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Packing in Orlicz sequence spaces

by

M. M. RAO (Riverside, Calif.) and
Z. D. REN (Riverside, Calif., and Suzhou)

Abstract. We show how one can, in a unified way, calculate the Kottman and the packing constants of the Orlicz sequence space defined by an N-function, equipped with either the gauge or Orlicz norms. The values of these constants for a class of reflexive Orlicz sequence spaces are found, using a quantitative index of N-functions and some interpolation theorems. The exposition is essentially selfcontained.

1. Introduction. We say that a sequence of balls with centers $\{x_i\}$ and radius r, $0 < r \le 1/2$, can be packed into the unit ball $U(X)$ of a Banach space X if $\|x_i\| \le 1 - r$, $i = 1, 2, \ldots$, and $\|x_i - x_j\| \ge 2r$, $i \ne j, i, j = 1, 2, \ldots$

DEFINITION 1.1. The *packing constant* of a Banach space X is defined as

$$(1)\qquad P(X) = \sup\{r > 0 : \text{infinitely many balls of radius } r \text{ can be packed into } U(X)\}.$$

It is clear that $P(X) = 0$ if $\dim X < \infty$. Burlak, Rankin and Robertson [1] proved that $P(\ell^1) = P(\ell^\infty) = 1/2$ and $P(\ell^p) = 1/(1 + 2^{1-1/p})$ for $1 < p < \infty$ (see also Example 2.4 below).

THEOREM 1.2 (Kottman [7]). *Let X be an infinite-dimensional Banach space. Define*

$$(2)\qquad K(X) = \sup\{\inf_{i \ne j} \|x_i - x_j\| : \{x_i\}_{i=1}^\infty \subset X,\ \|x_i\| = 1,\ i \ge 1\},$$

to be called the Kottman constant *of X. Then*

$$(3)\qquad P(X) = \frac{K(X)}{2 + K(X)}.$$

Remark 1.3. It is seen that $1 \le K(X) \le 2$. Furthermore, Elton and Odell [5] proved that if X is an infinite-dimensional Banach space, then there exists an ε $(= \varepsilon(X) > 0)$ such that $K(X) \ge 1 + \varepsilon$. Consequently, $1/3 <$

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$P(X) \le 1/2$. Hudzik [6] proved that $P(Y) = 1/2$ for every nonreflexive Banach lattice Y.

Let

$$(4)\qquad \Phi(u) = \int_0^{|u|} \phi(t)\,dt \quad\text{and}\quad \Psi(v) = \int_0^{|v|} \psi(s)\,ds$$

be a pair of complementary N-functions (see [8, p. 11]). The *Orlicz sequence space* ℓ^{Φ} is defined to be the set $\{x = (t_i) : \varrho_\Phi(\lambda x) = \sum_{i=1}^{\infty} \Phi(\lambda|t_i|) < \infty$ for some $\lambda > 0\}$. The *gauge norm* $\|\cdot\|_{(\Phi)}$ and the *Orlicz norm* $\|\cdot\|_\Phi$ are given by

$$\|x\|_{(\Phi)} = \inf\{c > 0 : \varrho_\Phi(x/c) \le 1\} \quad\text{and}\quad \|x\|_\Phi = \inf_{k>0} \frac{1}{k}[1 + \varrho_\Phi(kx)].$$

These norms are equivalent: $\|x\|_{(\Phi)} \le \|x\|_\Phi \le 2\|x\|_{(\Phi)}$. The closed separable subspace h^Φ of ℓ^Φ is the set $\{x \in \ell^\Phi : \varrho_\Phi(\lambda x) < \infty$ for all $\lambda > 0\}$.

A special role (for analysis) in an Orlicz (sequence) space is played by the rate of growth of the underlying N-function. An N-function $\Phi(u)$ is said to satisfy the Δ_2-*condition* near 0, in symbols $\Phi \in \Delta_2(0)$, if there exist $u_0 > 0$ and $K > 2$ such that $\Phi(2u) \le K\Phi(u)$ for $0 \le u \le u_0$. An N-function $\Phi(u)$ is said to satisfy the ∇_2-*condition* near 0, in symbols $\Phi \in \nabla_2(0)$, if there exist $u_0 > 0$ and $a > 1$ such that $\Phi(u) \le \frac{1}{2a}\Phi(au)$ for $0 \le u \le u_0$. The basic facts on N-functions and Orlicz spaces can be found in [8], [9] and [12]. For instance, $\Phi \in \nabla_2(0)$ if and only if $\Psi \in \Delta_2(0)$, where Ψ is the complementary N-function to Φ. The space ℓ^Φ is separable if and only if $\Phi \in \Delta_2(0)$, if and only if $\ell^\Phi = h^\Phi$. The space ℓ^Φ is reflexive if and only if $\Phi \in \Delta_2(0) \cap \nabla_2(0)$. For simplicity, we use the following notations:

$$\ell^{(\Phi)} = (\ell^\Phi, \|\cdot\|_{(\Phi)}), \quad \ell^\Phi = (\ell^\Phi, \|\cdot\|_\Phi), \quad h^{(\Phi)} = (h^\Phi, \|\cdot\|_{(\Phi)}), \quad h^\Phi = (h^\Phi, \|\cdot\|_\Phi).$$

The packing problem in Orlicz sequence spaces was investigated by Cleaver [3], Ye [16], Ye and Li [17], Wang [14], Wang and Liu [15], Domínguez Benavides and Rodríguez [4], Hudzik [6] and Ren [13, Section 4].

