STUDIA MATHEMATICA 126 (3) (1997)

On non-primary Fréchet Schwartz spaces

by

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Abstract. Let E be a Fréchet Schwartz space with a continuous norm and with a finite-dimensional decomposition, and let F be any infinite-dimensional subspace of E. It is proved that E can be written as $G \oplus H$ where G and H do not contain any subspace isomorphic to F. In particular, E is not primary. If the subspace F is not normable then the statement holds for other quasinormable Fréchet spaces, e.g., if E is a quasinormable and locally normable Köthe sequence space, or if E is a space of holomorphic functions of bounded type $\mathcal{H}_b(U)$, where U is a Banach space or a bounded absolutely convex open set in a Banach space.

Introduction. A Fréchet space E is said to be *primary* if whenever $E = G \oplus H$ then either G or H is isomorphic to E. This property has been thoroughly studied for Banach spaces, indeed classical Banach spaces are primary; but very little is known for non-Banach Fréchet spaces. In fact, it is folklore that the space $\omega = \mathbb{K}^{\mathbb{N}}$ is primary (actually, every infinite-dimensional closed subspace of ω is isomorphic to ω), but other examples arose quite recently. Thus, it has been proved that $X^{\mathbb{N}}$ is primary if X is: ℓ_p $(1 \le p \le \infty)$, c_0, or L_p $(1 \le p < \infty)$ (see [18], [23], [1] and [4]). The primary Fréchet spaces with a continuous norm known so far are: $\bigcap_{q>p} \ell_q$ $(1 \le p < \infty)$ [24], $\bigcap_{q<p} L_q$ $(1 < p \le \infty)$ [10], and the complementably universal element for the class of Fréchet spaces with a continuous norm and an unconditional basis (respectively, for the class of Köthe sequence spaces of order p, with $p \in [1, \infty) \cup \{0\}$), [11].

In this paper we prove that primariness does not occur in some rather large classes of non-Banach Fréchet spaces, e.g., Fréchet Schwartz spaces with continuous norm and finite-dimensional decomposition, quasinormable

1991 *Mathematics Subject Classification*: 46A04, 46A11, 46A45.
Key words and phrases: Fréchet spaces, primary spaces, Schwartz spaces, unconditional decompositions, spaces of Moscatelli type, holomorphic functions of bounded type.
The author is indebted to Pilar Rueda and Manuel Maestre for useful information concerning Fréchet spaces of holomorphic functions. He is also grateful to José Bonet and Paweł Domański for references and suggestions on Fréchet Schwartz spaces. The research has been partially supported by the DGICYT project no. PB94-0941.

locally normable Köthe sequence spaces, and Köthe sequence spaces of Moscatelli type. Our main tool is a topological invariant developed by Zahariuta (see [16], [17] and [27]). To avoid solecisms all spaces are assumed to be infinite-dimensional unless the contrary is stated.

The paper is divided into two sections. In Section 1 we prove that if E is a Fréchet Schwartz space with a continuous norm and a finite-dimensional decomposition, and if F is any subspace of E then we can write $E = G \oplus H$ in such a way that neither G nor H contains a copy of F. This is a consequence of a more general result, namely Theorem 1.4, which also applies, if F is not normable, to other classes of quasinormable Fréchet spaces, including spaces of holomorphic functions of bounded type on Banach spaces.

In Section 2 we consider Köthe sequence spaces $\lambda_1(A)$ of Moscatelli type and prove that they are not primary. This is somehow surprising because their structure is rather simple. Indeed, these spaces can be described in a natural way as $(\ell_1)^{\mathbb{N}} \cap \ell_1(\ell_1(a_1), \ell_1(a_2), \ldots)$, where $a_k = (a_k(n))_n$ are unbounded sequences with $a_k(n) \ge 1$ for $k, n \in \mathbb{N}$. Moreover, this class of spaces has a complementably universal element, which so far seemed a likely candidate to be primary. We should also observe that the crucial Corollary 2.5 fails for some Fréchet spaces of Moscatelli type, according to [3, Proof of Theorem 2.1].

The main results are obtained for Fréchet spaces with a continuous norm. When a nonnormable Fréchet space E with a continuous norm is denoted as $(E, (V_k))$, we assume that (V_k) is a decreasing sequence of absolutely convex closed 0-neighbourhoods such that $(k^{-1}V_k)$ is a 0-neighbourhood basis, the Minkowski functional associated with V_1 is a norm on E, and moreover V_k and V_{k+1} do not induce equivalent topologies for every $k \in \mathbb{N}$. The gauge of a 0-neighbourhood V (resp. V_k) is denoted by $\|\cdot\|_V$ (resp. $\|\cdot\|_k$).

Let us introduce notation of Köthe sequence spaces. The reader is also referred to [7]. Given a countable index set I, a matrix $A = (a_k(i))_{i\in I}$ is said to be a *Köthe matrix* if $0 < a_k(i) \le a_{k+1}(i)$, $k \in \mathbb{N}$, $i \in I$. For every $p \in [1, \infty] \cup 0$ we define the *Köthe sequence space of order* p as

$$\lambda_p(I, A) = \lambda_p(A) := \Big\{(x_i) : \|(x_i)\|_k := \Big(\sum_{i=1}^{\infty} |x_i|^p a_k(i)\Big)^{1/p} < \infty,\ \forall k \in \mathbb{N}\Big\}, \quad 1 \le p < \infty,$$

$$\lambda_\infty(I, A) = \lambda_\infty(A) := \{(x_i) : \|(x_i)\|_k := \sup\{|x_i| a_k(i) : i \in \mathbb{N}\} < \infty,\ \forall k \in \mathbb{N}\}.$$

The closed subspace of $\lambda_\infty(A)$ of the elements such that $(x_i a_k(i))_i$ converges to 0 for all $k \in \mathbb{N}$ is denoted by $\lambda_0(A)$. Given a subset $J \subset I$, the *sectional subspace* of $\lambda_p(I, A)$ with respect to J is

$$\lambda_p(J, A) = \lambda_p(A_J) := \{x = (x_i) \in \lambda_p(I, A) : x_i = 0\ \forall i \notin J\}.$$

The element with entry 1 in the ith component and 0 elsewhere is denoted by e_i.

Given sets A and B we write $A < B$ to indicate that $A \subset \alpha B$ for some $\alpha \ge 1$. The cardinal of a set A is denoted by $|A|$.

