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Strongly continuous integrated C-cosine operator functions

by

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Abstract. We extend some recent results for regularized semigroups to strongly continuous n-times integrated C-cosine operator functions. Several equivalent conditions for the existence and uniqueness of solutions of (ACP_2) are also presented.

0. Introduction. The Hille–Yosida-type theorem for regularized semigroups was proved by G. Da Prato [D] in 1966, but it has been generally overlooked for many years. Since 1987 when it was reestablished by Davies and Pang [Dav-P] and, at the same time, the related notion of integrated semigroups was introduced by W. Arendt [A], the theory and applications of the subject have been extensively developed and the related literature has been rapidly growing. The technique developed during these years, common now for a good number of mathematicians working in this field, is clearly presented in the 1994 monograph [deL] by R. deLaubenfels.

A parallel theory for regularized cosine operator functions and integrated cosine operator functions has also been attracting the attention of many mathematicians (see [deL-P], [L-S1], [L-S2], [S-L1], [S-L2], [W-Wa] etc.)

It is well known that, in the classical theory, all strongly continuous semigroups and strongly continuous cosine operator functions are exponentially bounded. But unfortunately, this is no longer true for regularized semigroups [deL, P. 4] and regularized cosine operator functions (see examples in §3).

This paper will concentrate on regularized cosine operator functions which are strongly continuous on a sequentially complete locally convex space without assuming exponential boundedness. For this case, some new techniques are needed. §1 offers several equivalent conditions for the existence and the uniqueness of mild solutions of (ACP_2). In §2, we study the relation between $2n$-times (resp. $(2n+1)$-times) integrated C-cosine operator functions and $(s^2-A)^{-n}C$-cosine (resp. $(s^2-A)^{-n-1}C$-cosine) operator functions. In our last Section 3, two illustrative examples are given.

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1. Basic properties of integrated C-cosine operator functions and (ACP_2). Throughout this paper, X is a sequentially complete locally convex space and $L(X)$ is the algebra of all continuous linear operators on X. $C \in L(X)$ is injective.

Consider the following abstract Cauchy problem:

$$(ACP_2) \qquad \begin{cases} u''(t,x,y) = Au(t,x,y), \\ u(0,x,y) = x, \quad u'(0,x,y) = y, \quad x,y \in X. \end{cases}$$

DEFINITION 1.1. A *solution* of (ACP_2) is a function $t \mapsto u(t,x,y)$ belonging to $C(\mathbb{R},[D(A)]) \cap C^2(\mathbb{R},X)$ and satisfying (ACP_2). An *n-times integrated mild solution* of (ACP_2) is a function $t \mapsto v(t,x,y)$ belonging to $C([0,\infty),X)$ such that for all $t \ge 0$, $\int_0^t (t-r)v(r,x,y)\,dr \in D(A)$, with

$$(*) \qquad A\int_0^t (t-r)v(r,x,y)\,dr = v(t,x,y) - \frac{t^n}{n!}x - \frac{t^{n+1}}{(n+1)!}y, \quad \forall x,y \in X.$$

A family $\{C(t)\}_{t\ge0} \subseteq L(X)$ is said to be *strongly continuous* if $C(\cdot)x$ is continuous on $[0,\infty)$ for every $x \in X$.

It is easily seen that the uniqueness of solutions of (ACP_2) and the uniqueness of n-times integrated mild solutions of (ACP_2) are equivalent.

DEFINITION 1.2 ([L-S1] and [S-L2]). Suppose $n \in \mathbb{N}$. A strongly continuous family $\{C(t)\}_{t\ge0}$ in $L(X)$ is called an *n-times integrated C-cosine operator function* if

(a) $CC(t) = C(t)C$ for every $t \ge 0$, and $C(0) = 0$;
(b) for $s,t \ge 0$ and $x \in X$,

$$\begin{aligned} 2C(t)C(s)x = \frac{1}{(n-1)!}\Big\{&(-1)^n \int_0^{|s-t|} (|s-t|-r)^{n-1}C(r)Cx\,dr \\ &+ \Big[\int_0^{s+t} - \int_0^t - \int_0^s\Big](s+t-r)^{n-1}C(r)Cx\,dr \\ &+ \int_0^t (s-t+r)^{n-1}C(r)Cx\,dr + \int_0^s (t-s+r)^{n-1}C(r)Cx\,dr\Big\}. \end{aligned}$$

A strongly continuous C-cosine operator function is called a *0-times integrated C-cosine operator function*.

The n-times integrated C-cosine operator function $\{C(t)\}_{t\ge0}$ is said to be *nondegenerate* if $C(t)x = 0$, for all $t \ge 0$, implies $x = 0$.

All n-times integrated C-cosine operator functions in this paper are assumed to be nondegenerate. It is easy to see that a strongly continuous integrated C-cosine operator function is locally equicontinuous. For convenience, we denote the right-hand side of the equality in Definition 1.2(b) by $J_n(t,s)x$. Thus we have

$$(1.1) \qquad 2C(t)C(s)x = J_n(t,s)x.$$

DEFINITION 1.3 ([S-L2]). Suppose A is closed and $\{C(t)\}_{t\ge0}$ is an n-times integrated C-cosine operator function. A is a *subgenerator* of $\{C(t)\}_{t\ge0}$ if

(a) $C(t)A \subseteq AC(t)$ for every $t \ge 0$;
(b) $\int_0^t (t-r)C(r)x\,dr \in D(A)$ and

$$(1.2) \qquad A\int_0^t (t-r)C(r)x\,dr = C(t)x - \frac{t^n}{n!}Cx, \quad \forall t \ge 0 \text{ and } x \in X.$$

We also say that $\{C(t)\}_{t\ge0}$ is an *n-times integrated C-cosine operator function for A*, or that A has an n-times integrated C-cosine operator function $\{C(t)\}_{t\ge0}$.

