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Complemented subspaces with a strong finite-dimensional decomposition of nuclear Köthe spaces have a basis

by

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Abstract. The following result is proved: Let E be a complemented subspace with an r-finite-dimensional decomposition of a nuclear Köthe space $\lambda(A)$. Then E has a basis.

Introduction. In [3] Djakov and Mityagin constructed a large class of nuclear Fréchet spaces without a basis. Most nuclear Köthe spaces have both subspaces and quotient spaces belonging to this class (cf. Dubinsky [4] and Dubinsky and Mityagin [6]), but this method does not apply to complemented subspaces. In fact, the so-called Pełczyński's Problem [10], whether complemented subspaces of nuclear Köthe spaces have a basis, is one of the outstanding open questions in the theory of nuclear spaces. Although there are positive solutions in many special cases (Krone [8] contains a survey) the problem is still unsolved—even in the case of the complemented subspaces of s or $s^{\mathbb{N}}$.

The construction of Djakov and Mityagin is based on finite-dimensional decompositions. A nuclear Fréchet space E has an *r-finite-dimensional decomposition* (r-FDD, where r is a positive integer) if and only if it has a decomposition into r-dimensional spaces (Definition 0.1). E has a *strong finite-dimensional decomposition* (SFDD) iff it has an r-FDD for some $r \in \mathbb{N}$. Djakov and Mityagin showed that for every r there are nuclear Fréchet spaces with an $(r+1)$-FDD and without an r-FDD, hence there are nuclear Fréchet spaces with a 2-FDD and without a basis. In view of Pełczyński's problem the question arises whether there are complemented subspaces of nuclear Köthe spaces with a 2-finite-dimensional decomposition (resp. strong finite-dimensional decomposition) and without a basis. In the present article this question is solved in the negative; in fact, we show the following result:

THEOREM. *Every complemented subspace with a strong finite-dimensional decomposition of a nuclear Köthe space has a basis.*

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This result is well known in two important special cases. Wagner [12] has proved the theorem for complemented subspaces of s. In [2] Djakov and Dubinsky have obtained the above result if the projection onto the complemented subspace and the projections onto the finite-dimensional decomposition spaces satisfy certain compatibility conditions.

Our proof is very different from both Wagner's and Djakov and Dubinsky's approach. It is based on the fact that the matrix trace of a finite-dimensional projection in a Köthe space is equal to the dimension of the range (cf. Lemma 1.1). For the sake of completeness we include an easy direct proof of this fact. Finite-dimensional projections hence always have diagonal elements that are in some sense "large". Using this idea we show the splitting of every complemented subspace of a nuclear Köthe space with an $(r+1)$-finite-dimensional decomposition into a space with an r-finite-dimensional decomposition and a space with a basis (cf. Lemma 1.2). This lemma implies inductively the above stated theorem.

Preliminaries. We use the standard notations for Fréchet spaces (see [5], [7], [9]) and nuclear spaces (see [11]). We consider linear spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Let I be an index set (in this paper $\mathbb{N}$). An infinite matrix $A = (a_{i,m})_{i\in I, m\in\mathbb{N}}$ with

$$0 \le a_{i,m} \le a_{i,m+1}, \qquad \sup_m a_{i,m} > 0$$

for all i, m is called a *Köthe matrix*. The *Köthe space* $\lambda(A)$ is defined as

$$\lambda(A) = \Big\{ x = (x_i)_{i\in I} : \|x\|_m = \sum_{i\in I} |x_i| a_{i,m} < \infty \text{ for all } m \Big\}.$$

Equipped with the seminorms $(\|\cdot\|_m)_m$ it is a Fréchet space. It is nuclear iff for every m there is an $s(m) \ge m$ such that

$$\sum_i \frac{a_{i,m}}{a_{i,s(m)}} < \infty$$

where $0/a = 0$ for $a \ge 0$ (Grothendieck–Pietsch criterion).

