STUDIA MATHEMATICA 126 (3) (1997)

BV coboundaries over irrational rotations

by

DALIBOR VOLNÝ (Praha)

Abstract. For every irrational rotation we construct a coboundary which is continuous except at a single point where it has a jump, is nondecreasing, and has zero derivative almost everywhere.

Let $\alpha \in (0,1)$ be an irrational number. Let T be the rotation of the unit circle $\mathbb{T}$ represented by the unit interval $[0,1)$, defined by $Tx = x+\alpha \bmod 1$. We denote by m the Lebesgue probability measure on $\mathbb{T}$.

Let F be an $\mathbb{R}$- or $\mathbb{T}$-valued function on $\mathbb{T}$. The transformation

$$T_F(x,y) = (Tx, y+F(x)), \quad x \in \mathbb{T},\ y \in \mathbb{R}\ (y \in \mathbb{T}),$$

of $\mathbb{T}\times\mathbb{R}$ ($\mathbb{T}\times\mathbb{T}$) onto itself is called a *skew product*; T_F preserves the product measure (see e.g. [9]). The function F is called a *cocycle*. If there exists a measurable function G such that $F = G - G\circ T$, then F is a *coboundary*. If F is $\mathbb{T}$-valued, it is called an *Anzai cocycle*.

If H is a continuous function on $[0,1)$ with $H(0)-\lim_{t\to 1_-} H(t) \in \mathbb{Z}$, then the factorization $F = H \bmod 1$ is a continuous Anzai cocycle. In this way we get all continuous Anzai cocycles. The difference $H(0)-\lim_{t\to 1_-} H(t)$ is called the (topological) *degree* of F.

Recall that by [2] every Lipschitz Anzai cocycle of nonzero degree is ergodic (hence not a coboundary) and by [3] every absolutely continuous Anzai cocycle of nonzero degree is ergodic. In [2], p. 583, H. Furstenberg stated that there exists a continuous Anzai cocycle of nonzero degree and bounded variation which is a coboundary. A first proof was, however, published probably by Iwanik, Lemańczyk, and Rudolph in [4]; in [1] even a Hölder continuous cocycle was found. In [4], [1] the constructions were not given for all irrational rotations: the existence of continuous coboundaries of nonzero degree and bounded variation was left as an open problem.

1991 *Mathematics Subject Classification*: 28D05, 11K50.
The research was partially supported by Charles University grant GAUK 6191.

For every irrational rotation we shall construct a (real) cocycle F which is monotone and continuous except at a single point. As the factorization of a real coboundary modulo $\mathbb{Z}$ gives an Anzai coboundary, this implies that for every irrational rotation there exists an Anzai cocycle which is continuous, has nonzero degree, is of bounded variation, and is a coboundary.

The construction is based on an idea similar to the proof that for an absolutely continuous cocycle F with $\int_0^1 |F'(x)|^{3+\delta}\,dx < \infty$ there exists a cohomologous Lipschitz one (see [10]). The proof of the general result is much longer than the proof for rotations with unbounded partial quotients in [4]. It seems that in the case of rotations with unbounded partial quotients the existence of some cocycles is easier to decide. For example the existence of an absolutely continuous cocycle of type III_0, or ergodic and squashable, in the case of bounded partial quotients remains an open problem (cf. [4], [11]).

THEOREM. *Let $0 < \alpha < 1$ be an irrational number. Then there exists a (real) cocycle which is continuous at all points except one at which it has a jump, is monotone, and is a coboundary.*

The cocycle has both one-sided limits at each point. Because a multiple of a real coboundary is again a real coboundary, the jump can have any real value; to get an Anzai cocycle we are interested in integer values.

In [7] there is a construction of stably ergodic cocycles of bounded variation (i.e. a small perturbation of the cocycle in the sense of the BV norm remains ergodic). In particular, the straight line $f(x) = x - 1/2$ is stably ergodic. Using the result from [4] one can show (see [7], Remark 5) that for a rotation with unbounded partial quotients there exists a continuous cocycle of bounded variation which is stably ergodic. Using the Theorem one can immediately extend the result to all irrational rotations.

I conjecture that using ideas from the proof of the Theorem one can construct, for every irrational rotation, a purely singular (i.e. continuous, with zero derivative almost everywhere) real cocycle which is stably ergodic.

Proof of the Theorem. We denote by $a_1, a_2, \ldots$ the partial quotients in the continued fraction expansion of α, and $q_1, q_2, \ldots$ are the denominators. For every $n = 1, 2, \ldots$,

$$q_{n+1} = a_{n+1} q_n + q_{n-1} \tag{1}$$

(cf. [5]).

As usual we denote by $[x]$ the integer part of a real number x, and $\{x\} = x - [x]$ denotes the fractional part. $\|x\| = \min(\{x\}, 1 - \{x\})$ is the distance of x from the integers.

Rokhlin towers. We shall use the Rokhlin towers for the rotation $Tx = x + \alpha \bmod 1$. Recall that for T two Rokhlin towers are defined:

For n even we have

- $[\{j\alpha\}, \{(j + q_{n-1})\alpha\})$, $j = 0, 1, \ldots, q_n - 1$ (the *bigger* tower), and
- $[\{q_n\alpha\}, 1)$, $[\{(j + q_n)\alpha\}, \{j\alpha\})$, $j = 1, \ldots, q_{n-1} - 1$ (the *smaller* tower);

for n odd we have

- $[\{q_{n-1}\alpha\}, 1)$, $[\{(j + q_{n-1})\alpha\}, \{j\alpha\})$, $j = 0, 1, \ldots, q_n - 1$ (the *bigger* tower), and
- $[\{j\alpha\}\{(j + q_n)\alpha\})$, $j = 0, 1, \ldots, q_{n-1} - 1$ (the *smaller* tower).

The two Rokhlin towers are disjoint and together they form a partition of $\mathbb{T}$. In fact, we shall use the bigger towers only.

For any positive integer n, the sets

$$I(n, i) = T^i[0, \|q_{n-1}\alpha\|), \quad i = 0, 1, \ldots, q_n - 1,$$

form thus a Rokhlin tower: for n even we get the bigger Rokhlin tower and for n odd the bigger Rokhlin tower is rotated by $q_{n-1}\alpha$.

Note that for any fixed $j = 0, 1, \ldots, q_{n-2} - 1$, the intervals $I(n, j + iq_{n-1})$, $i = 0, \ldots, a_n$, are adjacent; similarly for any $j = q_{n-2}, \ldots, q_{n-1} - 1$ the intervals $I(n, j + iq_{n-1})$, $i = 0, \ldots, a_n - 1$, are adjacent—in both cases on $\mathbb{T}$; on the interval $[0, 1)$ they can be separated by 0.

By (1) we have $q_{n+1} \ge q_n$, $q_{n+1} \ge 2q_{n-1}$, and hence for all $j = 1, 2, \ldots$,

$$q_{n+j} \ge 2^{[j/2]} q_n. \tag{2}$$

By [5], $1/(2q_n) \le 1/\|q_{n-1}\alpha\| \le 1/q_n$, hence

$$\frac{1}{2q_n} \le |I(n, 0)| \le \frac{1}{q_n} \quad \text{for all } n = 1, 2, \ldots. \tag{3}$$

Construction of the function F. We shall construct a function F on $\mathbb{T}$ which is continuous and nondecreasing on the interval $[0, 1)$ and $\lim_{t \to 1-} F(t) - F(0) = 1$.