For other unexplained functional analytic notions see [21] and [22].

1. Fréchet Schwartz spaces. If U and V are subsets of a vector space E and $\mathcal{E}_V$ is the set of all finite-dimensional subspaces of E *spanned by elements of* V we define

$$\beta(V, U) := \sup\{\dim L : L \in \mathcal{E}_V,\ L \cap U \subset V\}.$$

The definition of $\beta(\cdot,\cdot)$ has been given in [16] for absolutely convex sets U and V, but we need a more general notion. The next statement collects the basic properties of $\beta(\cdot,\cdot)$. The proofs are straightforward.

LEMMA 1.1. *Let E be a vector space and let $U, V \subset E$.*

(a) $\beta(A, B) \le \beta(V, U)$ *whenever* $A \subset V$ *and* $U \subset B$.

(b) *If T is an injective linear operator defined on E then* $\beta(T(V), T(U)) = \beta(V, U)$.

(c) $\beta(\alpha V, U) = \beta(V, \alpha^{-1}U)$ *for every scalar* $\alpha > 0$.

(d) *If S is a subspace of E then* $\beta(V \cap S, U \cap S) \le \beta(V, U)$.

LEMMA 1.2. *Let $(F, (V_k))$ be a nonnormable Fréchet space. For every scalar $\alpha > 0$ and for any $p < \min\{s, r\}$ there exists $t > 0$ such that*

$$\beta(V_p \cap tV_r, V_p \cup \alpha V_s) > 0.$$

Proof. Since the topologies induced by V_p and V_s are not equivalent there exists $x \in V_p \setminus \alpha V_s$. Hence, $\alpha V_s \cap [x] \subset V_p \cap [x]$ where $[x]$ is the subspace spanned by x. We take $t > 0$ such that $V_p \cap [x] \subset tV_r \cap [x]$. By Lemma 1.1(d),

$$\beta(V_p \cap tV_r, V_p \cup \alpha V_s) \ge \beta(V_p \cap tV_r \cap [x], (V_p \cup \alpha V_s) \cap [x]) = \beta(V_p \cap [x], V_p \cap [x]) = 1. \ \blacksquare$$

LEMMA 1.3. *Let $(F, (U_k))$ be a Fréchet space isomorphic to a subspace of $(E, (V_k))$. For every $k \in \mathbb{N}$ there exist $k < \sigma(k) < \tau(k) < \sigma(k+1) < \tau(k+1) < \sigma(k+2) < \tau(k+2)$ and $M = M(k) > 0$ such that for every couple of scalars s and t one has*

$$\beta(U_{\sigma(k)} \cap tU_{\sigma(k+2)}, U_{\sigma(k)} \cup sU_{\sigma(k+2)}) \le \beta(M(V_k \cap tV_{\tau(k+1)}), V_{\tau(k)} \cup sV_{\tau(k+2)}).$$

Proof. Let T denote an isomorphism from F into E. Given $k \in \mathbb{N}$ we can find $\sigma(i)$'s and $\tau(i)$'s, $i = k, k+1, k+2$, satisfying the order in the statement and such that

$$V_k \cap T(F) > T(U_{\sigma(k)}) > V_{\tau(k)} \cap T(F) > T(U_{\sigma(k+1)}) > V_{\tau(k+1)} \cap T(F) > T(U_{\sigma(k+2)}) > V_{\tau(k+2)} \cap T(F).$$

We take M, depending on k, such that the inclusion $A \subset M^{1/2}B$ holds for every couple of sets $A < B$ in the above chain. To finish, for arbitrary s and t we have

$$\begin{aligned}\beta(U_{\sigma(k)} &\cap tU_{\sigma(k+2)}, U_{\sigma(k)} \cup sU_{\sigma(k+2)}) \\ &= \beta(T(U_{\sigma(k)}) \cap tT(U_{\sigma(k+2)}), T(U_{\sigma(k)}) \cup sT(U_{\sigma(k+2)})) \\ &\le \beta(M^{1/2}(V_k \cap tV_{\tau(k+1)}) \cap T(F), M^{-1/2}(V_{\tau(k)} \cup sV_{\tau(k+2)}) \cap T(F)) \\ &\le \beta(M(V_k \cap tV_{\tau(k+1)}), V_{\tau(k)} \cup sV_{\tau(k+2)}). \ ■\end{aligned}$$

DEFINITION. A *(Schauder) decomposition* of a Fréchet space $(E,(V_k))$ is a sequence (P_n) of continuous linear projections defined on E such that $P_i \cdot P_j = \delta_{i,j} P_i$ $(i, j \in \mathbb{N})$, and $x = \sum_{n=1}^{\infty} P_n(x)$ for every $x \in E$. The decomposition is said to be: *unconditional* if the series converges unconditionally; *finite-dimensional* if $\dim P_n(E) < \infty$ for every n; *normable* if $P_n(E)$ is a normable subspace of E, for every n. A decomposition is said to have the *property* (S) if for every $k \in \mathbb{N}$ there exists $k' > k$ (we can assume $k' = k+1$) such that

$$(S) \qquad \lim_{n\to\infty} \sup\Big\{\Big\|x - \sum_{j=1}^{n} P_j(x)\Big\|_k : \|x\|_{k+1} \le 1\Big\} = 0.$$

Equivalently, for every $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that for every $n \ge m$ we have

$$\sum_{j=n}^{\infty} P_j(V_{k+1}) \subset \varepsilon V_k.$$

Finally, the decomposition is said to have the *property* (qn_0) if it is a normable decomposition and there exists a 0-neighbourhood, say V_1, such that $V_1 \cap P_n(E)$ induces the topology of $P_n(E)$ for every $n \in \mathbb{N}$. (Note that $(S)+(qn_0)$ implies that E is a quasinormable Fréchet space with continuous norm. Moreover, E is quasinormable by operators in the sense of Peris [25].)

Examples of Fréchet spaces with an unconditional decomposition with properties (S) and (qn_0) are given after the following theorem.

