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Added in proof (October 1997). We learnt in July 1997 that there is some overlap between this article and the work of Ioan Şerb. In particular, the main result of §3, the equivalence of (i) and (ii) in Theorem 3.6, was proved independently by him in *A Day-Nordlander theorem for the tangential modulus of a normed space*, J. Math. Anal. Appl. 209 (1997), 381–391. In *On the behaviour of the tangential modulus of a Banach space II*, Mathematica (Cluj) 38 (61) (1996), 199–207, he proved, earlier than we did, Theorem 2.4(ii).

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On the Djrbashian kernel of a Siegel domain

by

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Abstract. We establish an inversion formula for the M. M. Djrbashian & A. H. Karapetyan integral transform (cf. [6]) on the Siegel domain $\Omega_n = \{\zeta \in \mathbb{C}^n : \varrho(\zeta) > 0\}$, $\varrho(\zeta) = \mathrm{Im}(\zeta_1) - |\zeta'|^2$. We build a family of Kähler metrics of constant holomorphic curvature whose potentials are the ϱ^α-Bergman kernels, $\alpha > -1$, (in the sense of Z. Pasternak-Winiarski [20]) of Ω_n. We build an anti-holomorphic embedding of Ω_n in the complex projective Hilbert space $\mathbb{CP}(H^2_\alpha(\Omega_n))$ and study (in connection with work by A. Odzijewicz [18]) the corresponding transition probability amplitudes. The Genchev transform (cf. [9]) is shown to be well defined on $L^2(\Omega, \varrho^\alpha)$, for any strip $\Omega \subset \mathbb{C}$, and applied in a problem of approximation by holomorphic functions. Building on work by T. Mazur (cf. [15]) we prove the existence of a complete orthonormal system in $H^2_\alpha(\Omega_n)$ consisting of eigenfunctions of a certain explicitly defined operator V_a, $a \in B_n$.

1. Introduction. Let $\Omega \subset \mathbb{C}^n$ be an open set, $\Omega \neq \emptyset$. Let $W(\Omega)$ be the set of all *weights* on Ω (i.e. $\gamma \in W(\Omega)$ is a Lebesgue measurable function $\gamma : \Omega \to (0,\infty)$). For each $\gamma \in W(\Omega)$ let $L^2H(\Omega,\gamma)$ be the Hilbert space of all functions $f : \Omega \to \mathbb{C}$ for which $\|f\|_\gamma = (\int_\Omega |f|^2\gamma\, dm)^{1/2} < \infty$, where dm is the Lebesgue measure in $\mathbb{R}^{2n}$. Let $L^2H(\Omega,\gamma)$ be the set of all functions in $L^2(\Omega,\gamma)$ which are holomorphic in Ω. A weight $\gamma \in W(\Omega)$ is *admissible* if for any $z \in \Omega$ there is a neighbourhood V_z of z in Ω and a constant $C_z > 0$ so that $\|\delta_w\|_\gamma \le C_z$ for any $w \in V_z$ (cf. [19], p. 112). Here $\delta_z(f) = f(z)$, $f \in L^2H(\Omega,\gamma)$. The set of all admissible weights on Ω is denoted by $AW(\Omega)$. If $\gamma \in AW(\Omega)$ then (cf. Proposition 2.1 of [19], p. 113) $L^2H(\Omega,\gamma)$ is a closed subspace of $L^2(\Omega,\gamma)$ and the evaluation functional δ_z is continuous on $L^2H(\Omega,\gamma)$ for any $z \in \Omega$. Hence, by the Riesz representation theorem, there is a unique function $K_\gamma(\cdot,z) \in L^2H(\Omega,\gamma)$ (called the γ-*Bergman kernel* of Ω) so that

$$f(z) = \int_\Omega f(\zeta)\overline{K_\gamma(\zeta,z)}\gamma(\zeta)\, dm(\zeta)$$

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for any $f \in L^2H(\Omega,\gamma)$ and $z \in \Omega$. For $\gamma \equiv 1$ this is the Bergman kernel $K(\zeta, z)$ of Ω (cf. [2]–[3]). The main properties of γ-Bergman kernels have been investigated by Z. Pasternak-Winiarski (cf. [19]–[20]). His approach (as in classical complex analysis, e.g. [10], pp. 365–369) is based on the representation

$$K_\gamma(\zeta, z) = \sum_m \phi_m(\zeta)\overline{\phi_m(z)}$$

for any complete orthonormal system $\{\phi_m\}$ in $L^2H(\Omega,\gamma)$. The main inconvenience of this method is that (even in the simplest cases, e.g. [20], pp. 8–13, for $\Omega = B_1 = \{z \in \mathbb{C} : |z| < 1\}$ and the admissible weight $\gamma(z) = (\operatorname{Im} z)^2$) complete orthonormal systems are rather difficult to produce.

In the present paper, we look at the family of weights

$$\gamma_\alpha(\zeta) = (\operatorname{Im}\zeta_1 - |\zeta'|^2)^\alpha, \quad \alpha > -1,$$

on the Siegel domain $\Omega_n = \{\zeta \in \mathbb{C}^n : \operatorname{Im}\zeta_1 > |\zeta'|^2\}$. These turn out to be admissible and we write explicitly the γ_α-Bergman kernel of Ω_n. Note that $L^2H(\Omega_n,\gamma_\alpha)$ are precisely the function spaces $H^2_\alpha(\Omega_n)$ introduced in [6]. Our viewpoint is to make use of the representation theory of holomorphic functions (rather than of complete orthonormal systems in $H^2_\alpha(\Omega_n)$).

Using a result of S. Saitoh [22], we endow $H^2_\alpha(\Omega_n)$ with a complex 1-parameter family $(\ ,\ )_{H(K_\beta)}$, $\operatorname{Re}\beta > (\alpha-1)/2$, of inner products (in general not isometric to the $L^2(\Omega_n,\gamma_\alpha)$ inner product) and prove an inversion formula (cf. Theorem 1) for the Djrbashian–Karapetyan transform (1).

For any bounded domain $\Omega \subset \mathbb{C}^n$ there is a natural Kählerian metric on Ω of potential $K(z,z)$ (the *Bergman metric* of Ω). Although the arguments leading to the Bergman metric (cf. Proposition 3.4 of [10], pp. 368–369) break down for the case of an unbounded domain, we show (by using a result of T. Mazur [16]) that the γ_α-Bergman kernel of Ω_n gives rise to a Kählerian metric g_α on Ω_n of constant (negative) holomorphic curvature $-8\pi^n[(\alpha+1)\ldots(\alpha+n+1)]^{-1}$ (cf. Theorem 2).

In connection with work by A. Odzijewicz [18], we show that there is an anti-holomorphic embedding of Ω_n into the complex projective Hilbert space $\mathbb{CP}(H^2_\alpha(\Omega_n))$, hence one may introduce the transition probability amplitude $a_\alpha(\zeta,z)$ from ζ to z ($\zeta, z \in \Omega_n$), and establish (9) (cf. Section 4 for its interpretation).

The authors are grateful to the referee for several remarks which improved the first version of this paper, and in particular for drawing their attention to the work by M. Skwarczyński [24]–[25]. Indeed, one was able to show that, for a given strip $\Omega = \{z \in \mathbb{C} : b < \operatorname{Im} z < c\}$, the Genchev transform (cf. [9]) is well defined on $L^2H(\Omega,\gamma_\alpha)$, $\alpha > -1$, and furthermore elements of $L^2H(\Omega,\gamma_2)$ which are approximated by holomorphic functions in $H^2_2(\Omega_1)$ may be characterized in terms of the Genchev transform (cf. Theorem 5).

Building on work by T. Mazur [15], we prove the existence of a complete orthonormal system in $H^2_\alpha(\Omega_n)$ consisting of eigenfunctions of a certain explicitly defined operator V_a, $a \in B_n$ (cf. Theorem 6).