In the sequel, (n_k), $k = 1, 2, \ldots$, will be an increasing sequence of positive even integers which will be specified later. We shall suppose

$$n_{k+1} \ge n_k + k + 3, \quad k = 1, 2, \ldots \tag{4}$$

First we recursively define a sequence of sets $M_k \subset \{0, 1, \ldots, q_{n_k} - 1\}$, $k = 1, 2, \ldots$ Set $M_1 = \{0\}$. Suppose that M_j, $1 \le j \le k$, have been defined. Then we define

$$M_{k+1} = M_k \cup M_k + q_{n_{k+1}-1}.$$

We define

$$|M_k| = \max M_k, \quad k = 1, 2, \ldots$$

Later we shall use

$$(5)\qquad |M_k| \le q_{n_k} - 1,$$

$$(6)\qquad \text{for } k \ge 2, \quad |M_{k-1}| \le q_{n_k-k-2} - 1.$$

Indeed, from the definition we get $|M_k| = \sum_{j=2}^{k} q_{n_j-1}$, $|M_1| = 0 \le q_{n_1} - 1$ and if $|M_k| \le q_{n_k} - 1$ then $|M_{k+1}| \le q_{n_k} - 1 + q_{n_{k+1}-1}$. By (4) we have $n_{k+1} \ge n_k + 2$, therefore $q_{n_k} \le q_{n_{k+1}-2}$. By (1) we get $q_{n_{k+1}-2} + q_{n_{k+1}-1} \le q_{n_{k+1}}$. Therefore, $|M_{k+1}| \le q_{n_{k+1}} - 1$. This proves (5).

We have supposed that $n_{k-1} \le n_k - k - 2$. Hence, $|M_{k-1}| \le q_{n_{k-1}} - 1 \le q_{n_k-k-2} - 1$, which proves (6).

Next we recursively define a sequence of functions f_k on $\mathbb{T}$, $k = 1, 2, \ldots$ First, f_1 is constant on $I(n_2, 0)$, $\int_{I(n_2,0)} f_1\, dm = 1$, f_1 is zero elsewhere. Suppose that f_k, $k \ge 1$, have been defined and

- f_k is nonnegative, constant on the intervals $I(n_{k+1}, i)$, $i \in M_k$, zero elsewhere,
- $\int_0^1 f_k\, dm = 1$.

Then we define f_{k+1} so that:

- f_{k+1} is nonnegative and constant on the intervals $I(n_{k+2}, i)$, $i \in M_{k+1}$, and zero elsewhere,
- for $i \in M_k$ we have

$$\int_{I(n_{k+1},i)} f_k\, dm = \int_{I(n_{k+2},i)} f_{k+1}\, dm + \int_{I(n_{k+2},i+q_{n_{k+1}-1})} f_{k+1}\, dm,$$

$$\int_{I(n_{k+2},i)} f_{k+1}\, dm \Big/ \int_{I(n_{k+2},i+q_{n_{k+1}-1})} f_{k+1}\, dm = k + 1.$$

Notice that f_{k+1} is uniquely determined and that $\int_0^1 f_{k+1}\, dm = 1$.

For $0 \le i \le q_{n_k} - 1$ and $j = 1, 2, \ldots$ we define $u_j = n_{k+j}$, $v_j = i + \sum_{l=1}^{j-1} q_{n_{k+l}-1}$, and

$$(7)\qquad \overline{I}(n_k, i) = \bigcup_{j=1}^{\infty} I(u_j, v_j), \qquad \overline{I}^+(n_k, i) = \overline{I}(n_k, i + q_{n_k-1}).$$

Recall that n_k has been assumed to be even. The set $\overline{I}(n_k, i)$ is thus an interval and by (2)–(4) it is a subinterval of $I(n_k, i)$ with the same left end point; $I(n_k, i) \cup \overline{I}^+(n_k, i)$ is also an interval (on the circle).

From the properties of the Rokhlin towers we get $\|q_n\alpha\| \ge \sum_{i=1}^{\infty} \|q_{n+i}\alpha\|$. From this and the definition of $\overline{I}(n_k, 0)$ we get

$$(8)\qquad |\overline{I}(n_k, 0)| \le 2|I(n_{k+1}, 0)|.$$

LEMMA 1. *Let k be a positive integer. Then*

(9) *for each $i \in M_k$, the integrals $\int_{\overline{I}(n_k,i)} f_{k+j}\, dm$ have for all $j = 0, 1, \ldots$ the same value (which may depend on i), and*

$$\sum_{i \in M_k} \int_{\overline{I}(n_k,i)} f_{k+j}\, dm = 1,$$

(10) *for every $k \ge 3$, $i \in M_{k-1}$, $j = 0, 1, \ldots,$*

$$\int_{\overline{I}(n_k,i)} f_{k+j}\, dm \Big/ \int_{\overline{I}^+(n_k,i)} f_{k+j}\, dm = k,$$

(11) *for the same k, i, j, we have $f_{k+j} = 0$ on $[0, 1) \setminus \bigcup_{i \in M_k} \overline{I}(n_k, i)$.*

The proof follows directly from the definitions and is left to the reader.

On $[0, 1)$, for $k = 1, 2, \ldots$ we define

$$F_k(t) = \int_0^t f_k(x)\, dx - \int_0^1 \int_0^u f_k(x)\, dx\, du.$$

PROPOSITION 1. *Suppose that the sequence (n_k) increases sufficiently fast so that*

$$m\Big(\bigcup_{k=N}^{\infty} \bigcup_{i \in M_k} I(n_k, i)\Big) \underset{N\to\infty}{\longrightarrow} 0.$$

Then the uniform limit $F = \lim_{k\to\infty} F_k$ exists on $[0, 1)$, F is continuous, nondecreasing, $\int_0^1 F\, dm = 0$, $|F(0) - \lim_{x\to 1-} F(x)| = 1$. Moreover, F is singular on $[0, 1)$, i.e. it has zero derivative on a set of measure 1.

Proof. On the interval $[0, 1)$ we define auxiliary functions

$$\overline{F}_k(t) = \int_0^t f_k(x)\, dx.$$

For every $k = 1, 2, \ldots$ we define $A_k = [0, 1) \setminus \bigcup_{i \in M_k} \overline{I}(n_k, i)$. Lemma 1(9), (11) shows that $\overline{F}_k(t) = \overline{F}_{k+j}(t)$ for every $t \in A_k$ and $j \ge 0$. Because $\bigcup_{i \in M_{k+1}} \overline{I}(n_{k+1}, i) \subset \bigcup_{i \in M_k} \overline{I}(n_k, i)$, we have $A_k \subset A_{k+1}$. The functions $\overline{F}_k$ thus converge on $\bigcup_{k=1}^{\infty} A_k$.

We shall show that the functions F_k are uniformly continuous. The functions f_k are nonnegative and $\int_0^1 f_k\, dm = 1$, hence $\int_{I(n_1,1)} f_k\, dm \le 1$ for every $k = 1, 2, \ldots$ and $i = 1, \ldots, q_{n_1} - 1$.

