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A subsequence characterization of sequences spanning isomorphically polyhedral Banach spaces

by

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Abstract. Let (x_n) be a sequence in a Banach space X which does not converge in norm, and let E be an isomorphically precisely norming set for X such that

$$(*)\qquad \sum_n |x^*(x_{n+1}-x_n)| < \infty, \quad \forall x^* \in E.$$

Then there exists a subsequence of (x_n) which spans an isomorphically polyhedral Banach space. It follows immediately from results of V. Fonf that the converse is also true: If Y is a separable isomorphically polyhedral Banach space then there exists a normalized M-basis (x_n) which spans Y and there exists an isomorphically precisely norming set E for Y such that $(*)$ is satisfied. As an application of this subsequence characterization of sequences spanning isomorphically polyhedral Banach spaces we obtain a strengthening of a result of J. Elton, and an Orlicz–Pettis type result.

1. Introduction. In 1958 C. Bessaga and A. Pełczyński proved the following

THEOREM 1.1 ([BP]). *If (x_n) is a non-weakly convergent sequence in a Banach space X such that*

$$(1)\qquad \sup_{x^*\in \mathrm{Ba}(X^*)} \sum_n |x^*(x_{n+1}-x_n)| < \infty$$

then there exists a subsequence of (x_n) which is equivalent to the summing basis (s_n) of c_0.

Recall that the *summing basis* (s_n) of c_0 is defined by $s_n = e_1 + \ldots + e_n$, for $n \in \mathbb{N}$, where (e_n) denotes the unit vector basis of c_0. In 1981 J. Elton was able to eliminate the assumption "non-weakly convergent" and relax the condition (1) and still show that c_0 embeds in the closed linear span $[x_n]$ of (x_n). The result of J. Elton can be stated as follows:

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THEOREM 1.2 ([E2]). *If* (x_n) *is a seminormalized basic sequence in a Banach space* X *such that*

$$\sum |x^*(x_n)| < \infty, \quad \forall x^* \in \operatorname{ext} \mathrm{Ba}(X^*) \tag{2}$$

(*where* $\operatorname{ext}\mathrm{Ba}(X^*)$ *denotes the set of the extreme points of the dual ball*) *then* c_0 *embeds in* $[x_n]$.

In order to prove this result, J. Elton first showed that there exists a polyhedral Banach space which embeds in $[x_n]$ (for the definition, examples and properties of the polyhedral Banach spaces see the next section). Then the result of Theorem 1.2 follows from the following theorem of V. Fonf:

THEOREM 1.3 ([F3]). *Every polyhedral Banach space* X *contains an isomorph of* c_0, *and if in addition* X *is separable, then* X^* *is separable.*

We prove a result stronger than Theorem 1.2 by eliminating the condition of having a basic sequence, by replacing the set of the extreme points in condition (2) by any isomorphically precisely norming set, and finally by obtaining the precise way that a polyhedral Banach space embeds in $[x_n]$. Our main result can be stated as follows:

THEOREM 1.4. *If* (x_n) *is a sequence in a Banach space* X *which does not converge in norm, and* E *is an isomorphically precisely norming set for* X *such that*

$$\sum_n |x^*(x_{n+1} - x_n)| < \infty, \quad \forall x^* \in E, \tag{3}$$

then there exists a subsequence of (x_n) *which spans an isomorphically polyhedral Banach space. Conversely, if* Y *is a separable isomorphically polyhedral Banach space then there exists an M-basis* (x_n) *in* Y, *with* $\|x_n\| = 1$ *for all* n, *and an isomorphically precisely norming set* E *for* Y *such that* $[x_n] = Y$ *and* (3) *holds.*

We recall the following terminology:

DEFINITION 1.5. Let $(X, \|\cdot\|)$ be a Banach space.

A set $E \subset X^*$ is called *isomorphically precisely norming* for $(X, \|\cdot\|)$, (the terminology is due to H. Rosenthal [R]) if there exists $C \geq 1$ such that

(a) $E \subseteq C \cdot \mathrm{Ba}(X^*)$,
(b) $C^{-1}\|x\| \leq \sup_{e\in E} |e(x)|$ for $x \in X$, and
(c) for each $x \in X$ there is $e_0 \in E$ such that $|e_0(x)| = \sup_{e\in E} |e(x)|$.

If E satisfies (a), (b), and (c) for $C = 1$ then E is called *precisely norming* (or *boundary*) for $(X, \|\cdot\|)$.

A sequence (v_i) of vectors in X is called a *complete minimal system in* X *with dual system* (v_i^*) if

(a) the finite linear combinations of $\{v_i\}_{i\in\mathbb{N}}$ are dense in X, and
(b) $v_i^*(v_j) = \delta_{ij}$ for all $i, j \in \mathbb{N}$.

An *M-basis* for the Banach space X is a complete minimal system $(v_i)_{i\in\mathbb{N}}$ for X with dual system $(v_i^*)_{i\in\mathbb{N}}$ such that whenever $v_i^*(x) = 0$ for all $i \in \mathbb{N}$ then $x = 0$.

Recall that the set of the norm achieving extreme points of the dual ball of a Banach space X is defined as follows:

$$\operatorname{next}\mathrm{Ba}(X^*) = \{x^* \in \operatorname{ext}\mathrm{Ba}(X^*) : \exists x \in \mathrm{Ba}(X)\ |x^*(x)| = 1\}.$$

The set $\operatorname{next}\mathrm{Ba}(X^*)$ is an example of a precisely norming set for X.

Theorem 1.4 is a strengthening of the following remark which can be easily derived from a result of V. Fonf [F4].

Remark 1.6. Under the same hypotheses of Theorem 1.4 there exist a sequence $(\varepsilon_n) \in \{\pm 1\}^{\mathbb{N}}$ and an increasing sequence (l_k) of positive integers such that $[(\sum_{i=1}^{l_k} \varepsilon_i(x_i - x_{i-1}))_k]$ is an i.p. space.

We sketch the proof of Remark 1.6 at the end of Section 3.