2. A reproducing kernel Hilbert space. If $\zeta = (\zeta_1,\ldots,\zeta_n) \in \mathbb{C}^n$ we set $\zeta' = (\zeta_2,\ldots,\zeta_n)$. Let $\alpha \in \mathbb{R}$, $\alpha > -1$, and $\beta \in \mathbb{C}$, $\operatorname{Re}\beta > (\alpha-1)/2$. Consider the linear operator T_β given by

$$(1)\qquad (T_\beta f)(w) = 2^{n-1+\beta} c_{n,\beta} \int_{\Omega_n} \frac{f(\zeta)(\operatorname{Im}\zeta_1 - |\zeta'|^2)^\beta\, dm(\zeta)}{[i(\overline{\zeta}_1 - w_1) - 2\langle w', \zeta'\rangle]^{n+1+\beta}}$$

for any $f \in L^2_\alpha(\Omega_n) = L^2(\Omega_n,\gamma_\alpha)$ and $w \in \Omega_n$ (cf. (2.15) in [6], p. 98). Here $c_{n,\beta} = \pi^{-n}(\beta+1)\ldots(\beta+n)$. By Theorems 2.1 and 3.1 of [6], T_β is a continuous linear operator from $L^2_\alpha(\Omega_n)$ onto $H^2_\alpha(\Omega_n)$ (referred to hereafter as the *Djrbashian–Karapetyan transform*). We shall need the following:

LEMMA 1. *For any $z, \zeta \in \Omega_n$ set*

$$(2)\qquad h_z(\zeta) = 2^{n-1+\overline{\beta}} \frac{\overline{c}_{n,\beta}(\operatorname{Im}\zeta_1 - |\zeta'|^2)^{\overline{\beta}-\alpha}}{[i(\overline{z}_1 - \zeta_1) - 2\langle\zeta', z'\rangle]^{n+1+\overline{\beta}}}.$$

Then $h_z \in L^2_\alpha(\Omega_n)$.

Proof. We have

$$\begin{aligned}\|h_z\|^2_{2,\alpha} &= \int_{\Omega_n} |h_z(\zeta)|^2(\operatorname{Im}\zeta_1 - |\zeta'|^2)^\alpha\, dm(\zeta)\\ &= |2^{n-1+\beta}c_{n,\beta}|^2 \int_{\Omega_n} \frac{(\operatorname{Im}\zeta_1 - |\zeta'|^2)^{2(\operatorname{Re}\beta-\alpha)+\alpha}}{|i(\overline{\zeta}_1 - z_1) - 2\langle z', \zeta'\rangle|^{2(n+1+\operatorname{Re}\beta)}}\\ &\qquad \times \exp(2\operatorname{Im}(\beta)\arg(i(\overline{\zeta}_1 - z_1) - 2\langle z', \zeta'\rangle))\, dm(\zeta)\\ &\le \text{const}\cdot e^{2\pi|\operatorname{Im}\beta|} \int_{\Omega_n} \frac{(\operatorname{Im}\zeta_1 - |\zeta'|^2)^{2\operatorname{Re}\beta-\alpha}\, dm(\zeta)}{|i(\overline{\zeta}_1 - z_1) - 2\langle z', \zeta'\rangle|^{2(n+1+\operatorname{Re}\beta)}}.\end{aligned}$$

By Lemma 2.2 of R. R. Coifman & R. Rochberg [4], if $t > -1$ and $c > 0$ then an integral of the form

$$J_{t,c}(z) = \int_{\Omega_n} \frac{(\operatorname{Im}\zeta_1 - |\zeta'|^2)^t\, dm(\zeta)}{[i(\overline{\zeta}_1 - z_1) - 2\langle z', \zeta'\rangle]^{n+1+t+c}}$$

may be computed as

$$J_{t,c}(z) = \frac{\text{const}}{(\operatorname{Im} z_1 - |z'|^2)^c}$$

where the constant depends only on n, t and c. To end the proof of Lemma 1, set $t = 2\operatorname{Re}\beta - \alpha$ and $c = n+1+\alpha$. Then $t > -1$, $c > 0$ and we may conclude that

$$\|h_z\|_{2,\alpha}^2 \le \frac{\text{const}}{(\operatorname{Im} z_1 - |z'|^2)^{n+1+\alpha}} < \infty$$

for any $z \in \Omega_n$.

S. Saitoh has devised (cf. Theorem 2.1 of [22], p. 75) a fairly general method for organizing the range of a linear operator (induced by a Hilbert space valued function) as a Hilbert space with reproducing kernel (in the sense of [1]). We briefly recall its essentials and apply it to the Djrbashian–Karapetyan transform.

Let $E \neq \emptyset$ be a set and $\mathcal{F}(E)$ the linear space of all functions $f : E \to \mathbb{C}$. Let $(\mathcal{H}, (\ ,\ )_{\mathcal{H}})$ be a Hilbert space. Given a function $h : E \to \mathcal{H}$ consider the linear map $L : \mathcal{H} \to \mathcal{F}(E)$ given by $(LF)(p) = (F, h(p))_{\mathcal{H}}$ for any $F \in \mathcal{H}$ and $p \in E$. The range $\mathcal{R}(L)$ of L is a Hilbert space with the inner product $(f, g)_{\mathcal{R}(L)} = (PF, PG)_{\mathcal{H}}$ for some $F \in L^{-1}(f)$ and $G \in L^{-1}(g)$. Here $P : \mathcal{H} \to \mathcal{H} \ominus \mathcal{N}(L)$ is the natural projection and $\mathcal{N}(L)$ the null space of L. Then $\|f\|_{\mathcal{R}(L)} = \inf\{\|F\|_{\mathcal{H}} : F \in L^{-1}(f)\}$ and $K(p, q) = (h(q), h(p))_{\mathcal{H}}$ is a reproducing kernel for $\mathcal{R}(L)$. Also L is an isometry of $\mathcal{H}$ onto $\mathcal{R}(L)$ iff $\{h(p) : p \in E\}$ is complete in $\mathcal{H}$. Cf. also [23], p. 51.

Set $K_\beta(\zeta, z) = (h_z, h_\zeta)_{2,\alpha}$ (by Lemma 1, K_β is well defined). Let $P_\beta : L_\alpha^2(\Omega_n) \to L_\alpha^2(\Omega_n) \ominus \mathcal{N}(T_\beta)$ be the orthogonal projection. Note that $P_\beta h_z = h_z$ for any $z \in \Omega_n$. As $(T_\beta f)(\zeta) = (f, h_\zeta)_{2,\alpha}$ for any $f \in L_\alpha^2(\Omega_n)$ and $\zeta \in \Omega_n$, it follows that (i) $K_\beta(\cdot, \zeta) \in \mathcal{R}(T_\beta)$ and (ii) $F(\zeta) = (F, K_\beta(\cdot, \zeta))_{\mathcal{R}(T_\beta)}$. Then $\mathcal{R}(T_\beta) = H_\alpha^2(\Omega_n)$ (thought of as a Hilbert space with the reproducing kernel K_β) will be denoted by $H(K_\beta)$. On the other hand, by a result of M. M. Djrbashian & A. H. Karapetyan (cf. Proposition 4.3 of [6], p. 107), $\mathcal{N}(T_\alpha) = L_\alpha^2(\Omega_n) \ominus H_\alpha^2(\Omega_n)$, hence $H_\alpha^2(\Omega_n)$ is a closed subspace of $L_\alpha^2(\Omega_n)$.