Suppose that for all $i = 0, 1, \ldots, q_{n_k} - 1$ and all $j = 0, 1, \ldots$ we have

$$\int_{\overline{I}(n_k,i)} f_{k+j}\, dm \le c.$$

By Lemma 1 it follows that

$$\int_{\bar I(n_{k+1},i)} f_{k+1+j}\,dm \le c\Big(1-\frac{1}{k+1}\Big)$$

for all $i=0,1,\ldots,q_{n_{k+1}}-1$ and all $j=0,1,\ldots$ By recursion it follows that for any $k=1,2,\ldots$, $i\in\{0,1,\ldots,q_{n_k}-1\}$, and $j=0,1,\ldots$ we have

$$\int_{\bar I(n_k,i)} f_{k+j}\,dm \le \prod_{l=2}^{k}\Big(1-\frac{1}{l}\Big).$$

It follows that for $|x-y|<|I(n_{k+1},i)|$ and for every $j\ge 0$ we have

$$|\bar F_{k+j}(x)-\bar F_{k+j}(y)| \le \prod_{j=2}^{k}\frac{j-1}{j}=\frac{1}{k}.$$

The functions $\bar F_k$, $k=1,2,\ldots$, are thus uniformly continuous.

By the assumptions of the proposition,

$$m\Big(\bigcup_{k=1}^{\infty}A_k\Big)=\lim_{k\to\infty}m(A_k)=1,$$

hence $\bigcup_{k=1}^{\infty}A_k$ is a dense subset of $[0,1)$. Because the functions $\bar F_k$ are uniformly continuous and converge on a dense subset of $[0,1)$, they uniformly converge to a continuous limit $\bar F$ on $[0,1)$.

Let A_k° denote the interior of A_k. We have $A_k^\circ\subset A_{k+1}^\circ$, $m(A_k^\circ)=m(A_k)\to 1$ for $k\to\infty$, and the functions $\bar F_{k+j}$, $j\ge 0$, are constant on A_k°. Therefore, $\bar F$ has derivative 0 on $\bigcup_{k=1}^{\infty}A_k^\circ$ and $m(\bigcup_{k=1}^{\infty}A_k^\circ)=1$.

The functions $F_k=\bar F_k-\int_0^1\bar F_k\,dm$ are thus continuous and converge uniformly to the limit $F=\bar F-\int_0^1\bar F\,dm$. All the remaining properties of F are easy to see. ∎

Proof that F is a coboundary. It is well known that a cocycle F is a coboundary if and only if the partial sums

$$S_n(F)=\sum_{i=0}^{n-1}F\circ T^{-i}$$

are stochastically bounded (see [9]), and F is a coboundary $G-G\circ T$ with a square integrable transfer function G if and only if the norms $\|S_n(F)\|_2$ are bounded (see e.g. [8]).

The rest of the proof of the theorem will thus consist in proving the next proposition:

PROPOSITION 2. *There exists a real number K such that for every positive integer N,*

$$\|S_N(F)\|_2\le K.$$

Hence, F is a coboundary with a square integrable transfer function.

Proof. For any positive integer N we have the Ostrowski decomposition

$$N=\sum_{j=1}^{n}c_jq_{j-1}$$

where $0\le c_j\le a_j$ are integers and a_j are the partial quotients and q_j the denominators in the continued fraction development of α. We thus have

$$S_N(F)=S_{c_1q_0}(F)+S_{c_2q_1}(F)\circ T^{-c_1q_0}+\ldots \\ +S_{c_nq_{n-1}}(F)\circ T^{-c_1q_0-\ldots-c_{n-1}q_{n-2}}.$$

Define

$$N_1=\sum\{c_jq_{j-1}:\exists k,\ |j-n_k|\le k\},\qquad N_2=N-N_1.$$

We have $\|S_N(F)\|_2\le\|S_{N_1}(F)\|_2+\|S_{N_2}(F)\circ T^{-N_1}\|_2=\|S_{N_1}(F)\|_2+\|S_{N_2}(F)\|_2$. We shall show that the sums of the norms of the summands in $S_{N_2}(F)$ are bounded (independently of N). The norms of the sums $\sum\{c_jq_{j-1}:|j-n_k|\le k\}$, $k=1,2,\ldots$, will be estimated by $1/k$ and shown to be almost orthogonal, which will prove the boundedness of the norms $\|S_{N_1}(F)\|_2$.

The rest of the proof will be devoted to carrying out this plan.

Next, we shall suppose that the number N is fixed. The function F will be approximated by an absolutely continuous function.

For $0<\delta<1$ let $\tilde f_\delta$ be a function on $[0,1)$ such that

$$\tilde f_\delta\le 0,\qquad \tilde f_\delta=0\quad\text{on }[\delta,1),\qquad \int_0^\delta\tilde f_\delta\,dm=-1.$$

For each $k=1,2,\ldots$ and for $0<\delta<1$ (which will be chosen very small) we define

$$\hat f_{\delta,k}=\tilde f_\delta+f_k;$$

when δ is fixed, we shall also use the notation $\hat f_k=\hat f_{\delta,k}$.

For positive integers N, k, $0<\delta<1$, and $a\in[0,1)$ we define

$$\hat F_{\delta,k,N,a}(t)=\int_a^t S_N(\hat f_{\delta,k})\,dm,\qquad t\in[0,1).$$

When there is no danger of confusion we shall omit some of the indices.

LEMMA 2. *For every $N = 1, 2, \ldots$ and $\varepsilon > 0$ we can choose k and δ so that for every $a \in [0,1)$,*

$$\|S_N(F)\|_2 \le \|\widehat{F}_{N,a}\|_2 + \varepsilon,$$

$$\Big\|S_N(F) - \Big(\widehat{F}_{N,a} - \int_0^1 \widehat{F}_{N,a}\, dm\Big)\Big\|_2 < \varepsilon.$$

Proof. We define

$$\widetilde{F}_{\delta,k}(t) = \int_a^t \widehat{f}_k(x)\,dx - \int_0^1\int_a^u \widehat{f}_k(x)\,dx\,du.$$

Recall the definition $F_k(t) = \int_0^t f_k(x)\,dx - \int_0^1\int_0^u f_k(x)\,dx\,du$ from the proof of Proposition 1. Now, F_k and $\widetilde{F}_{\delta,k}$ are absolutely continuous on $[0,1)$ and have the same derivatives on $[\delta,1)$, hence there exists $b \in \mathbb{R}$ such that $F_k\chi_{[\delta,1)} = b\chi_{[\delta,1)} + \widetilde{F}_{\delta,k}\chi_{[\delta,1)}$. Because $|F_k|, |\widetilde{F}_{\delta,k}| \le 1$ and $\int_0^1 F_k\,dm = 0 = \int_0^1 \widetilde{F}_{\delta,k}\,dm$, we have $b(1-\delta) \le 2\delta$ and hence $b \le 2\delta$. The functions F_k converge uniformly to F, therefore

$$\lim_{\delta\searrow 0}\lim_{k\to\infty} \widetilde{F}_{\delta,k}(t) = F(t) \quad \text{for all } 0 < t \le 1.$$

Let N be fixed. The functions $\widetilde{F}_{\delta,k}$ are uniformly bounded, hence we have

$$\lim_{\delta\searrow 0}\lim_{k\to\infty} \|S_N(\widetilde{F}_{k,\delta}) - S_N(F)\|_2 = 0.$$

The function $S_N(\widetilde{F}_{\delta,k})$ has zero mean and the derivatives of $S_N(\widetilde{F}_{\delta,k})$ and $\widehat{F}_{\delta,k,N,a}$ are equal, hence

$$S_N(\widetilde{F}_{\delta,k}) = \widehat{F}_{\delta,k,N,a} - \int_0^1 \widehat{F}_{\delta,k,N,a}\,dm.$$

This proves the second inequality.