THEOREM 1.4. *Let $(E,(V_k))$ be a Fréchet space with an unconditional decomposition (P_n) having the properties (S) and (qn_0). Let F be any non-normable subspace of E. Then the space E can be written as a direct sum $G \oplus H$ where neither G nor H contains a copy of F. In particular, E is not primary.*

Proof. Define $Q_n = \sum_{j=1}^{n} P_j$ and $R_n = \mathrm{id} - Q_n$. We can assume that

$$(1) \qquad Q_n(V_k) \subset V_k, \quad R_n(V_k) \subset V_k, \quad \forall k, n \in \mathbb{N}.$$

(In the other case we define new seminorms

$$|||x|||_k := \sup\Big\{\Big\|\sum_{j=n}^{m} P_j(x)\Big\|_k : n < m\Big\}.$$

The sequence $(|||\cdot|||_k)$ of seminorms induces the same topology as $(\|\cdot\|_k)$, the properties (S) and (qn_0) still hold, and (1) is satisfied.) We denote by (U_k) a 0-neighbourhood basis of F. Now we select sequences (m_n), (s_n) and (t_n) of integers, with $m_n < m_{n+1}$ and $s_n < t_n$, such that

$$(2) \qquad nV_1 \cap Q_{m_n}(E) \subset \tfrac{1}{3} s_n V_{n^2} \cap Q_{m_n}(E),$$

$$(3) \qquad \beta(U_j \cap t_n U_{nj}, U_j \cup s_n U_{j+1}) > 0, \quad \forall j = 1, \ldots, n,$$

$$(4) \qquad nt_n V_{q+1} \cap R_{m_{n+1}}(E) \subset \tfrac{1}{3} V_q \cap R_{m_{n+1}}(E), \quad \forall q \le n+1.$$

For $n = 1$ take $m_1 = 1$. In this case, inclusion (2) is obvious with $s_1 = 3$. By Lemma 1.2 we choose $t_1 > s_1$ to obtain (3). Then we use the property (S) to find a suitable $m_2 > 1$ for the expression (4). Assume that we have already found s_i, t_i and m_{i+1} for $i \le n-1$. By the property (qn_0), the neighbourhood V_1 induces the topology of $Q_{m_n}(E)$, thus we can choose s_n which fulfils (2). Then we use Lemma 1.2 to fix $t_n > s_n$ such that (3) holds. To finish, by (S) there exists $m_{n+1} > m_n$ such that the inclusions in (4) are satisfied. The construction proceeds by induction.

Since the decomposition is unconditional we see that

$$\sum_{i=1}^{\infty} Q_{m_{2i}} - Q_{m_{2i-1}} \quad \text{and} \quad Q_{m_1} + \sum_{i=1}^{\infty} Q_{m_{2i+1}} - Q_{m_{2i}}$$

are well defined continuous projections onto subspaces G and H, respectively. Clearly, $E = G \oplus H$. Let us check that F is not isomorphic to a subspace of G. The other case is analogous.

By contradiction, if F is isomorphic to a subspace of G, given any $k \in \mathbb{N}$, there are $\sigma(i)$'s, $\tau(i)$'s $(i = k, k+1, k+2)$ and M fulfilling the assertion of Lemma 1.3. We select an odd integer n such that

$$(5) \qquad n \ge \sup\{M, \sigma(k), \tau(k), \sigma(k+2)/\sigma(k), \tau(k+2)/k\}.$$

On the one hand, by (5) and (3),

$$\begin{aligned}\beta(U_{\sigma(k)} &\cap t_n U_{\sigma(k+2)}, U_{\sigma(k)} \cup s_n U_{\sigma(k+2)}) \\ &\ge \beta(U_{\sigma(k)} \cap t_n U_{n\sigma(k)}, U_{\sigma(k)} \cup s_n U_{\sigma(k)+1}) > 0.\end{aligned}$$

On the other hand,

$$\beta(M(V_k \cap t_n V_{\tau(k+1)}) \cap G, (V_{\tau(k)} \cup s_n V_{\tau(k+2)}) \cap G) \le \beta(n(V_k \cap t_n V_{\tau(k)+1}) \cap G, (V_{\tau(k)} \cup s_n V_{nk}) \cap G).$$

There is a contradiction with Lemma 1.3 if we prove that the right hand side of the last inequality is equal to zero. In fact, assume on the contrary that there is $x \in G$, $x \neq 0$, such that

$$[x] \cap (V_{\tau(k)} \cup s_n V_{nk}) \subset n(V_k \cap t_n V_{\tau(k)+1}). \tag{6}$$

Set $\alpha := \sup\{\gamma : \gamma x \in V_{\tau(k)} \cup s_n V_{nk}\}$. By (6), αx belongs to $n(V_k \cap t_n V_{\tau(k)+1})$. Since x belongs to G and n has been chosen to be odd one has $(Q_{m_n} - Q_{m_{n-1}})(x) = 0$. Therefore we can write

$$\alpha x = Q_{m_{n-1}}(\alpha x) + R_{m_n}(\alpha x).$$

On account of (1), (4) and (5),

$$R_{m_n}(\alpha x) \in n(V_k \cap t_n V_{\tau(k)+1}) \cap R_{m_n}(E) \subset n t_n V_{\tau(k)+1} \cap R_{m_n}(E) \subset \tfrac{1}{3} V_{\tau(k)} \cap R_{m_n}(E) \subset \tfrac{1}{3}(V_{\tau(k)} \cup s_n V_{nk}).$$

In the same way, by (1), (2) and (5),

$$Q_{m_{n-1}}(\alpha x) \in n(V_k \cap t_n V_{\tau(k)+1}) \cap Q_{m_{n-1}}(E) \subset n V_k \cap Q_{m_{n-1}}(E) \subset n V_1 \cap Q_{m_{n-1}}(E) \subset \tfrac{1}{3} s_n V_{n^2} \cap Q_{m_{n-1}}(E) \subset \tfrac{1}{3} s_n V_{nk} \subset \tfrac{1}{3}(V_{\tau(k)} \cup s_n V_{nk}).$$

Altogether we see that αx belongs to $\frac{2}{3}(V_{\tau(k)} \cup s_n V_{nk})$, contrary to the choice of α. This finishes the proof. ∎

We state separately the main particular cases of Theorem 1.4.