PROPOSITION 1. *$H(K_\beta) = H_\alpha^2(\Omega_n)$, i.e. the identity is an isometry, if and only if $\mathcal{N}(T_\beta) = \mathcal{N}(T_\alpha)$.*

Proof. Let $Q_\beta : L_\alpha^2(\Omega_n) \to \mathcal{N}(T_\beta)$ be the orthogonal projection. If $(F, G)_{H(K_\beta)} = (F, G)_{2,\alpha}$ for any $F, G \in H_\alpha^2(\Omega_n)$ then $Q_\beta F = 0$ (because, by Theorem 2.1 of [6], T_β reproduces the holomorphic functions), hence $H_\alpha^2(\Omega_n) \subseteq L_\alpha^2(\Omega_n) \ominus \mathcal{N}(T_\beta)$. Conversely, let $f \in L_\alpha^2(\Omega_n) \ominus \mathcal{N}(T_\beta)$ and set $F = T_\beta f$. Then $F \in H_\alpha^2(\Omega_n)$, hence $f - F \in L_\alpha^2(\Omega_n) \ominus \mathcal{N}(T_\beta)$. Finally, note that $T_\beta(f - F) = 0$. ■

Let $F \in H_\alpha^2(\Omega_n)$. Then $\|F\|_{H(K_\beta)} \le \|f\|_{2,\alpha}$ for any $f \in L_\alpha^2(\Omega_n)$ with $T_\beta f = F$. Next (by Theorem 2.1 of [6]), $T_\beta F = F$. Yet (in view of Proposition 1) in general F is not the element of minimum $\|\ \|_{2,\alpha}$ norm in the fibre of T_β over F.

By Lemma 1 we may define $h_\beta : \Omega_n \times \Omega_n \to \mathbb{C}$ by setting $h_\beta(\zeta, z) = h_z(\zeta)$ where h_z is given by (2). We refer to $h_\beta(\zeta, z)$ as the *Djrbashian kernel* of Ω_n.

We adopt the following notations. Let $r > 0$ and $B_{n,r} = \{z \in \mathbb{C}^n : |z| < r\}$. Let $\varphi : B_n \to \Omega_n$ be the *Cayley transform*, i.e.

$$\varphi(z_1, \ldots, z_n) = \left(i\frac{1 + z_1}{1 - z_1}, i\frac{z_2}{1 - z_1}, \ldots, i\frac{z_n}{1 - z_1}\right),$$

and set $\Omega_{n,r} = \varphi(B_{n,r})$. We now state the following:

THEOREM 1. *Let $\alpha > -1$ and $\beta \in \mathbb{C}$, $\operatorname{Re}\beta > (\alpha - 1)/2$. Then $H_\alpha^2(\Omega_n)$ is a Hilbert space $H(K_\beta)$ with the reproducing kernel*

$$(3)\quad K_\beta(\zeta, z) = |2^{n-1+\beta} c_{n,\beta}|^2 \times \int_{\Omega_n} \frac{(\operatorname{Im}\omega_1 - |\omega'|^2)^{2\operatorname{Re}\beta - \alpha}\, dm(\omega)}{[i(\overline{\omega}_1 - \zeta_1) - 2\langle \zeta', \omega'\rangle]^{n+1+\beta}[i(\overline{z}_1 - \omega_1) - 2\langle \omega', z'\rangle]^{n+1+\overline{\beta}}}$$

Let $(r_N)_{N \ge 1}$ be a sequence of positive numbers so that $r_N \uparrow 1$ as $N \to \infty$. Set $D_N = \Omega_{n,r_N}$, $N \ge 1$. For any $F \in H_\alpha^2(\Omega_n)$ the unique $f^* \in L_\alpha^2(\Omega_n)$ so that $T_\beta f^* = F$ and $\|F\|_{H(K_\beta)} = \|f^*\|_{2,\alpha}$ is given by

$$(4)\qquad f^*(\zeta) = \lim_{N \to \infty} \int_{D_N} F(z) h_\beta(\zeta, z)(\operatorname{Im} z_1 - |z'|^2)^\alpha\, dm(z)$$

in the sense of $L_\alpha^2(\Omega_n)$ convergence.

Proof. Set

$$d\mu_\alpha(\zeta) = (\operatorname{Im}\zeta_1 - |\zeta'|^2)^\alpha\, dm(\zeta)$$

for simplicity. Note that $\{D_N\}_{N \ge 1}$ is an exhaustion of Ω_n with μ_α-measurable sets satisfying (i) $D_1 \subset D_2 \subset \ldots$, and (ii) $\bigcup_{N=1}^\infty D_N = \Omega_n$. The unique $f^* \in L_\alpha^2(\Omega_n)$ in the statement of Theorem 1 is $f^* = P_\beta F$. By a result of S. Saitoh (cf. Theorem 4.3 of [23], p. 56) to prove (4) one needs to check that

$$(5)\qquad \int_{D_N} K_\beta(\zeta, \zeta)\, d\mu_\alpha(\zeta) < \infty$$

and

$$(6)\qquad \int_{D_N} F(z) h_\beta(\cdot, z) d\mu_\alpha(z) \in L_\alpha^2(\Omega_n)$$

for any $N \ge 1$. To prove (5) note that by (3),

$$K_\beta(\zeta, \zeta) = |2^{n+1-\beta} c_{n,\beta}|^2 \int_{\Omega_n} \frac{(\operatorname{Im}\omega_1 - |\omega'|^2)^{2\operatorname{Re}\beta - \alpha}\, dm(\omega)}{|(i(\overline{\omega}_1 - \zeta_1) - 2\langle\zeta', \omega'\rangle)^{n+1+\beta}|^2}$$

$$= |2^{n+1-\beta} c_{n,\beta}|^2 \int_{\Omega_n} \frac{(\operatorname{Im}\omega_1 - |\omega'|^2)^{2\operatorname{Re}\beta - \alpha}}{|i(\overline{\omega}_1 - \zeta_1) - 2\langle\zeta', \omega'\rangle|^{2(n+1+\operatorname{Re}\beta)}} \times \exp(2\operatorname{Im}(\beta)\arg(i(\overline{\omega}_1 - \zeta_1) - 2\langle\zeta', \omega'\rangle))\, dm(\omega)$$

$$\le e^{2\pi|\operatorname{Im}\beta|}2^{2(n+1-\operatorname{Re}\beta)}|c_{n,\beta}|^2 \int_{\Omega_n} \frac{(\operatorname{Im}\omega_1 - |\omega'|^2)^{2\operatorname{Re}\beta-\alpha}\, dm(\omega)}{|i(\overline{\omega}_1 - \zeta_1) - 2\langle\zeta',\omega'\rangle|^{2(n+1+\operatorname{Re}\beta)}},$$

hence

$$K_\beta(\zeta,\zeta) \le \text{const}\cdot J_{t,c}(\zeta)$$

with $t = 2\operatorname{Re}\beta - \alpha$ and $c = n+1+\alpha$. Again by Lemma 2.2 of R. R. Coifman & R. Rochberg [4], the integral $J_{t,c}(\zeta)$ may be explicitly computed (as $t > -1$, $c > 0$) so that

$$K_\beta(\zeta,\zeta) \le \frac{\text{const}}{(\operatorname{Im}\zeta_1 - |\zeta'|^2)^{n+1+\alpha}}. \tag{7}$$

LEMMA 2.

$$\int_{\Omega_{n,r}} \frac{dm(\zeta)}{|\overline{\zeta}_1 - i|^{2(n+1)}} = 4^{-n} m(B_{n,r}).$$

Proof. Set $\zeta = \varphi(z)$ and recall that the complex Jacobian of the Cayley transform is $J_\varphi(z) = 2i^n(1-z_1)^{-(n+1)}$.