The last section is devoted to applications of Theorem 1.4. One application is given in $C(K)$ spaces. If K is a compact metric space then $\mathrm{DSC}(K)$ denotes the class of bounded differences of semicontinuous functions on K (the definition appears in Section 4). An immediate corollary of Theorem 1.4 is the following:

THEOREM 1.7. *Let* $f \in \mathrm{DSC}(K) \setminus C(K)$, *where* K *is a compact metric space. Then* f *strictly governs the class of (separable) polyhedral Banach spaces.*

This theorem was the main motivating result for this research. The definitions of the terms "strictly governs" and "governs" appear in Section 4. This generalizes the following theorem of J. Elton which was also proved by R. Haydon, E. Odell and H. Rosenthal:

THEOREM 1.8 ([E2], [HOR]). *Let* $f \in \mathrm{DSC}(K) \setminus C(K)$, *where* K *is a compact metric space. Then* f *governs* $\{c_0\}$.

Another application is the following Orlicz–Pettis type result:

THEOREM 1.9. *Let* (y_n) *be a sequence in a Banach space* X *and let* E *be an isomorphically precisely norming set for* X. *If* c_0 *does not embed isomorphically in the closed linear span* $[y_n]$ *of* (y_n) *and*

$$\sum_n |x^*(y_n)| < \infty, \quad \forall x^* \in E,$$

then $\sum_n y_n$ *converges unconditionally.*

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2. Isomorphically polyhedral Banach spaces. Polyhedral Banach spaces were introduced by V. Klee [K]. An infinite-dimensional Banach space is called *polyhedral* if the ball of any of its finite-dimensional subspaces is a polyhedron, i.e. it has finitely many extreme points. c_0 is an example of a polyhedral Banach space. A Banach space will be called *isomorphically polyhedral* (i.p. in short) if it is polyhedral under some equivalent norm. We are interested in isomorphic theory and therefore in i.p. Banach spaces. Examples of i.p. Banach spaces are: c (the space of convergent sequences), the ℓ_1 preduals [F4], the spaces $C(\alpha)$ for any ordinal α (see [F2]), c_0-sums of separable i.p. spaces (easy to prove using Theorem 2.1), finite-dimensional extensions of i.p. spaces (easy to prove), the Orlicz sequence space h_M where M is a non-degenerate Orlicz function satisfying $\lim_{t\to 0} M(Kt)/M(t) = \infty$ for some $K > 1$ (see [L]). The following characterization of the separable i.p. Banach spaces was proved by V. Fonf (note that if $(X, \|\cdot\|)$ is a Banach space, $|\cdot|$ is an equivalent norm and $C \ge 1$ then we say that these norms are *C-equivalent* if $C^{-1}\|x\| \le |x| \le C\|x\|$ for all $x \in X$):

THEOREM 2.1 ([F3], [F4], [F5]). *Let $(X, \|\cdot\|)$ be a separable Banach space. The following are equivalent*:

(1) *For every $\varepsilon > 0$ there exists a $(1+\varepsilon)$-equivalent norm $|\cdot|$ on X such that $(X, |\cdot|)$ is polyhedral.*

(2) *For every $\varepsilon > 0$ there exists a $(1+\varepsilon)$-equivalent norm $|||\cdot|||$ on X such that the set* $\operatorname{next}\operatorname{Ba}(X, |||\cdot|||)^*$ *is countable.*

The next two lemmata give sufficient conditions for a Banach space to be an i.p. space. We start with some notation: If X is a Banach space and K is a subset of the unit dual ball then the space $[X, \|\cdot\|_K]$ is the completion of the space X in the norm

$$\|x\|_K = \sup\{|f(x)| : f \in K\}$$

for all $x \in X$. Note that if X is separable then w*-cl(K) is a compact metric space in the weak* topology and $[X, \|\cdot\|_K]$ is isometric to a subspace of $C(\text{w*-cl}(K))$ and hence it is separable.

LEMMA 2.2. *Let X be a separable Banach space having a boundary K with $K = \bigcup_{i=1}^{\infty} K_i$ such that for all i we have $K_i \subset K_{i+1}$ and $X_i = [X, \|\cdot\|_{K_i}]$ is an i.p. space. Then X is an i.p. space.*

Proof. Take a decreasing sequence $(\varepsilon_i)_{i\in\mathbb{N}}$ of positive numbers, and using the main result of [DFH] (Theorem 1 and Proposition 1-2) find an approximating polytope V_i for the unit ball $\operatorname{Ba}(X_i)$ of X_i such that

$$V_i \subset \operatorname{Ba}(X_i) \subset (1+\varepsilon_i)V_i$$

and V_i is a closed absolutely convex body, i.e. there is a $(1+\varepsilon_i)$-equivalent norm $\|\cdot\|_{V_i}$ whose unit ball is the set V_i. Moreover, the unit dual ball V_i^* has a countable boundary $\{h_j^i\}_{j=1}^{\infty}$ with the property that no weak*-approximation point of $\{h_j^i\}_{j=1}^{\infty}$ attains its supremum on V_i, where the set of *weak*-approximation points* of a set A is defined as the set of points of the weak* closure of A which do not belong to A:

$$\text{w*-ap}(A) = \text{w*-cl}(A) \setminus A.$$

It is clear that for all $i \in \mathbb{N}$,

$$V_i^* \supset \operatorname{Ba}(X_i^*) \supset \frac{1}{1+\varepsilon_i} V_i^*.$$

For $i \in \mathbb{N}$ consider the natural restriction map $T_i : X \to X_i$ and note that $T_i^*(\operatorname{Ba}(X_i^*)) \supset K_i$.