Quantitative indices of $\Phi(u)$, in the theory of Orlicz sequence spaces, are provided by the following two constants:

$$(5)\qquad \alpha_\Phi^0 = \liminf_{u\to 0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)} \quad\text{and}\quad \beta_\Phi^0 = \limsup_{u\to 0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}.$$

It is clear that $1/2 \le \alpha_\Phi^0 \le \beta_\Phi^0 \le 1$. The following result will play a key role in the analysis below.

THEOREM 1.4. (i) $\Phi \notin \Delta_2(0)$ *if and only if* $\beta_\Phi^0 = 1$;
(ii) $\Phi \notin \nabla_2(0)$ *if and only if* $\alpha_\Phi^0 = 1/2$.

Proof. The proof is similar to that of Theorem 3 in Rao and Ren [12, pp. 23–26]. ∎

Other quantitative indices of $\Phi(u)$ at 0 are provided by the following two constants:

$$(6)\qquad A_\Phi^0 = \liminf_{t\to 0} \frac{t\phi(t)}{\Phi(t)} \quad\text{and}\quad B_\Phi^0 = \limsup_{t\to 0} \frac{t\phi(t)}{\Phi(t)},$$

where ϕ is the right derivative of Φ. Clearly, $1 \le A_\Phi^0 \le B_\Phi^0 \le \infty$. It is also known that $\Phi \notin \Delta_2(0) \Leftrightarrow B_\Phi^0 = \infty$, $\Phi \notin \nabla_2(0) \Leftrightarrow A_\Phi^0 = 1$ and that

$$(7)\qquad \frac{1}{A_\Phi^0} + \frac{1}{B_\Psi^0} = 1 = \frac{1}{A_\Psi^0} + \frac{1}{B_\Phi^0},$$

where Ψ is the complementary N-function to Φ (see Lindenstrauss and Tzafriri [10]). The relation between the indices (5) and (6) is as follows.

PROPOSITION 1.5. *Let Φ be an N-function. Then*

$$(8)\qquad 2^{-1/A_\Phi^0} \le \alpha_\Phi^0 \le \beta_\Phi^0 \le 2^{-1/B_\Phi^0}.$$

Proof. If $B_\Phi^0 = \infty$, then clearly $\beta_\Phi^0 \le 1 = 2^{-1/B_\Phi^0}$. Assume that $B_\Phi^0 < \infty$. For any given $\varepsilon > 0$, by (6) there exists $t_0 > 0$ such that $t\phi(t)/\Phi(t) < B_\Phi^0 + \varepsilon$ for all $0 < t \le t_0$. If $0 < t_1 < t_2 \le t_0$, we have

$$\ln\frac{\Phi(t_2)}{\Phi(t_1)} = \int_{t_1}^{t_2} \frac{\phi(t)}{\Phi(t)}\,dt \le \int_{t_1}^{t_2} \frac{B_\Phi^0 + \varepsilon}{t}\,dt = \ln\left(\frac{t_2}{t_1}\right)^{B_\Phi^0+\varepsilon}.$$

Letting $t_1 = \Phi^{-1}(u)$ and $t_2 = \Phi^{-1}(2u)$ in the above, we get for $0 < 2u \le \Phi(t_0)$,

$$\frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)} \le 2^{-1/(B_\Phi^0+\varepsilon)},$$

which proves the right side of (8). Similarly, one can prove the left side of (8). ∎

Sections 1 and 2 of this paper are adapted from Ren [13, Section 4]. The main results of this paper are Theorems 4.3 and 4.8. Some illustrative examples are included.

2. Estimates for $K(\ell^{(\Phi)})$ and $P(\ell^{(\Phi)})$. We start with Ye's result.

LEMMA 2.1 (Ye [16]). *Let Φ be an N-function.*

(i) *If $\Phi \notin \Delta_2(0)$, then $K(\ell^{(\Phi)}) = 2$ and $P(\ell^{(\Phi)}) = 1/2$.*
(ii) *If $\Phi \in \Delta_2(0)$, then*

$$(9)\qquad K(\ell^{(\Phi)}) = \sup_{\|x\|_{(\Phi)}=1} \{c > 0 : \varrho_\Phi(x/c) = 1/2\}.$$

The proof of Lemma 2.1(i) can be found in Orlicz [10, p. 180]. Formula (9) is a special case of a general theorem given in Wang and Liu [15].