COROLLARY 1.5. *Let E be a Fréchet Schwartz space with a continuous norm and a finite-dimensional decomposition. If F is any subspace of E then we can write $E = G \oplus H$ where neither G nor H contains a subspace isomorphic to F.*

Proof. It follows from [5, Theorem 2] that E has an unconditional finite-dimensional decomposition (P_n). The decomposition has the property (S) because E is Schwartz. (Incidentally, a Fréchet space with a finite-dimensional decomposition is Schwartz if and only if the decomposition satisfies (S).) The property (qn_0) holds since E has a continuous norm and the decomposition is finite-dimensional. ∎

Remark. The proof of Theorem 1.4 does not work for Fréchet spaces without a continuous norm. Therefore we do not know if Corollary 1.5 holds for a countable product of Fréchet Schwartz spaces. In particular, if s denotes the space of rapidly decreasing sequences, we do not know if the space $s^{\mathbb{N}}$, which is universal for the class of nuclear Fréchet spaces, is primary or not.

DEFINITION. A Fréchet space E is *locally normable* if there is a continuous norm on E such that the topology induced by this norm and the topology of the space coincide on every bounded subset of E.

The local normability condition was introduced by Terzioğlu and Vogt [26] to characterize the Köthe sequence spaces of order one whose bidual does not have a continuous norm. Köthe sequence spaces which are quasinormable and locally normable were characterized in [8, Corollary 8]. Every non-locally normable space $\lambda_p(B)$ is universal for the class of Köthe sequence spaces of order p (see [12]).

COROLLARY 1.6. *Let $\lambda_p(I,A)$ be a quasinormable Köthe space of order $p \in [1,\infty] \cup \{0\}$. If $\lambda_p(I,A)$ is not normable then the following conditions are equivalent:*

(a) *$\lambda_p(I,A)$ is locally normable.*

(b) *For every nonnormable subspace $F \subset \lambda_p(I,A)$ there exists a subset $J \subset I$ such that neither $\lambda_p(A_J)$ nor $\lambda_p(A_{I\setminus J})$ contains a copy of F.*

(c) *There exists $J \subset I$ such that neither $\lambda_p(A_J)$ nor $\lambda_p(A_{I\setminus J})$ contains a copy of $\lambda_p(I,A)$.*

Proof. (a)⇒(b). Since $\lambda_p(I,A)$ is quasinormable and locally normable there is a sequence (J_n) of pairwise disjoint sets with $I = \bigcup_{n\ge1} J_n$ such that (P_n) is an unconditional decomposition of $\lambda_p(I,A)$ which satisfies (S) and (qn_0), where P_n is the canonical projection onto $\lambda_p(J_n, A)$ (see [8, Corollary 8]). (We observe that Corollary 8 of [8] is stated for $p \in [1,\infty) \cup \{0\}$ but it extends to $p = \infty$.) Thus, we can apply Theorem 1.4. Moreover, it follows from its proof that the subspaces G and H are sectional subspaces of $\lambda_p(I,A)$.

(b)⇒(c) is clear.

(c)⇒(a). Given $J \subset I$, if $\lambda_p(I,A)$ is not locally normable then $\lambda_p(A_J)$ or $\lambda_p(A_{I\setminus J})$ is not locally normable. Assume that $\lambda_p(A_J)$ is not locally normable. By [12, Proposition 4] the space $\lambda_p(A_J)$ is universal for the class of Köthe sequence spaces of order p, hence $\lambda_p(A_J)$ contains a copy of $\lambda_p(I,A)$. ∎

Remark. We can improve Corollary 1.6 if the space $\lambda_p(I,A)$ is isomorphic to $\ell_p(\lambda_p(I,A))$ and $p \neq \infty$. Under these hypotheses, if $\lambda_p(I,A) \cong F \oplus G$ then there exists $J \subset I$ such that F contains a complemented copy of $\lambda_p(A_J)$ and G contains a complemented copy of $\lambda_p(A_{I\setminus J})$ (see [11, Proposition 5]). Consequently, we can add the following statement to Corollary 1.6.

(d) *The space $\lambda_p(I,A)$ can be written as a direct sum $F \oplus G$ such that neither F nor G contains a copy of $\lambda_p(I,A)$.*

Unconditional decompositions with properties stronger than (S) have been widely used in infinite-dimensional holomorphy. (These decompositions are usually defined by canonical projections onto spaces of continuous n-homogeneous polynomials, $n \in \mathbb{N}$.) E.g., see the notion of $\mathcal{S}$-absolute decomposition in [14, Chapter 3] or [15, Definition 1.1], $\mathcal{R}$-Schauder decomposition in [19], and normal decomposition in [20]. Moreover, the property (qn_0) also occurs in the instances which appear in the next corollary (see [19, Examples 4 and 6] or [20, Examples]). Therefore, our next result is a direct consequence of Theorem 1.4.

COROLLARY 1.7. *Let U denote a Banach space X or a bounded absolutely convex open set of X, and let E denote one of the following Fréchet spaces:*

(a) *the space $\mathcal{H}_b(U)$ of all holomorphic functions of bounded type on U, endowed with the topology of uniform convergence on U-bounded sets;*

(b) *the subspace $\mathcal{H}_{\omega u}(U)$ (resp., $\mathcal{H}_{\omega^*}(U)$, if X is a dual space) of all holomorphic functions of bounded type on U which are weakly uniformly continuous (resp., weak*-uniformly continuous) on all U-bounded sets.*

If F is any nonnormable subspace of E then we can write $E = G \oplus H$ where neither G nor H has a subspace isomorphic to F.

To finish this section we show two methods which give Fréchet spaces with an unconditional decomposition with properties (S) and (qn_0). The statement and the proof of part (1) are similar to a prior result of Peris ([25, Proposition 3.4]).

PROPOSITION 1.8. *Let X and X_n denote Banach spaces, $n \in \mathbb{N}$. Let E be a Fréchet space with an unconditional decomposition (P_n) with properties (S) and (qn_0). Let λ be a Fréchet Schwartz space with an unconditional basis (e_n). The following spaces have an unconditional decomposition with properties (S) and (qn_0):*

(1) *the tensor product $X \widehat{\otimes}_\tau E$ for $\tau = \varepsilon$ or π;*

(2) *the vector-valued sequence space*

$$\lambda(X_n) := \Big\{(x_n) \in \prod_{n\in\mathbb{N}} X_n : \sum_n \|x_n\| e_n \in \lambda\Big\}.$$

Proof. We consider the decomposition $(\mathrm{id} \otimes P_n)$ in case (1). In case (2), we take as P_n the projection onto the nth component, for every $n \in \mathbb{N}$. The properties of the decompositions can be readily checked by the reader. ∎