Now put

$$W^* = \text{w*-cl}\operatorname{co}\{(1+\varepsilon_i)T_i^* h_j^i : i, j \in \mathbb{N}\}$$

and for $x \in X$ define

$$|||x||| = \sup\{|f(x)| : f \in W^*\}.$$

We first show that $|||\cdot|||$ is an equivalent norm on X. Indeed, for every $x \in X$ there exists $x^* \in K$ such that $\|x\| = |x^*(x)|$. There exist $i \in \mathbb{N}$ and $y^* \in \operatorname{Ba}(X_i^*)$ such that $x^* = T_i^* y^*$. Thus

$$\|x\| = |y^*(T_i x)|.$$

Since $y^* \in V_i^*$ and $\{h_j^i\}_{j=1}^{\infty}$ is a boundary for V_i^*, there exists $j \in \mathbb{N}$ with

$$|y_i^*(T_i x)| \le |h_j^i(T_i x)| < (1+\varepsilon_i)|(T_i^* h_j^i)(x)| \le |||x|||.$$

Also, since

$$T_i^*(h_j^i) \subset (1+\varepsilon_i)T_i^*(\operatorname{Ba}(X_i^*)) \subset (1+\varepsilon_i)\operatorname{Ba}(X^*)$$

we have

$$|||x||| \le (1+\varepsilon_1)^2\|x\|,$$

which proves the equivalence of the norms.

We now claim that for every $x \in X \setminus \{0\}$,

$$\sup\{|(T_i^* h_j^i)(x)| : i, j \in \mathbb{N}\} < \sup\{(1+\varepsilon_i)|(T_i^* h_j^i)(x)| : i, j \in \mathbb{N}\}. \tag{4}$$

Indeed, let $x \in X \setminus \{0\}$. Let $x^* \in K$ be such that $\|x\| = |x^*(x)|$ and let $i_0 \in \mathbb{N}$ with $x^* \in K_{i_0}$. Note that

$$\begin{aligned}\sup\{|(T_i^*h_j^i)(x)| : i > i_0,\ j \in \mathbb{N}\} &\le \sup\{(1+\varepsilon_i)|(T_i^*y^*)(x)| : i > i_0,\ y^* \in \mathrm{Ba}(X_i^*)\}\\ &\le (1+\varepsilon_{i_0+1})\sup\{|(T_i^*y^*)(x)| : i > i_0,\ y^* \in \mathrm{Ba}(X_i^*)\}\\ &\le (1+\varepsilon_{i_0+1})\sup\{|y^*(x)| : y^* \in \mathrm{Ba}(X^*)\} = (1+\varepsilon_{i_0+1})\|x\|\\ &= (1+\varepsilon_{i_0+1})\sup\{|y^*(x)| : y^* \in K_{i_0}\} = (1+\varepsilon_{i_0+1})\|T_{i_0}x\|_{K_{i_0}}\\ &\le (1+\varepsilon_{i_0+1})\sup\{|y^*(T_{i_0}x)| : y^* \in V_{i_0}^*\}\\ &= (1+\varepsilon_{i_0+1})\sup\{|h_j^{i_0}(T_{i_0}x)| : j \in \mathbb{N}\}\\ &< \sup\{(1+\varepsilon_{i_0})|(T_{i_0}^*h_j^{i_0})(x)| : j \in \mathbb{N}\}\\ &\le \sup\{(1+\varepsilon_i)|(T_i^*h_j^i)(x)| : i,j \in \mathbb{N}\}.\end{aligned}$$

Also for every $i \in \mathbb{N}$ there exists $i' \in \mathbb{N}$ such that

$$\sup\{|h_j^i(T_ix)| : j \in \mathbb{N}\} = |h_{i'}^i(T_ix)|.$$

Thus

$$\begin{aligned}\sup\{|(T_i^*h_j^i)(x)| : i \le i_0,\ j \in \mathbb{N}\} &= \max\{|(T_i^*h_{i'}^i)(x)| : i \le i_0\}\\ &< \max\{(1+\varepsilon_i)|(T_i^*h_{i'}^i)(x)| : i \le i_0\}\\ &\le \sup\{(1+\varepsilon_i)|(T_i^*h_j^i)(x)| : i,j \in \mathbb{N}\},\end{aligned}$$

which finishes the proof of (4).

Obviously,

$$\operatorname{ext}\mathrm{Ba}(X, |\!|\!|\cdot|\!|\!|)^* \subset \text{w}^*\text{-cl}\{(1+\varepsilon_i)T_i^*h_j^i : i,j \in \mathbb{N}\}.$$

We claim that

$$\text{(5)}\qquad \operatorname{next}\mathrm{Ba}(X, |\!|\!|\cdot|\!|\!|)^* = \{(1+\varepsilon_i)T_i^*h_j^i : i,j \in \mathbb{N}\},$$

which will finish the proof of the lemma by Theorem 2.1. In order to prove (5) it is enough to show that no

$$x^* \in \text{w}^*\text{-ap}\{(1+\varepsilon_i)T_i^*h_j^i : i,j \in \mathbb{N}\}$$

achieves its supremum on $\mathrm{Ba}(X, |\!|\!|\cdot|\!|\!|)$. Indeed, for such an x^* there exists a sequence

$$((1+\varepsilon_{i(n)})T_{i(n)}^*h_{j(n)}^{i(n)})_{n\in\mathbb{N}}$$

which converges weak* to x^*. If there exists an infinite subsequence of $(i(n))_{n\in\mathbb{N}}$ which is constant, then the result follows by the choice of $(h_j^i)_{j\in\mathbb{N}}$ for each $i \in \mathbb{N}$. Otherwise, we can assume that $i(n) \to \infty$. Since $\varepsilon_{i(n)} \to 0$ we see that

$$T_{i(n)}^*h_{j(n)}^{i(n)} \to x^* \quad \text{weak}^*,$$

i.e.

$$x^* \in \text{w}^*\text{-cl}\{T_i^*h_j^i : i,j \in \mathbb{N}\}.$$

If there exists $x \in X$ with $|\!|\!|x|\!|\!| = 1$ and $|x^*(x)| = 1$ then (4) gives a contradiction. ∎

The following lemma is just a combination of Lemma 1.5 from [DFH] and Theorem 2.1.

LEMMA 2.3. *Let X be a Banach space having a boundary which may be covered by a countable union of norm-compact sets. Then X is an i.p. space.*

The next lemma gives sufficient conditions for detecting norm-precompact sets.