LEMMA 2.2. (i) *Let Φ be an N-function. Then*

$$\frac{1}{\alpha_\Phi^0} \le K(\ell^{(\Phi)}), \tag{10}$$

or equivalently,

$$\frac{1}{1+2\alpha_\Phi^0} \le P(\ell^{(\Phi)}). \tag{11}$$

(ii) *If $\Phi \notin \nabla_2(0)$, then $K(\ell^{(\Phi)}) = 2$ and $P(\ell^{(\Phi)}) = 1/2$.*

Proof. (i) By (5), there exist $1/2 > u_n \searrow 0$ such that

$$\lim_{n\to\infty} \Phi^{-1}(u_n)/\Phi^{-1}(2u_n) = \alpha_\Phi^0.$$

For any given $1 > \varepsilon > 0$, there is an $n_0 \ge 1$ such that $\Phi^{-1}(u_n)/\Phi^{-1}(2u_n) < \alpha_\Phi^0 + \varepsilon$ for all $n \ge n_0$, or equivalently,

$$\Phi[(\alpha_\Phi^0+\varepsilon)\Phi^{-1}(2u_n)] > u_n, \quad n \ge n_0. \tag{12}$$

Let $k_n = [1/(2u_n)]$ (the integer part). Then $k_n \le 1/(2u_n) < k_n + 1$. Choose $c_n \ge 0$ such that $k_n 2u_n + \Phi(c_n) = 1$. Since $\Phi(c_n) < (k_n+1)2u_n - k_n 2u_n = 2u_n \searrow 0$ as $n \to \infty$, we may assume that $\Phi(c_n) < \varepsilon$ for all $n \ge n_0$. For simplicity, we set $v_i = u_{n_0+i}$, $m_i = k_{n_0+i}$ and $d_i = c_{n_0+i}$, $i = 1, 2, \ldots$ Let $Z_i = (0, 0, \ldots, 0)$ and $X_i = (\Phi^{-1}(2v_i), \Phi^{-1}(2v_i), \ldots, \Phi^{-1}(2v_i), d_i)$ be (m_i+1)-component vectors. Define $\{x_i\}_{i=1}^\infty \subset \ell^{(\Phi)}$ by

$$x_i = (Z_1, Z_2, \ldots, Z_{i-1}, X_i, Z_{i+1}, \ldots).$$

Then $\varrho_\Phi(x_i) = 1$, and so $\|x_i\|_{(\Phi)} = 1$, $i \ge 1$. For $i \ne j$, from (12) we have

$$\begin{aligned}
\varrho_\Phi\left[\frac{\alpha_\Phi^0+\varepsilon}{1-\varepsilon}(x_i-x_j)\right]
&> \frac{1}{1-\varepsilon}\varrho_\Phi[(\alpha_\Phi^0+\varepsilon)(x_i-x_j)]\\
&\ge \frac{1}{1-\varepsilon}\{m_i\Phi[(\alpha_\Phi^0+\varepsilon)\Phi^{-1}(2v_i)] + m_j\Phi[(\alpha_\Phi^0+\varepsilon)\Phi^{-1}(2v_j)]\}\\
&> \frac{1}{1-\varepsilon}\{m_i v_i + m_j v_j\} = \frac{1}{2(1-\varepsilon)}\{[1-\Phi(d_i)] + [1-\Phi(d_j)]\}\\
&> \frac{1}{2(1-\varepsilon)}[(1-\varepsilon)+(1-\varepsilon)] = 1,
\end{aligned}$$

which implies that

$$\|x_i - x_j\|_{(\Phi)} > \frac{1-\varepsilon}{\alpha_\Phi^0+\varepsilon}.$$

We have thus proved (10) since ε is arbitrary. Finally, (11) follows from (10) and (3).

(ii) If $\Phi \notin \nabla_2(0)$, then $\alpha_\Phi^0 = 1/2$ by Theorem 1.4(ii), and so $2 \le K(\ell^{(\Phi)})$ by (10). Since $K(\ell^{(\Phi)}) \le 2$ is always true, we obtain the desired conclusion. ■

Lemma 2.2 sharpens Theorem 4 of Rao and Ren [12, p. 256] when the Orlicz function space is a sequence space; and (10) will be used in Section 4. A more complete discussion of packing in Orlicz function spaces for nonatomic measures is given in Ren [13]. In view of Lemmas 2.1(i) and 2.2(ii), if $\ell^{(\Phi)}$ is nonreflexive, then $K(\ell^{(\Phi)}) = 2$ and $P(\ell^{(\Phi)}) = 1/2$. Therefore, we only need to estimate $K(\ell^{(\Phi)})$ and $P(\ell^{(\Phi)})$ for a reflexive space $\ell^{(\Phi)}$. In view of (3), it suffices to consider one of these.

