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The parametric Weierstrass integral over a BV curve as a length functional

by

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Abstract. The constructive definition of the Weierstrass integral through only one limit process over finite sums is often preferable to the more sophisticated definition of the Serrin integral, especially for approximation purposes. By proving that the Weierstrass integral over a BV curve is a length functional with respect to a suitable metric, we discover a further natural reason for studying the Weierstrass integral. This characterization was conjectured by Menger.

1. Introduction. The parametric integral of the Calculus of Variations was introduced as a Weierstrass integral (**W**-integral) over a variety, in a very general setting, by Cesari [16, 17] in terms of a suitable Burkill–Cesari integral.

Cesari considered the following setting:

Let A be a topological space, $\{I\}$ be a collection of subsets of A, $(T,\gg)$ be a directed set, and $(D_t)_{t\in T}$ be a net of finite systems of elements of $\{I\}$; given a continuous variety $x : A \to K \subset \mathbb{R}^n$, a set function $\phi : \{I\} \to \mathbb{R}^n$, and a function $f : K \times \mathbb{R}^n \to \mathbb{R}$ with the properties:

(1.1) f is bounded and uniformly continuous on $K \times \mathbb{R}^n$;

(1.2) $f(p,tq) = tf(p,q)$ for all $t \geq 0$, $p \in K$, $q \in \mathbb{R}^n$,

consider the set function

$$\Phi(I) = f(x(\tau), \phi(I)), \qquad I \in \{I\},$$

where τ is an arbitrary point in I.

Cesari proved that, if ϕ is quasiadditive (q.a.) (see Section 2), then the following limit exists:

$$\lim_{T} \sum_{I\in D_t} \Phi(I),$$

and Cesari called it the **W**-*integral* of Φ over the variety x with respect to the q.a. function ϕ (i.e. $\mathbf{W}(f,x,\phi)$); moreover, if ϕ is q.a. and of bounded

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variation (BV) (see Section 2), then $\mathbf{W}(f,x,\phi)$ can be extended to a regular measure μ on the Borel sets of A, and has a representation in terms of a Lebesgue–Stieltjes integral:

$$\mathbf{W}(f,x,\phi)=\int_A f\left(x,\frac{d\mu}{d|\mu|}\right)d|\mu|.$$

Many authors studied the parametric $\mathbf{W}$-integral in the setting proposed by Cesari: we just mention Breckenridge [14], Warner [20, 21], and in particular Brandi & Salvadori [3, 4, 6–8, 10–12] who proved Cesari's results in abstract spaces and, especially, weakened the hypotheses on the variety x; more precisely, Brandi & Salvadori [8–13] extended the definition of the $\mathbf{W}$-integral over BV varieties not necessarily continuous.

The Weierstrass integral has a simple and constructive definition through only one limit process over finite sums; a well known advantage in the study of the $\mathbf{W}$-integral is the definition of a satisfactory length functional.

Consider, for example, the interval function $\psi(I)=|\Delta x|$; the finite Weierstrass sums of ψ, related to a finite system $\{[t_{i-1},t_i]:i=1,\ldots,n\}$, are equal to the Euclidean length of the inscribed polygon $\{x(t_1),\ldots,x(t_n)\}$; thus, the $\mathbf{W}$-integral of ψ over x exists for every curve x, and its value is the length of the curve x in the classical sense. On the contrary, the length functional in terms of a Lebesgue integral $\int_a^b|\dot{x}|\,dt$ does not exist, in general, unless $\dot{x}$ exists a.e. in $[a,b]$; furthermore, it is equal to the Jordan length of x iff x is absolutely continuous.

There are some more "geometrical" reasons for studying the integrals of the Calculus of Variations as Weierstrass integrals, rather than as Lebesgue ones. The present paper proves that the $\mathbf{W}$-integral over a BV curve, unlike the Lebesgue one, always represents the measure of a geometric quantity connected to the curve. Menger [18] conjectured that the $\mathbf{W}$-integral is always a length functional with respect to a suitable metric d.