According to Bessaga [1] we will assume without restriction for a nuclear Köthe space $\lambda(A)$ that the matrix has the following property: for every $k \in \mathbb{N}$ there exist $l \in \mathbb{N}$ and $C > 0$ such that

$$i^2 a_{i,k} \le C a_{i,l} \qquad \text{for all } i \in \mathbb{N}.$$

Definition 0.1. Let E be a nuclear Fréchet space.

(1) E has a *finite-dimensional decomposition* (FDD) iff there is a sequence $(P^n)_{n\in\mathbb{N}}$ of continuous linear operators (projections) $P^n : E \to E$ with $\dim P^n(E) < \infty$, $P^n \circ P^m = \delta_{n,m} P^n$ and $x = \sum_{n=1}^\infty P^n x$ for all $x \in E$.

(2) E has an *r-finite-dimensional decomposition* (r-FDD), $r \in \mathbb{N}$, iff E has a FDD and the sequence $(P^n)_n$ in (1) can be chosen such that $\dim P^n(E) \le r$.

(3) E has a *strong finite-dimensional decomposition* (SFDD) iff there exists an $r \in \mathbb{N}$ such that E has an r-FDD.

Let $(\|\cdot\|_k)_k$ be a fundamental system of seminorms which induces the topology of a nuclear Fréchet space E with a FDD. We define $\|x\|_k^\sim := \sum_n \|P^n x\|_k$. Then $(\|\cdot\|_k^\sim)_k$ is equivalent to $(\|\cdot\|_k)_k$. (The nuclearity of E implies that $\|x\|_k^\sim$ is finite for every $x \in E$. Since E is barreled the unit ball corresponding to $\|\cdot\|_k^\sim$ is a neighborhood of 0 in E; cf. [5] for the above definition and results.)

1. First we prove that the matrix trace of a finite-dimensional projection in a Köthe space is equal to the dimension of the range.

Lemma 1.1. *Let $\lambda(A)$ be a Köthe space and P be a continuous projection in $\lambda(A)$ with finite-dimensional range. Then*

$$\sum_{i=1}^\infty P_{ii} = \dim \operatorname{im} P.$$

Proof. Let $f^1, \dots, f^n$ be a basis of $\operatorname{im} P$ and $(e_i)_{i\in\mathbb{N}}$ the canonical basis of $\lambda(A)$. For $x = \sum_{i=1}^n x_i f^i \in \operatorname{im} P$ let $y^i(x) := x_i$ be the coordinate functional; we have $y^i \in (\operatorname{im} P)'$. We define $z^i := y^i \circ P \in \lambda(A)'$. We get

$$\begin{aligned} P_{jj} = e_j^*(P(e_j)) &= e_j^*\Big(\sum_{i=1}^n y^i(P(e_j)) f^i\Big) \\ &= e_j^*\Big(\sum_{i=1}^n z^i(e_j) f^i\Big) = \sum_{i=1}^n z^i(e_j) e_j^*(f^i). \end{aligned}$$

For $i = 1, \dots, n$ we have

$$1 = y^i(f^i) = y^i(P(f^i)) = z^i(f^i),$$

and

$$\sum_{j=1}^\infty e_j^*(f^i) e_j = f^i.$$

We apply z^i to the last equality and get

$$\sum_{j=1}^\infty e_j^*(f^i) z^i(e_j) = z^i(f^i) = 1.$$

We conclude

$$\sum_{j=1}^{\infty}\sum_{i=1}^{n} e_j^*(f^i)z^i(e_j) = \sum_{j=1}^{\infty} P_{jj} = n.$$

Now we prove the main lemma of this paper, which shows that every complemented subspace with an r-FDD of a nuclear Köthe space is the direct sum of a complemented subspace with an $(r-1)$-FDD and a complemented subspace with a basis.