2. Köthe spaces of Moscatelli type. In this section we deal with the projective limits $\mathrm{proj}_k(X_k, I_{k,k+1})$ of the Banach spaces

$$X_k = \ell_1(\ell_1(a_1), \ldots, \ell_1(a_k), \ell_1, \ell_1, \ldots),$$

where $a_k = (a_k(n))$ are unbounded weights with $a_k(n) \geq 1$ for $k, n \in \mathbb{N}$, and linking maps $I_{k,k+1} : X_{k+1} \to X_k$ defined as the canonical inclusion on the $(k+1)$th component and the identity on the rest. These are Fréchet spaces of Moscatelli type in the terminology of Bonet and Dierolf [9]. We write these spaces as Köthe sequence spaces $\lambda_1(\mathbb{N}^2, A)$ such that

(1) $a_k(i,j) = 1 \ \forall j, k \in \mathbb{N} \ \forall i > k$,
(2) $\sup_j a_k(k,j) = \infty \ \forall k \in \mathbb{N}$,
(3) $a_p(i,j) = a_q(i,j) \ \forall i, j \in \mathbb{N} \ \forall p, q \geq i$.

The classical nondistinguished space due to Köthe and Grothendieck [22, 31.7] is of this kind. For that reason, these spaces are called (KG) *spaces* in the sequel. The conditions (1), (2) and (3) of the (KG) spaces are widely used without further reference.

Our interest in the structure of (KG) spaces is due to several known facts:

(i) Every Köthe sequence space of order one which is isomorphic to a complemented subspace of a (KG) space is normable or isomorphic to a (KG) space. This is a consequence of Kondakov's Lemma [6, Propositions 5.2 and 5.3].

(ii) Any (KG) space is a universal element for the class of Köthe sequence spaces of order one [12].

(iii) No complemented subspace of a (KG) space is Montel [2].

(iv) The class of (KG) spaces contains a complementably universal element. Indeed, if $\lambda_1(G)$ denotes Köthe–Grothendieck's nondistinguished space, then it is readily checked that $\ell_1(\lambda_1(G))$ contains any other (KG) space as a complemented subspace. By Pełczyński's decomposition method this is (up to isomorphism) the only complementably universal element for the class of (KG) spaces. Thus, the space $\ell_1(\lambda_1(G))$ was a good candidate to be primary.

Our main result in this section is that no (KG) space is primary (Corollary 2.5). It is derived from Theorem 2.3 which provides a characterization of the Köthe spaces $\lambda_1(A)$ which are complemented in a given (KG) space $\lambda_1(B)$.

Once more, our main tool is the topological invariant $\beta(\cdot,\cdot)$ and the circle of ideas handled in [16]. For the sake of completeness we collect some basic facts about $\beta(\cdot,\cdot)$ in the framework of Köthe sequence spaces. Let $\mathcal{A}$ be the set of all sequences with positive terms. For any $a, b \in \mathcal{A}$ we set

$$ab = (a_i b_i), \quad a \wedge b = (\min(a_i, b_i)),$$
$$a^\alpha = (a_i^\alpha), \quad a \vee b = (\max(a_i, b_i)).$$

For any $a \in \mathcal{A}$ we define

$$U_a := \Big\{ (x_n) \in \omega : \|x\|_a := \sum_{n=1}^{\infty} |x_n| a_n \le 1 \Big\}.$$

We put $U_a^\alpha U_b^{1-\alpha} := U_{a^\alpha b^{1-\alpha}}$. Denote by $\overline{\operatorname{conv}}(B)$ the closed absolutely convex hull of B. The proof of the following properties can be seen in [16, Lemmas 4 and 5].

LEMMA 2.1. *Let $a, b \in \mathcal{A}$.*

(1) *$U_{a\vee b} \subset U_a \cap U_b \subset 2U_{a\vee b}$ and $U_{a\wedge b} = \overline{\operatorname{conv}}(U_a \cup U_b)$.*
(2) *If $\lambda_1(A)$ is a Köthe sequence space then*

$$\beta(U_a \cap \lambda_1(A), U_b \cap \lambda_1(A)) = |\{ i : a_i / b_i \le 1 \}|.$$

(3) *Let $\lambda_1(A)$ and $\lambda_1(B)$ be Köthe sequence spaces and let $T : \lambda_1(A) \to \lambda_1(B)$ be a linear operator such that*

$$T(U_a \cap \lambda_1(A)) \subset MU_c \quad \text{and} \quad T(U_b \cap \lambda_1(A)) \subset MU_d$$

for some $a, b, c, d \in \mathcal{A}$, and for some $M > 0$. Then for any $\alpha \in (0,1)$ we have

$$T(U_a^\alpha U_b^{1-\alpha} \cap \lambda_1(A)) \subset MU_c^\alpha U_d^{1-\alpha}.$$

One should note that in [16] the authors use a strict inequality to define U_a, but the above properties do not change with our definition. We also need the following elementary fact.

LEMMA 2.2. *Let F be a complemented subspace of a Fréchet space E, and let $P : E \to F$ be a continuous projection. If A_j and B_j are subsets of E such that $P(B_j) \subset A_j$ $(j \in J)$, then*

$$\Big(\overline{\operatorname{conv}}\Big(\bigcup_j B_j\Big)\Big) \cap F \subset \overline{\operatorname{conv}}\Big(\bigcup_j (A_j \cap F)\Big).$$

DEFINITION. A Köthe space $\lambda_1(I, A)$ is said to be *diagonally complemented* into a Köthe space $\lambda_1(I, B)$ if there exist an injective mapping $\gamma : I \to I$ and a continuous and open linear operator $T : \lambda_1(A) \to \lambda_1(B)$ such that $T(e_i) = t_i e_{\gamma(i)}$ for some $t_i > 0$ and every $i \in I$. If $\gamma(\cdot)$ is a bijection then we say that $\lambda_1(A)$ and $\lambda_1(B)$ are *diagonally isomorphic.*

THEOREM 2.3. *Let $\lambda_1(A)$ and $\lambda_1(B)$ be two (KG) spaces. The following conditions are equivalent*:

(a) *$\lambda_1(A)$ is diagonally complemented into $\lambda_1(B)$.*
(b) *$\lambda_1(A)$ is isomorphic to a complemented subspace of $\lambda_1(B)$.*
(c) *There exist increasing sequences $(\sigma(n))$ and $(\tau(n))$ of integers and a sequence of scalars $M_n \ge 1$ such that for every $k \in \mathbb{N}$ and every $M_k < s < t$, one has*

$$\begin{aligned} &|\{(i,j) : \sigma(k) < i \le \sigma(k+2),\ s \le a_{\sigma(k+2)}(i,j) \le t\}| \\ &\qquad \le |\{(i,j) : \tau(k) < i \le \tau(k+2),\ s/M_k \le b_{\tau(k+2)}(i,j) \le tM_k\}|. \end{aligned}$$