Menger's argument is the following: Let C be a compact and metrizable space, Γ be the class of continuous curves $x:[a,b]\to C$, and L be a nonnegative functional defined on Γ and having the following properties:

(i) (Additivity) Given two curves $x_1,x_2\in\Gamma$ with $x_1(b)=x_2(a)$, and denoting by $\widetilde{x}$ the compound curve, we have

$$L(\widetilde{x})=L(x_1)+L(x_2).$$

(ii) (Semicontinuity) Given a sequence $(x_n)_n\subset\Gamma$ converging pointwise to x_0, we have

$$\lim_n L(x_n)\ge L(x_0).$$

(iii) (Regularity) If, for every $n\in\mathbb{N}$, x_n is a curve with $x_n(b)=p_0$, and $\lim_n L(x_n)=0$, then $\lim_n x_n(a)=p_0$.

Menger asserted that, if the space C is L-connected (1) and L-nonatomic (2), then L is the length functional with respect to the metric

$$d_L(p,q)=\inf_{x\in\Gamma_{p,q}}L(x),$$

where $\Gamma_{p,q}=\{x\in\Gamma:x(a)=p,\ x(b)=q\}$.

Thus, if the $\mathbf{W}$-integral satisfies the hypotheses (i), (ii), and (iii) and $K\subset\mathbb{R}^n$ is $\mathbf{W}$-connected and $\mathbf{W}$-nonatomic, then the $\mathbf{W}$-integral is the length functional with respect to the metric $d_{\mathbf{W}}$.

Following the outlines of Menger's conjecture, we prove here that, given a general class Ω of functions (see Section 2), endowed with an abstract convergence σ, and a nonnegative functional L, defined on Ω and having the properties (i), (ii) (3), and (iii), L is the length functional with respect to the metric d_L; furthermore, as an application of this result, if f does not depend explicitly on the curve and satisfies classical hypotheses, then the Weierstrass integral over a BV curve, possibly discontinuous, is the length functional with respect to $d_{\mathbf{W}}$.

2. An abstract result. Let $\mathcal{K}$ be a metrizable space, $\mathcal{D}$ be the collection of all finite systems of subintervals of $[a,b]$, and $\delta:\mathcal{D}\to\mathbb{R}^+$ be a mesh function. Let Ω be a class of functions $x:[a,b]\to\mathcal{K}$, σ be a convergence on Ω which separates the points of Ω (i.e. if $(x_n)_n\subset\Omega$, $x_n\xrightarrow{\sigma}x$ and $x_n\xrightarrow{\sigma}y$ with $x,y\in\Omega$ then $x=y$), and L be a nonnegative functional defined on Ω. We shall consider classes Ω of functions with the following properties:

(iv) (Compactness) For any $\tau\in\mathbb{R}^+$, the level set $\{x\in\Omega:L(x)\le\tau\}$ is sequentially σ-compact in Ω.

(v) (Approximation) Given a function $x\in\Omega$ and a sequence $(D_n)_n$ of finite systems with the property that D_{n+1} is a refinement of D_n and $\lim_n\delta(D_n)=0$, for every sequence $(x_n)_n\subset\Omega$ such that x_n and x_{n+1} coincide with x at the endpoints of the intervals of D_n, we have $\sigma\text{-}\lim_n x_n=x$.

The consistency of the property (v) can be easily verified by taking as Ω, for example, a Sobolev space endowed with the weak topology.

Theorem 2.1. *Assume that*:

(2.3) L *is σ-lower semicontinuous and has the properties* (i) *and* (iii);

(1) C is *L-connected* if, for every pair $(p,q)\in C\times C$, there is a curve y joining p to q such that $L(y)<\infty$.

(2) C is *L-nonatomic* if, given $(p_n)_n\subset C$ with $\lim_n p_n=p_0$, there is a sequence $(x_n)_n\subset\Omega$, each x_n joining p_n to p_0, such that $\lim_n L(x_n)=0$. Menger called this property *local L-connectivity*.

(3) In this case, the property (ii) is modified by replacing the pointwise convergence with the σ-convergence.