LEMMA 2.4. *Let $\{v_i\}_{i=1}^\infty$ be a complete minimal system in a Banach space X with dual system $\{v_i^*\}_{i=1}^\infty$. If $D \subset \mathrm{Ba}(X^*)$ has the property*

$$\sum_{i=1}^\infty \|v_i^*\| \sup_{d\in D} |d(v_i)| < \infty$$

then D is $\|\cdot\|$-precompact.

Proof. Take $\varepsilon > 0$ and let $n \in \mathbb{N}$ be such that

$$\sum_{i=n+1}^\infty \|v_i^*\| \sup_{d\in D} |d(v_i)| < \varepsilon/4.$$

Without loss of generality we assume that D is weak* compact, so that the restriction $D|[v_i]_{i=1}^n$ of D to the (closed) linear span $[v_i]_{i=1}^n$ (where D is now considered as a subset of X^{**}) is norm-compact. Choose $\{d_j\}_{j=1}^l \subset D$ such that $\{d_j|[v_i]_{i=1}^n\}_{j=1}^l$ is a δ-net for $D|[v_i]_{i=1}^n$ where $\delta = \frac{1}{2}\varepsilon(\sum_{i=1}^n \|v_i^*\|\cdot\|v_i\|)^{-1}$.

We claim that $\{d_j\}_{j=1}^l$ is a finite ε-net for D, which finishes the proof. Indeed, for $d \in D$ find $j \in \{1,\ldots,l\}$ such that $\|(d-d_j)|[v_i]_{i=1}^n\| < \delta$. For every finite linear combination $x = \sum_{i=1}^m x_iv_i$ with $\|x\| \le 1$ we have

$$\begin{aligned}|(d-d_j)(x)| &\le \sum_{i=1}^m |x_i|\cdot|(d-d_j)(v_i)|\\ &\le \sum_{i=1}^n \|v_i^*\|\delta\|v_i\| + \sum_{i=n+1}^m \|v_i^*\|\cdot 2\sup_{d'\in D}|d'(v_i)| < \varepsilon,\end{aligned}$$

which proves that $\{d_j\}_{j=1}^l$ is an ε-net for D since the finite linear combinations of $\{v_i\}$ are dense in X. ∎

Finally, the last ingredient of the proof is a technical lemma which makes repeated use of diagonal arguments.

LEMMA 2.5. *Let K be a set which can be written as an increasing union $K=\bigcup_{m=1}^{\infty} K_m$ of sets and let $(x_n)_{n=1}^{\infty}$ be a sequence in $\ell_\infty(K)$. Suppose that for each $m\in\mathbb{N}$ and for each subsequence (y_n) of (x_n) we have*

$$\inf_{p\neq q}\|y_p-y_q\|_{K_m}=0,\quad \textit{where}\quad \|y\|_{K_m}=\sup_{k\in K_m}|k(y)|$$

for $y\in\ell_\infty(K)$. Then there exists a subsequence (z_n) of (x_n) such that

$$\sum_{n=1}^{\infty} n\|z_{n+1}-z_n\|_{K_m}<\infty$$

for each $m\in\mathbb{N}$.

Proof. We begin with the following claim: For every subsequence (y_n) of (x_n), for every $m\in\mathbb{N}$, and for every $\varepsilon>0$ there exists a subsequence (z_n) of (y_n) such that

$$\|z_1-z_n\|_{K_m}<\varepsilon,\qquad \forall n\in\mathbb{N}.$$

Indeed, assume that the claim is false. Thus, if we set

$$I_1=\{n\in\mathbb{N}:\|y_1-y_n\|_{K_m}<\varepsilon\},$$

then I_1 is finite. Set $i_1=\max I_1+1$. Also, the set

$$I_2=\{n>i_1:\|y_{i_1}-y_n\|_{K_m}<\varepsilon\}$$

is finite. Set $i_2=\max I_2+1$. We continue similarly. Then the subsequence (y_{i_n}) of (y_n) satisfies

$$\inf_{p\neq q}\|y_{i_p}-y_{i_q}\|_{K_m}>\varepsilon,$$

which is a contradiction. The claim is proved.

Note that if (z_n) satisfies the previous claim then

$$\|z_p-z_q\|_{K_m}<2\varepsilon\qquad \text{for all } p,q\in\mathbb{N}.$$

For $m=1$, using this remark and a diagonal argument we can choose a subsequence (z_n^1) of (x_n) such that

$$\|z_p^1-z_q^1\|_{K_1}<\frac{1}{2^n}\qquad \text{for all } n\in\mathbb{N} \text{ and } p,q\geq n.$$

Take $m=2$ and similarly find a subsequence (z_n^2) of (z_n^1) such that

$$\|z_p^2-z_q^2\|_{K_2}<\frac{1}{2^n}\qquad \text{for all } n\in\mathbb{N} \text{ and } p,q\geq n.$$

We continue in the same manner. It is easy to verify that the diagonal sequence (z_n^n) satisfies the statement of the lemma. ■

3. The proof of the main result.

Before we present the proof of Theorem 1.4, we give some more preliminary ingredients. We use the following subsequence dichotomy for the c_0 basis, due to J. Elton:

THEOREM 3.1 ([E1]). *Every seminormalized weakly null sequence which does not have a semiboundedly complete subsequence, has a subsequence equivalent to the unit vector basis of c_0.*

Recall that a sequence (x_n) is called *semiboundedly complete* if for every sequence $(\lambda_n)\subset\mathbb{R}$ we have

$$\sup_m\Big\|\sum_{n=1}^{m}\lambda_n x_n\Big\|<\infty\ \Rightarrow\ \lambda_n\to 0.$$

Our main result will follow from

THEOREM 3.2. *If (x_n) is a basic sequence in a Banach space X with $\inf_n\|x_n\|>0$, and E is an isomorphically precisely norming set for X such that*

$$\sum_n|x^*(x_{n+1}-x_n)|<\infty,\qquad \forall x^*\in E,$$

then there exists a subsequence of (x_n) which spans an isomorphically polyhedral Banach space.