Proof. (b)⇒(c). Denote by $T : \lambda_1(A) \to \lambda_1(B)$ an isomorphism onto a subspace F, complemented in $\lambda_1(B)$. Let $P : \lambda_1(B) \to F$ be a continuous projection. By induction we choose increasing sequences $(\sigma(n))$ and $(\tau(n))$ of integers such that

$$\begin{aligned} T(U_{\sigma(k-1)}) &> V_{\gamma(k)} \cap F > V_{\tau(k)} \cap F > T(U_{\sigma(k)}) \\ &> T(U_{\sigma(k+1)}) > V_{\gamma(k+1)} \cap F > V_{\tau(k+1)} \cap F > T(U_{\sigma(k+2)}) \\ &> T(U_{\sigma(k+3)}) > V_{\gamma(k+2)} \cap F > V_{\tau(k+2)} \cap F > T(U_{\sigma(k+4)}), \end{aligned}$$

where $V_{\tau(i)}$ is selected, after choosing $V_{\gamma(i)}$, in such a way that $P(V_{\tau(i)}) < V_{\gamma(i)}$ for $i = k, k+1, k+2$. We now take M_k such that, for every couple of sets $A < B$ in the chain before, one has $A \subset (M_k/4)^{1/3} B$. Given $M_k < s < t$, we fix $0 < \alpha < 1$ such that $(tM_k/2)^{1/\alpha} < tM_k$. Note that

$$T^{-1} P(V_{\tau(k)}^\alpha V_{\tau(k+2)}^{1-\alpha}) \subset (M_k/4)^{1/3} (U_{\sigma(k-1)}^\alpha U_{\sigma(k+2)}^{1-\alpha})$$

by Lemma 2.1(3). By the properties of $\beta(\cdot,\cdot)$ and by Lemma 2.2 we have

$$\begin{aligned} &\beta(U_{\sigma(k)} \cap tU_{\sigma(k+4)}, \overline{\operatorname{conv}}(U_{\sigma(k)} \cup U_{\sigma(k-1)}^\alpha U_{\sigma(k+2)}^{1-\alpha} \cup sU_{\sigma(k+2)})) \\ &= \beta(T(U_{\sigma(k)}) \cap tT(U_{\sigma(k+4)}), \\ &\qquad \overline{\operatorname{conv}}(T(U_{\sigma(k)}) \cup T(U_{\sigma(k-1)}^\alpha U_{\sigma(k+2)}^{1-\alpha}) \cup sT(U_{\sigma(k+2)}))) \\ &\le \beta((M_k/4)^{1/3}(V_{\tau(k)} \cap tV_{\tau(k+2)}) \cap F, \\ &\qquad (M_k/4)^{-1/3} \overline{\operatorname{conv}}((V_{\gamma(k+1)} \cap F) \cup (P(V_{\tau(k)}^\alpha V_{\tau(k+2)}^{1-\alpha}) \cap F) \cup (sV_{\gamma(k+2)} \cap F))) \\ &\le \beta((M_k/4)^{1/3}(V_{\tau(k)} \cap tV_{\tau(k+2)}) \cap F, \\ &\qquad (M_k/4)^{-2/3} \overline{\operatorname{conv}}(V_{\tau(k+1)} \cup V_{\tau(k)}^\alpha V_{\tau(k+2)}^{1-\alpha} \cup sV_{\tau(k+2)}) \cap F) \\ &\le \beta((M_k/4)(V_{\tau(k)} \cap tV_{\tau(k+2)}), \overline{\operatorname{conv}}(V_{\tau(k+1)} \cup V_{\tau(k)}^\alpha V_{\tau(k+2)}^{1-\alpha} \cup sV_{\tau(k+2)})). \end{aligned}$$

Therefore, by Lemma 2.1(1), (2), and the basic properties of $\beta(\cdot,\cdot)$,

$$\begin{aligned} (1) \quad &\Big| \Big\{ (i,j) : \\ &\qquad \frac{\max\{a_{\sigma(k)}(i,j), a_{\sigma(k+4)}(i,j)/t\}}{\min\{a_{\sigma(k)}(i,j), a_{\sigma(k-1)}^\alpha(i,j) a_{\sigma(k+2)}^{1-\alpha}(i,j), a_{\sigma(k+2)}(i,j)/s\}} \le 1 \Big\} \Big| \\ &\le \Big| \Big\{ (i,j) : \\ &\qquad \frac{\max\{b_{\tau(k)}(i,j), b_{\tau(k+2)}(i,j)/t\}}{\min\{b_{\tau(k+1)}(i,j), b_{\tau(k)}^\alpha(i,j) b_{\tau(k+2)}^{1-\alpha}(i,j), b_{\tau(k+2)}(i,j)/s\}} \le \frac{M_k}{2} \Big\} \Big|. \end{aligned}$$

We now calculate the left hand side of (1), and estimate the right hand side. Depending on i, the weight $\max\{a_{\sigma(k)}(i,j), a_{\sigma(k+4)}(i,j)/t\}$ takes the following values:

$$\begin{cases} a_{\sigma(k)}(i,j) & \text{if } i \le \sigma(k),\\ \max\{1, a_{\sigma(k+4)}(i,j)/t\} & \text{if } \sigma(k) < i \le \sigma(k+4),\\ 1 & \text{if } i > \sigma(k+4).\end{cases}$$

On the other hand, the weight

$$\min\{a_{\sigma(k)}(i,j), a^{\alpha}_{\sigma(k-1)}(i,j)a^{1-\alpha}_{\sigma(k+2)}(i,j), a_{\sigma(k+2)}(i,j)/s\},$$

depending on i, is defined as

$$\begin{cases} a_{\sigma(k-1)}(i,j)/s & \text{if } i \le \sigma(k-1),\\ \min\{a^{1-\alpha}_{\sigma(k)}(i,j), a_{\sigma(k)}(i,j)/s\} & \text{if } \sigma(k-1) < i \le \sigma(k),\\ \min\{1, a_{\sigma(k+2)}(i,j)/s\} & \text{if } \sigma(k) < i \le \sigma(k+2),\\ 1/s & \text{if } i > \sigma(k+2).\end{cases}$$