(2.4) *Ω has the properties* (iv) *and* (v);

(2.5) *𝒦 is L-connected and L-nonatomic.*

Set

$$d_L(p,q) = \inf_{x\in\Omega_{p,q}} L(x), \qquad p,q\in\mathcal{K},$$

where $\Omega_{p,q} = \{x\in\Omega : x(a)=p,\ x(b)=q\}$. *Then L is the length functional with respect to the metric* d_L, *i.e.*

$$L(x) = \mathcal{L}_{d_L}(x) = \sup_{D\in\mathcal{D}} \sum_{I\in D} d_L(x(t_1^I), x(t_2^I)) \ (^4),$$

for every $x\in\Omega$.

Proof. For every $p,q\in\mathcal{K}$, $\Omega_{p,q}\neq\emptyset$ and $d_L(p,q)$ is finite since $\mathcal{K}$ is L-connected. In order to prove that d_L is a metric over $\mathcal{K}$, let $p,q,r\in\mathcal{K}$; by definition of d_L, for fixed $\varepsilon>0$ there are $x^*,y^*\in\Omega_{p,q}$ with

$$d_L(p,q)+\varepsilon/2 \ge L(x^*) \quad \text{and} \quad d_L(q,r)+\varepsilon/2\ge L(y^*);$$

from the additivity property (i) and the arbitrariness of ε, we get

$$d_L(p,r)\le d_L(p,q)+d_L(q,r).$$

Since $\mathcal{K}$ is L-nonatomic, $d_L(p,p)=0$ for every $p\in\mathcal{K}$.

Now, let $d_L(p,q)=0$ for some $p,q\in\mathcal{K}$; then there is a sequence $(x_n)_n\subset\Omega_{p,q}$ such that $\lim_n L(x_n)=0$. Thus, from the regularity property (iii), we get $p=q$. This proves that d_L is a metric on $\mathcal{K}$.

Now, given $x\in\Omega$ and $D\in\mathcal{D}$, we have

$$L(x)=\sum_{I\in D} L(z_I),$$

where $z_I\in\Omega_{x(t_1^I),x(t_2^I)}$ is defined by

$$z_I(t) = x\left(t_1^I + \frac{t_2^I - t_1^I}{b-a}(t-a)\right), \quad t\in[a,b].$$

From the definition of d_L, we get

$$L(x)=\sum_{I\in D} L(z_I) \ge \sum_{I\in D} d_L(x(t_1^I), x(t_2^I));$$

therefore,

$$L(x)\ge \sup_{D\in\mathcal{D}} \sum_{I\in D} d_L(x(t_1^I), x(t_2^I)) = \mathcal{L}_{d_L}(x).$$

For the other inequality, let $(D_n)_n$ be a sequence of finite systems with the properties that D_{n+1} is a refinement of D_n for every $n\in\mathbb{N}$ and $\lim_n \delta(D_n)=0$. From the definition of d_L, for every $I\in D_n$ there is a $z_I^n\in\Omega_{x(t_1^I),x(t_2^I)}$ such that

$$\sum_{I\in D_n} d_L(x(t_1^I), x(t_2^I)) \ge \sum_{I\in D_n} L(z_I^n) - \frac{1}{2^n}.$$

Let x_n be the function defined by

$$x_n(t) = z_I^n\left(a+(b-a)\frac{t-t_1^I}{t_2^I-t_1^I}\right), \qquad t\in I;$$

we have

$$\sum_{I\in D_n} d_L(x(t_1^I), x(t_2^I)) \ge L(x_n) - \frac{1}{2^n};$$

therefore,

$$\mathcal{L}_{d_L}(x)+1\ge L(x_n) \qquad \text{for every } n\in\mathbb{N}.$$

From the compactness property (iv), we may assume, passing to subsequences if necessary, that there is an $x_0\in\Omega$ with $x_n \xrightarrow{\sigma} x_0$. Taking into account the approximation property (v) and the unicity of the limit in the σ-convergence, we get $x_0=x$. Finally, by the σ-lower semicontinuity of L, we get

$$\mathcal{L}_{d_L}(x) = \lim_n \sum_{I\in D_n} d_L(x(t_1^I), x(t_2^I)) \ge \liminf_n L(x_n) \ge L(x). \ ■$$

(⁴) For every interval I, we write $I=[t_1^I, t_2^I]$.

