.3) that

$$\varphi'(x) = a_{n+1}(x)\varphi'(F_\alpha^{n+1} x) - \sum_{k=0}^{n} a_k(x)\gamma'(F_\alpha^k x)$$

where

$$a_k(x) = \prod_{j=0}^{k-1} F'_\alpha(F_\alpha^j x).$$

Due to the factor $F'_\alpha(F_\alpha^n x)$, we have $a_{n+1}(F_\alpha^{-n} x_s) = 0$, hence

$$\varphi'(F_\alpha^{-n} x_s) = -\sum_{k=0}^{n} a_k(F_\alpha^{-n} x_s)\gamma'(F_\alpha^{k-n} x_s). \tag{6.3}$$

Here $a_n(F_\alpha^{-n} x_s) \neq 0$ because of (6.1). Let

$$s_n = \left[-\frac{1}{2\pi\alpha} F_\alpha^{-n} 0\right]$$

where $[\cdot]$ means the integer part. Obviously, $s_n \to \infty$ but

$$|F_\alpha^{-n} x_{s_n}| = |x_{s_n} + F_\alpha^{-n} 0| \le 2\pi\alpha. \tag{6.4}$$

Now let γ be a smooth function satisfying

$$|\gamma'(x_{s_n})| \ge \frac{1}{|a_n(F_\alpha^{-n} x_{s_n})|}\Big\{n + \sum_{k=0}^{n-1} |a_k(F_\alpha^{-n} x_{s_n})| \cdot |\gamma'(F_\alpha^{k-n} x_{s_n})|\Big\}.$$

Then it follows from (6.3) that

$$|\varphi'(F_\alpha^{-n} x_{s_n})| \ge n.$$

This means that φ' is unbounded on the interval $[-2\pi\alpha, 2\pi\alpha]$ because of (6.4). Hence, equation (6.2) has no smooth solutions.

Remark 6.3. The set of α is uncountable. Indeed, (6.1) holds if and only if $\alpha \notin \bigcup Z_{n,s}$, where $Z_{n,s}$ is the zero set of the analytic function $\theta_{n,s}(\alpha) = F_\alpha^n 0 - 2s\pi\alpha$.

Remark 6.4. If (6.1) does not hold, then it may happen that the corresponding c.e. is C^k-totally solvable in the class of C^k-functions, $k \ge 1$. For instance, if $\alpha = 1/(2\pi)$ then $c(F_\alpha) = \mathbb{Z}$. This set is invariant, hence (6.1) is violated. In this case one can prove the C^k-total solvability for all $k \ge 1$.

The authors thank the referee for valuable remarks.

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Ben-Gurion University of the Negev
84105 Beer-Sheva, Israel
E-mail: genrich@cs.bgu.ac.il

Technion
32000 Haifa, Israel
E-mail: lyubich@technix.technion.ac.il

Received November 4, 1996
Revised version June 17, 1997 (3768)