It is readily checked that the inequality $\max\{\cdot,\cdot\} \le \min\{\cdot,\cdot,\cdot\}$ does not occur whenever $i \le \sigma(k)$ or $i > \sigma(k+2)$. If $\sigma(k) < i \le \sigma(k+2)$ then we obtain the inequality for the indices j such that $s \le a_{\sigma(k+2)}(i,j) \le t$. Hence, the left hand side of (1) is

$$(2) \qquad |\{(i,j) : \sigma(k) < i \le \sigma(k+2),\ s \le a_{\sigma(k+2)}(i,j) \le t\}|.$$

Let us estimate the right hand side of (1). As before, the weight

$$\max\{b_{\tau(k)}(i,j), b_{\tau(k+2)}(i,j)/t\}$$

takes the following values:

$$\begin{cases} b_{\tau(k)}(i,j) & \text{if } i \le \tau(k),\\ \max\{1, b_{\tau(k+2)}(i,j)/t\} & \text{if } \tau(k) < i \le \tau(k+2),\\ 1 & \text{if } i > \tau(k+2),\end{cases}$$

while $\min\{b_{\tau(k+1)}(i,j), b^{\alpha}_{\tau(k)}(i,j)b^{1-\alpha}_{\tau(k+2)}(i,j), b_{\tau(k+2)}/s\}$ is

$$\begin{cases} b_{\tau(k)}(i,j)/s & \text{if } i \le \tau(k),\\ \min\{b^{1-\alpha}_{\tau(k+2)}(i,j), b_{\tau(k+2)}/s\} & \text{if } \tau(k) < i \le \tau(k+1),\\ \min\{1, b_{\tau(k+2)}(i,j)/s\} & \text{if } \tau(k+1) < i \le \tau(k+2),\\ 1/s & \text{if } i > \tau(k+2).\end{cases}$$

Since $M_k/s < 1$, the inequality $\max\{\cdot,\cdot\} \le (M_k/2)\min\{\cdot,\cdot,\cdot\}$ does not hold if $i \le \tau(k)$ or $i > \tau(k+2)$. Two cases remain: (1) If $\tau(k+1) < i \le \tau(k+2)$, and the inequality holds, then $2s/M_k \le b_{\tau(k+2)}(i,j) \le tM_k$; (2) If $\tau(k) < i \le \tau(k+1)$, the inequality implies that $b_{\tau(k+2)}(i,j) \ge 2s/M_k$ and $b^{1-\alpha}_{\tau(k+2)}(i,j) \ge 2b_{\tau(k+2)}(i,j)/(tM_k)$. The latter inequality is equivalent to $b^{\alpha}_{\tau(k+1)}(i,j) \le tM_k/2$. Consequently, by the choice of α (recall that $tM_k/2 \le (tM_k)^{\alpha}$), the set of indices where the inequality holds is contained in

$$(3) \qquad \{(i,j) : \tau(k) < i \le \tau(k+2),\ s/M_k \le b_{\tau(k+2)} \le tM_k\}.$$

We obtain part (c) on account of (1), (2) and (3).

(c)$\Rightarrow$(a). We define the following subsets of $\mathbb{N}^2$:

$$J_0 := \{(i,j) : i \le \sigma(2)\},$$
$$J := \bigcup_{k\ge 1}\{(i,j) : \sigma(2k) < i \le \sigma(2k+2),\ M_{2k} < a_{\sigma(2k+2)}(i,j)\}.$$

Note that $\lambda_1(A_{J_0})$ is normable and that $a_k(i,j)$ is bounded on $\mathbb{N}^2 \setminus (J \cup J_0)$ for $k \ge \sigma(4)$. Hence, $\lambda_1(A_{\mathbb{N}^2\setminus J})$ is normable. Therefore, since $\lambda_1(B)$ can be written as $\ell_1 \oplus \lambda_1(B)$, it suffices to show that $\lambda_1(A_J)$ is diagonally complemented into $\lambda_1(B)$. By condition (c) and by [16, Lemma 2], for every $k \in \mathbb{N}$ there is an injective mapping

$$\varphi_k : (\{i : \sigma(2k) < i \le \sigma(2k+2)\} \times \mathbb{N}) \cap J \to \{i : \tau(2k) < i \le \tau(2k+2)\} \times \mathbb{N}$$

such that

$$\frac{a_{\sigma(2k+2)}(i,j)}{M_k^2} \le b_{\tau(2k+2)}(\varphi_k(i,j)) \le M_k^2 a_{\sigma(2k+2)}(i,j).$$

Consequently, there is a continuous and open linear operator $T : \lambda_1(A_J) \to \lambda_1(B)$ such that for every $(i,j) \in J$, given $k \in \mathbb{N}$ with $\sigma(2k) < i \le \sigma(2k+2)$, we have $T(e_{i,j}) = e_{\varphi_k(i,j)}$. This proves that $\lambda_1(A_J)$ is diagonally complemented into $\lambda_1(B)$. ∎

COROLLARY 2.4. *Let $\lambda_1(A)$ and $\lambda_1(B)$ be two (KG) spaces. The following conditions are equivalent*:

(a) *They are diagonally isomorphic.*
(b) *They are isomorphic.*
(c) *They contain each other as complemented subspaces.*

Proof. Only (c)$\Rightarrow$(a) needs a proof. By Theorem 2.3, both spaces contain each other as diagonally complemented subspaces, which actually implies that they are diagonally isomorphic (see [16, Lemma 1]). ∎

So far, there was some hope (in the opinion of the author) to find new primary spaces in the class of (KG) spaces. As mentioned before, the complementably universal element of the class (KG) was a firm candidate to be primary. But we can prove that no (KG) space is primary as an application of Theorem 2.3.