THEOREM 2.3. *Let Φ be an N-function and let $\Phi \in \Delta_2(0) \cap \nabla_2(0)$. Then*

$$1 < 2^{1/B_\Phi^0} \le \frac{1}{\alpha_\Phi^0} \le K(\ell^{(\Phi)}) \le \frac{1}{\widetilde{\alpha}_\Phi} \le 2^{1/\widetilde{A}_\Phi} < 2, \tag{13}$$

or equivalently,

$$\begin{aligned}
\frac{1}{3} < \frac{1}{1+2^{1-1/B_\Phi^0}} \le \frac{1}{1+2\alpha_\Phi^0} \le P(\ell^{(\Phi)})\\
\le \frac{1}{1+2\widetilde{\alpha}_\Phi} \le \frac{1}{1+2^{1-1/\widetilde{A}_\Phi}} < \frac{1}{2},
\end{aligned} \tag{14}$$

where

$$\widetilde{\alpha}_\Phi = \inf\left\{\frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)} : 0 < u \le \frac{1}{2}\right\}, \tag{15}$$

and

$$\widetilde{A}_\Phi = \inf\left\{\frac{t\phi(t)}{\Phi(t)} : 0 < t \le \Phi^{-1}(1)\right\}. \tag{16}$$

Proof. The left side of (13) follows immediately from (10), (8) and the fact that $B_\Phi^0 < \infty$ if $\Phi \in \Delta_2(0)$. By using (9), Ye [16] proved

$$K(\ell^{(\Phi)}) \le 1/\widetilde{\alpha}_\Phi. \tag{17}$$

We now assert

$$1/\widetilde{\alpha}_\Phi \le 2^{1/\widetilde{A}_\Phi}. \tag{18}$$

By (16), $t\phi(t)/\Phi(t) \ge \widetilde{A}_\Phi$ if $0 < t \le \Phi^{-1}(1)$. Therefore, if $0 < t_1 < t_2 \le \Phi^{-1}(1)$ one has

$$\ln\frac{\Phi(t_2)}{\Phi(t_1)} = \int_{t_1}^{t_2} \frac{\phi(t)}{\Phi(t)}\,dt \ge \int_{t_1}^{t_2} \frac{\widetilde{A}_\Phi}{t}\,dt = \ln\left(\frac{t_2}{t_1}\right)^{\widetilde{A}_\Phi}.$$

Letting $t_1 = \Phi^{-1}(u)$ and $t_2 = \Phi^{-1}(2u)$ for $0 < u \le 1/2$ in the above and noting that Φ is strictly increasing, we obtain

$$\frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)} \ge 2^{-1/\widetilde{A}_\Phi},$$

which implies (18). Finally, to prove the last part of (13) we must show

$$(19)\qquad 1 < \widetilde{A}_{\Phi}.$$

Since $\Phi \in \nabla_2(0)$, there exist $a > 1$ and $t_0 > 0$ such that $t\phi(t)/\Phi(t) \ge a$ if $0 < t \le t_0$. We may assume that $t_0 < \Phi^{-1}(1)$. Therefore, for all $t \in [t_0, \Phi^{-1}(1)]$ we have

$$\frac{t\phi(t)}{\Phi(t)} = 1 + \frac{\Psi[\phi(t)]}{\Phi(t)} \ge 1 + \Psi[\phi(t_0)] > 1,$$

where Ψ is the complementary N-function to Φ. Thus, (19) holds. ∎

It should be noted that Ye and Li [17] first showed that $\ell^{(\Phi)}$ is reflexive if and only if $P(\ell^{(\Phi)}) < 1/2$, with a longer proof. This result was reproved in Hudzik [6, Theorem 3] by using Banach lattice techniques. Also under the assumption $\Phi(1) = 1$, Domínguez Benavides and Rodríguez [4, Theorem 5 and Corollary 2] gave some lower and upper bounds for $K(\ell^{(\Phi)})$ and $P(\ell^{(\Phi)})$, and these are all contained in Theorem 2.3. The referee has informed us that the leftmost inequality in (13) was also obtained by H. Hudzik and T. Landes, *Packing constant in Orlicz spaces equipped with the Luxemburg norm*, Boll. Un. Mat. Ital. A (7) 9 (1995), 225–237.

EXAMPLE 2.4 ([1]). If $\Phi_p(u) = |u|^p$, $1 < p < \infty$, then $\ell^p = \ell^{(\Phi_p)}$ and $\|\cdot\|_{(\Phi_p)} = \|\cdot\|_p$. Since $\alpha^0_{\Phi_p} = \widetilde{\alpha}_{\Phi_p} = 2^{-1/p}$, from (13) and (3) one has

$$(20)\qquad K(\ell^p) = 2^{1/p}, \quad P(\ell^p) = \frac{1}{1+2^{1-1/p}}.$$

EXAMPLE 2.5. Let $M_r(u) = e^{|u|^r} - 1$ with $1 < r < \infty$ and $\Phi(u) = |u|^p + |u|^{p+a}$ with $1 < p < \infty$ and $a > 0$. Then

$$(21)\qquad K(\ell^{(M_r)}) = 2^{1/r} = K(\ell^r), \quad P(\ell^{(M_r)}) = P(\ell^r)$$

and

$$(22)\qquad K(\ell^{(\Phi)}) = K(\ell^p), \quad P(\ell^{(\Phi)}) = P(\ell^p).$$

Note that $M_r^{-1}(u) = [\ln(1+u)]^{1/r}$ for $u \ge 0$ and

$$\alpha^0_{M_r} = \beta^0_{M_r} = \lim_{u\to 0} \frac{M_r^{-1}(u)}{M_r^{-1}(2u)} = 2^{-1/r},$$