Let $c = \int_0^1 \widehat{F}_{\delta,k,N,a}\,dm$. Then

$$\|\widehat{F}_N\|_2^2 = \int_0^1 (S_N(\widetilde{F}_{\delta,k}) + c)^2\,dm = \int_0^1 (S_N(\widetilde{F}_{\delta,k}))^2\,dm + c^2 \ge \int_0^1 (S_N(\widetilde{F}_{k,\delta}))^2\,dm,$$

which proves the first inequality. ∎

By l we shall denote nonnegative integers such that

$$n_{k-1} < n_k - l.$$

We define $I_j(i) = I(n_k - l, j + iq_{n_k-l-1})$, $\bar{I}_j^+(i) = \bar{I}^+(n_k, j + iq_{n_k-l-1})$ (cf. (7)) and

$$J_j = \begin{cases} \bigcup_{i=0}^{a_{n_k-l}} I_j(i) \cup \bar{I}_j^+(a_{n_k-l}), & 0 \le j \le q_{n_k-l-2} - 1,\\ \bigcup_{i=0}^{a_{n_k-l}-1} I_j(i) \cup \bar{I}_j^+(a_{n_k-l} - 1), & q_{n_k-l-2} \le j \le q_{n_k-l-1} - 1. \end{cases}$$

The sets J_j are mutually disjoint intervals:

For $0 \le j \le q_{n_k-l-2} - 1$ the sets $I_j(0), \ldots, I_j(a_{n_k-l})$ are adjacent and similarly for $q_{n_k-l-2} \le j \le q_{n_k-l-1} - 1$ the sets $I_j(0), \ldots, I_j(a_{n_k-l} - 1)$ are adjacent. The sets $I_j(i)$ form the bigger Rokhlin tower (if $n_k - l$ is even) or the bigger Rokhlin tower rotated by $q_{n_k-l-1}\alpha$ (for $n_k - l$ odd), respectively. For $l \ge 1$ the intervals $\bar{I}_j^+(a_{n_k-l})$ are subsets of $I_j(a_{n_k-l})$, $0 \le j \le q_{n_k-l-2} - 1$, and $\bar{I}_j^+(a_{n_k-l}-1)$ are subsets of $I_j(a_{n_k-l}-1)$, $q_{n_k-l-2} \le j \le q_{n_k-l-1}-1$. The intervals $\bar{I}_j^+(a_{n_k})$ are adjacent to $I_j(a_{n_k})$, $0 \le j \le q_{n_k-2}-1$, and $\bar{I}_j^+(a_{n_k}-1)$ are adjacent to $I_j(a_{n_k} - 1)$, $q_{n_k-2} \le j \le q_{n_k-1} - 1$. They are subsets of the smaller Rokhlin tower, hence the intervals J_j, $0 \le j \le q_{n_k-l-1} - 1$, are disjoint. ∎

Recall that $|M_k|$ denotes $\max M_k$.

LEMMA 3. *Let c be an integer, $0 \le c \le a_{n_k}$. Then*

$$\|S_{cq_{n_k-1}}(F)\|_2^2 \le \Big(1 + \frac{1}{k}\Big)^2 \frac{|\bar{I}(n_k,0)|}{|I(n_k,0)|} + 4\frac{|M_{k-1}|}{q_{n_k-1}} + \frac{1}{k^2}. \tag{12}$$

Let $l \ge 1$, $0 \le c \le a_{n_k-l}$, $|M_{k-1}| \le q_{n_k-l-2}$. Then

$$\|S_{cq_{n_k-l-1}}(F)\|_2^2 \le 4\frac{|M_{k-2}|}{q_{n_k-l-1}} + \frac{4}{k^2}\cdot\frac{|M_{k-1}|}{q_{n_k-l-1}} + 4\frac{|\bar{I}(n_k,0)|}{|I(n_k - l,0)|} + \frac{1}{k^2}\cdot\frac{|I(n_k,0)|}{|I(n_k - l,0)|}. \tag{13}$$

(14) *In both cases* (12), (13) *there exists $a \in [0,1)$ such that the function $\widehat{F}_{cq_{n_k-l-1},a}$ can be expressed by*

$$\widehat{F}_{cq_{n_k-l-1},a} = \sum_{j=0}^{q_{n_k-l-1}-1} F^* \circ T^{-j}$$

where F^* is zero outside J_0.

Before proving Lemma 3 we introduce some notions and derive several auxiliary statements.

All the time we shall suppose that k is fixed.

Recall the definition of the intervals $\bar{I}(n_k,i)$ and $\bar{I}^+(n_k,i) = \bar{I}(n_k, i+q_{n_k-1})$, and recall that n_k is supposed to be even.

The number $\delta = \delta_k > 0$ will be considered negligible with respect to the length of $I(n_{k+1},0)$ so that the function $\widehat{f}_k$ is zero outside the intervals $\bar{I}(n_k,i)$, $\bar{I}^+(n_k,i)$, $i \in M_{k-1}$.

Let $L > q_{n_{k+2}}$ be an integer. We define

$$f = f_L, \quad \widehat{f} = \widehat{f}_{\delta,L}, \quad \widetilde{f} = \widetilde{f}_\delta, \quad \widehat{F} = \widehat{F}_{\delta,L,cq_{n_k\pm l-1},a}.$$

Recall that l denotes nonnegative integers such that $n_{k-1} < n_k - l$ in the expressions cq_{n_k-l-1}, and c denotes an integer satisfying

$$0 \le c \le a_{n_k-l} - 1.$$

Set

$$A = \bigcup_{i=0}^{|M_{k-1}|} \bar{I}(n_k,i), \quad B = \bigcup_{i=0}^{|M_{k-1}|} \bar{I}^+(n_k,i),$$
$$f_{(1)} = f\chi_A, \quad f_{(2)} = f\chi_B.$$

Notice that $A\cap B = \emptyset$ and $f = f_{(1)} + f_{(2)}$. From Lemma 1 we get

$$\int_0^1 f_{(1)}\,dm = 1 - \frac{1}{k}, \quad \int_0^1 f_{(2)}\,dm = \frac{1}{k}. \tag{15}$$

For $x \in [0,1)$ we have

$$S_{cq_{n_k-l-1}}(\widetilde{f})(x) \ne 0 \Rightarrow x \in \bigcup_{i=0}^{cq_{n_k-l-1}-1} T^i[0,\delta),$$
$$S_{cq_{n_k-l-1}}(f_{(1)})(x) \ne 0 \Rightarrow x \in \bigcup_{i=0}^{cq_{n_k-l-1}-1} T^i A, \tag{16}$$
$$S_{cq_{n_k-l-1}}(f_{(2)})(x) \ne 0 \Rightarrow x \in \bigcup_{i=0}^{cq_{n_k-l-1}-1} T^i B.$$

For every integrable function g and every measurable set I we have $\int_I g\circ T^{-1}\,dm = \int_{T^{-1}I} g\,dm$, hence

$$\int_I S_{cq_{n_k-l-1}}(g)\,dm = \sum_{i=0}^{cq_{n_k-l-1}-1} \int_{T^{-i}I} g\,dm.$$