To end the proof of (5) note that

$$1 - |\varphi^{-1}(\zeta)|^2 = \frac{4(\operatorname{Im}\zeta_1 - |\zeta'|^2)}{|\overline{\zeta}_1 - i|^2}$$

for any $\zeta \in \Omega_n$. Also,

$$\frac{1}{(1-|\varphi^{-1}(\zeta)|^2)^{n+1}} < \frac{1}{(1-r^2)^{n+1}}$$

for any $\zeta \in \Omega_{n,r}$. Using (7) and Lemma 2 we may perform the estimates

$$\begin{aligned}\int_{\Omega_{n,r}} K_\beta(\zeta,\zeta)\, d\mu_\alpha(\zeta) &\le \text{const}\cdot \int_{\Omega_{n,r}} \frac{dm(\zeta)}{(\operatorname{Im}\zeta_1 - |\zeta'|^2)^{n+1}} \\ &= \text{const}\cdot \int_{\Omega_{n,r}} \frac{4^{n+1}\, dm(\zeta)}{|\overline{\zeta}_1 - i|^{2(n+1)}(1-|\varphi^{-1}(\zeta)|^2)^{n+1}} \\ &< \frac{\text{const}}{(1-r^2)^{n+1}} \int_{\Omega_{n,r}} \frac{dm(\zeta)}{|\overline{\zeta}_1 - i|^{2(n+1)}} \\ &= \text{const}\cdot \frac{m(B_{n,r})}{(1-r^2)^{n+1}} < \infty.\end{aligned}$$

Next, to prove (6) we perform the estimates

$$\begin{aligned}&\int_{\Omega_n} \Big| \int_{D_N} F(z) h_\beta(\zeta,z)\, d\mu_\alpha(z)\Big|^2 d\mu_\alpha(\zeta) \\ &\quad\le \int_{\Omega_n} \Big[\int_{D_N} |F(z)|^2\, d\mu_\alpha(z)\Big]\Big[\int_{D_N} |h_\beta(\zeta,z)|^2\, d\mu_\alpha(z)\Big] d\mu_\alpha(\zeta)\end{aligned}$$

$$\begin{aligned}&\le \|F\|^2_{2,\alpha} \int_{D_N} \Big[\int_{\Omega_n} |h_z(\zeta)|^2\, d\mu_\alpha(\zeta)\Big] d\mu_\alpha(z) \\ &= \|F\|^2_{2,\alpha} \int_{D_N} \|h_z\|^2_{2,\alpha}\, d\mu_\alpha(z) \\ &\le \text{const}\cdot \|F\|^2_{2,\alpha} \int_{D_N} \frac{d\mu_\alpha(z)}{(\operatorname{Im} z_1 - |z|')^{n+1+\alpha}} \\ &< \text{const}\cdot \|F\|^2_{2,\alpha} m(B_{n,r_N})(1-r_N^2)^{-(n+1)} < \infty.\end{aligned}$$

3. The γ_α-Bergman kernel. Recall that $H^2_\alpha(\Omega_n)$ is closed in $L^2_\alpha(\Omega_n)$. On the other hand,

$$|\delta_z F| = |(T_\beta F)(z)| = |(F,h_z)_{2,\alpha}| \le \|F\|_{2,\alpha}\|h_z\|_{2,\alpha}$$

so that the evaluation functional $\delta_z : H^2_\alpha(\Omega_n) \to \mathbb{C}$ is continuous. Thus (by Theorem 2.2 of [20], p. 4), $\gamma_\alpha \in AW(\Omega_n)$. In view of

$$(T_\beta f)(z) = \int_{\Omega_n} f(\zeta)\overline{h_\beta(\zeta,z)}\, d\mu_\alpha(\zeta)$$

and of Theorem 2.1 of [6], p. 101, the γ_α-Bergman kernel of Ω_n may be identified among the Djrbashian kernels $h_\beta(\zeta,z)$, $\operatorname{Re}\beta > (\alpha-1)/2$, as the one corresponding to $\beta = \alpha$. Indeed,

$$h_\alpha(\zeta,z) = \frac{2^{n-1+\alpha} c_{n,\alpha}}{[i(\overline{z}_1 - \zeta_1) - 2\langle\zeta',z'\rangle]^{n+1+\alpha}}$$

is holomorphic in ζ and hence, by the uniqueness statement in the Riesz representation theorem, $h_\alpha(\zeta,z)$ is the γ_α-Bergman kernel of Ω_n. Also, again because of $\overline{\partial}_\zeta h_\alpha(\zeta,z) = 0$, and by the reproducing property of $K_\alpha(\zeta,z)$, we actually have $K_\alpha(\zeta,z) = h_\alpha(\zeta,z)$, $\alpha > -1$. Indeed, as for $\beta = \alpha$ one has $h_z \in H^2_\alpha(\Omega_n)$, it follows that

$$K_\alpha(\zeta,z) = (h_z,h_\zeta)_{2,\alpha} = (T_\alpha h_z)(\zeta) = h_z(\zeta) = h_\alpha(\zeta,z).$$

Let g_α be the real $(0,2)$-tensor field on Ω_n given by

$$g_\alpha = \operatorname{Re}\{L_{\alpha|\mathcal{X}(\Omega_n)\times\mathcal{X}(\Omega_n)}\}$$

where

$$L_\alpha = \sum_{1\le j,k\le n} \frac{\partial^2 \log K_\alpha(z,z)}{\partial z_j \partial \overline{z}_k} dz_j \otimes d\overline{z}_k$$

and $\mathcal{X}(\Omega_n)$ is the $C^\infty(\Omega_n)$-module of all tangent vector fields on Ω_n. We now state the following:

THEOREM 2. *Let $\alpha > -1$ and consider the weights $\gamma_\alpha \in W(\Omega_n)$ given by $\gamma_\alpha(\zeta) = (\operatorname{Im}\zeta_1 - |\zeta'|^2)^\alpha$, $\zeta \in \Omega_n$. Then each γ_α is admissible and the* *corresponding* γ_α*-Bergman kernel of* Ω_n *is*

$$K_\alpha(\zeta,z)=\frac{2^{n-1+\alpha}c_{n,\alpha}}{[i(\overline{z}_1-\zeta_1)-2\langle\zeta',z'\rangle]^{n+1+\alpha}}. \tag{8}$$

Also,

$$(g_\alpha)_{j\overline{k}}=\frac{\partial^2\log K_\alpha(z,z)}{\partial z_j\partial\overline{z}_k}$$

defines a family $\{g_\alpha\}_{\alpha>-1}$ *of Kähler metrics of constant (negative) holomorphic sectional curvature* $-8/(\pi c_{n+1,\alpha})$.

Proof. The arguments leading to Theorem 1 of [7], p. 151, fail to apply (because Ω_n is unbounded). We restate a result of [16], p. 135, as it applies to our situation:

LEMMA 3. *Let* $\alpha>-1$. *Assume that* (i) *for any* $\zeta\in\Omega_n$ *there is* $F\in H^2_\alpha(\Omega_n)$ *so that* $F(\zeta)\neq 0$, *and* (ii) *for any* $\zeta\in\Omega_n$ *and any* $Z\in T^{1,0}(\Omega_n)_\zeta$, $Z\neq 0$, *there is* $F\in H^2_\alpha(\Omega_n)$ *so that* $F(\zeta)=0$ *and* $Z(F)\neq 0$. *Then* g_α *is a Kählerian metric on* Ω_n.

Cf. also S. Kobayashi [12], p. 271, and M. Skwarczyński [24], p. 18. Here $T^{1,0}(\Omega_n)$ is the holomorphic tangent bundle over Ω_n (i.e. the span of $\partial/\partial z_j$, $1\leq j\leq n$).