which shows that $M_r \in \Delta_2(0) \cap \nabla_2(0)$ in view of Theorem 1.4. By (15), one has

$$\widetilde{\alpha}_{M_r} = \inf\{[F(u)]^{1/r} : 0 < u \le 1/2\},$$

where $F(u) = \ln(1+u)/\ln(1+2u)$. Let

$$G(u) = (1+2u)\ln(1+2u) - 2(1+u)\ln(1+u).$$

Then $G(0) = 0$ and $G'(u) = 2[\ln(1+2u) - \ln(1+u)] > 0$ for $u > 0$. Thus, for $u > 0$ we have

$$F'(u) = \frac{G(u)}{(1+u)(1+2u)[\ln(1+2u)]^2} > 0,$$

and so $\widetilde{\alpha}_{M_r} = \alpha^0_{M_r}$. Finally, (21) follows from Theorem 2.3.

It remains to establish (22). Since

$$\frac{t\phi(t)}{\Phi(t)} = p + \frac{at^a}{1+t^a}, \quad 0 < t < \infty,$$

we have $\widetilde{A}_{\Phi} = B^0_{\Phi} = p$, which implies (22) in view of Theorem 2.3.

EXAMPLE 2.6. Let $M(u) = e^{|u|} - |u| - 1$ and $N(v) = (1+|v|)\ln(1+|v|) - |v|$. Then

$$(23)\qquad K(\ell^{(M)}) = \sqrt{2}, \quad P(\ell^{(M)}) = \frac{1}{1+\sqrt{2}}$$

and

$$(24)\qquad 1.41 \approx 2^{1/2} \le K(\ell^{(N)}) \le 2^{1/(e-1)} \approx 1.50,$$

or equivalently,

$$\frac{1}{1+2^{1/2}} \le P(\ell^{(N)}) \le \frac{1}{1+2^{(e-2)/(e-1)}}.$$

In fact, since $tM'(t)/M(t) = t(e^t-1)/(e^t-t-1)$ for $t > 0$, we have $B^0_M = 2$ and

$$\frac{d}{dt}\left[\frac{tM'(t)}{M(t)}\right] = \frac{(e^t-1)^2 - t^2e^t}{(e^t-t-1)^2} > 0$$

if $t > 0$. Therefore, $\widetilde{A}_M = B^0_M$ and (23) follows from Theorem 2.3.

As for (24), note that $B^0_N = \lim_{t\to 0} tN'(t)/N(t) = 2$. We assert that for $t > 0$,

$$(25)\qquad \frac{d}{dt}\left[\frac{tN'(t)}{N(t)}\right] = \frac{[\ln(1+t)]^2 - t^2/(1+t)}{[(1+t)\ln(1+t) - t]^2} < 0.$$

Consider the function $f(v) = \sqrt{v} - 1/\sqrt{v} - \ln v$ for $v \ge 1$. One has $f(1) = 0$ and if $v > 1$,

$$f'(v) = \frac{(\sqrt{v}-1)^2}{2v\sqrt{v}} > 0.$$

Putting $v = 1+t$ in the above, we see that $t - \sqrt{1+t}\ln(1+t) > 0$ if $t > 0$, which proves (25). Since $N^{-1}(1) = e-1$, from (25) and (16) we have

$$\widetilde{A}_N = \left[\frac{tN'(t)}{N(t)}\right]_{t=e-1} = e-1.$$

Finally, (24) follows from Theorem 2.3.

PROBLEM 2.7. Find the value of $K(\ell^{(N)})$, where the N-function $N(v)$ is given above in Example 2.6.

3. Estimates for $K(\ell^\Phi)$ and $P(\ell^\Phi)$. For the Orlicz norm, we start with Wang's result.

LEMMA 3.1 (Wang [14]). *Let Φ be an N-function.*

(i) *If $\Phi \notin \Delta_2(0)$, then $K(\ell^\Phi) = 2$ and $P(\ell^\Phi) = 1/2$.*

(ii) *If $\Phi \in \Delta_2(0)$, then*

$$K(\ell^\Phi) = \sup_{\|x\|_\Phi=1} \inf_{k_x>1} \left\{ c > 0 : \varrho_\Phi\left(\frac{k_x x}{c}\right) = \frac{k_x - 1}{2} \right\}. \tag{26}$$

The proof of (26) can also be found in Wang and Liu [15].

LEMMA 3.2. (i) *Let Φ be an N-function. Then*

$$2\beta_\Psi^0 \le K(\ell^\Phi), \tag{27}$$

or equivalently,

$$\frac{1}{1 + 1/\beta_\Psi^0} \le P(\ell^\Phi), \tag{28}$$

where Ψ is the complementary N-function to Φ.