Let

$$0 \le j \le q_{n_k-l-1} - 1,$$
$$0 \le i \le \begin{cases} a_{n_k-l}, & 0 \le j \le q_{n_k-l-2} - 1, \\ a_{n_k-l} - 1, & q_{n_k-l-2} \le j \le q_{n_k-l-1} - 1, \end{cases}$$
$$I_1 = T^{j+iq_{n_k-l-1}}\bar{I}(n_k,0),$$
$$I_2 = T^{j+iq_{n_k-l-1}}\bar{I}^+(n_k,0),$$
$$I = [\{(j+iq_{n_k-l-1})\alpha\}, \{(j+iq_{n_k-l-1})\alpha\} + \delta).$$

We have

$$\int_I S_{cq_{n_k-l-1}}(\widetilde{f}_\delta)\,dm = \sum_{u=0}^{cq_{n_k-l-1}-1} \int_{T^{-u}I} \widetilde{f}_\delta\,dm = \begin{cases} -1, & 0 \le i \le c-1, \\ 0, & c \le i \le a_{n_k}, \end{cases} \tag{17}$$

and we shall prove that for $c \ge 1$,

$$\int_{I_1} S_{cq_{n_k-l-1}}(f_{(1)})\,dm = \begin{cases} v, & i = 0, \\ 1-1/k, & 1 \le i \le c-1, \\ 1-1/k-v, & i = c, \\ 0, & i \ge c+1, \end{cases}$$
$$\int_{I_2} S_{cq_{n_k-l-1}}(f_{(2)})\,dm = \begin{cases} w, & i = 0, \\ 1/k, & 1 \le i \le c-1, \\ 1/k-w, & i = c, \\ 0, & i \ge c+1, \end{cases} \tag{18}$$

where $0 \le v \le 1 - 1/k$ and $0 \le w \le 1/k$.

Proof of (18). We need the following auxiliary proposition:

For every $0 \le c \le a_{n_k-l} - 1$ and $1 \le i \le cq_{n_k-l-1}$,

$$T^{-i}\bar{I}(n_k,0)\cap A = \emptyset, \quad T^{-i}\bar{I}^+(n_k,0)\cap B = \emptyset. \tag{19}$$

The sets $\bar{I}(n_k,i)$, $\bar{I}^+(n_k,i)$, $0 \le i \le cq_{n_k-1}$, are Rokhlin towers. We have $x \in T^{-i}\bar{I}(n_k,0)\cap A$ if and only if $T^i x \in \bar{I}(n_k,0)\cap \bigcup_{u=i}^{i+|M_{k-1}|}\bar{I}(n_k,u)$. By (6), $i + |M_{k-1}| \le q_{n_k} - 1$, hence the intersection is empty.

The proof of $\bar{I}^+(n_k,0)\cap B = \emptyset$ can be done in the same way. ■

By (15) and Lemma 1(9),

$$\sum_{u\in M_{k-1}} \int_{\bar{I}(n_k,u)} f_{(1)}\,dm = 1 - \frac{1}{k},$$
$$\sum_{u\in M_{k-1}} \int_{\bar{I}^+(n_k,u)} f_{(2)}\,dm = \frac{1}{k}. \tag{20}$$

By (19) and Lemma 1(9), for $e=1,2$ and $c\ge 1$ we have

(21)
$$\int_{I_e} S_{cq_{n_k-l-1}}(f_{(e)})\,dm = \sum_{u=0}^{cq_{n_k-l-1}-1}\ \int_{T^{-u}I_e} f_{(e)}\,dm$$
$$= \sum\Big\{\int_{T^{-u}I_e} f_{(e)}\,dm : j+iq_{n_k-l-1}-u\in M_{k-1},\ 0\le u\le cq_{n_k-l-1}-1\Big\}.$$

It follows that $\int_{I_1} S_{cq_{n_k-l-1}}(f_{(1)})\,dm$ equals $0\le v'\le 1-1/k$ for $i=0$, $1-1/k$ for $1\le i\le c-1$, $0\le v''\le 1-1/k$ for $i=c$, and 0 for $i\ge c+1$; and similarly for I_2 and $f_{(2)}$.

From the assumption $l\le n_k-n_{k-1}-1$ and (5) we get $|M_{k-1}|\le q_{n_k-l-1}-1$. Now, M_{k-1} is the disjoint union of $\{j-u\in M_{k-1}: 0\le u\le cq_{n_k-l-1}-1\}$ and $\{j+cq_{n_k-l-1}-u\in M_{k-1}: 0\le u\le cq_{n_k-l-1}-1\}$, hence $v'+v''=1-1/k$ and the corresponding equality holds for I_2 and $f_{(2)}$. ∎

(22) If $j\ge|M_{k-1}|$ then $v=1-1/k$, $w=1/k$.
If $j\ge|M_{k-2}|+1$ then
$$v\ge 1-\frac{2}{k},\qquad w\ge\frac{1}{k}\Big(1-\frac{1}{k-1}\Big).$$

Proof. The first statement follows immediately from (21) and (20).

By Lemma 1, for $j\le|M_{k-2}|$ we have
$$\int_{\bar I(n_{k-1},j)} f\,dm\Big/\int_{\bar I^+(n_{k-1},j)} f\,dm = k-1,$$
$$\int_{\bar I(n_k,j)} f\,dm\Big/\int_{\bar I^+(n_k,j)} f\,dm = k$$
and
$$\int_{\bar I(n_{k-1},j)} f\,dm = \int_{\bar I(n_k,j)\cup\bar I^+(n_k,j)} f\,dm,$$
$$\int_{\bar I^+(n_{k-1},j)} f\,dm = \int_{\bar I(n_k,j+q_{n_{k-1}-1})\cup\bar I^+(n_k,j+q_{n_{k-1}-1})} f\,dm,$$
and by Lemma 1(9),
$$\sum_{j\in M_{k-2}}\ \int_{\bar I(n_{k-2},j)} f\,dm = 1,$$
hence
$$\sum_{j\in M_{k-2}}\ \int_{\bar I(n_k,j)} f_{(1)}\,dm = \Big(1-\frac{1}{k}\Big)\Big(1-\frac{1}{k-1}\Big),$$

$$\sum_{j\in M_{k-2}}\ \int_{\bar I^+(n_k,j)} f_{(2)}\,dm = \frac{1}{k}\Big(1-\frac{1}{k-1}\Big).$$

From this and (21) we get
$$v\ge\Big(1-\frac{1}{k}\Big)\Big(1-\frac{1}{k-1}\Big)=1-\frac{2}{k}\quad\text{and}\quad w\ge\frac{1}{k}\Big(1-\frac{1}{k-1}\Big).\ ∎$$

We have $|M_{k-1}|\le q_{n_k-l-1}-1$ (from $l\le n_k-n_{k-1}-1$ and (5)), hence
$$\bigcup_{i=0}^{a_{n_k-l}q_{n_k-l-1}-1} T^i(A\cup B)$$
$$=\bigcup_{i=0}^{a_{n_k-l}q_{n_k-l-1}-1}\ \bigcup_{u\in M_{k-1}} T^i(\bar I(n_k,u)\cup\bar I^+(n_k,u))\subset\bigcup_{j=0}^{q_{n_k-l-1}-1} J_j.$$