3. The Burkill–Cesari integral. Let $(A,\mathcal{G})$ be a topological space and denote by $\mathcal{M}$ the family of all subsets of A and by $\mathcal{B}(A)$ the σ-algebra generated by $\mathcal{G}$. We consider a subfamily $\{I\}\subset\mathcal{M}$ and call the sets I *intervals*. A *finite system* $D=[I_1,\ldots,I_N]$ is a finite collection of nonoverlapping intervals, i.e.

$$\mathring{I}_i\neq\emptyset \quad \text{and} \quad \mathring{I}_i\cap\bar{I}_j=\emptyset, \quad i\neq j,\ i,j=1,\ldots,N,$$

where $\mathring{I}$ and $\bar{I}$ denote the $\mathcal{G}$-interior and $\mathcal{G}$-closure of I, respectively.

Let $(T,\gg)$ be a directed set and let $(D_t)_{t\in T}$ be a net of finite systems. Let $s:\mathcal{M}\times\mathcal{M}\to\{0,1\}$ be the function defined by

$$s(H,K)=\begin{cases}1 & \text{if } H\subset K,\\ 0 & \text{otherwise.}\end{cases}$$

Let $\Phi:\{I\}\to\mathbb{R}^m$ be an interval function. The function Φ is said to be *Burkill–Cesari integrable* (BC-integrable) over $M\in\mathcal{M}$ (see Cesari [16]) if the limit below exists:

$$\lim_T \sum_{I\in D_t} s(I,M)\Phi(I);$$

in this case, we shall denote its value by BC-$\int_M \Phi$.

The function Φ is said to be of *bounded variation* (BV) over $M \in \mathcal{M}$ if

$$V(\Phi, M) = \limsup_{T} \sum_{I \in D_t} s(I, M)|\Phi(I)| < \infty.$$

The function Φ is said to be *quasiadditive* (q.a.) (see Cesari [16]) over M if

(q.a.) given $\varepsilon > 0$ there is $t_1 = t_1(M, \varepsilon)$ such that for every $t_0 \gg t_1$ there is $t_2 = t_2(M, \varepsilon, t_0)$ such that if $t \gg t_2$ then

$$\text{(j)} \sum_I s(I, M) \Big| \sum_J s(J, I)\Phi(J) - \Phi(I) \Big| < \varepsilon,$$

$$\text{(jj)} \sum_J s(J, M)\Big[1 - \sum_I s(J, I) s(I, M)\Big] |\Phi(J)| < \varepsilon,$$

where $D_{t_0} = [I]$ and $D_t = [J]$.

The following results are well known (see Cesari [16], Breckenridge [14], Warner [21], Brandi & Salvadori [3]):

- If Φ is q.a. on M, then it is BC-integrable over M.
- If Φ is q.a. and BV on M, then $|\Phi|$ is q.a. on M.
- If Φ is q.a. and BV on A, then Φ is q.a. on M for every $M \in \mathcal{M}$.

Consider functions $f : \mathbb{R}^m \to \mathbb{R}_0^+$ and $\phi : \{I\} \to \mathbb{R}^m$, and denote by $\Phi : \{I\} \to \mathbb{R}_0^+$ the set function defined by

$$\Phi(I) = f(\phi(I)).$$

Following Cesari [16], the BC-integral of the function Φ, when it exists, will be called the *parametric Weierstrass integral of the Calculus of Variations* (**W**-integral) and denoted by $\mathbf{W}(f, \Phi, \phi)$.

Thus, any set of conditions guaranteeing that Φ is q.a. and BV yields an existence theorem for $\mathbf{W}(f, \Phi, \phi)$.

Throughout the paper, we will suppose that the integrand f satisfies the following conditions:

(F_1) f is continuous on $\{x \in \mathbb{R}^m : |x| = 1\}$;
(F_2) $f(tx) = tf(x)$ for every $t \geq 0$, $x \in \mathbb{R}^m$.

3.1. *The parametric* **W**-*integral over a BV curve.* In this section we recall some known results about the existence, representation, semicontinuity, and approximation for the parametric Weierstrass integral over a BV curve.