(ii) *If $\Phi \notin \nabla_2(0)$, then $K(\ell^\Phi) = 2$ and $P(\ell^\Phi) = 1/2$.*

Proof. (i) By (5), for any given $0 < \varepsilon < 1/2$ there exist $1/2 > v_n \searrow 0$ such that

$$\frac{\Psi^{-1}(v_n)}{\Psi^{-1}(2v_n)} > \beta_\Psi^0 - \frac{\varepsilon}{2}, \quad n \ge 1. \tag{29}$$

Let $k_n = [1/(2v_n)]$. Then $1/(k_n + 1) < 2v_n \le 1/k_n$. Note tht $[\Psi^{-1}(v)/v] \nearrow \infty$ as $v \searrow 0$. For all $n \ge 1$ we have

$$2(k_n + 1)\Psi^{-1}\left(\frac{1}{2(k_n+1)}\right) > \frac{\Psi^{-1}(v_n)}{v_n} \tag{30}$$

and

$$\frac{\Psi^{-1}(2v_n)}{2v_n} \ge k_n \Psi^{-1}\left(\frac{1}{k_n}\right). \tag{31}$$

Put

$$b_n = \frac{1}{k_n \Psi^{-1}(1/k_n)}$$

and

$$c_n = 2(k_n+1)\Psi^{-1}\left(\frac{1}{2(k_n+1)}\right) - 2k_n\Psi^{-1}\left(\frac{1}{2k_n}\right).$$

Then $b_n \to 0$ and

$$0 < c_n < 2(k_n+1)\Psi^{-1}\left(\frac{1}{2k_n}\right) - 2k_n\Psi^{-1}\left(\frac{1}{2k_n}\right) = 2\Psi^{-1}\left(\frac{1}{2k_n}\right) \to 0$$

as $n \to \infty$. Choose $n_0 \ge 1$ such that $b_n c_n < \varepsilon$ for all $n \ge n_0$. For simplicity, set $v_0 = v_{n_0}$, $k_0 = k_{n_0}$, $b_0 = b_{n_0}$ and $c_0 = c_{n_0}$. Put $Z_0 = (0, 0, \ldots, 0)$ and $X_0 = (b_0, b_0, \ldots, b_0)$, each with k_0 components. Define $\{x_i\}_{i=1}^\infty \subset \ell^\Phi$ by

$$x_i = (\overbrace{Z_0, Z_0, \ldots, Z_0}^{i-1}, X_0, Z_0, \ldots).$$

Then $\|x_i\|_\Phi = 1$ for all $i \ge 1$. For $i \ne j$, from (30), (29) and (31) we have

$$\begin{aligned}
\|x_i - x_j\|_\Phi &= b_0 2k_0 \Psi^{-1}\left(\frac{1}{2k_0}\right) = b_0\left[2(k_0+1)\Psi^{-1}\left(\frac{1}{2(k_0+1)}\right) - c_0\right] \\
&> b_0\left[\frac{\Psi^{-1}(v_0)}{v_0} - c_0\right] > b_0\left[\frac{1}{v_0}\left(\beta_\Psi^0 - \frac{\varepsilon}{2}\right)\Psi^{-1}(2v_0) - c_0\right] \\
&\ge 2\left(\beta_\Psi^0 - \frac{\varepsilon}{2}\right) b_0 k_0 \Psi^{-1}\left(\frac{1}{k_0}\right) - b_0 c_0 \\
&= 2\beta_\Psi^0 - \varepsilon - b_0 c_0 > 2(\beta_\Psi^0 - \varepsilon),
\end{aligned}$$

which shows that $K(\ell^\Phi) > 2(\beta_\Psi^0 - \varepsilon)$. Since ε is arbitrary, we obtain (27).

(ii) If $\Phi \notin \nabla_2(0)$, then $\Psi \notin \Delta_2(0)$ and by Theorem 1.4, $\beta_\Psi^0 = 1$. It follows from (27) that $2 \le K(\ell^\Phi)$, as desired. ■

Lemma 3.2 will be used in Section 4. In view of Lemmas 3.1(i) and 3.2(ii), if ℓ^Φ is nonreflexive, then $K(\ell^\Phi) = 2$ and $P(\ell^\Phi) = 1/2$. Therefore, we only need to estimate the Kottman constants and the packing constants of reflexive Orlicz sequence spaces ℓ^Φ. For this we use the following.

PROPOSITION 3.3 (Wang [14]). *Let Φ be an N-function. Define*

$$q_\Phi = \inf_{\|x\|_\Phi=1} \left\{ k_x > 1 : \|x\|_\Phi = \frac{1}{k_x}[1 + \varrho_\Phi(k_x x)] \right\} \tag{32}$$

and

$$Q_\Phi = \sup_{\|x\|_\Phi=1} \left\{ k_x > 1 : \|x\|_\Phi = \frac{1}{k_x}[1 + \varrho_\Phi(k_x x)] \right\}. \tag{33}$$

If $\Phi \in \Delta_2(0) \cap \nabla_2(0)$, then $1 < q_\Phi \le Q_\Phi < \infty$.