From this and
$$\bigcup_{i=0}^{cq_{n_k-l-1}-1} T^i[0,\delta)\subset\bigcup_{j=0}^{q_{n_k-l-1}-1} J_j,$$
by (16) we derive
$$f(T^{-i}x)=0,\qquad i=0,\dots,a_{n_k-l}q_{n_k-l-1}-1,\ \text{for } x\in[0,1)\setminus\bigcup_{j=0}^{q_{n_k-l-1}-1} J_j,$$
hence

(23)
$$S_{cq_{n_k-l-1}}(f)(x)=0\qquad(0\le c\le a_{n_k-l}).$$

Proof of Lemma 3

Proof of (12) *and of* (14) *for* $l=0$. Recall that by our assumptions, n_k is even, hence 0 is the left end point of J_0. Let $0\le j\le q_{n_k-1}-1$ and let z be the left end point of J_j, i.e. $z=\{j\alpha\}$. Recall the notation
$$\widehat F(t)=\widehat F_{cq_{n_k-1},z}(t)=\int_z^t S_{cq_{n_k-1}}(\widehat f)\,dm.$$

We shall study $\widehat F$ on the interval J_j.

By definition $\widehat F(z)=0$. On the interval $I=[z,z+|\bar I(n_k,0)|)=\bar I(n_k,j)$ the function $\widehat F$ first decreases by
$$\int_I S_{cq_{n_k-1}}(\widetilde f)\,dm=-1$$
(see (17)), and following (18) it grows by
$$\int_{\bar I(n_k,j)} S_{cq_{n_k-1}}(\widehat f_{(1)})\,dm=v$$

to the value $\widehat{F} = v - 1$ and remains constant on the interval $[z + |\bar{I}(n_k, j)|, z + |I(n_k, j)|) = I(n_k, j) \setminus \bar{I}(n_k, j)$ (cf. (16)).

Suppose that $c \geq 2$. On the interval $\bar{I}^+(n_k, j) = \bar{I}(n_k, j + q_{n_k - 1})$ the function $\widehat{F}$ decreases by

$$-1 = \int_{\bar{I}^+(n_k,j)} S_{cq_{n_k-1}}(\widetilde{f})\,dm$$

(cf. (17)), then it increases by

$$1 - \frac{1}{k} = \int_{\bar{I}(n_k, j+q_{n_k-1})} S_{cq_{n_k-1}}(f_{(1)})\,dm$$

and by

$$w = \int_{\bar{I}^+(n_k,j)} S_{cq_{n_k-1}}(f_{(2)})\,dm$$

(cf. (18). At its end it thus reaches the value $v - 1 + w - 1/k$ and keeps it until the end of $I(n_k, j + q_{n_k-1})$ (cf. (16)).

On the intervals $I(n_k, j + iq_{n_k-1})$, $1 \leq i \leq c - 1$, the function $\widehat{F}$ behaves similarly: on $\bar{I}(n_k, j + iq_{n_k-1})$ we first have a decrease by 1 (due to $\widetilde{f}_\delta$) and an increase by $1 - 1/k$ (influence of $f_{(1)}$) and by $1/k$ (influence of $f_{(2)}$), then $\widehat{F}$ keeps the value $v - 1 + w - 1/k$.

On the interval $I(n_k, j + cq_{n_k-1}) \cup \bar{I}^+(n_k, j + cq_{n_k-1})$ the function $\widetilde{f}_\delta$ does not contribute any more (cf. (17)).

If $q_{n_k-2} \leq j \leq q_{n_k-1} - 1$, the functions $f_{(1)}$ and $f_{(2)}$ do not contribute either (on J_j) and we have $v = 1 - 1/k$, $w = 1/k$ (cf. (22)); on $\bar{I}(n_k, j + cq_{n_k-1}) = \bar{I}^+(n_k, j + (c-1)q_{n_k-1})$ the function $f_{(1)}$ does not contribute either, while $f_{(2)}$ contributes $1/k$. Hence, $\widehat{F} = 0$ at the end of J_j.

Let $0 \leq j \leq q_{n_k-2} - 1$. On $\bar{I}(n_k, j + cq_{n_k-1}) = \bar{I}^+(n_k, j + (c-1)q_{n_k-1})$, $f_{(1)}$ contributes $1 - 1/k - v$ and $f_{(2)}$ adds $1/k$, while on $\bar{I}^+(n_k, j + cq_{n_k-1})$ the function $f_{(2)}$ contributes $1/k - w$. Hence, at the right end point of J_j the function $\widehat{F}$ reaches the value $v-1+w-1/k+1-1/k-v+1/k+1/k-w = 0$ and $\widehat{F} = 0$ at the end of J_j as well.

In particular, if $c = 1$ we have

$$\int_{\bar{I}^+(n_k,j)} S_{cq_{n_k-1}}(\widetilde{f})\,dm = 0,$$

$$\int_{\bar{I}^+(n_k,j)} S_{cq_{n_k-1}}(f_{(1)})\,dm = 1 - \frac{1}{k} - v,$$

$$\int_{\bar{I}^+(n_k,j)} S_{cq_{n_k-1}}(f_{(2)})\,dm = w,$$

$$\int_{\bar{I}^+(n_k, j+q_{n_k-1})} S_{cq_{n_k-1}}(f_{(2)})\,dm = \frac{1}{k} - w.$$

Hence, at the right end point of J_j the function $\widehat{F}$ reaches the value $v - 1 + w - 1/k + 1 - 1/k - v + 1/k + 1/k - w = 0$.

By (23) we have $S_{cq_{n_k-1}}(f)(x) = 0$ for all $x \in [0,1) \setminus \bigcup_{j=0}^{q_{n_k-1}-1} J_j$. Therefore, $\widehat{F}_{cq_{n_k-1},z}(t) = \widehat{F}_{cq_{n_k-1},0}(t)$ for every $t \in J_j$ and on $[0,1) \setminus \bigcup_{j=0}^{q_{n_k-1}-1} J_j$ we have $\widehat{F}_{cq_{n_k-1},0}(t) = 0$.

Let $\widehat{F} = \widehat{F}_{cq_{n_k-1},0}$ on $[0,1)$. We can see that (14) is fulfilled.

We have

- $|S_{cq_{n_k-1}}(\widehat{F})| \leq 2$ on the sets J_j with $0 \leq j \leq |M_{k-1}|$, hence on a set of measure smaller than $|M_{k-1}|/q_{n_k-1}$,
- $|S_{cq_{n_k-1}}(\widehat{F})| \leq 1 + 1/k$ on the sets $\bar{I}(n_k, j + iq_{n_k-1})$, $|M_{k-1}| + 1 \leq j \leq q_{n_k-1} + 1$, $0 \leq i \leq c - 1$, hence on a set of measure smaller than $|\bar{I}(n_k, 0)|/|I(n_k, 0)|$,
- $|S_{cq_{n_k-1}}(\widehat{F})| \leq 1/k$ on the rest of $[0,1)$.