LEMMA 1.4. *Suppose that A is a subgenerator of $\{C(t)\}_{t\ge0}$. If $x \in D(A)$, then $C(t)x$ is differentiable and*

$$\frac{d}{dt}C(t)x = \int_0^t C(r)Ax\,dr + \frac{t^{n-1}}{(n-1)!}Cx.$$

Proof. By closedness of A and Definition 1.3(b). ■

The following theorem asserts that a strongly continuous operator family $\{C(t)\}_{t\ge0}$ satisfying (a) and (b) of Definition 1.3 is automatically an n-times integrated C-cosine operator function.

THEOREM 1.5. *Suppose A is closed and $\{C(t)\}_{t\ge0}$ is a strongly continuous family of linear continuous operators. Then $\{C(t)\}_{t\ge0}$ is an n-times integrated C-cosine operator function with A as a subgenerator if and only if one of the following conditions is true.*

(I) (a) *and* (b) *of Definition* 1.3 *hold.*
(II) (i) $CA \subseteq AC$;
  (ii) (b) *of Definition* 1.3 *holds*;
  (iii) *all solutions of* (ACP_2) *are unique.*

Proof. (I)⇒(II). We only have to prove that (i) and (iii) of (II) hold. Let $x \in D(A)$. From $C(t)A \subseteq AC(t)$ and

$$\begin{aligned} C(t)Ax - \frac{t^n}{n!}CAx &= A\int_0^t (t-r)C(r)Ax\,dr \\ &= A^2\int_0^t (t-r)C(r)x\,dr = AC(t)x - \frac{t^n}{n!}ACx, \end{aligned}$$

we have $\mathcal{C}Ax = A\mathcal{C}x$. Now we prove (iii). It suffices to prove that the n-times integrated mild solutions of (ACP_2) are unique. To do this, we prove that the function identically equal to zero is the only solution to the equation

$$A\int_0^t (t-r)v(r)\,dr = v(t), \quad v(t)\in C([0,\infty),X). \tag{1.3}$$

Let $v(t)$ be a solution of (1.3). Then, by Lemma 1.4,

$$\begin{aligned}\frac{d}{ds}\Big[C(t-s)\int_0^s (s-r)v(r)\,dr\Big] &= -A\int_0^{t-s} C(\alpha)\,d\alpha\int_0^s (s-r)v(r)\,dr \\ &\quad - \frac{(t-s)^{n-1}}{(n-1)!}\mathcal{C}\int_0^s (s-r)v(r)\,dr + C(t-s)\int_0^s v(r)\,dr \\ &= -\int_0^{t-s} C(\alpha)v(s)\,d\alpha - \frac{(t-s)^{n-1}}{(n-1)!}\mathcal{C}\int_0^s (s-r)v(r)\,dr \\ &\quad + C(t-s)\int_0^s v(r)\,dr.\end{aligned} \tag{1.4}$$

Integrate (1.4) in s from 0 to t to obtain

$$\begin{aligned}0 = &-\int_0^t\int_0^{t-s} C(\alpha)v(s)\,d\alpha\,ds - \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}\mathcal{C}\int_0^s (s-r)v(r)\,dr\,ds \\ &+\int_0^t C(t-s)\int_0^s v(r)\,dr\,ds.\end{aligned}$$

Since

$$-\int_0^t\int_0^{t-s} C(\alpha)v(s)\,d\alpha\,ds + \int_0^t C(t-s)\int_0^s v(r)\,dr\,ds = 0,$$

we have

$$\int_0^t \frac{(t-s)^{n-1}}{(n-1)!}\mathcal{C}\int_0^s (s-r)v(r)\,dr\,ds = 0.$$

Differentiating the above equality $n+1$ times in t, we get $v(t)\equiv 0$.

Now assume (II) holds. We prove that $\{C(t)\}_{t\ge 0}$ is an n-times integrated $\mathcal{C}$-cosine operator function with A as a subgenerator.

We first prove that $C(t)\mathcal{C} = \mathcal{C}C(t)$ for all $t\ge 0$. From $\mathcal{C}A\subseteq A\mathcal{C}$ and

from (1.2), we get

$$A\int_0^t (t-r)\mathcal{C}C(r)x\,dr = \mathcal{C}A\int_0^t (t-r)C(r)x\,dr = \mathcal{C}C(t)x - \frac{t^n}{n!}\mathcal{C}^2x.$$

Condition (1.2) also implies that

$$A\int_0^t (t-r)C(r)\mathcal{C}x\,dr = C(t)\mathcal{C}x - \frac{t^n}{n!}\mathcal{C}^2x.$$

Hence $C(s)\mathcal{C} = \mathcal{C}C(s)$ by the uniqueness of solutions.