We postpone the proof of Theorem 3.2 for the moment. We first give a proof of Theorem 1.4 using the result of Theorem 3.2.

DEFINITION 3.3. Let $(X,\|\cdot\|)$ be a Banach space and Y be a linear (not necessarily closed) subspace of X^*. Then Y is a *norming subspace* if there exists $C>0$ such that

$$\frac{1}{C}\|x\|\leq\sup_{y\in Y,\,\|y\|=1}|y(x)|\leq C\|x\|\qquad \text{for every } x\in X.$$

The following criterion for extracting basic sequences will be used:

CRITERION ([KP], see also [M]). *Let $(X,\|\cdot\|)$ be a Banach space, Y be a norming subspace of X^*, and (x_n) be a sequence in X such that $\inf_n\|x_n\|>0$. In each of the following cases (x_n) has a basic subsequence.*

(a) *$y(x_n)\to 0$ for all $y\in Y$.*

(b) *$(y(x_n))$ is a Cauchy sequence for all $y\in Y$ yet there is no x in X with $y(x_n-x)\to 0$ for all $y\in Y$.*

Proof of Theorem 1.4. Let (x_n) be a sequence in a Banach space X which does not converge in norm, and let E be an isomorphically precisely norming set for X such that (3) holds. We define the (not necessarily closed) subspace $Y=\operatorname{span}(E)$ of X^*. Then Y is norming. If (b) of the above criterion applies then (x_n) has a basic subsequence, and the result follows from Theorem 3.2. If (b) does not apply then there exists x in X such that $y(x_n-x)\to 0$. Since (x_n) does not converge in norm, there exists a subsequence (x_{n_k}) of (x_n) with $\inf\|x_{n_k}-x\|>0$. Thus (a) of the above criterion gives that there exists a subsequence $(x_{n_{k_l}})$ of (x_{n_k}) such that $(x_{n_{k_l}} - x)$ is a basic sequence. Since

$$\sum_{l=1}^{\infty} |x^*[(x_{n_{k_{l+1}}} - x) - (x_{n_{k_l}} - x)]| < \infty,$$

Theorem 3.2 gives the existence of a subsequence (y_n) of (x_n) such that $[(y_n - x)_n]$ is isomorphically polyhedral. Thus the 1-dimensional extension $[(y_n - x)_n] + [x]$ is an i.p. space, and therefore so is its subspace $[y_n]$.

Conversely, consider a separable isomorphically polyhedral Banach space Y. By Theorem 2.1 there exists a countable isomorphically precisely norming set $E = \{f_1, f_2, \ldots\}$ of non-zero functionals which are finitely linearly independent, i.e. $\dim[f_i]_{i=1}^n = n$ for all n. Using [M] find an M-basis (x_n) of X with dual system (x_n^*) such that $[x_i^*]_{i=1}^n = [f_i]_{i=1}^n$, and $\|x_n\| = 1$ for all n. It is trivial that (3) holds. ■

Proof of Theorem 3.2. We can assume without loss of generality that X is separable (e.g. by considering $X = [x_n]$). For every $x \in X$ we define

$$\|x\| = \sup_{e\in E} |e(x)|.$$

This defines an equivalent norm on X, and E is a precisely norming set for $(X, \|\cdot\|)$. Also, the weak* topology is metrizable on $\mathrm{Ba}(X^*)$, and let $d(\cdot,\cdot)$ denote the induced metric. For $m \in \mathbb{N}$ we define (set $x_0 = 0$)

$$K_m = \Big\{x^* \in \mathrm{Ba}(X, \|\cdot\|)^* : \sum_{n=1}^{\infty} |x^*(x_n - x_{n-1})| \le m\Big\}.$$

Then K_m is a weak* closed subset of $\mathrm{Ba}(X^*)$ for every $m \in \mathbb{N}$, $K_1 \subseteq K_2 \subseteq \ldots$, and $K := \bigcup_{m=1}^{\infty} K_m \supseteq E$. Define $f : K \to \mathbb{R}$ by

$$f(k) = \lim_n k(x_n), \qquad \forall k \in K.$$

We separate the following cases:

CASE 1: Assume that there exists $m \in \mathbb{N}$ such that the restriction $f|K_m$ is not continuous (K_n will always be equipped with the weak* topology of X^*, for every $n \in \mathbb{N}$).

We claim that for every $m' \ge m$ there exists a sequence $(x_n^{m'})_n$ satisfying:

- $(x_n^m)_n$ is a subsequence of (x_n).
- $(x_n^{m'+1})_n$ is a subsequence of $(x_n^{m'})_n$.
- $[(x_n^{m'}|K_{m'})_n]$ is an i.p. Banach space (where $[(x_n^{m'}|K_{m'})_n]$ denotes the completion of the normed space $\mathrm{span}(x_n^{m'}|K_{m'})_n$).

Indeed, for $m' = m$ we have

$$\sup\Big\{\sum_n |x^*(x_n - x_{n-1})| : x^* \in \mathrm{Ba}([(x_n|K_m)_n], \|\cdot\|_{C(K_m)})^*\Big\} \le m$$

and $(x_n|K_m)_n$ is non-weakly convergent in $C(K_m)$. Thus by Theorem 1.1 there exists a subsequence $(x_n^m)_n$ of (x_n) such that $(x_n^m|K_m)_n$ is equivalent to the summing basis. Thus $[(x_n^m|K_m)_n]$ is an i.p. Banach space. The proof of the inductive step is a repetition of the same argument, since the hypothesis that $f|K_m$ is not continuous gives that $f|K_{m'}$ is not continuous for every $m' \ge m$. The proof of the claim is complete.

Set $y_n = x_n^n$ for every $n \ge m$. Then $(y_n)_{n\ge m}$ is a subsequence of (x_n) and satisfies the assumptions of Lemma 2.2, therefore $[y_n]$ is an i.p. space.