Let $x : [a, b] \to \mathbb{R}^m$ be a bounded variation curve in the generalized sense (BV) (see Cesari [15]). It is well known (Boni [1], Salvadori [19]) that $\operatorname{esslim}_{t \to c^\pm} x(t) = x^\pm(c)$ exists for every $c \in [a, b]$ and if we take $E_x = \{c \in]a, b[: x(c) = x^+(c) = x^-(c)\}$ and $S_x = \{c \in]a, b[: x^+(c) \neq x^-(c)\}$, then $[a, b] \setminus E_x$ is a null set and S_x is at most denumerable.

Let $\{I\}$ be the family of all closed subintervals of $[a, b]$ whose endpoints belong to E_x and let $\mathcal{D}_x$ be the collection of all finite subdivisions $D = [I_1, \ldots, I_N]$, with $I_i \in \{I\}$ and $\bigcup_{i=1}^N I_i = [\alpha^D, \beta^D]$.

Consider the mesh function $\delta : \mathcal{D}_x \to \mathbb{R}^+$ defined by

$$\delta(D) = \max\{\alpha^D - a, b - \beta^D, |I| : I \in D\},$$

which makes $\mathcal{D}_x$ a directed set.

Consider the function $\Delta x : \{I\} \to \mathbb{R}^m$ defined by

$$\Delta x(I) = \Delta x([\alpha, \beta]) = x^-(\beta) - x^+(\alpha).$$

Let $V^*(x)$ be the generalized variation of x, that is,

$$V^*(x) = V(\Delta x, E_x).$$

The BC-integrals of the interval functions Δx and $|\Delta x|$ can be extended to regular measures (see Cesari [17], Breckenridge [14], Brandi & Salvadori [7]); more precisely, there is a regular measure $\mu : \mathcal{B}([a, b]) \to \mathbb{R}^m$ of bounded variation such that

$$\mu(G) = \text{BC-}\int_G \Delta x \quad \text{and} \quad |\mu|(G) = \text{BC-}\int_G |\Delta x|, \quad G \subset [a, b],$$

where $|\mu|$ denotes the total variation of μ. For further properties of μ and $|\mu|$, we refer to Cesari [16, 17] and Brandi & Salvadori [6, 7, 9].

THEOREM 3.1 (Existence and Representation, see Cesari [16], Warner [20], Brandi & Salvadori [6, 12]). *Let $f : \mathbb{R}^m \to \mathbb{R}_0^+$ be a function satisfying (F_1) and (F_2), and let $x : [a, b] \to \mathbb{R}^m$ be a BV function. Then the interval function $\Phi : \{I\} \to \mathbb{R}_0^+$ defined by*

$$\Phi(I) = f(\Delta x(I))$$

is q.a. and BV with respect to $\mathcal{D}_x$ and δ; moreover,

$$\mathbf{W}(x) = \int_a^b f\left(\frac{d\mu}{d|\mu|}\right) d|\mu| \ (^5).$$

In particular, if x is absolutely continuous in the generalized sense (AC) (see Cesari [15]), then

$$\mathbf{W}(x) = \int_a^b f(\dot{x}(t))\, dt.$$

THEOREM 3.2 (Semicontinuity, see Brandi & Salvadori [11]). *Suppose that f is convex and satisfies (F_1) and (F_2). Let $x_n : [a, b] \to \mathbb{R}^m$, $n \geq 0$,*

(5) When there is no risk of confusion, we write $\mathbf{W}(x)$ instead of $\mathbf{W}(f, \Phi, \Delta x)$.

be a sequence of BV curves such that $\sup_{n\geq 0} V^*(x_n) \in \mathbb{R}$ *and* $x_n \to x_0$ *pointwise a.e. in* $[a,b]$. *Then*

$$\liminf_{n\to\infty} \mathbf{W}(x_n) \geq \mathbf{W}(x_0).$$

THEOREM 3.3 (Approximation, see also Brandi & Salvadori [10]). *Under the hypotheses of Theorem* 2.1, *for every* $x \in BV$ *there is a sequence* $(P_n)_n$ *of broken lines such that*

$$P_n \to x \quad in\ L_1([a,b])$$

and

$$\lim_n \mathbf{W}(P_n) = \lim_n \int_a^b f(\dot{P}_n(t))\,dt = \mathbf{W}(x).$$

Proof. Let $D_n \in \mathcal{D}_x$ be such that $\lim_n \delta(D_n) = 0$ and assume that D_{n+1} is a refinement of D_n.