LEMMA 1.2. *Let $\lambda(A)$ be a nuclear Köthe space and E be a complemented subspace with an r-FDD of $\lambda(A)$. Then $E \cong \lambda(B) \oplus F$ and F has an $(r-1)$-FDD.*

Proof. Let Q be the projection of $\lambda(A)$ onto E and P^n the projection of E according to the r-FDD. Then $x = \sum_n P^n(x)$ for every $x \in E$ and $P^i \circ P^j = \delta_{i,j}P^j$. We define

$$Q^n := P^n \circ Q \in L(\lambda(A), E) \subset L(\lambda(A)).$$

We get $Q^n \circ Q^m = \delta_{n,m}Q^m$. Let $(e_i)_{i\in\mathbb{N}}$ be the canonical basis of $\lambda(A)$. We put $Q_{ij}^n := e_i^*(Q^n(e_j))$ and we deduce that for all k there exist $C_k > 0$ and m_k such that

$$\sum_n \|Q^n(e_j)\|_k = \sum_n\sum_i a_{ik}|Q_{ij}^n| \le C_k a_{jm_k} \quad \forall j.$$

From Lemma 1.1 it follows that

$$\sum_{i=1}^{\infty} Q_{ii}^n \ge 1.$$

From the fact

$$\frac{1}{2}\sum_{i=1}^{\infty}\frac{1}{i^2} = \frac{1}{2}\frac{\pi^2}{6} < 1$$

we conclude that for every n there exists i_n such that

$$|Q_{i_n i_n}^n| \ge \frac{1}{2i_n^2}.$$

We define

$$T(x) := \sum_{n=1}^{\infty}\lambda_n e_{i_n}^*(Q^n(x))Q^n(e_{i_n}) \quad\text{with}\quad \lambda_n = \frac{1}{Q_{i_n i_n}^n}.$$

We divide the rest of the proof into 4 steps.

(1) We show that T is well defined and continuous as an operator from $\lambda(A)$ to $\lambda(A)$ and also as an operator from E to E. We estimate

$$\begin{aligned}\|T(e_j)\|_k &\le \sum_n |\lambda_n|\cdot|e_{i_n}^*(Q^n(e_j))|\cdot\|Q^n(e_{i_n})\|_k\\ &\le \sum_n 2i_n^2|e_{i_n}^*(Q^n(e_j))|C_k a_{i_n m_k}\\ &\le \sum_n |e_{i_n}^*(Q^n(e_j))|\widetilde{C}_k a_{i_n\widetilde{m}_k}\\ &\le \sum_n \widetilde{C}_k\|Q^n(e_j)\|_{\widetilde{m}_k} \le C_k' a_{jm_k'}.\end{aligned}$$

The second inequality holds by the choice of the i_n and the equicontinuity of the projections Q^n. These estimates show that T is a linear continuous operator from $\lambda(A)$ to $\lambda(A)$. We have $\operatorname{im} T \subset E$ and so we can also consider T as a continuous operator from E to E.

(2) We show that T is a projection in E:

$$\begin{aligned}T(T(x)) &= \sum_{n=1}^{\infty}\lambda_n e_{i_n}^*\Big(Q^n\Big(\sum_{m=1}^{\infty}\lambda_m e_{i_m}^*(Q^m(x))Q^m(e_{i_m})\Big)\Big)Q^n(e_{i_n})\\ &= \sum_{n=1}^{\infty}\lambda_n e_{i_n}^*\Big(\sum_{m=1}^{\infty}\lambda_m e_{i_m}^*(Q^m(x))Q^n(Q^m(e_{i_m}))\Big)Q^n(e_{i_n})\\ &= \sum_{n=1}^{\infty}\lambda_n e_{i_n}^*(\lambda_n e_{i_n}^*(Q^n(x))Q^n(e_{i_n}))Q^n(e_{i_n})\\ &= \sum_{n=1}^{\infty}\lambda_n^2 e_{i_n}^*(Q^n(x))e_{i_n}^*(Q^n(e_{i_n}))Q^n(e_{i_n})\\ &= \sum_{n=1}^{\infty}\lambda_n e_{i_n}^*(Q^n(x))Q^n(e_{i_n}) = T(x).\end{aligned}$$

The equality in the last line holds because

$$\lambda_n = \frac{1}{Q_{i_n i_n}^n} = \frac{1}{e_{i_n}^*(Q^n(e_{i_n}))}.$$

(3) $(f^n)_n = (Q^n(e_{i_n}))_n$ is a basis in $\operatorname{im} T$.