COROLLARY 2.5. *Let $\lambda_1(A)$ be any (KG) space. There exists $J \subset \mathbb{N}^2$ such that neither $\lambda_1(A_J)$ nor $\lambda_1(A_{\mathbb{N}^2\setminus J})$ contains a complemented subspace isomorphic to $\lambda_1(A)$. In particular, $\lambda_1(A)$ is not primary.*

Proof. We take $s_1 = 1$ and select $t_1 > s_1$ such that

$$|\{(1,j) : s_1 \le a_1(1,j) \le t_1\}| > 0.$$

Set $I_1 := \{(1,j) : s_1 \le a_1(1,j) \le t_1\}$. Take now s_2 with $s_2/2 > t_1$, and choose $t_2 > s_2$ such that

$$|\{(i,j) : s_2 \le a_i(i,j) \le t_2\}| > 0, \quad i = 1,2.$$

We put

$$I_2 := \{(i,j) : s_2/2 \le a_2(i,j) \le 2t_2,\ i = 1,2\}.$$

Note that I_1 and I_2 are disjoint. By induction, if we have already constructed

$$I_{n-1} = \left\{(i,j) : \frac{s_{n-1}}{n-1} \le a_{n-1}(i,j) \le (n-1)t_{n-1},\ 1 \le i \le n-1\right\},$$

then we fix s_n with $(n-1)t_{n-1} < s_n/n$, and choose t_n such that

(1) $$|\{(i,j) : s_n \le a_i(i,j) \le t_n\}| > 0, \quad \forall 1 \le i \le n.$$

Then we set

(2) $$I_n := \{(i,j) : s_n/n \le a_n(i,j) \le nt_n,\ 1 \le i \le n\}.$$

By the choice of s_n, the index set I_n is disjoint from I_j for all $j < n$. We define $J := \bigcup_{n\ge1} I_{2n-1}$. Let us check that $\lambda_1(A)$ is not isomorphic to any complemented subspace of $\lambda_1(A_J)$. Indeed, we prove that for every $k, \tau(k) \in \mathbb{N}$ and every $M \ge 1$, there are s and t such that

(3) $$|\{(k,j) : s \le a_k(k,j) \le t\}| > 0,$$

while

(4) $$|\{(i,j) \in J : i \le \tau(k),\ s/M \le a_{\tau(k)}(i,j) \le tM\}| = 0.$$

By Theorem 2.3, this is enough to conclude that $\lambda_1(A)$ is not isomorphic to a complemented subspace of $\lambda_1(A_J)$. Given any $k, \tau(k) \in \mathbb{N}$ and $M \ge 1$ we fix $p \in \mathbb{N}$ with $2p \ge \max\{k, \tau(k), M\}$. By (1) we have

$$|\{(k,j) : s_{2p} \le a_k(k,j) \le t_{2p}\}| > 0.$$

On the other hand, by (2),

$$\{(i,j) : i \le \tau(k),\ s_{2p}/M \le a_{\tau(k)}(i,j) \le Mt_{2p}\}$$
$$\subseteq \{(i,j) : s_{2p}/(2p) \le a_{2p}(i,j) \le 2pt_{2p},\ 1 \le i \le 2p\} = I_{2p} \subset \mathbb{N}^2 \setminus J,$$

which implies (4). To prove that $\lambda_1(A)$ is not isomorphic to a complemented subspace of $\lambda_1(A_{\mathbb{N}^2\setminus J})$ we just take $p \in \mathbb{N}$ with

$$2p - 1 \ge \max\{k, \tau(k), M\},$$

and proceed as before. ∎

The first part of Corollary 2.5 does not hold for some Fréchet spaces of Moscatelli type. In fact, for the space $E := (\ell_p)^{\mathbb{N}} \cap \ell_q(\ell_q)$, with $1 \le p < q \le \infty$, Albanese and Moscatelli [3, Proof of Theorem 2.1] have proved that if $E = G \oplus H$ then G or H contains a complemented copy of E.

As a further consequence of Theorem 2.3 we characterize the (KG) spaces $\lambda_1(A)$ which are isomorphic to their cartesian square. This property is important for structural reasons. Moreover, it ensures that the space of n-homogeneous polynomials and the space of n-linear forms, defined on $\lambda_1(A)$, are isomorphic (see [13]).

PROPOSITION 2.6. *A (KG) space $\lambda_1(A)$ is isomorphic to its cartesian square if and only if there exist increasing sequences $(\sigma(k))$, $(\tau(k))$ and (M_k) such that for every $M_k < s < t$, one has*

$$2|\{(i,j) : \sigma(k) < i \le \sigma(k+2),\ s \le a_{\sigma(k+2)}(i,j) \le t\}|$$
$$\le |\{(i,j) : \tau(k) < i \le \tau(k+2),\ s/M_k \le a_{\tau(k+2)}(i,j) \le tM_k\}|.$$

Proof. By Corollary 2.4, it suffices to show that the stated condition holds if and only if $(\lambda_1(A))^2$ is isomorphic to a complemented subspace of $\lambda_1(A)$. The space $(\lambda_1(A))^2$ can be written as $\lambda_1(\mathbb{N}^2, B)$ where

$$b_k(i,2j) = b_k(i,2j+1) = a_k(i,j)$$

for every $i,j,k \in \mathbb{N}$. Then our assertion can be readily obtained as a particular case of Theorem 2.3. ∎

EXAMPLE (A (KG) space which is not isomorphic to its cartesian square). We construct a matrix A with the following property: For every $k, \tau(k) \in \mathbb{N}$ and for every $M \ge 1$ there are s and t such that

$$|\{(i,j) : s/M \le a_{\tau(k)}(i,j) \le tM\}| \le |\{(k,j) : s \le a_k(k,j) \le t\}|.$$

We write $\mathbb{N}$ as the disjoint union of a countable family $(N_k)_{k\ge1}$ of infinite subsets. The elements of N_k are labelled as $\{j_k : j \in \mathbb{N}\}$. We define the weights

$$a_k(i,j) := j_i! \quad \text{if } i \le k, \qquad a_k(i,j) := 1 \quad \text{if } i > k.$$

Given any $k \in \mathbb{N}$ and $M \ge 1$, we choose $j_k \in N_k$ such that $j_k > M$. Note that, for any $i \in \mathbb{N}$, if $(j_k! - 1)/M \le i! \le (j_k! + 1)M$ then $i = j_k$. Therefore, putting $s = j_k! - 1$ and $t = j_k! + 1$ we have

$$|\{(k,j) : s \le a_k(k,j) \le t\}| = 1,$$

while for every $\tau(k)$,

$$|\{(i,j) : s/M \le a_{\tau(k)}(i,j) \le tM\}|$$

is one if $\tau(k) \ge k$ and zero if $\tau(k) < k$.

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Received December 16, 1996 (3803)
Revised version June 9, 1997