Denote by P_n the broken line with vertices at those points of the graph of x which are the endpoints of the intervals of D_n. By Theorem 2 of Boni [1] and Lemma 4 of Boni & Brandi [2], we have $P_n \to x$ in $L_1([a,b])$.

Assuming that $D = [J]$ is a refinement of $D_n = [I]$, we have

$$\frac{\Delta P_n(I)}{|I|} = \frac{\Delta P_n(J)}{|J|} \quad \text{for every } J \subset I.$$

Now, if D is a refinement of D_n, we get

$$\begin{aligned}|\mathbf{W}(x) - \mathbf{W}(P_n)| &\leq \Big|\mathbf{W}(x) - \sum_{I\in D_n} f(\Delta x(I))\Big| \\ &\quad + \Big|\sum_{I\in D_n} f(\Delta x(I)) - \sum_{I\in D} f(\Delta P_n(I))\Big| \\ &\quad + \Big|\sum_{I\in D} f(\Delta P_n(I)) - \mathbf{W}(P_n)\Big| \\ &\leq \Big|\mathbf{W}(x) - \sum_{I\in D_n} f(\Delta x(I))\Big| \\ &\quad + \sum_{I\in D_n} \Big|f(\Delta P_n(I)) - \sum_{J\subset I} f(\Delta P_n(J))\Big| \\ &\quad + \Big|\sum_{I\in D} f(\Delta P_n(I)) - \mathbf{W}(P_n)\Big|;\end{aligned}$$

therefore, $\lim_n \mathbf{W}(P_n) = \mathbf{W}(x)$. ■

3.2. $\mathbf{W}(x)$ *as a length functional.* Since the result of this section is a straightforward consequence of Theorem 2.1, we use here the same notation of Section 2.

Let $\mathcal{K} \subset \mathbb{R}^m$ be connected and bounded. For every $p,q \in \mathcal{K}$, set

$$\mathbf{BV}^{\mathcal{K}}_{p,q} = \{x : [a,b] \to \mathcal{K},\ BV : x^+(a) = p,\ x^-(b) = q\}.$$

Put

$$\mathbf{BV}^{\mathcal{K}} = \bigcup_{p,q\in\mathcal{K}} \mathbf{BV}^{\mathcal{K}}_{p,q}.$$

Let σ be the following convergence on $\mathbf{BV}^{\mathcal{K}}$: a sequence $(x_n)_n \subset \mathbf{BV}^{\mathcal{K}}$ σ-converges to $x_0 \in \mathbf{BV}^{\mathcal{K}}$ if

- $x_n \to x_0$ pointwise a.e. in $[a,b]$;
- $\sup_n V^*(x_n) < \infty$.

In order to prove that the $\mathbf{W}$-integral over a $\mathbf{BV}^{\mathcal{K}}$ curve is a length functional as a consequence of Theorem 2.1, we should make some remarks. The functional $L(x)$ defined in Section 2 is meant in the classical sense, while $\mathbf{W}(x)$ is meant in a generalized sense (it makes no sense to consider the value $x(t_1^I)$ or $x(t_2^I)$). However, by the definitions of Section 3.1, each division $D \in \mathcal{D}_x$ is formed by points of essential continuity for x, therefore the value $x(t)$ for $t \in D$ might be meant as the essential limit of x at t. With this fact in mind, Theorem 2.1 still holds in a space of generalized functions such as $\mathbf{BV}^{\mathcal{K}}$. Therefore, we only have to check that $\mathbf{W}$, $\mathbf{BV}^{\mathcal{K}}$, and $\mathcal{K}$ satisfy the hypotheses of Theorem 2.1.

THEOREM 3.4. *Suppose that* f *is convex and satisfies* (F_1) *and* (F_2); *assume that the following coercivity condition holds*:

(F_3) $\quad\mathbf{W}(x) \geq MV^*(x)$ *for every* $x \in \mathbf{BV}^{\mathcal{K}}$, *with* $M \in \mathbb{R}^+$.