To check that (i)–(ii) of Lemma 3 do hold in our case, we state:

LEMMA 4. *Let* $\zeta_0\in\Omega_n$ *and* $z_0=\varphi^{-1}(\zeta_0)$. *Fix* $w\in\mathbb{C}^n-\{0\}$ *and consider the holomorphic function* $f:B_n\to\mathbb{C}, f(z)=\langle z-z_0,w\rangle$, $z\in B_n$. *Let* $g(\zeta)=f(\varphi^{-1}(\zeta))(\zeta_1+i)^{-(n+1+\alpha)}$, $\zeta\in\Omega_n$. *Then* $g\in H^2_\alpha(\Omega_n)$.

Proof. Clearly $\overline{\partial}g=0$. Moreover,

$$\begin{aligned}\int_{B_n}|f(z)|^2(1-|z|^2)^\alpha\,dm(z)&\leq\int_{B_n}|z-z_0|^2|w|^2(1-|z|^2)^\alpha\,dm(z)\\&\leq 4|w|^2\int_{B_n}(1-|z|^2)^\alpha\,dm(z)\\&=4|w|^2\int_0^1 d\varrho\int_{|z|=\varrho}(1-|z|^2)^\alpha\,dS_z\\&=4\omega_{2n}|w|^2\int_0^1\varrho^{2n-1}(1-\varrho^2)^\alpha\,d\varrho\end{aligned}$$

where ω_{2n} is the measure of $S^{2n-1}\subset\mathbb{C}^n$. As $\alpha>-1$ the last integral is convergent, hence $\int_{B_n}|f(z)|^2(1-|z|^2)^\alpha\,dm(z)<\infty$. Thus (by 2) of Lemma 1.2 in [6], p. 95), $g\in L^2_\alpha(\Omega_n)$. ∎

The function $g\in H^2_\alpha(\Omega_n)$ furnished by Lemma 4 satisfies $g(\zeta_0)=0$. Given $Z\in T^{1,0}(\Omega_n)_{\zeta_0}$, $Z=\sum_{j=1}^n\lambda_j(\partial/\partial z_j)_{\zeta_0}$, we have to choose $w\in\mathbb{C}^n-\{0\}$ so that $Z(g)\neq 0$. Since $\varphi^{-1}:\Omega_n\to B_n$ is given by

$$\varphi^{-1}(\zeta_1,\ldots,\zeta_n)=\left(\frac{\zeta_1-i}{\zeta_1+i},\frac{2\zeta_2}{\zeta_1+i},\ldots,\frac{2\zeta_n}{\zeta_1+i}\right)$$

we have

$$Z(g)=\frac{2\lambda_1}{(\zeta_{0,1}+i)^{n+3+\alpha}}[i\overline{w}_1-\langle\zeta_0',w'\rangle]+\frac{2}{(\zeta_{0,1}+i)^{n+1+\alpha}}\langle\lambda',w'\rangle.$$

At this point we choose $w'=\lambda'$ and $w_1=i(\overline{\zeta}_{0,1}-i)^2\lambda_1+i\langle\lambda',\zeta_0'\rangle$ so that

$$Z(g)=\frac{2}{(\zeta_{0,1}+i)^{n+1+\alpha}}|\lambda|^2,$$

hence $Z\neq 0$ yields $Z(g)\neq 0$ and (ii) of Lemma 3 is checked.

Finally, (i) follows from

LEMMA 5. *Let* $\zeta_0\in\Omega_n$ *and* $z_0=\varphi^{-1}(\zeta_0)$. *Fix* $w\in\mathbb{C}^n-\{0\}$ *and set*

$$f(z)=\begin{cases}\langle z+z_0,z_0\rangle & \text{if } z_0\neq 0,\\ \langle z+w,w\rangle & \text{if } z_0=0,\end{cases}\qquad z\in B_n,$$

$$g(\zeta)=\frac{f(\varphi^{-1}(\zeta))}{(\zeta_1+i)^{n+1+\alpha}}.$$

Then $g\in H^2_\alpha(\Omega_n)$ *and* $g(\zeta_0)\neq 0$.

The proof is similar to that of Lemma 4 and thus omitted. P. F. Klembeck [11] has computed the curvature of the Bergman metric of a bounded domain near its boundary, by using Fefferman's asymptotic formula for the Bergman kernel. While this is not available for Ω_n and K_α, we may (due to the explicit expression (8) of K_α) perform a direct calculation of the curvature tensor $(R_\alpha)_{j\overline{k}l\overline{m}}$ of the Kähler metric $(g_\alpha)_{j\overline{k}}$. It is given by

$$\begin{aligned}-\frac{1}{2}(R_\alpha)_{j\overline{k}l\overline{m}}&=(g_\alpha)_{j\overline{k}}(g_\alpha)_{l\overline{m}}+(g_\alpha)_{j\overline{m}}(g_\alpha)_{l\overline{k}}\\&\quad-K_\alpha^{-2}\{K_\alpha(K_\alpha)_{j\overline{k}l\overline{m}}-(K_\alpha)_{jl}(K_\alpha)_{\overline{k}\overline{m}}\}\\&\quad+K_\alpha^{-4}(g_\alpha)^{\overline{p}q}\{K_\alpha(K_\alpha)_{jl\overline{p}}-(K_\alpha)_{jl}(K_\alpha)_{\overline{p}}\}\\&\quad\times\{K_\alpha(K_\alpha)_{\overline{k}\overline{m}q}-(K_\alpha)_{\overline{k}\overline{m}}(K_\alpha)_q\}\end{aligned}$$

where K_α is short for $K_\alpha(z,z)$ (we adopt the conventions of [12], p. 275). Yet (by (8)) we have $-\partial^2(\log\varrho)/\partial z_j\partial\overline{z}_k=4[c_{n,\alpha}(n+1+\alpha)]^{-1}\partial^2(\log K_\alpha)/\partial z_j\partial\overline{z}_k$ where $\varrho(z)=\operatorname{Im}z_1-|z'|^2$. Hence the curvature of $(g_\alpha)_{j\overline{k}}$ will be $c_{n,\alpha}(n+1+\alpha)/2$ times the tensor

$$\begin{aligned}R_{j\overline{k}l\overline{m}}&=h_{j\overline{k}}h_{l\overline{m}}+h_{j\overline{m}}h_{l\overline{k}}-\varrho^{-2}\{\varrho\varrho_{j\overline{k}l\overline{m}}-\varrho_{jl}\varrho_{\overline{k}\overline{m}}\}\\&\quad+\varrho^{-4}h^{\overline{p}q}\{\varrho\varrho_{jl\overline{p}}-\varrho_{jl}\varrho_{\overline{p}}\}\{\varrho\varrho_{\overline{k}\overline{m}q}-\varrho_{\overline{k}\overline{m}}\varrho_q\}\end{aligned}$$

with $h_{j\overline{k}} = \partial^2(\log \varrho)/\partial z_j \partial \overline{z}_k$. Therefore

$$(R_\alpha)_{j\overline{k}l\overline{m}} = \frac{4}{c_{n,\alpha}(n+1+\alpha)}\{(g_\alpha)_{j\overline{k}}(g_\alpha)_{l\overline{m}} - (g_\alpha)_{j\overline{m}}(g_\alpha)_{l\overline{k}}\},$$

hence g_α is a Kähler metric of constant holomorphic curvature $-8/[c_{n,\alpha}(n+1+\alpha)]$. In particular, g_α is Kähler–Einstein. Our Theorem 2 is proved.

4. Transition probability amplitudes. A. Odzijewicz [18], while studying the quantization of a mechanical system whose phase space is a complex manifold M, pointed out a deep interrelation between the theory of reproducing kernel Hilbert spaces, the complex Monge–Ampère equations, and the calculation of transition probability amplitudes from one coherent state to another. Cf. also [19], pp. 110–111. To fix the notation and terminology, we briefly recall the essentials of [18].