Remark 3.4. Under the assumption of Proposition 3.3, we have the following estimates for Q_Φ, which will be used in the proof of Theorem 3.5:

$$1 < 1 + \frac{1}{C} \le q_\Phi \le Q_\Phi \le \frac{C}{D-1}, \tag{34}$$

where

$$C=\sup\left\{\frac{\Phi(2u)}{\Phi(u)}:0<u\le\Phi^{-1}(1)\right\}, \tag{35}$$

$$D=\inf\left\{\frac{t\phi(t)}{\Phi(t)}:0<t\le\psi[\Psi^{-1}(1)]\right\}, \tag{36}$$

with $\Psi(v)=\int_0^{|v|}\psi(s)\,ds$ being the complementary N-function to $\Phi(u)$.

Proof of (34). Since $\Phi\in\Delta_2(0)$, we have $C<\infty$ and $\Phi(2u)\le C\Phi(u)$ for all $0\le u\le\Phi^{-1}(1)$. If $x=(t_i)\in S(\ell^\Phi)$, then $\sum_{i=1}^\infty\Phi(|t_i|)=\varrho_\Phi(x)\le\|x\|_{(\Phi)}<\|x\|_\Phi=1$, so that $|t_i|\le\Phi^{-1}(1)$, where $S(X)$ denotes the unit sphere of a Banach space X. Therefore, we have

$$C\varrho_\Phi(x)=C\sum_{i=1}^\infty\Phi(|t_i|)\ge\sum_{i=1}^\infty\Phi(2|t_i|)=\varrho_\Phi(2x)\ge\|2x\|_{(\Phi)}\ge\|x\|_\Phi=1$$

and hence,

$$1+\frac1C\le1+\inf_{\|x\|_\Phi=1}\varrho_\Phi(x)\le\inf_{\|x\|_\Phi=1}\{k_x>1:k_x=1+\varrho_\Phi(k_xx)\}=q_\Phi.$$

Next we consider the right side of (34). Since $\Phi\in\nabla_2(0)$, one has $D>1$. For any given $x=(t_i)\in S(\ell^\Phi)$, assume that $k_x>1$ satisfies

$$1=\|x\|_\Phi=\frac1{k_x}[1+\varrho_\Phi(k_xx)]$$

(see [12, p. 69]). For any given $\varepsilon>0$ satisfying $k_x-\varepsilon\ge q_\Phi-\varepsilon>1$, we have

$$1\ge\sum_{i=1}^\infty\Psi\{\phi[(k_x-\varepsilon)|t_i|]\}=\sum_{i=1}^\infty\{(k_x-\varepsilon)|t_i|\phi[(k_x-\varepsilon)|t_i|]-\Phi[(k_x-\varepsilon)|t_i|]\}$$
$$\ge(D-1)\sum_{i=1}^\infty\Phi[(k_x-\varepsilon)|t_i|]\ge(D-1)(k_x-\varepsilon)\varrho_\Phi(x)\ge\frac{D-1}{C}(k_x-\varepsilon),$$

and this proves the right side of (34), since ε is arbitrary. ■

THEOREM 3.5. *Let Φ be an N-function and let $\Phi\in\Delta_2(0)\cap\nabla_2(0)$. Then*

$$1<2^{1/B_\Phi^0}\le2\beta_\Psi^0\le K(\ell^\Phi)\le\frac1{\alpha_\Phi^*}\le2^{1/A_\Phi^*}<2, \tag{37}$$

or equivalently,

$$\frac13<\frac1{1+2^{1-1/B_\Phi^0}}\le\frac1{1+1/\beta_\Psi^0}\le P(\ell^\Phi)\le\frac1{1+2\alpha_\Phi^*}\le\frac1{1+2^{1-1/A_\Phi^*}}<\frac12, \tag{38}$$

where

$$\alpha_\Phi^*=\inf\left\{\frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}:0<u\le\frac{Q_\Phi-1}{2}\right\}, \tag{39}$$

$$A_\Phi^*=\inf\left\{\frac{t\phi(t)}{\Phi(t)}:0<t\le\Phi^{-1}(Q_\Phi-1)\right\}, \tag{40}$$

with Q_Φ given by (33).

Proof. Assume that $\Phi\in\Delta_2(0)\cap\nabla_2(0)$. The left side of (37) follows immediately from (27), (8) and (7):

$$1<2^{1/B_\Phi^0}=2^{1-1/A_\Psi^0}\le2\alpha_\Psi^0\le2\beta_\Psi^0\le K(\ell^\Phi).$$

To prove the right side of (37) we first show

$$K(\ell^\Phi)\le1/\alpha_\Phi^*. \tag{41}$$

Put $\xi(u)=\Phi^{-1}(u)/\Phi^{-1}(2u)$ for $u>0$. Then $\Phi[\xi(u)\Phi^{-1}(2u)]=u$. For any given $x=(t_1,t_2,\ldots)\in\ell^\Phi$ with $\|x\|_\Phi=1$, there exists $k_x>1$ such that

$$1=\|x\|_\Phi=\frac1{k_x}[1+\varrho_\Phi(k_xx)].$$

Let $u_i=\frac12\Phi(k_x|t_i|)$ for all $t_i\ne0$. Then $\Phi(k_x|t_i|)\le\varrho_\Phi(k_xx)=k_x-1\le Q_\Phi-1$, and so $u_i\le\frac12(Q_\Phi-1)$. It follows from (39) that

$$\varrho_\Phi(\alpha_\Phi^*k_xx)=\sum_{i=1}^\infty\Phi(\alpha_\Phi^*k_x|t_i|)\le\sum_{i=1}^\infty\Phi\left[\xi\left(\frac12\Phi(k_x|t_i|)\right)k_x|t_i|\right]$$
$$=\frac12\sum_{i=1}^\infty\Phi(k_x|t_i|)=\frac12(k_x-1).$$