By (i) and (ii) of (II), for $s\ge 0$,

$$A\int_0^t (t-r)C(r)C(s)x\,dr = C(t)C(s)x - \frac{t^n}{n!}\mathcal{C}C(s)x. \tag{1.5}$$

We now claim that $J_n(t,s)$ in (1.1), as a function of t, satisfies

$$A\int_0^t (t-r)J_n(r,s)x\,dr = J_n(t,s)x - \frac{t^n}{n!}2\mathcal{C}C(s)x. \tag{1.6}$$

We prove (1.6) only for $s\ge t$ and $n\ge 2$. The case of $n=1$ is much easier. Write

$$\int_0^t (t-\alpha)J_n(\alpha,s)x\,d\alpha \equiv I_1+\ldots+I_6$$

where

$$I_1 \equiv \frac{1}{(n-1)!}\int_0^t\int_0^{s-\alpha}(t-\alpha)(s-\alpha-r)^{n-1}\mathcal{C}C(r)x\,dr\,d\alpha;$$

$$I_2 \equiv \frac{1}{(n-1)!}\int_0^t\int_0^{s+\alpha}(t-\alpha)(s+\alpha-r)^{n-1}\mathcal{C}C(r)x\,dr\,d\alpha;$$

$$I_3 \equiv -\frac{1}{(n-1)!}\int_0^t\int_0^{\alpha}(t-\alpha)(s+\alpha-r)^{n-1}\mathcal{C}C(r)x\,dr\,d\alpha;$$

$$I_4 \equiv -\frac{1}{(n-1)!}\int_0^t\int_0^{s}(t-\alpha)(s+\alpha-r)^{n-1}\mathcal{C}C(r)x\,dr\,d\alpha;$$

$$I_5 \equiv \frac{1}{(n-1)!}\int_0^t\int_0^{\alpha}(t-\alpha)(s-\alpha-r)^{n-1}\mathcal{C}C(r)x\,dr\,d\alpha;$$

$$I_6 \equiv \frac{1}{(n-1)!}\int_0^t\int_0^{s}(t-\alpha)(\alpha-s+r)^{n-1}\mathcal{C}C(r)x\,dr\,d\alpha.$$

Define

$$K(t) \equiv CC(t)x - \frac{t^n}{n!}C^2x.$$

We have

$$I_1 = \frac{1}{n!}\int_0^s t(s-r)^n CC(r)x\,dr + \frac{1}{(n+1)!}\int_0^{s-t}(s-t-r)^{n+1}CC(r)x\,dr - \frac{1}{(n+1)!}\int_0^s (s-r)^{n+1}CC(r)x\,dr.$$

(Simple proof by differentiating in t and changing the order of integration.) Thus, by (i)–(ii),

$$AI_1 = \frac{1}{(n-2)!}\int_0^s t(s-r)^{n-2}K(r)\,dr + \frac{1}{(n-1)!}\int_0^{s-t}(s-t-r)^{n-1}K(r)\,dr - \frac{1}{(n-1)!}\int_0^s (s-r)^{n-1}K(r)\,dr$$

Moreover,

$$I_2 = -\frac{1}{n!}\int_0^s t(s-r)^n CC(r)x\,dr + \frac{1}{(n+1)!}\int_0^{s+t}(s+t-r)^{n+1}CC(r)x\,dr - \frac{1}{(n+1)!}\int_0^s (s-r)^{n+1}CC(r)x\,dr$$

and

$$AI_2 = -\frac{1}{(n-2)!}\int_0^s t(s-r)^{n-2}K(r)\,dr + \frac{1}{(n-1)!}\int_0^{s+t}(s+t-r)^{n-1}K(r)\,dr - \frac{1}{(n-1)!}\int_0^s (s-r)^{n-1}K(r)\,dr.$$

By definition,

$$I_3 + I_5 = \frac{1}{(n-1)!}\sum_{i=0}^{n-1}\binom{n-1}{i}\frac{-1+(-1)^i}{(i+1)(i+2)}s^{n-1-i}\int_0^t (t-r)^{i+2}CC(r)\,dr$$

and

$$A(I_3+I_5) = \frac{1}{(n-1)!}\int_0^t [(s-t+r)^{n-1} - (s+t-r)^{n-1}]K(r)\,dr.$$

Finally, by definition again,

$$I_4 + I_6 = \frac{1}{(n-1)!}\sum_{i=0}^{n-1}\binom{n-1}{i}\frac{-1+(-1)^i}{(n-i)(n-i+1)}t^{n-i+1}\int_0^s (s-r)^i CC(r)\,dr$$

and

$$A(I_4+I_6) = -\frac{t^n}{n!}K(s) + \frac{1}{(n-1)!}\int_0^s [(t-s+r)^{n-1} - (s+t-r)^{n-1}]K(r)\,dr + \frac{2}{(n-1)!}\int_0^s (s-r)^{n-1}K(r)\,dr.$$

Therefore,

$$A\sum_{i=1}^{6} I_i = J_n(t,s)x - \frac{t^n}{n!}2CC(s)x - \frac{1}{(n-1)!n!}\Big\{\int_0^{s-t}(s-t-r)^{n-1}r^n\,dr + \Big[\int_0^{s+t} - \int_0^t - \int_0^s\Big](s+t-r)^{n-1}r^n\,dr + \int_0^t (s-t+r)^{n-1}r^n\,dr + \int_0^s (t-s+r)^{n-1}r^n\,dr\Big\}C^2x + 2\frac{t^n s^n}{(n!)^2}C^2x.$$

A direct computation yields that the coefficient of C^2x equals 0. Hence $J_n(t,s)x$, as a function of t, satisfies (1.6). This, combined with (1.5) and the uniqueness of solutions to $(*)$, shows that (1.1) holds. Since $C(0) = 0$ is an immediate consequence of (1.2), it remains to prove that

$$C(t)A \subseteq AC(t) \quad \forall t \geq 0.$$

For every $x \in D(A)$, define

$$\widetilde{C}(t)x = \int_0^t (t-r)C(r)Ax\,dr + \frac{t^n}{n!}Cx.$$

By (i) and (ii) of (II), Cx, $\int_0^t (t-r)C(r)x\,dr$ and $\int_0^t (t-\alpha)\widetilde{C}(\alpha)x\,d\alpha$ belong to $D(A)$, and

$$A\int_0^t (t-\alpha)\widetilde{C}(\alpha)x\,d\alpha = \int_0^t (t-\alpha)A\int_0^\alpha (\alpha-r)C(r)Ax\,dr\,d\alpha + \frac{t^{n+2}}{(n+2)!}CAx = \int_0^t (t-r)C(r)Ax\,dr = \widetilde{C}(t)x - \frac{t^n}{n!}Cx.$$