Let $E \to M$ be a holomorphic line bundle over a complex n-dimensional manifold M. Let H be a Hermitian metric on E whose Chern connection ∇ has a nonsingular curvature form $\omega = i\,\mathrm{curv}(\nabla)$. Let $\Lambda^{n,0}(M)$ be the canonical bundle of M ($\eta \in \Lambda^{n,0}(M)$ is a complex form of type $(n,0)$ on M).

The *space of quantum states* is the complex Hilbert space $\mathcal{M}$ of all $s \in H^0(M, \mathcal{O}(E \otimes \Lambda^{n,0}(M)))$ with $\langle s,s\rangle < \infty$, where the inner product is given by $\langle s,t\rangle = i^{n^2}\int_M H^*(s,t)$, for any E-valued holomorphic n-forms s,t on M. Cf. also [8]. Here H^* is the metric induced by H on $E \otimes \Lambda^{n,0}(M)$, hence $H^*(s,t)$ is an (n,n)-form on M.

The *quantization of classical states* is an embedding $\mathcal{K} : M \to \mathbb{CP}(\mathcal{M})$ of M (the *classical phase space* of the system) into the complex projective Hilbert space $\mathbb{CP}(\mathcal{M})$. If $z \in M$ then $\mathcal{K}(z)$ is a *coherent state*. Identifying a classical state $z \in M$ with the coherent state $\mathcal{K}(z) \in \mathbb{CP}(\mathcal{M})$ one defines the *transition probability amplitude* from ζ to z by $a(\zeta,z) = \langle \mathcal{K}(\zeta), \mathcal{K}(z)\rangle$. Next, the transition probability amplitude from z to w *with simultaneous transition through* $\zeta \in M$ is by definition $a(\zeta,w)a(z,\zeta)$.

Now a natural question is whether averaging $a(\zeta,w)a(z,\zeta)$ over $\zeta \in M$ one retrieves the transition probability amplitude from z to w. In other words, as the natural measure on the phase space M is the *Liouville measure* $d\mu_L = (-i)^n \det[\omega_{j\overline{k}}] d\zeta_1 \wedge \ldots \wedge d\zeta_n \wedge d\overline{\zeta}_1 \wedge \ldots \wedge d\overline{\zeta}_n$, one asks whether

$$\int_M a(\zeta,w)a(z,\zeta)\, d\mu_L(\zeta) = a(z,w), \tag{9}$$

possibly with $d\mu_L$ multiplied by some constant $c > 0$. Here $\omega_{j\overline{k}}$ is the (local manifestation of the) curvature 2-form of (E,H) with respect to a local trivialization of E and a local coordinate system $(\zeta_1,\ldots,\zeta_n)$ on M.

Our result in this section is that (9) holds when $M = \Omega_n$. Precisely, thinking of Ω_n as the classical phase space of some mechanical system, let $E = \Omega_n \times \mathbb{C}$ be the trivial line bundle over Ω_n with the Hermitian metric H_α given by $H_\alpha(s^0,s^0) = \varrho^\alpha$, $\varrho(\zeta) = \operatorname{Im}\zeta_1 - |\zeta'|^2$, where the holomorphic frame $s^0 : \Omega_n \to E$ is given by $s^0(\zeta) = (\zeta,1)$. We establish the following:

THEOREM 3. *Let $\alpha > -1$. Then $H^2_\alpha(\Omega_n)$ is the space of quantum states of Ω_n. There is an anti-holomorphic embedding $\mathcal{K}_\alpha$ of Ω_n into $\mathbb{CP}(H^2_\alpha(\Omega_n))$. Assume that (n,α) satisfies one of the following conditions*:

(i) $n = \mathcal{M}_4$, $\alpha \in (-1,0) \cup (0,\infty)$,
(ii) $n = \mathcal{M}_4 + 1$, $\alpha \in (0,\infty)$,
(iii) $n = \mathcal{M}_4 + 3$, $\alpha \in (-1,0)$,

where $\mathcal{M}_4 = 4k$ for some $k \in \mathbb{N}$. Then the corresponding transition probability amplitude $a_\alpha(\zeta,z) = \langle \mathcal{K}_\alpha(\zeta), \mathcal{K}_\alpha(z)\rangle$, $\zeta,z \in \Omega_n$, satisfies the rule

$$\int_{\Omega_n} a_\alpha(\zeta,w)a_\alpha(z,\zeta)c\, d\mu_L(\zeta) = a_\alpha(z,w)$$

for some constant $c > 0$ (depending only on n and α).

In $H^2_\alpha(\Omega_n) - \{0\}$ one may consider the equivalence relation $f \sim g$ if $g = \lambda f$ for some $\lambda \in \mathbb{C} - \{0\}$. The quotient space

$$\mathbb{CP}(H^2_\alpha(\Omega_n)) = (H^2_\alpha(\Omega_n) - \{0\})/\!\sim$$

is a complete metric space with the distance

$$d_\alpha([f],[g]) = \inf_{a,b\in\mathbb{R}} \left\| \frac{e^{ia}f}{\|f\|_{2,\alpha}} - \frac{e^{ib}g}{\|g\|_{2,\alpha}} \right\|_{2,\alpha} \tag{10}$$

(cf. e.g. [24], p. 20). We organize the proof of Theorem 3 in several steps, as follows.

STEP 1. *Let $K_\alpha(\zeta,z)$ be the γ_α-Bergman kernel of Ω_n. The map $\mathcal{K}_\alpha : \Omega_n \to \mathbb{CP}(H^2_\alpha(\Omega_n))$, $\mathcal{K}_\alpha(z) = [K_\alpha(\cdot,z)]$, $z \in \Omega_n$, is an anti-holomorphic embedding.*

If $\mathcal{K}_\alpha(z) = \mathcal{K}_\alpha(w)$ then (by (8))

$$\frac{i(\overline{z}_1 - \zeta_1) - 2\langle \zeta', z'\rangle}{i(\overline{w}_1 - \zeta_1) - 2\langle \zeta', w'\rangle} = \text{const}$$

with respect to $\zeta \in \Omega_n$. Differentiate this with respect to ζ_1 to get

$$i\overline{\omega}_1 - 2\sum_{j=2}^n \zeta_j \overline{\omega}_j = 0$$

where $\omega = z - w$. Next, differentiation with respect to ζ_j, $j \geq 2$, gives $\omega = 0$. Thus $\mathcal{K}_\alpha$ is injective. The quadratic form (2.16) in [18], p. 582, and our Kähler metric g_α actually coincide. Therefore, we may apply Propositions 2 and 3 of [18], pp. 582–583, to end the proof of Step 1.

STEP 2. *The identity*

$$a_\alpha(z,w) = \int_{\Omega_n} a_\alpha(\zeta,w)\overline{a_\alpha(\zeta,z)}K_\alpha(\zeta,\zeta)\gamma_\alpha(\zeta)\,dm(\zeta) \tag{11}$$

holds for any $z,w \in \Omega_n$.

Note first that

$$a_\alpha(\zeta,z) = \frac{K_\alpha(\zeta,z)}{K_\alpha(z,z)^{1/2}K_\alpha(\zeta,\zeta)^{1/2}}$$

so that $a_\alpha(\zeta,\zeta) = 1$ and $\overline{a_\alpha(\zeta,z)} = a_\alpha(z,\zeta)$. Then (11) follows from the reproducing property of $K_\alpha(\zeta,z)$.