CASE 2: Assume that $f|K_m$ is continuous for every $m \in \mathbb{N}$. We separate two cases:

SUBCASE 2.1: Assume that there exists a subsequence (y_n) of (x_n) and there exists $m \in \mathbb{N}$ such that

$$\inf_n \|(y_n - f)|K_m\|_{C(K_m)} > 0$$

and therefore for every $m' \ge m$ we have

$$\inf_n \|(y_n - f)|K_{m'}\|_{C(K_{m'})} > 0.$$

Thus for every $m' \ge m$, $((y_n - f)|K_{m'})_n$ is a weakly null seminormalized sequence (by the definition of $K_{m'}$, note that $\|y_n|K_{m'}\|_{C(K_{m'})} \le m'$ for all $n \in \mathbb{N}$).

For every subsequence (z_n) of (y_n) and for every $m' = m, m+1, \ldots$ we find that $((z_n - f)|K_{m'})_n$ is not semiboundedly complete.

Indeed, for every $n \in \mathbb{N}$ we have

$$\begin{aligned}\|[(z_1 - f) - (z_2 - f) + \ldots + (-1)^{n+1}(z_n - f)]|K_{m'}\|_{C(K_{m'})} \\ \le \|[z_1 - z_2 + \ldots + (-1)^{n+1} z_n]|K_{m'}\|_{C(K_{m'})} + \|f|K_{m'}\|_{C(K_{m'})}.\end{aligned}$$

There exists $k \in K_{m'}$ such that

$$\begin{aligned}\|[z_1 - z_2 + \ldots + (-1)^{n+1} z_n]|K_{m'}\|_{C(K_{m'})} &= |(z_1 - z_2 + \ldots + (-1)^{n+1} z_n)(k)| \\ &\le |(z_1 - z_2)(k)| + |(z_3 - z_4)(k)| + \ldots + m' \\ &\le \sum_i |k(x_i - x_{i-1})| + m' \le 2m'.\end{aligned}$$

Thus

$$\sup_n \|[(z_1-f)-(z_2-f)+\ldots+(-1)^{n+1}(z_n-f)] \mid K_{m'}\|_{C(K_{m'})}$$
$$\le 2m' + \|f \mid K_{m'}\|_{C(K_{m'})}.$$

Therefore, the sequence $((z_n-f)|K_{m'})_n$ is not semiboundedly complete since the sequence $((-1)^{n+1})_n$ does not converge to zero.

We claim that for every $m' \ge m$ there exists a sequence $(y_n^{m'})_n$ satisfying:

- (y_n^m) is a subsequence of (y_n).
- $(y_n^{m'+1})_n$ is a subsequence of $(y_n^{m'})_n$.
- $([(y_n^{m'}|K_{m'})_n], \|\cdot\|_{C(K_{m'})})$ is an i.p. Banach space.

Indeed, for $m'=m$, $((y_n-f)|K_m)_n$ is a weakly null seminormalized sequence which does not have any semiboundedly complete subsequence (by Claim B). By Theorem 3.1 there exists a subsequence $(y_n^m)_n$ of (y_n) such that $((y_n^m-f)|K_m)_n$ is equivalent to the unit vector basis of c_0. Thus $([((y_n^m-f)|K_m)_n], \|\cdot\|_{C(K_m)})$ is an i.p. Banach space. Hence $[((y_n^m-f)|K_m)_n]+[f|K_m]$ is an i.p. Banach space, and therefore so is its subspace $[(y_n^m|K_m)_n]$. The proof of the inductive step is a repetition of the same argument. The proof of Claim C is complete and the proof of Subcase 2.1 finishes identically as in Case 1.

SUBCASE 2.2: Assume that for every subsequence (y_n) of (x_n), and for every $m \in \mathbb{N}$, we have

$$\inf_n \|(y_n-f)|K_m\|_{C(K_m)} = 0.$$

It is clear that in this case for every subsequence (y_n) of (x_n), and for every $m \in \mathbb{N}$, we have

$$\inf_{n\neq n'} \|(y_n-y_{n'})|K_m\|_{C(K_m)} = 0.$$

Using Lemma 2.5 find a subsequence (z_n) of (x_n) such that

$$\sum_{n=1}^{\infty} n\|(z_{n+1}-z_n)|K_m\|_{C(K_m)} < \infty, \quad m=1,2,\ldots$$

Since (x_n) is a basic sequence with $\inf_n \|x_n\| > 0$, the sequence of the biorthogonal functionals is bounded:

$$\sup_n \|x_n^*\| = C < \infty.$$

Define

$$v_n = z_{n+1}-z_n, \quad Y = [v_n]_{n=1}^{\infty}, \quad v_n^* = -\sum_{i=1}^{n} z_i^*|Y, \quad n=1,2,\ldots$$

Then (v_n) is a complete minimal system for Y with dual system (v_i^*). We have

$$\|v_n^*\| \le Cn, \quad n=1,2,\ldots$$

It is clear that for each $m \in \mathbb{N}$,

$$\sum_n \|v_n^*\| \cdot \|v_n|K_m\|_{C(K_m)} < \infty$$

and therefore by Lemma 2.4 each K_m is $\|\cdot\|$-precompact (actually, $\|\cdot\|$-compact). Using Lemma 2.3 we conclude that Y is an i.p. space, as well as $[z_n]_{n=1}^{\infty} = Y + [z_1]$. The proof of Theorem 3.2 is complete. ∎

Using Theorem 1 of [F4] we can give an easy proof of the following result weaker than Theorem 1.4.

Remark 1.6. Under the same hypotheses of Theorem 1.4 there exist a sequence $(\varepsilon_n) \in \{\pm 1\}^{\mathbb{N}}$ and an increasing sequence (l_k) of positive integers such that $[(\sum_{i=1}^{l_k} \varepsilon_i(x_i-x_{i-1}))_k]$ is an i.p. space.