A direct computation shows that

$$\|S_{cq_{n_k-1}}(\widehat{F}_0)\|_2^2 \leq \left(1 + \frac{1}{k}\right)^2 \frac{|\bar{I}(n_{k+1}, 0)|}{|I(n_k, 0)|} + 4\frac{|M_{k-1}|}{q_{n_k-1}} + \frac{1}{k^2}.$$

From this and from Lemma 2 we get (12). ∎

If n_k is odd, the intervals J_j are reversed (in the order of $[0,1)$ the interval $I(n_k, 0)$ is on the right of $I(n_k, q_{n_k-1})$) and by a similar computation we get the same estimate.

Proof of (13) *and of* (14) *for* $1 \leq l \leq n_k - n_{k-1} - 1$. We consider the Rokhlin tower $I(n_k - l, i)$, $i = 0, \ldots, q_{n_k-l} - 1$, and the intervals J_j, $0 \leq j \leq q_{n_k-l-1} - 1$.

For simplicity suppose that l is even; the other case is similar.

Let z be the left end point of the interval J_j, and

$$\widehat{F}(t) = \widehat{F}_{cq_{n_k-l},z}(t) = \int_z^t S_{cq_{n_k-l}}(\widehat{f})\,dm, \quad t \in J_j.$$

As in the proof of (12), on the interval $I = [z, z + |\bar{I}(n_k, 0)|) = \bar{I}(n_k, j)$ the function $\widehat{F}$ first decreases by $\int_I S_{cq_{n_k-1}}(\widetilde{f})\,dm = -1$ (see (17)), and following (18) it grows by $\int_{\bar{I}(n_k,j)} S_{cq_{n_k-1}}(\widetilde{f}_{(1)})\,dm = v$ to the value $\widehat{F} = v - 1$; by (16) it is constant on the interval $[z + |\bar{I}(n_k, j)|, z + |I(n_k, j)|) = I(n_k, j) \setminus \bar{I}(n_k, j)$.

On the interval $\bar{I}^+(n_k, j)$ the function $\widehat{F}$ increases by

$$w = \int\limits_{\bar{I}^+(n_k,j)} S_{cq_{n_k-1}}(f_{(2)})\,dm$$

(cf. (18)). At its end it thus reaches the value $v-1+w$ and by (16) it remains constant until the end of $I(n_k - l, j + q_{n_k-l-1})$.

On the intervals $I(n_k - l, j + iq_{n_k-l-1})$ the function $\widehat{F}$ behaves similarly: Suppose that $1 \le i \le c-1$. On $\bar{I}(n_k, j + iq_{n_k-l-1})$ we first have a decrease by 1 (due to $\widetilde{f_\delta}$) and an increase by $1 - 1/k$ (influence of $f_{(1)}$); on $\bar{I}^+(n_k, j + iq_{n_k-l-1})$ we have an increase by $1/k$ (influence of $f_{(2)}$), then by (16), $\widehat{F}$ remains equal to $v-1+w$ until the end of $I(n_k - l, j + iq_{n_k-l-1})$.

Let $i = c$. On the interval $I(n_k - l, j + cq_{n_k-l-1})$ the function $\widetilde{f_\delta}$ does not contribute (cf. (17)).

If $q_{n_k-l-2} \le j \le q_{n_k-l-1} - 1$, the functions $f_{(1)}$ and $f_{(2)}$ do not contribute either (on J_j) and we have $v = 1 - 1/k$, $w = 1/k$ (cf. (22)), hence $\widehat{F} = 0$ at the end of J_j.

Let $0 \le j \le q_{n_k-l-2} - 1$. On $\bar{I}(n_k, j + cq_{n_k-1})$ the function $f_{(1)}$ contributes $1 - 1/k - v$, and on $\bar{I}^+(n_k, j + cq_{n_k-1})$, $f_{(2)}$ contributes $1/k - w$ (cf. (18)). Hence, at the right end point of J_j the function $\widehat{F}$ reaches the value 0.

Define

$$\widehat{F}(t) = \widehat{F}_{cq_{n_k-l-1},0}(t) = \int_0^t S_{cq_{n_k-l-1}}(\widehat{f})\,dm \quad \text{on } [0,1).$$

0 is the left end point of the interval J_0 (this is not the case for l odd, $c > 1$!). By the previous computation the values on both ends of each of the intervals J_j, $0 \le j \le q_{n_k-l-1} - 1$, are the same and by (23), $\widehat{F}(t)$ is constant (hence zero) on $[0,1) \setminus \bigcup_{j=0}^{q_{n_k-1}-1} J_j$. Thus (14) is fulfilled.

If $|M_{k-1}| + 1 \le j \le q_{n_k-1} - 1$, we have (cf. (22))

- $|\widehat{F}| \le 1$ on the sets $\bar{I}(n_k, j + iq_{n_k-l-1})$, $0 \le i \le c-1$,
- $\widehat{F} = -1/k$ on $I(n_k, j + iq_{n_k-l-1}) \setminus \bar{I}(n_k, j + iq_{n_k-l-1})$, $0 \le i \le c-1$,
- $0 \ge |\widehat{F}| \ge -1/k$ on $\bar{I}^+(n_k, j + iq_{n_k-l-1})$, $0 \le i \le c-1$,
- $\widehat{F} = 0$ on the rest of J_j.

Let $|M_{k-2}| + 1 \le j \le |M_{k-1}|$. By (22) we then get

- $|\widehat{F}| \le 1 + 2/k$ on the sets $\bar{I}(n_k, j + iq_{n_k-l-1})$, $0 \le i \le c-1$ (≤ 1 on $\bar{I}(n_k, j)$),
- $-1/k \ge \widehat{F} \ge -2/k$ on $I(n_k, j + iq_{n_k-l-1}) \setminus \bar{I}(n_k, j + iq_{n_k-l-1})$, $0 \le i \le c-1$,
- $|\widehat{F}| \le 2/k$ on the whole of J_j, and on its end it equals 0.

By direct computation we get the required estimate. ∎

Recall the Ostrowski decomposition $N = \sum_{j=1}^n c_j q_{j-1}$, which implies

$$S_N(F) = S_{c_1q_0}(F) + S_{c_2q_1}(F) \circ T^{-c_1q_0} + \ldots + S_{c_nq_{n-1}}(F) \circ T^{-c_1q_0 - \ldots - c_{n-1}q_{n-2}},$$

and define

$$\overline{D}_k = S_{c_{n_k-k}q_{n_k-k-1}}(F) + S_{c_{n_k-k+1}q_{n_k-k}}(F) \circ T^{-c_{n_k-k}q_{n_k-k-1}} + \ldots + S_{c_{n_k+k}q_{n_k+k-1}}(F) \circ T^{-(c_{n_k-k}q_{n_k-k-1} + \ldots + c_{n_k+k-1}q_{n_k+k-2})},$$

$$D_k = \overline{D}_k \circ T^{-(c_1q_0 + \ldots c_{n_k-k-1}q_{n_k-k-2})}, \quad k = 1, 2, \ldots$$

We thus have

$$\begin{aligned} S_N(F) = {} & S_{c_1q_0}(F) + S_{c_2q_1}(F) \circ T^{-c_1q_0} + \ldots \\ & + S_{c_{n_1-2}q_{n_1-3}}(F) \circ T^{-(c_1q_0 + \ldots + c_{n_1-2}q_{n_1-3})} \\ & + D_1 + S_{c_{n_1+2}q_{n_1+1}}(F) \circ T^{-(c_1q_0 + \ldots + c_{n_1+1}q_{n_1})} + \ldots \\ & + S_{c_{n_k-k-1}q_{n_k-k-2}}(F) \circ T^{-(c_1q_0 + \ldots + c_{n_k-k-2}q_{n_k-k-3})} + D_k \\ & + S_{c_{n_k+k+1}q_{n_k+k}}(F) \circ T^{-(c_1q_0 + \ldots + c_{n_k+k}q_{n_k+k-1})} + \ldots \end{aligned}$$