Hence $\widetilde{C}(t)x = C(t)x$, and

$$C(t)x = \int_0^t (t-r)C(r)Ax\,dr + \frac{t^n}{n!}Cx.$$

Thus

$$A\int_0^t (t-r)C(r)x\,dr = \int_0^t (t-r)C(r)Ax\,dr.$$

Since A is closed, differentiating the equality twice, we have

$$AC(t)x = C(t)Ax.$$

Thus, $\{C(t)\}_{t\ge 0}$ is an n-times integrated C-cosine operator function with A as a subgenerator.

In order to complete the proof, it is enough to note that if $\{C(t)\}_{t\ge0}$ is an n-times integrated C-cosine operator function with A as a subgenerator, then (I) is automatically true. ∎

PROPOSITION 1.6. *Assume that A is a subgenerator of the n-times integrated C-cosine operator function $\{C(t)\}_{t\ge0}$. Then $\{C(t)\}_{t\ge0}$ is uniquely determined by A.*

Proof. The proof is similar to that of the uniqueness of solutions to (ACP_2) given in the part (I)⇒(II) of Theorem 1.5. We give it here for completeness. Suppose A is a subgenerator of n-times integrated C-cosine operator functions $\{C_i(t)\}_{t\ge0}$ for $i=1,2$. For any $x\in X$ and $s,t\ge 0$,

$$(1.7)\quad \frac{d}{ds}\Big[C_1(t-s)\int_0^s (s-\alpha)C_2(\alpha)x\,d\alpha\Big]$$
$$= C_1(t-s)\int_0^s C_2(r)x\,dr - A\int_0^{t-s} C_1(r)\int_0^s (s-\alpha)C_2(\alpha)x\,d\alpha\,dr$$
$$- \frac{(t-s)^{n-1}}{(n-1)!}C\int_0^s (s-\alpha)C_2(\alpha)x\,d\alpha$$
$$= C_1(t-s)\int_0^s C_2(r)x\,dr - \int_0^{t-s} C_1(r)C_2(s)x\,dr$$
$$+ \frac{s^n}{n!}\int_0^{t-s} C_1(r)Cx\,dr - \frac{(t-s)^{n-1}}{(n-1)!}\int_0^s (s-r)C_2(r)Cx\,dr.$$

Integrate (1.7) in s from 0 to t to obtain

$$(1.8)\quad 0 \doteq \int_0^t\int_0^s C_1(t-s)C_2(r)x\,dr\,ds - \int_0^t\int_0^{t-s} C_1(r)C_2(s)x\,dr\,ds$$
$$+ \frac{1}{n!}\int_0^t\int_0^{t-s} s^n C_1(r)Cx\,dr\,ds$$
$$- \frac{1}{(n-1)!}\int_0^t\int_0^s (t-s)^{n-1}(s-r)C_2(r)Cx\,dr\,ds.$$

Since

$$\int_0^t\int_0^s C_1(t-s)C_2(r)x\,dr\,ds = \int_0^t\int_r^t C_1(t-s)C_2(r)x\,ds\,dr$$
$$= \int_0^t\int_0^{t-r} C_1(s)C_2(r)x\,ds\,dr,$$

(1.8) implies

$$\int_0^t \frac{(t-r)^{n+1}}{(n+1)!}C_1(r)Cx\,dr - \int_0^t \frac{(t-r)^{n+1}}{(n+1)!}C_2(r)Cx\,dr = 0.$$

Differentiating the above equality $n+2$ times in t, we get

$$C[C_1(t) - C_2(t)]x = 0.$$

This implies $C_1(t) = C_2(t)$. ∎

We close this section with a remark on the existence and uniqueness of n-times integrated mild solutions of (ACP_2).

The following definition is an analogy of [deL, Definition 2.3].

DEFINITION 1.7. A *mild n-times integrated C-existence family of second order for A* is a strongly continuous family of operators $\{C(t)\}_{t\ge0}\subseteq L(X)$ such that for any $x\in X$ and $t\ge0$, $\int_0^t (t-r)C(r)x\,dr \in D(A)$ with

$$A\int_0^t (t-r)C(r)x\,dr = C(t)x - \frac{t^n}{n!}Cx.$$

PROPOSITION 1.8. *Suppose A is a closed operator, C is injective and $CA\subseteq AC$. Then the following are equivalent:*

(a) (ACP_2) *has a unique n-times integrated mild solution for all $x,y\in$* $\mathrm{Im}(C)$;

(b) *all n-times integrated mild solutions of* (ACP_2) *are unique and there exists a mild n-times integrated C-existence family of second order for A;*

(c) *there exists a mild n-times integrated C-existence family of second order for A such that $C(t)A \subseteq AC(t)$ for all $t \geq 0$;*

(d) *there exists an n-times integrated C-cosine operator function with A as a subgenerator.*

Proof. (a)⇒(b). For every $x \in X$, define

$$C(t)x = v(t, Cx, 0),$$

where $v(t, Cx, 0)$ is the n-times integrated mild solution of (ACP_2) with x, y replaced by Cx and 0, respectively. As in [deL, Theorem 4.13], $C(t)$ is the desired family.