Indeed, the proof of Theorem 1 in [F4] shows the following:

Let $(X, \|\cdot\|)$ be a Banach space, $K_1 \subset K_2 \subset \ldots$ be subsets of $\mathrm{Ba}(X^*)$ and let (w_n) be a sequence in X. If (w_n) is basic, $\inf_n \|w_n\| > 0$, $\sum_n \|w_n|K_n\| < \infty$ and $\bigcup_n K_n$ is an isomorphically precisely norming set, then $[w_n]$ is an i.p. Banach space.

Now, the proof of the assertion of the remark can be sketched as follows: If there is no subsequence of (x_n) equivalent to the summing basis, then there exists a sequence $(\varepsilon_n) \in \{\pm 1\}^{\mathbb{N}}$ such that

$$\Big(\sum_{i=1}^{n} \varepsilon_i(x_i-x_{i-1})\Big)_n \text{ is not bounded.}$$

Therefore there exists an increasing sequence (n_k) of integers such that

$$\Big\|\sum_{i=1}^{n_k} \varepsilon_i(x_i-x_{i-1})\Big\| \ge 2^k k, \quad \forall k \in \mathbb{N}.$$

Set $z_k = \sum_{i=1}^{n_k} \varepsilon_i(x_i-x_{i-1})$ for every $k \in \mathbb{N}$. Since (z_k) does not converge in norm, and $(y(z_k))$ is Cauchy for every $y \in \operatorname{span} E$, we deduce (as in the proof of Theorem 1.4) that there exists $z \in X$ (z can also be zero) and an increasing sequence (m_k) of integers such that $(z_{m_k}-z)$ is a basic sequence. Set

$$K_m = \Big\{x^* \in \mathrm{Ba}(X^*) : \sum_{n=1}^{\infty} |x^*(x_n-x_{n-1})| \le m\Big\}, \quad \forall m \in \mathbb{N}$$

(where $x_0=0$). We easily see that

$$\sum_k \Big\|\frac{z_{m_k}-z}{\|z_{m_k}-z\|}\Big| K_k\Big\| < \infty.$$

Thus, by the above mentioned Theorem 1 of [F4] we conclude that

$$\Big[\Big(\sum_{i=1}^{n_{m_k}} \varepsilon_i(x_i-x_{i-1})\Big)_k\Big] \text{ is an i.p. space.}$$

4. Applications. As a first application we strengthen a corollary of Theorem 1.2 which was also proved in a different way by R. Haydon, E. Odell and H. Rosenthal [HOR]. First we need some definitions. Let K be a compact metric space. $B_1(K)$ denotes the class of bounded Baire-1 functions on K, i.e. pointwise limits of uniformly bounded sequences of continuous functions on K. $\mathrm{DSC}(K)$ denotes the space of bounded Differences of SemiContinuous functions on K, i.e.

$$\mathrm{DSC}(K) = \Big\{f : K \to \mathbb{R} : \text{there exists a uniformly bounded sequence } (f_n)_{n=1}^{\infty} \subset C(K) \text{ such that } \lim_n f_n(k) = f(k) \text{ and } \sum_{n=1}^{\infty} |f_{n+1}(k) - f_n(k)| < \infty \text{ for all } k \in K\Big\}.$$

Let f be a non-continuous function on $B_1(K)$ and $\mathcal{C}$ be a non-empty class of Banach spaces. Using the terminology introduced by R. Haydon, E. Odell and H. Rosenthal [HOR], we say that f *governs* $\mathcal{C}$ if for every uniformly bounded sequence (f_n) of continuous functions on K which converges pointwise to f on K, there exists $X \in \mathcal{C}$ which embeds isomorphically in the closed linear span $[f_n]$ of (f_n) equipped with the supremum norm. We say that f *strictly governs* $\mathcal{C}$ if for every uniformly bounded sequence (f_n) of continuous functions on K which converges pointwise to f on K there exists a convex block sequence (g_n) of (f_n) such that the closed linear span $[g_n]$ of (g_n) is isomorphic to some $X \in \mathcal{C}$. A corollary of Theorem 1.2 which was proved in a different way by R. Haydon, E. Odell and H. Rosenthal can be stated as follows:

THEOREM 1.8 ([E2], [HOR]). *Let $f \in \mathrm{DSC}(K) \setminus C(K)$, where K is a compact metric space. Then f governs $\{c_0\}$.*

A generalization of this result is the following:

THEOREM 1.7. *Let $f \in \mathrm{DSC}(K) \setminus C(K)$, where K is a compact metric space. Then f strictly governs the class of (separable) polyhedral Banach spaces.*

To deduce Theorem 1.7 from Theorem 1.4 we need the next well known remark. We first fix some terminology: If A is a subset of a Banach space X then $\widetilde{A}$ denotes the weak* closure of A in X^{**}. Also if A, B are non-empty subsets of $(X, \|\cdot\|)$ then the minimum distance between A and B is defined by

$$\mathrm{md}(A, B) = \inf\{\|a - b\| : a \in A,\ b \in B\}.$$

Remark 4.1. If A, B are convex subsets of a Banach space, then $\mathrm{md}(A, B) = \mathrm{md}(\widetilde{A}, \widetilde{B})$.

Thus, if $f \in \mathrm{DSC}(K) \setminus C(K)$ and (f_n) is a bounded sequence of continuous functions which converges pointwise to f on K, then by Remark 4.1 there exists a convex block sequence (g_n) of (f_n) such that

$$\sum_{n=1}^{\infty} |g_{n+1}(k) - g_n(k)| < \infty, \quad \forall k \in K.$$

Since $f \notin C(K)$, we can also assume (by considering an appropriate subsequence) that (g_n) is a seminormalized basic sequence. Thus Theorem 1.4 gives that some subsequence of (g_n) spans an i.p. Banach space, which proves Theorem 1.7.

As a second application we obtain an Orlicz–Pettis type result:

THEOREM 1.9. *Let (y_n) be a sequence in a Banach space X and let E be an isomorphically precisely norming set for X. If c_0 does not embed isomorphically in the closed linear span $[y_n]$ of (y_n) and*

$$\sum_n |x^*(y_n)| < \infty, \quad \forall x^* \in E,$$

then $\sum_n y_n$ converges unconditionally.