Consider the sums

$$\sum \{ \|S_{c_jq_{j-1}}(F)\|_2 : \forall k,\ |j - n_k| > k \}.$$

Define

$$r_k = \sum \{ \|S_{c_jq_{j-1}}(F)\|_2 : n_{k-1} + k \le j \le n_k - k \}.$$

By Lemma 3 we have

$$\|S_{c_{n_k-l}q_{n_k-l-1}}(F)\|_2 \le 2\sqrt{\frac{|M_{k-2}|}{q_{n_k-l-1}}} + 2\sqrt{\frac{|\bar{I}(n_k,0)|}{|I(n_k-l,0)|}} + \frac{2}{k}\sqrt{\frac{|M_{k-1}|}{q_{n_k-l-1}}} + \frac{1}{k}\sqrt{\frac{|I(n_k,0)|}{|I(n_k-l,0)|}}.$$

From this and (5), (2), (3), (8) it follows that there exists a constant K_1 such that for any N,

$$r_k \le \frac{K_1}{2^{k/4}} \left(\sqrt{\frac{q_{n_{k-2}}}{q_{n_k-1}}} + \sqrt{\frac{q_{n_k}}{q_{n_{k+1}}}} + \frac{1}{k}\sqrt{\frac{q_{n_{k-1}}}{q_{n_{k-1}}}} + \frac{1}{k}\sqrt{\frac{q_{n_k}}{q_{n_k}}} \right) \le \frac{4K_1}{2^{k/4}},$$

and hence there exists $K_2 < \infty$ such that for every N,

$$\sum_{k=1}^{\infty} r_k \le K_2. \tag{24}$$

To finish the proof of Proposition 2 it thus suffices to show that the sums $\sum_k D_k$ are uniformly bounded in L^2.

If the sequence (n_k) grows sufficiently fast (which we assume), using Lemma 3, (5), (2), (3), and (8) in the same way as above we can prove that there exists a constant K_3 such that $\|D_k\|_2 \le K_3/k$ for every N and every k.

Let $0 < k' < k''$ be positive integers, l', l'' integers for which $|l'| \le k'$, $|l''| \le k''$, $0 \le c' \le a_{n_{k'}}$, $0 \le c'' \le a_{n_{k''}}$, $N' = c' q_{n_{k'}+l'-1}$, $N'' = c'' q_{n_{k''}+l''-1}$. By Lemma 2 for any $\varepsilon > 0$ there exist $\delta', \delta'' > 0$ sufficiently small and integers K', K'' sufficiently large so that for every $a', a'' \in [0,1)$,

$$\|S_{N'}(F) - (\widehat{F}_{\delta',K',N',a'} - E\widehat{F}_{\delta',K',N',a'})\|_2 < \varepsilon,$$

$$\|S_{N''}(F) - (\widehat{F}_{\delta'',K',N'',a''} - E\widehat{F}_{\delta'',K'',N'',a''})\|_2 < \varepsilon.$$

Suppose that the numbers N', N'' are given. Choose $\varepsilon = 1/(k'k''2^{k'+k''})$. Fix δ', K', a'. Since $\widehat{F}_{\delta',K',N',a'}$ is a continuous function, we can find a step function $\widehat{F}_1$ for which $\|\widehat{F}_1 - \widehat{F}_{\delta',K',N',a'}\|_2 < \varepsilon$. In Lemma 3(14) we showed that the function $\widehat{F}_{\delta'',K'',N'',a''}$ has the same values on subintervals of the intervals J_j, $0 \le j \le q_{n_{k''}+l'-1} - 1$. Because the sequence $(\{j\alpha\})$ is uniformly distiributed (cf. [6]), for every fixed interval $I \subset [0,1)$, and for $n_{k''}$ sufficiently large, the fraction $\#\{J : J_j \subset I\}/q_{n_{k''}+l'-1}$ is close enough to $m(I)$. Therefore, if $n_{k'+1} \le n_{k''}$ is sufficiently large, we have

$$|E(\widehat{F}_{\delta',K',N',a'} - E\widehat{F}_{\delta',K',N',a'})(\widehat{F}_{\delta'',K',N'',a''} - E\widehat{F}_{\delta'',K'',N'',a''})| < \varepsilon.$$

From this we deduce that if the sequence (n_k) grows sufficiently fast, then there exists a constant M such that for every k, $\sum_{i=1}^{k-1} |ED_iD_k| < M2^{-k}$ (independently of N).

Therefore, the sums $\sum_{k=1}^{\infty} D_k$ are uniformly bounded in L^2 for all N.

From this and (24) it follows that the norms $\|S_N(F)\|_2$ are uniformly bounded independently of N, which proves Proposition 2. This finishes the proof of the Theorem. ∎

A substantial part of the paper was written during my stay at the University of Tours; I thank my colleagues, especially E. Lesigne, for their hospitality. I also thank M. Lemańczyk and F. Parreau for helpful discussions.

I thank the referee for helpful remarks.

References

[1] W. Bułatek, M. Lemańczyk and D. Rudolph, *Constructions of cocycles over irrational rotations*, Studia Math. 125 (1997), 1–11.

[2] H. Furstenberg, *Strict ergodicity and transformations of the torus*, Amer. J. Math. 89 (1961), 573–601.

[3] P. Gabriel, M. Lemańczyk et P. Liardet, *Ensemble d'invariants pour les produits croisés de Anzai*, Mém. Soc. Math. France 47 (1991).

[4] A. Iwanik, M. Lemańczyk and D. Rudolph, *Absolutely continuous cocycles over irrational rotations*, Israel J. Math. 83 (1993), 73–95.

[5] A. Ya. Khintchine, *Continued Fractions*, Noordhoff, Groningen, 1963.

[6] L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences*, Wiley, New York, 1974.

[7] M. Lemańczyk, F. Parreau and D. Volný, *Ergodic properties of real cocycles and pseudo-homogeneous Banach spaces*, Trans. Amer. Math. Soc., to appear.

[8] W. Parry and S. Tuncel, *Classification Problems in Ergodic Theory*, London Math. Soc. Lecture Note Ser. 67, Cambridge Univ. Press, Cambridge, 1982.

[9] K. Schmidt, *Cocycles of Ergodic Transformation Groups*, Macmillan Lectures in Math. 1, Macmillan of India, 1977.

[10] D. Volný, *Cohomology of Lipschitz and absolutely continuous functions over irrational circle rotations*, submitted for publication.

[11] —, *Constructions of smooth and analytic cocycles over irrational circle rotations*, Comment. Math. Univ. Carolin. 36. (1995), 745–764.

Mathematical Institute
Charles University
Sokolovská 83
186 00 Praha 8, Czech Republic
E-mail: dvolny@karlin.mff.cuni.cz

Current address:
Département de Mathématiques
Université de Rouen
76821 Mont-Saint-Aignan Cedex, France
E-mail: Dalibor.Volny@univ-rouen.fr

Received September 23, 1996
Revised version January 13, 1997
(3742)