STEP 3. *Let* (n,α) *satisfy one of the assumptions* (i)–(iii) *of Theorem* 3. *There is a constant* $C > 0$ *(depending only on* n *and* α*) so that the weight* $\gamma_\alpha(\zeta) = (\operatorname{Im}\zeta_1 - |\zeta'|^2)^\alpha$ *satisfies the complex Monge–Ampère equation*

$$\det\left[\frac{\partial^2\log\gamma(\zeta)}{\partial\zeta_j\partial\overline{\zeta}_k}\right] = (-1)^{n(n+1)/2}C\frac{1}{n!}\gamma(\zeta)K_\gamma(\zeta,\zeta)$$

where K_γ *is the* γ*-Bergman kernel.*

Indeed, a calculation shows that

$$\det\left[\frac{\partial^2\log\gamma_\alpha(\zeta)}{\partial\zeta_j\partial\overline{\zeta}_k}\right] = (-1)^n\frac{\alpha^n}{4\varrho(\zeta)^{n+1}},$$

hence (by taking into account (8)) one obtains

$$C = (-1)^{n(n-1)/2}\frac{n!\alpha^n\pi^n}{(\alpha+1)\dots(\alpha+n)}$$

and Step 3 is proved. Note that $n = \mathcal{M}_4 + 2$ yields $C \le 0$. Finally, by a result of A. Odzijewicz ([18], p. 584) and by Step 3 one has

$$d\mu_L(\zeta) = CK_\alpha(\zeta,\zeta)\gamma_\alpha(\zeta)dm(\zeta),$$

hence (11) is equivalent to (9) with $d\mu_L$ replaced by $C^{-1}d\mu_L$ and Theorem 3 is proved.

We end this section with the following remark. For each $\alpha > -1$, let $d_{\Omega_n,\alpha}$ be the pullback of (10) by $\mathcal{K}_\alpha : \Omega_n \to \mathbb{CP}(H^2_\alpha(\Omega_n))$. Then $d_{\Omega_n,\alpha}$ is a family of distances on Ω_n given by

$$d_{\Omega_n,\alpha}(\zeta,z) = \sqrt{2}\,(1-|a_\alpha(\zeta,z)|)^{1/2},$$

$$a_\alpha(\zeta,z) = \left[\frac{2\sqrt{\varrho(\zeta)\varrho(z)}}{i(\overline{z}_1-\zeta_1)-2\langle\zeta',z'\rangle}\right]^{n+1+\alpha}.$$

By analogy with [24], pp. 22–27, one may ask whether $(\Omega_n, d_{\Omega_n,\alpha})$ is complete.

5. The Genchev transform. Let $J \subseteq \mathbb{R}$ be an interval (possibly unbounded) and $\Omega = \{z \in \mathbb{C} : \operatorname{Im} z \in J\}$. We shall need the following:

LEMMA 6. *Let* $f \in L^2H(\Omega,\gamma_\alpha)$ *and* $y \in J$. *Set* $g_y(x) = f(x+iy)$, $x \in \mathbb{R}$. *Then* $g_y \in L^2(\mathbb{R})$.

For $\alpha = 0$ this is Lemma 1 of [25], p. 121. Cf. also [5] and [9]. Fix $x \in \mathbb{R}$. Set $h(u+iv) = f(u+x+iv)$. Given $y \in J$ let $\varepsilon > 0$ so that $(y-\varepsilon, y+\varepsilon) \subset J$. Then h is holomorphic on a domain D containing $\{(u,v) : |u| \le \varepsilon,\ |v-y| \le \varepsilon\}$, hence $|h|^s$ is subharmonic in D, for any $s > 0$ (e.g. [13], p. 75). Let $a > 0$ and set $p = (1+a)/a$, $q = 1+a$. Then (see e.g. [13], p. 71)

$$\begin{aligned}
&|h(iy)|^{2/p} \\
&\le \frac{1}{\operatorname{vol}(B(iy,\varepsilon))}\int_{B(iy,\varepsilon)}|h(u+iv)|^{2/p}\,du\,dv \\
&\le \frac{1}{\pi\varepsilon^2}\int_{B(iy,\varepsilon)}|f(u+x+iv)|^{2/p}\gamma(u+x+iv)^{1/p}\gamma(u+x+iv)^{-1/p}\,du\,dv \\
&\le \frac{1}{\pi\varepsilon^2}\Big(\int_{B(iy,\varepsilon)}|f(u+x+iv)|^2\gamma(u+x+iv)\,du\,dv\Big)^{1/p} \\
&\quad\times\Big(\int_{B(iy,\varepsilon)}\gamma(u+x+iv)^{-q/p}\,du\,dv\Big)^{1/q},
\end{aligned}$$

hence

$$\begin{aligned}
&|f(x+iy)|^2 \\
&\le (\pi\varepsilon^2)^{-p}\Gamma_{\varepsilon,a}(y)^{1/a}\int_{|u|<\varepsilon,\,|v-y|<\varepsilon}|f(u+x+iv)|^2\gamma(u+x+iv)\,du\,dv
\end{aligned} \tag{12}$$

where

$$\Gamma_{\varepsilon,a}(y) = \int_{|u|<\varepsilon,\,|v-y|<\varepsilon}\gamma(u+x+iv)^{-a}\,du\,dv$$

for any $\gamma \in W(\Omega)$. When $\gamma = \gamma_\alpha$ one has $\Gamma_{\varepsilon,a}(y) < \infty$ and $\Gamma_{\varepsilon,a}(y)$ does not depend on x. If this is the case $(\gamma_\alpha(\zeta) = (\operatorname{Im}\zeta)^\alpha,\ \alpha > -1)$ then integration of (12) with respect to x gives

$$\begin{aligned}
\int_{-\infty}^{\infty}|g_y(x)|^2\,dx &\le \frac{2\varepsilon^{1-2p}}{\pi^p}\Gamma_{\varepsilon,a}(y)^{1/a}\int_{y-\varepsilon}^{y+\varepsilon}\Big(\int_{-\infty}^{\infty}|f(x+iy)|^2\,dx\Big)v^\alpha\,dv \\
&\le \frac{2\varepsilon^{1-2p}}{\pi^p}\Gamma_{\varepsilon,a}(y)^{1/a}\|f\|^2_{2,\alpha}
\end{aligned}$$

and Lemma 6 is proved.

Let $\mathcal{F}$ be the Fourier transform. If $f \in L^2H(\Omega,\gamma_\alpha)$ then $e^{-2\pi ty}\mathcal{F}(g_y)(t)$ does not depend upon the choice of $y \in J$ simply because (by following the idea in [25], p. 121) we may represent it by a complex line integral (of a holomorphic function):

$$e^{-2\pi ty}\mathcal{F}(g_y)(t) = \int_{\operatorname{Im} z=y} e^{2\pi itz} f(z)\,dz$$

and apply the Cauchy theorem. Hence we may define the *Genchev transform* $G_\alpha(f)$ of $f \subset L^2H(\Omega,\gamma_\alpha)$ by setting

$$G_\alpha(f)(t) = e^{-2\pi ty}\mathcal{F}(g_y)(t), \quad t \in \mathbb{R}.$$

This was originally defined on $L^2H(\Omega)$ (cf. T. Genchev [9] for the case $\alpha = 0$). We now state

THEOREM 4. *Let $\alpha > -1$. The Genchev transform G_α defines a unitary isomorphism of $L^2H(\Omega,\gamma_\alpha)$ onto $L^2(\mathbb{R}, w_{J,\alpha})$ where $w_{J,\alpha}(t) = \int_J y^\alpha e^{4\pi ty}\,dy$.*

This generalizes a result of [5], [9] (cf. also Theorem 1 of [25], p. 122). To prove Theorem 4, let $J = (b,c)$ and $f \in L^2H(\Omega,\gamma_\alpha)$. Then (by the Plancherel theorem)

$$\int_\Omega |f(x+iy)|^2 y^\alpha\,dx\,dy = \int_b^c \|g_y\|^2_{L^2(\mathbb{R})} y^\alpha dy = \int_b^c \|\mathcal{F}(g_y)\|^2_{L^2(\mathbb{R})} y^\alpha\,dy$$
$$= \int_b^c y^\alpha \int_{-\infty}^{\infty} |e^{2\pi ty} G_\alpha(f)(t)|\,dt\,dy = \int_{-\infty}^{\infty} |G_\alpha(f)(t)|^2 w_{J,\alpha}(t)\,dt.$$

Finally, the image of G_α contains a dense subset of $L^2(\mathbb{R}, w_{J,\alpha})$ because for any bounded $\phi \in L^2(\mathbb{R}, w_{J,\alpha})$ which vanishes off a compact subset of $\mathbb{R}$ one has $G_\alpha(f) = \phi$, where $f(z) = \int_{-\infty}^{\infty} e^{-2\pi itz}\phi(t)\,dt$.