Proof. For $(\eta_i) \in \{\pm 1\}^{\mathbb{N}}$ define the sequence (x_n) by

$$x_n = \sum_{i=1}^{n} \eta_i y_i, \quad \forall n \in \mathbb{N}.$$

We see that the sequence (x_n) satisfies (3). Since c_0 does not embed isomorphically in $[y_n] = [x_n]$, the conclusion of Theorem 1.4 fails. Thus the sequence (x_n) converges in norm. Hence $\sum_n y_n$ converges unconditionally. ■

As a final application of Theorem 1.4 we prove the following immediate corollary which has been proved previously by V. Fonf [F4].

COROLLARY 4.2. *Let X be a Banach space which does not contain an isomorph of c_0. Let A be a subset of X, and let B be an isomorphically precisely norming subset of X^*. If for every $b \in B$ the set $\{b(a) : a \in A\}$ is bounded, then A is bounded.*

Proof. If A is not bounded, we can find a sequence $(a_n) \subset A$ such that $\|a_n\| > 2^n$ for all $n \in \mathbb{N}$. Set

$$\alpha_n = \sum_{i=1}^{n} \frac{a_i}{\|a_i\|}, \quad \forall n \in \mathbb{N}.$$

Thus

$$\sum |b(\alpha_{n+1} - \alpha_n)| < \infty, \quad \forall b \in B.$$

Since X does not contain an isomorph of c_0, by Theorem 1.4 we deduce that (α_n) converges in norm, which is a contradiction. ■

Remark 4.3. It can be proved that if $\|\cdot\|$ is a Gateaux differentiable and locally uniformly convex norm on c_0, and B is an isomorphically precisely norming set for $(c_0, \|\cdot\|)$ then for any $A \subset c_0$ with $\{b(a) : a \in A\}$ bounded for every $b \in B$, the set A is bounded.

References

[BP] C. Bessaga and A. Pełczyński, *On bases and unconditional convergence of series in Banach spaces*, Studia Math. 17 (1958), 151–164.

[D] J. Diestel, *Sequences and Series in Banach Spaces*, Grad. Texts in Math. 92, Springer, 1984.

[DFH] R. Deville, V. Fonf and P. Hájek, *Analytic and polyhedral approximation of convex bodies in separable polyhedral Banach spaces*, preprint.

[E1] J. Elton, Thesis, Yale University, New Haven, Ct.

[E2] —, *Extremely weakly unconditionally convergent series*, Israel J. Math. 40 (1981), 255–258.

[F1] V. P. Fonf, *One property of Lindenstrauss–Phelps spaces*, Functional Anal. Appl. 13 (1979), 66–67.

[F2] —, *Some properties of polyhedral Banach spaces*, Funktsional. Anal. i Prilozhen. 14 (4) (1980), 89–90 (in Russian).

[F3] —, *Polyhedral Banach spaces*, Mat. Zametki 30 (1981), 627–634 (in Russian).

[F4] —, *Weakly extreme properties of Banach spaces*, Math. Notes 45 (1989), 488–494.

[F5] —, *Three characterizations of polyhedral Banach spaces*, Ukrainian Math. J. 42 (1990), 1145–1148.

[HOR] R. Haydon, E. Odell and H. Rosenthal, *On certain classes of Baire-1 functions with applications to Banach space theory*, in: Lecture Notes in Math. 1470, Springer, Berlin, 1991, 1–35.

[KF] M. Kadets and V. Fonf, *Some properties of the set of extreme points of the unit ball of a Banach space*, Math. Notes 20 (1976), 737–739.

[KP] M. I. Kadets and A. Pełczyński, *Basic sequences, biorthogonal systems and norming sets in Banach and Fréchet spaces*, Studia Math. 25 (1965), 297–323.

[K] V. L. Klee, *Polyhedral sections of convex bodies*, Acta Math. 4 (1966), 235–242.

[L] D. Leung, *Some isomorphically polyhedral Orlicz sequence spaces*, Israel J. Math. 87 (1994), 117–128.

[M] V. D. Milman, *Geometric theory of Banach spaces, I, The theory of bases and minimal systems*, Russian Math. Surveys 25 (3) (1970), 111–170.

[R] H. Rosenthal, *Some aspects of the subspace structure of infinite-dimensional Banach spaces*, in: Approximation Theory and Functional Analysis, C. Chuy (ed.), Academic Press, New York, 1991, 151–176.

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The Abel equation and total solvability of linear functional equations

by

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Abstract. We investigate the solvability in continuous functions of the Abel equation $\varphi(Fx) - \varphi(x) = 1$ where F is a given continuous mapping of a topological space X. This property depends on the dynamics generated by F. The solvability of all linear equations $P(x)\psi(Fx) + Q(x)\psi(x) = \gamma(x)$ follows from solvability of the Abel equation in case F is a homeomorphism. If F is noninvertible but X is locally compact then such a total solvability is determined by the same property of the cohomological equation $\varphi(Fx) - \varphi(x) = \gamma(x)$. The smooth situation can also be considered in this way.

1. Introduction. Results and applications. The *Abel equation* (A.e.) is a special kind of functional equation, namely,

$$\varphi(Fx) - \varphi(x) = 1 \quad (x \in X) \tag{1.1}$$

where $F : X \to X$ is a given continuous mapping of a given arbitrary topological space X, and $\varphi : X \to \mathbb{C}$ is an unknown function. N. H. Abel [1] (pp. 36–39) considered this equation on an interval $[0, a) \subset \mathbb{R}$.

We say that (1.1) is *solvable* if this equation has a continuous solution φ. Note that if the A.e. has a solution φ then the real part of φ is also a solution which is continuous since φ is. Therefore the solvability of the A.e. over $\mathbb{C}$ is equivalent to its solvability over $\mathbb{R}$.

Being written in the form

$$\varphi(Fx) = \varphi(x) + 1$$

equation (1.1) means that we have the commutative diagram

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