(b)⇒(c)⇒(d) follow from Theorem 1.5 and its proof. (d)⇒(a) is clear. ∎

2. Integrated C-cosine operator functions and $(s^2 - A)^{-n}C$-cosine operator functions. [S-L2, Theorem 4.1] shows that A has a $2n$-times integrated C-cosine operator function if and only if A has an $(s^2 - A)^{-n}C$-cosine operator function, under the assumption of exponential equicontinuity. In this section, we remove this condition. To do this, we have to follow a quite different path. In order to find this path, let us return to the exponentially equicontinuous case. Assume $\{C_{2n}(t)\}_{t\geq 0}$ is a $2n$-times integrated C-cosine operator function. Define

$$P_k(t) \equiv \frac{t^k}{k!} \quad (k = 0, 1, \ldots, n), \qquad h_s(t) \equiv e^{st},$$

$$H_n(t) \equiv \sum_{k=1}^{n} (-1)^{n-k} \frac{d}{dt}[(P_{n-k}h_s) * (P_{n-k}h_{-s})](t)(s^2 - A)^{-k}C.$$

For $r > s > 0$ sufficiently large, from

$$(r^2 - A)^{-1}(s^2 - A)^{-n}Cx = (-1)^n \sum_{k=0}^{n} \binom{n}{k} \frac{s^{2k}}{(r^2 - s^2)^k} \left[\frac{1}{r^{2n}}(r^2 - A)^{-1}Cx\right] + \sum_{k=0}^{n} (-1)^{n-k} \frac{1}{(r^2 - s^2)^{n+1-k}} (s^2 - A)^{-k}Cx,$$

from

$$\frac{1}{(r \pm s)^k} = \int_0^\infty \frac{t^{k-1}}{(k-1)!} e^{-(r\pm s)t}\, dt,$$

and from

$$(r^2 - A)^{-1}Cx = \frac{1}{r} \int_0^\infty e^{-rt} C_{2n}(t)x\, dt,$$

we have

$$\begin{aligned}(r^2 - A)^{-1}(s^2 - A)^{-n}Cx &= (-1)^n \sum_{k=0}^{n} \binom{n}{k} s^{2k} \int_0^\infty \frac{t^{k-1}}{(k-1)!} e^{-(r-s)t}\, dt \left(\int_0^\infty \frac{t^{k-1}}{(k-1)!} e^{-(r+s)t}\, dt\right) \\ &\quad \times \left(\frac{1}{r} \int_0^\infty e^{-rt} C_{2n}(t)x\, dt\right) + (-1)^n \frac{1}{r} \int_0^\infty e^{-rt} C_{2n}(t)x\, dt \\ &\quad + \sum_{k=0}^{n} (-1)^{n-k} \left(\int_0^\infty \frac{t^{n-k}}{(n-k)!} e^{-(r-s)t}\, dt\right) \\ &\quad \times \left(\int_0^\infty \frac{t^{n-k}}{(n-k)!} e^{-(r+s)t}\, dt\right)(s^2 - A)^{-k}Cx \\ &= \frac{1}{r} \int_0^\infty e^{-rt} \left\{\left[(-1)^n \sum_{k=1}^{n} \binom{n}{k} s^{2k} (P_{k-1}h_s) * (P_{k-1}h_{-s})\right] * C_{2n}\right\}(t)x\, dt \\ &\quad + (-1)^n \frac{1}{r} \int_0^\infty e^{-rt} C_{2n}(t)x\, dt \\ &\quad + \frac{1}{r} \int_0^\infty e^{-rt} \sum_{k=1}^{n} (-1)^{n-k} \frac{d}{dt}[(P_{n-k}h_s) * (P_{n-k}h_{-s})](t)\, dt\, (s^2 - A)^{-k}Cx.\end{aligned}$$

For $x \in X$, if we define

$$\begin{aligned}(2.1)\quad C_0(t)x = &\left\{\left[(-1)^n \sum_{k=1}^{n} \binom{n}{k} s^{2k} (P_{k-1}h_s) * (P_{k-1}h_{-s})\right] * C_{2n}\right\}(t)x \\ &+ (-1)^n C_{2n}(t)x \\ &+ \sum_{k=1}^{n} (-1)^{n-k} \frac{d}{dt}[(P_{n-k}h_s) * (P_{n-k}h_{-s})](t)(s^2 - A)^{-k}Cx\end{aligned}$$

then

$$(r^2 - A)^{-1}(s^2 - A)^{-n}Cx = \frac{1}{r} \int_0^\infty e^{-rt} C_0(t)x\, dt.$$

From [L-S1, Theorem 2.10], it follows that $\{C_0(t)\}_{t\geq 0}$ is an $(s^2 - A)^{-n}C$-cosine operator function. Therefore, if we remove the assumption of exponential equicontinuity to define $\{C_0(t)\}_{t\geq 0}$ in terms of $\{C_{2n}(t)\}_{t\geq 0}$, (2.1) is possibly a proper choice, since it does not contain the Laplace transform. The following theorem confirms it.

THEOREM 2.1. *Let A be a closed operator such that $s^2 - A$ is injective and $\mathrm{Im}(C) \subset D((s^2 - A)^{-n-1})$ for some $s \in \mathbb{C}$. Then A is a subgenerator* *of the $2n$-times (resp. $(2n+1)$-times) integrated C-cosine operator function $\{C_{2n}(t)\}_{t\ge 0}$ (resp. $\{C_{2n+1}(t)\}_{t\ge 0}$) if and only if A is a subgenerator of the $(s^2-A)^{-n}C$-cosine (resp. $(s^2-A)^{-n-1}C$-cosine) operator function $\{C_0(t)\}_{t\ge0}$, where $\{C_0(t)\}_{t\ge0}$ is defined in (2.1) if $\{C_{2n}(t)\}_{t\ge0}$ is given in advance, and*

$$C_{2n}(t)=(s^2-A)^n\frac{1}{(2n-1)!}\int_0^t (t-r)^{2n-1}C_0(r)x\,dr, \tag{2.2}$$

if $\{C_0(t)\}_{t\ge0}$ is given in advance.