Then $\mathbf{W}$ *is the length functional on* $\mathbf{BV}^{\mathcal{K}}$ *with respect to the metric*

$$d_{\mathbf{W}}(p,q) = \inf_{x\in\mathbf{BV}^{\mathcal{K}}_{p,q}} \mathbf{W}(x).$$

Remark 3.5. Condition (F_3) is satisfied in the following remarkable case:

$$f(w) \geq M|w| \quad \text{for every } w \in \mathbb{R}^m, \text{ with } M \in \mathbb{R}^+.$$

Proof (of Theorem 3.4). The class $\mathbf{BV}^{\mathcal{K}}$ satisfies the approximation condition (v) by virtue of Theorem 2 of Boni [1] and Lemma 4 of Boni & Brandi [2]; furthermore, from (F_3) and the boundedness of $\mathcal{K}$, the level sets of $\mathbf{W}$ are σ-sequentially compact due to the Helly compactness theorem.

From Theorem 3.1 and the connectivity of $\mathcal{K}$, $\mathcal{K}$ is $\mathbf{W}$-connected. In order to prove that $\mathcal{K}$ is $\mathbf{W}$-nonatomic, let $(p_n)_n \subset \mathcal{K}$ with $\lim_n p_n = p_0$, and $(x_n)_n \subset \mathbf{BV}^{\mathcal{K}}$ be defined by

$$x_n(t) = \begin{cases} p_n & \text{if } t \in [a,(a+b)/2], \\ p_0 & \text{if } t \in \,](a+b)/2, b]; \end{cases}$$

we get

$$\mathbf{W}(x_n) = f(p_n - p_0)$$

and, since $f(0) = 0$,

$$\lim_n \mathbf{W}(x_n) = 0.$$

From Theorem 3.2, $\mathbf{W}$ is σ-l.s.c., and (F_3) easily implies the regularity property (iii).

It remains to prove that $\mathbf{W}$ is additive on $\mathbf{BV}^{\mathcal{K}}$, that is, if $x \in \mathbf{BV}^{\mathcal{K}}_{p,q}$ and $y \in \mathbf{BV}^{\mathcal{K}}_{q,r}$ then the function

$$\text{(a)} \qquad z(t) = \begin{cases} x(2t-a) & \text{if } t \in [a, (a+b)/2[, \\ q & \text{if } t = (a+b)/2, \\ y(2t-b) & \text{if } t \in](a+b)/2, b], \end{cases}$$

belongs to $\mathbf{BV}^{\mathcal{K}}_{p,r}$ and $\mathbf{W}(z) = \mathbf{W}(x) + \mathbf{W}(y)$.

To this end, given $D \in \mathcal{D}_z$ with $(a+b)/2 \in D$, we get

$$\begin{aligned} \sum_{I \in D} f(\Delta z(I)) &= \sum_{I\in D,\, t_2^I \le (a+b)/2} f(\Delta x(2I-a)) \\ &\quad + \sum_{I\in D,\, t_1^I \ge (a+b)/2} f(\Delta y(2I-b)) \\ &= \sum_{I\in D_x} f(\Delta x(I)) + \sum_{I \in D_y} f(\Delta y(I)) \\ &\quad + f(\Delta x(2[\beta^{D_x}, b] - a)) + f(\Delta y(2[a, \alpha^{D_y}] - b)) \end{aligned}$$

where $2I - a = \{t \in [a,b] : (t+a)/2 \in I\}$, $2I - b = \{t \in [a,b] : (t+b)/2 \in I\}$, and $D_x = \{I \in D : t_2^I < (a+b)/2\}$, $D_y = \{I \in D : t_1^I > (a+b)/2\}$.

Since $\lim_n \delta(D^n) = 0$, we get $\lim_n \delta(D^n_x) = 0$ and $\lim_n \delta(D^n_y) = 0$, and from the existence of $\mathbf{W}(x), \mathbf{W}(y)$, and $\mathbf{W}(z)$, we get $\mathbf{W}(z) = \mathbf{W}(x) + \mathbf{W}(y)$. ∎

Remark 3.6. We underline that, if $\mathbf{BV}^{\mathcal{K}}$ is replaced with $\mathbf{AC}^{\mathcal{K}} = \{x : [a,b] \to \mathcal{K},\ \text{AC}\}$, the boundedness of $\mathcal{K}$ can be dropped.

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