Let $c \in (0,\infty)$ and $\Omega = \{z \in \mathbb{C} : 0 < \operatorname{Im} z < c\}$. For any $f \in H^2_\alpha(\Omega_1)$ one has $f_{|\Omega} \in L^2H(\Omega,\gamma_\alpha)$. Also $\gamma_{\alpha|\Omega} \in AW(\Omega)$. Indeed, there is $a > 0$ so that $\gamma_\alpha^{-a} \in L^1_{\rm loc}(\Omega)$, hence one may apply Corollary 3.1 of [20], p. 6. Therefore $L^2H(\Omega,\gamma_\alpha)$ is closed in $L^2(\Omega,\gamma_\alpha)$, hence we may define the subspace $L^2_+H(\Omega,\gamma_\alpha)$ of $L^2H(\Omega,\gamma_\alpha)$ consisting of all $f : \Omega \to \mathbb{C}$ which are the $L^2(\Omega,\gamma_\alpha)$ limits of sequences $f_k \in H^2_\alpha(\Omega_1)$, $k \geq 1$. We now state

THEOREM 5. *Let $f \in L^2H(\Omega,\gamma_2)$. Then $f \in L^2_+H(\Omega,\gamma_2)$ if and only if its Genchev transform vanishes a.e. in $(0,\infty)$.*

A calculation shows that 1) if $J = (b,c)$ then $w_{J,2}(t) = (4\pi t)^{-1}\{e^{4\pi ct}Q_t(c) - e^{4\pi bt}Q_t(b)\}$, 2) if $J = (b,\infty)$ then $w_{J,2}(t) = -(4\pi t)^{-1}e^{4\pi bt}Q_t(b)$ for $t < 0$ and $w_{J,2}(t) = \infty$ for $t > 0$, and 3) if $J = (-\infty, c)$ then $w_{J,2}(t) = \infty$ for $t < 0$ and $w_{J,2}(t) = (4\pi t)^{-1}e^{4\pi ct}Q_t(c)$ for $t > 0$, where $Q_t(y) = y^2 - y/(2\pi t) + 1/(8\pi^2 t^2)$.

To prove Theorem 5, let $f \in L^2_+H(\Omega,\gamma_2)$ and $f_k \in H^2_2(\Omega_1)$ so that $f_k \to f$ as $k \to \infty$. Then $G(f_k) = 0$ on $(0,\infty)$. If $g \in H^2_2(\Omega_1)$ then $G(g_{|\Omega})(t) = G(g)(t)$. As (by Theorem 4) $G(f_k) \to G(f)$ as $k \to \infty$ ($L^2(\mathbb{R}, w_{(0,c),2})$ convergence) one obtains $|G(f)|^2 w_{(0,c),2} = 0$ a.e. in $(0,\infty)$, hence $G(f) = 0$ a.e. in $(0,\infty)$ (as $w_{(0,c),2}$ has at most two zeros).

We end this section with the following remark. By a result of M. Skwarczyński [25], p. 124, if $\Omega = \{\zeta \in \mathbb{C} : |\operatorname{Im}\zeta| < \pi\}$ then the Bergman kernel K of Ω is given by

$$K(\zeta,z) = \sum_{k=1}^{\infty} \frac{1}{K_0(\zeta, z + 4i(k-1)\pi)} + \sum_{k=1}^{\infty} \frac{1}{K_0(\zeta, z - 4ik\pi)} \tag{13}$$

where K_0 is obtained from (8) for $n = 1$ and $\alpha = 0$. It is an open question whether the γ_α-Bergman kernel $K_{\gamma_\alpha}(\cdot, z) \in L^2H(\Omega,\gamma_\alpha)$ is related to the Djrbashian kernel of the half-plane Ω_1 (i.e. we ask for a weighted analogue of (13)).

6. Canonical isometries. Let $a \in B_n$ and $\phi_a \in \operatorname{Aut}(\Omega_n)$ be given by $\phi_a = \varphi \circ \widetilde{\phi}_a \circ \varphi^{-1}$ where $\varphi : B_n \to \Omega_n$ is the Cayley map and

$$\widetilde{\phi}_a(z) = \frac{a - P_a z - s_a Q_a z}{1 - \langle z, a\rangle}$$

for any $z \in B_n$ (cf. notations and conventions in [21], p. 25). We establish the following:

THEOREM 6. *Let $\alpha > -1$ and $a_1 = u + iv \in B_1$. Let $V_a : H^2_\alpha(\Omega_n) \to H^2_\alpha(\Omega_n)$ be given by*

$$(V_a f)(\zeta) = \left[\frac{s_a}{\zeta_1(u-1) - v}\right]^{n+1+\alpha} f(\phi_a(\zeta))$$

where $a = (a_1, 0)$ and $s_a = (1 - |a|^2)^{1/2}$. Then $H^2_\alpha(\Omega_n)$ admits a complete orthonormal system consisting of eigenfunctions of V_a.

The main ingredient in the proof of Theorem 6 is a result of T. Mazur [15]. Cf. also [17] for its unweighted version. Note that μ_α is absolutely continuous and has a strictly positive Radon–Nikodym derivative with respect to m (the Lebesgue measure). Let $G(\mu_\alpha) \subset \operatorname{Aut}(\Omega_n)$ be the subgroup of all automorphisms leaving μ_α invariant modulo a holomorphic change of gauge (cf. the terminology in [15], p. 304). We shall need:

LEMMA 7. *If $a = (a_1, 0)$ with $a_1 \in B_1$, then $\phi_a \in G(\mu_\alpha)$.*

Proof. We have to find a holomorphic function $\psi_a : \Omega_n \to \mathbb{C}$ so that

$$\mu_\alpha(\phi_a(\Omega)) = \int_\Omega |\psi_a|^2\,d\mu_\alpha \tag{14}$$

for any domain $\Omega \subset \Omega_n$. A calculation shows that

$$\phi_a(\zeta) = \left(\frac{\zeta_1 v + u + 1}{\zeta_1(u-1) - v}, \frac{i s_a \zeta'}{\zeta_1(u-1) - v}\right)$$

for any $\zeta \in \Omega_n$. Also,

$$\varrho(\phi_a(\zeta)) = \frac{s_a^2}{|\zeta_1(u-1) - v|^2}\varrho(\zeta),$$

$$J_{\phi_a}(\zeta) = i^{n-1}\left(\frac{s_a}{\zeta_1(u-1) - v}\right)^{n+1}.$$

Next, (14) may be written as

$$\int_\Omega [|J_{\phi_a}(\zeta)|^2 \varrho(\phi_a(\zeta))^\alpha - |\psi_a(\zeta)|^2 \varrho(\zeta)^\alpha]\, dm(\zeta) = 0,$$

hence we may take ψ_a to be

$$\psi_a(\zeta) = \left(\frac{s_a}{\zeta_1(u-1) - v}\right)^{n+1+\alpha}.$$

Clearly ψ_a is holomorphic and satisfies (14). ∎

Finally, note that ϕ_a has (exactly) one fixed point. Hence, we may use our Lemma 7 together with Theorem 2 of [15], p. 304, to end the proof of Theorem 6.

The authors hope that the present paper may contribute to a better understanding of the function spaces $H^2_\alpha(\Omega_n)$.

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