Proof. We first assume $\{C_{2n}(t)\}_{t\ge0}$ is a $2n$-times integrated C-cosine operator function with A as a subgenerator and prove that $\{C_0(t)\}_{t\ge0}$ defined in (2.1) is an $(s^2-A)^{-n}C$-cosine operator function with A as a subgenerator. We have

$$\begin{aligned}
A\int_0^t (t-\alpha)C_0(\alpha)x\,d\alpha
&=(-1)^n\sum_{k=1}^n\frac{\binom{n}{k}s^{2k}}{[(k-1)!]^2}A\int_0^t(t-\alpha)\int_0^\alpha\int_0^r u^{k-1}(r-u)^{k-1}\\
&\quad\times e^{s(2u-r)}C_{2n}(\alpha-r)x\,du\,dr\,d\alpha+(-1)^nA\int_0^t(t-\alpha)C_{2n}(\alpha)x\,d\alpha\\
&\quad+\sum_{k=1}^n\frac{(-1)^{n-k}}{[(n-k)!]^2}\int_0^t\int_0^r u^{n-k}(r-u)^{n-k}e^{s(2u-r)}A(s^2-A)^{-k}Cx\,du\,dr\\
&=(-1)^n\sum_{k=1}^n\frac{\binom{n}{k}s^{2k}}{[(k-1)!]^2}\int_0^t\Big[\int_0^r u^{k-1}(r-u)^{k-1}e^{s(2u-r)}\,du\Big]\\
&\quad\times A\int_r^t(t-\alpha)C_{2n}(\alpha-r)x\,d\alpha\,dr+(-1)^nA\int_0^t(t-\alpha)C_{2n}(\alpha)x\,d\alpha\\
&\quad+\sum_{k=1}^n\frac{(-1)^{n-k}}{[(n-k)!]^2}\int_0^t\int_0^r u^{n-k}(r-u)^{n-k}e^{s(2u-r)}A(s^2-A)^{-k}Cx\,du\,dr.
\end{aligned}$$

Since $A(s^2-A)^{-1}C=s^2(s^2-A)^{-1}C-C$, the above expression equals

$$\begin{aligned}
&C_0(t)x-(-1)^n\sum_{k=1}^n\frac{\binom{n}{k}s^{2k}}{[(k-1)!]^2}\int_0^t\int_0^r u^{k-1}(r-u)^{k-1}
\end{aligned}$$

$$\begin{aligned}
&\times e^{s(2u-r)}\frac{(t-r)^{2n}}{(2n)!}Cx\,du\,dr-(-1)^n\frac{t^{2n}}{(2n)!}Cx\\
&-\sum_{k=1}^n\frac{(-1)^{n-k}}{[(n-k)!]^2}\frac{d}{dt}\int_0^t u^{n-k}(t-u)^{n-k}e^{s(2u-t)}(s^2-A)^{-k}Cx\,du\\
&+\sum_{k=1}^n\frac{(-1)^{n-k}}{[(n-k)!]^2}\int_0^t\int_0^r u^{n-k}(r-u)^{n-k}e^{s(2u-r)}s^2(s^2-A)^{-k}Cx\,du\,dr\\
&-\sum_{k=1}^n\frac{(-1)^{n-k}}{[(n-k)!]^2}\int_0^t\int_0^r u^{n-k}(r-u)^{n-k}e^{s(2u-r)}(s^2-A)^{-k+1}Cx\,du\,dr\\
&=C_0(t)-(s^2-A)^{-n}Cx\\
&\quad-(-1)^n\sum_{k=1}^n\frac{\binom{n}{k}s^{2k}}{[(k-1)!]^2}\int_0^t\int_0^r u^{k-1}(r-u)^{k-1}e^{s(2u-r)}\frac{(t-r)^{2n}}{(2n)!}Cx\,du\,dr\\
&\quad-(-1)^n\frac{t^{2n}}{(2n)!}Cx+\frac{(-1)^n}{[(n-1)!]^2}\int_0^t\int_0^r u^{n-1}(r-u)^{n-1}e^{s(2u-r)}Cx\,du\,dr\\
&\quad-\sum_{k=1}^{n-1}\frac{(-1)^{n-k}}{[(n-k)!]^2}\int_0^t[(n-k)u^{n-k}(t-u)^{n-k-1}-su^{n-k}(t-u)^{n-k}]\\
&\quad\times e^{s(2u-t)}(s^2-A)^{-k}Cx\,du\\
&\quad+\sum_{k=1}^{n-1}\frac{(-1)^{n-k}}{[(n-k)!]^2}\int_0^t\int_0^r u^{n-k}(r-u)^{n-k}e^{s(2u-r)}s^2(s^2-A)^{-k}Cx\,du\,dr\\
&\quad-\sum_{k=1}^{n-1}\frac{(-1)^{n-k-1}}{[(n-k-1)!]^2}\int_0^t\int_0^r u^{n-k-1}(r-u)^{n-k-1}e^{s(2u-r)}(s^2-A)^{-k}Cx\,du\,dr\\
&=C_0(t)x-(s^2-A)^{-n}Cx+f_1(t)+f_2(t),
\end{aligned}$$