(4) We show that $\operatorname{im}(\mathrm{id}-T)$ has an $(r-1)$-FDD, where we consider $\mathrm{id}-T$ as an operator from E to E. We define

$$R^n(x) := \lambda_n e_{i_n}^*(Q^n(x))Q^n(e_{i_n}).$$

From $Q^n \circ Q^m = \delta_{n,m} Q^m$ and the definitions we conclude that

$$\begin{aligned} R^n(x) &= Q^n \circ T(x) = T(Q^n(x)),\\ R^n \circ Q^m &= \delta_{n,m} R^m = Q^n \circ R^m,\\ R^n \circ R^m &= 0 \quad \text{for } n \neq m,\\ R^n \circ R^n(x) &= (Q^n \circ T) \circ (T \circ Q^n)(x) = Q^n \circ T \circ Q^n(x)\\ &= Q^n \circ Q^n \circ T(x) = Q^n \circ T(x) = R^n(x). \end{aligned}$$

We proved that R^n is a projection in E. For $S^n := Q^n - R^n$ we have $S^n = Q^n - T \circ Q^n = (\mathrm{id} - T) Q^n$ and therefore $\operatorname{im} S^n \subset \operatorname{im}(\mathrm{id} - T)$. We get

$$\begin{aligned} S^n \circ S^m &= (Q^n - R^n)(Q^m - R^m) = Q^n Q^m - R^n Q^m - Q^n R^m + R^n R^m\\ &= \delta_{n,m}(Q^m - R^m - R^m + R^m) = \delta_{n,m}(Q^m - R^m) = \delta_{n,m} S^m, \end{aligned}$$

$$(\mathrm{id} - T)(x) = \sum_n (\mathrm{id} - T)(Q^n(x)) = \sum_n S^n(x).$$

This implies that $(S^n)_n$ defines a FDD on $\operatorname{im}(\mathrm{id} - T)$ and it remains to show that $\dim \operatorname{im} S^n \leq r - 1$.

Since R^n, S^n and Q^n can be considered as projections in $\lambda(A)$ with finite-dimensional range, we may apply Lemma 1.1. Since $S^n = Q^n - R^n$ and $0 \neq Q^n_{i_n, i_n} = e^*_{i_n}(Q^n(e_{i_n}))$ we obtain $\dim \operatorname{im} S^n = \dim \operatorname{im} Q^n - \dim \operatorname{im} R^n \leq r - 1$. So we proved that $\operatorname{im}(\mathrm{id} - T)$ has an $(r-1)$-FDD.

Now we are ready to prove the main result of this paper.

THEOREM 1.3. *Let $\lambda(A)$ be a nuclear Köthe space and E be a complemented subspace with an r-FDD of $\lambda(A)$. Then E has a basis.*

Proof (induction on r). For $r = 1$ the statement is trivial because E has a 1-FDD if and only if E has a basis.

Now we come to the induction step and assume that the theorem is true for all F with an $(r-1)$-FDD. Let E have an r-FDD. Then Lemma 1.2 implies $E \cong \lambda(B) \oplus F$ and F has an $(r-1)$-FDD and therefore a basis, which means $F \cong \lambda(C)$, $\lambda(C)$ a suitable nuclear Köthe space. We conclude

$$E \cong \lambda(B) \oplus \lambda(C),$$

hence E has a basis.

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