where

$$\begin{aligned}
f_1(t)&=-(-1)^n\sum_{k=1}^n\frac{\binom{n}{k}s^{2k}}{[(k-1)!]^2}\int_0^t\int_0^r u^{k-1}(r-u)^{k-1}\\
&\quad\times e^{s(2u-r)}\frac{(t-r)^{2n}}{(2n)!}Cx\,du\,dr-(-1)^n\frac{t^{2n}}{(2n)!}Cx\\
&\quad+\frac{(-1)^n}{[(n-1)!]^2}\int_0^t\int_0^r u^{n-1}(r-u)^{n-1}e^{s(2u-r)}Cx\,du\,dr,
\end{aligned}$$

and

$$f_2(t) = -\sum_{k=1}^{n-1} \frac{(-1)^{n-k}}{[(n-k)!]^2} \int_0^t [(n-k)u^{n-k}(t-u)^{n-k-1} - su^{n-k}(t-u)^{n-k}] \times e^{s(2u-t)}(s^2-A)^{-k}Cx\,du$$
$$+ \sum_{k=1}^{n-1} \frac{(-1)^{n-k}}{[(n-k)!]^2} \int_0^t\int_0^r u^{n-k}(r-u)^{n-k} e^{s(2u-r)} s^2 (s^2-A)^{-k}Cx\,du\,dr$$
$$- \sum_{k=1}^{n-1} \frac{(-1)^{n-k-1}}{[(n-k-1)!]^2} \int_0^t\int_0^r u^{n-k-1}(r-u)^{n-k-1} \times e^{s(2u-r)}(s^2-A)^{-k}Cx\,du\,dr.$$

Choose $\lambda > \max\{\operatorname{Re} s, 0\}$. Then, by integration by parts,

$$\int_0^\infty e^{-\lambda t} f_1(t)\,dt = -\frac{(-1)^n}{\lambda^{2n+1}}\left(\frac{\lambda^2}{\lambda^2-s^2}\right)^n Cx + \frac{(-1)^n}{\lambda(\lambda^2-s^2)^n}Cx = 0.$$

Hence $f_1(t) \equiv 0$ by the uniqueness of the Laplace transform. Since $f_2'(t) \equiv 0$, we have $f_2(t) \equiv f_2(0) = 0$. Therefore

$$A\int_0^t (t-\alpha)C_0(\alpha)x\,d\alpha = C_0(t)x - (s^2-A)^{-n}Cx.$$

From Proposition 1.8(c), (d), we conclude that $\{C_0(t)\}_{t\ge 0}$ is an $(s^2-A)^{-n}C$-cosine operator function with A as a subgenerator.

If $\{C_{2n+1}(t)\}_{t\ge 0}$ is a $(2n+1)$-times integrated C-cosine operator function with A as a subgenerator, then

$$C_{2n+2}(t)x = \int_0^t C_{2n+1}(s)x\,ds$$

defines a $(2n+2)$-times integrated C-cosine operator function $\{C_{2n+2}(t)\}_{t\ge 0}$ with A as a subgenerator. In this case it suffices to replace n by $n+1$ in (2.1).

Now assume $\{C_0(t)\}_{t\ge 0}$ is an $(s^2-A)^{-n}C$-cosine operator function with A as a subgenerator. Then the proof of (2.2) is contained in the following lemma.

LEMMA 2.2. *Let A be a closed operator such that $s^2 - A$ is injective and $\operatorname{Im}(C) \subset D((s^2-A)^{-n-1})$ for some $s > 0$. If A is a subgenerator of the $(s^2-A)^{-n}C$-cosine operator function $\{C_0(t)\}_{t\ge 0}$, then for each $0 \le m \le n$, A is also a subgenerator of the $2m$-times (resp. $(2m+1)$-times) integrated $(s^2-A)^{-n+m}C$-cosine operator function $\{C_{2m}(t)\}_{t\ge 0}$ defined by (2.2) with n replaced by m (resp. $\{C_{2m+1}(t)\}_{t\ge 0}$ defined by $C_{2m+1}(t)x = \int_0^t C_{2m}(r)x\,dr$).*

Proof. Let $C_{2m}(t)$ $(t \ge 0)$ be defined in (2.2) with n replaced by m. We claim that $C_{2m}(t)$ $(t \ge 0)$ is a $2m$-times integrated $(s^2-A)^{-n+m}C$-cosine operator function. From

$$A\int_0^t (t-r)C_{2m}(r)x\,dr$$
$$= (s^2-A)^m \frac{1}{(2m-1)!} A\int_0^t (t-r)\int_0^r (r-\alpha)^{2m-1}C_0(\alpha)x\,d\alpha\,dr$$
$$= (s^2-A)^m \frac{1}{(2m+1)!} A\int_0^t (t-\alpha)^{2m+1}C_0(\alpha)x\,ds$$
$$= (s^2-A)^m \frac{1}{(2m-1)!} \int_0^t (t-\alpha)^{2m-1}C_0(s)x\,d\alpha - \frac{1}{(2m)!}t^{2m}(s^2-A)^{-n+m}Cx$$
$$= C_{2m}(t)x - \frac{1}{(2m)!}t^{2m}(s^2-A)^{-n+m}Cx,$$

and from Theorem 1.5, the conclusion follows. ∎

COROLLARY 2.3. *Let A be a closed operator such that $s^2 - A$ is injective and $\operatorname{Im}(C) \subset D((s^2-A)^{-n-1})$ for some $s \in \mathbb{C}$. If there exists some $0 \le m \le n$ such that A is a subgenerator of the $2m$-times integrated $(s^2-A)^{-n+m}C$-cosine operator function $\{C_{2m}(t)\}_{t\ge 0}$, then for every $0 \le k \le n$, A is a subgenerator of the $2k$-times integrated $(s^2-A)^{-n+k}C$-cosine operator function $\{C_{2k}(t)\}_{t\ge 0}$.*

3. Examples. In this section, we give two examples which are nonexponentially bounded regularized cosine operator functions. Example 3.1 is a version of [deL, Examples 4.10].

EXAMPLE 3.1. Let $\Omega \equiv \{z = x+iy \mid y \ge 0,\ e^{y^2} \le x \le e^{2y^2}\}$ and let $X \equiv BC(\Omega)$ be the space of all complex-valued bounded continuous functions on Ω. Equipped with the norm

$$\|f\| = \sup\{|f(z)| \mid z \in \Omega\},$$

X is a Banach space. Define $(Af)(z) = -z^2 f(z)$, on X, with maximal domain. Then A generates the A^{-1}-regularized cosine operator function

$\{C(t)\}_{t\ge 0}$ with

$$[C(t)f](z)=\frac{z^{-2}(e^{itz}+e^{-itz})}{2}f(z),\qquad \forall f\in X.$$

We claim that $\{C(t)\}_{t\ge0}$ is not exponentially bounded. Set $f_0(z)\equiv 1$. Then

$$\begin{aligned}\|C(t)\| &\ge \|C(t)f_0\|\sup\left\{\frac{|z^{-2}(e^{itz}+e^{-itz})|}{2}\;\middle|\; z\in\Omega\right\}\\ &\ge \sup\left\{\frac{e^{ty}-e^{-ty}}{2(x^2+y^2)}\;\middle|\; z=x+iy\in\Omega\right\}\\ &\ge M\sup\{e^{ty-4y^2}\mid y\ge 0\}=Me^{t^2/16},\end{aligned}$$

for some $M>0$.

EXAMPLE 3.2. Let $X\equiv L^1(\mathbb{R})$. Then X is a Banach algebra with the convolution product

$$(f*g)(s)=\int_{\mathbb{R}} f(s-u)g(u)\,du,\qquad f,g\in X.$$

Let $\{f_\alpha\}$ $(\alpha\in\mathcal{A})$ be an approximate identity of X, i.e. (i) $\|f_\alpha\|=1$ for all $\alpha\in\mathcal{A}$; (ii) $\lim_\alpha f_\alpha * f=f$ for all $f\in X$.

Denote by ch the hyperbolic cosine and set

$$G_t=\mathcal{F}(u\mapsto(\operatorname{ch} tu)\exp(-u^2)),$$

where $\mathcal{F}$ is the Fourier transform

$$\mathcal{F}(f)(s)=\int_{\mathbb{R}} e^{-isu}f(u)\,du.$$

Then

$$(C(t)f)(s)=(G_t*f)(s)$$

defines a $C(0)$-cosine operator function $\{C(t)\}_{t\ge0}$ with generator $A=-(d/ds)^2$. From

$$\|C(t)\|\ge\|C(t)f_\alpha\|=\|G_t*f_\alpha\|\to\|G_t\|$$

and

$$\begin{aligned}\|G_t\|&=\int_{\mathbb{R}}\frac{e^{tu}+e^{-tu}}{2}e^{-u^2}\,du\\ &=e^{t^2/4}\int_{\mathbb{R}} e^{-u^2}\,du=\sqrt{\pi}\,e^{t^2/4},\end{aligned}$$

it follows that

$$\|C(t)\|\ge\sqrt{\pi}\,e^{t^2/4}.$$

$\{C(t)\}_{t\ge0}$ is thus not exponentially bounded.

References

[A] W. Arendt, *Vector-valued Laplace transforms and Cauchy problems*, Israel J. Math. 59 (1987), 327–352.

[D] G. Da Prato, *Semigruppi regolarizzabili*, Ricerche Mat. 15 (1966), 223–248.

[Dav-P] E. B. Davies and M. M. Pang, *The Cauchy problem and a generalization of the Hille–Yosida theorem*, Proc. London Math. Soc. 55 (1987), 181–208.

[deL] R. deLaubenfels, *Existence Families, Functional Calculi and Evolution Equations*, Lecture Notes in Math. 1570, Springer, 1994.

[deL-P] R. deLaubenfels and S. Piskarev, *Growth rate of cosine families*, J. Math. Anal. Appl. 196 (1995), 442–451.

[L-S1] Y.-C. Li and S.-Y. Shaw, *On generators of integrated C-semigroups and C-cosine functions*, Semigroup Forum 47 (1993), 29–35.

[L-S2] —, —, *Integrated C-cosine functions and the abstract Cauchy problem*, preprint.

[S-L1] S.-Y. Shaw and Y.-C. Li, *On N-times integrated C-cosine functions*, in: Evolution Equations, Marcel Dekker, New York, 1994, 393–406.

[S-L2] —, —, *N-times integrated C-semigroups and the abstract Cauchy problem*, Taiwanese J. Math. 1 (1997), 75–102.

[So] M. Sova, *Cosine operator functions*, Rozprawy Mat. 49 (1966), 1–47.

[W-Wa] H. Wang and S. Wang, *C-cosine operator functions and the applications to the second order abstract Cauchy problem*, in: Functional Analysis in China, Kluwer Acad. Publ., Dordrecht, 1996, 333–350.

[Wa] S. Wang, *Mild integrated C-existence families*, Studia Math. 112 (1995), 251–266.

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