Because of (26), we obtain (41). Similar to (18) and (19), we can prove

$$1/\alpha_\Phi^*\le2^{1/A_\Phi^*}<2. \tag{42}$$

Finally, the right side of (37) follows from (41) and (42). ■

It should be observed that Wang [14] first proved that ℓ^Φ is reflexive if and only if $P(\ell^\Phi)<1/2$.

EXAMPLE 3.6. Let $M_r(u)$ and $\Phi(u)$ be defined as in Example 2.5. Then $P(\ell^{M_r})=P(\ell^r)$ and $P(\ell^\Phi)=P(\ell^p)$.

EXAMPLE 3.7. Let $M(u)$ and $N(v)$ be as in Example 2.6. Then $A_M^*=B_M^0=2$ and $K(\ell^M)=K(\ell^2)$. By using (34)–(37), (39) and (40) we can verify that

$$1.41\approx2^{1/2}\le K(\ell^N)\le2^{1/A_N^*}\approx1.56.$$

PROBLEM 3.8. Find the value of $K(\ell^N)$, where the N-function $N(v)$ is given in Example 2.6.

It does not appear simple to find the values of these geometric constants for an arbitrary reflexive Orlicz sequence space. However, we can do this for a subclass of such spaces defined by certain intermediate N-functions (see (43) below).

4. The values of $P(\ell^{(\Phi_s)})$ and $P(\ell^{\Phi_s})$. In 1966, Rao [11] obtained a Riesz–Thorin type convexity theorem between Orlicz spaces equipped with the Orlicz norm (see also [12, p. 226]). In 1972, Cleaver [2] extended that theorem to the ℓ^p-product of Orlicz spaces (see also [12, p. 240]). In 1985, Ren proved that these interpolation theorems are still valid for the Orlicz spaces equipped with gauge norms (see [12, p. 226, p. 256]). Since the counting measure is σ-finite, we deduce the following.

LEMMA 4.1. *Let Φ be an N-function. Suppose that $\Phi_0(u)=u^2$, $0\le s\le 1$ and $\Phi_s(u)$ is defined to be the inverse of*

$$\Phi_s^{-1}(u)=[\Phi^{-1}(u)]^{1-s}[\Phi_0^{-1}(u)]^s \quad (u\ge 0). \tag{43}$$

Then, for any collection $\{x_i: 1\le i\le n\}\subset h^{(\Phi_s)}$ or $\subset h^{\Phi_s}$ and any $\{c_j\ge 0\}_{j=1}^n$ with $\sum_{j=1}^n c_j=1$, we have

$$\sum_{i=1}^n\sum_{j=1}^n c_ic_j\|x_i-x_j\|_{(\Phi_s)}^{2/(2-s)}\le 2c^{2(1-s)/(2-s)}\sum_{i=1}^n c_i\|x_i\|_{(\Phi_s)}^{2/(2-s)} \tag{44}$$

and

$$\sum_{i=1}^n\sum_{j=1}^n c_ic_j\|x_i-x_j\|_{\Phi_s}^{2/(2-s)}\le 2c^{2(1-s)/(2-s)}\sum_{i=1}^n c_i\|x_i\|_{\Phi_s}^{2/(2-s)} \tag{45}$$

where $c=\max(1-c_j:1\le j\le n)$.

It should be noted that (45) was originally given in Cleaver [3, Theorem 3.2].

LEMMA 4.2. *Let Φ be an N-function and let $\Phi_s(u)$ be the inverse of (43). If $0<s\le 1$, then $\Phi_s\in\Delta_2(0)\cap\nabla_2(0)$.*

Proof. Note that $1/2\le\alpha_\Phi^0\le\beta_\Phi^0\le 1$, $0<s\le 1$ and for $u>0$,

$$\frac{\Phi_s^{-1}(u)}{\Phi_s^{-1}(2u)}=\left[\frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}\right]^{1-s}\left[\frac{\sqrt{u}}{\sqrt{2u}}\right]^s.$$

Therefore, we have

$$\alpha_{\Phi_s}^0=(\alpha_\Phi^0)^{1-s}2^{-s/2}\ge 2^{s/2-1}>\frac12$$

and

$$\beta_{\Phi_s}^0=(\beta_\Phi^0)^{1-s}2^{-s/2}\le 2^{-s/2}<1.$$

Thus the conclusion follows from Theorem 1.4. ■

The first main result of this paper is as follows.

THEOREM 4.3. *Let Φ be an N-function and let $\Phi_s(u)$ be the inverse of (43). If $0<s\le 1$, then*

$$1/\alpha_{\Phi_s}^0\le K(\ell^{(\Phi_s)})\le 2^{1-s/2}, \tag{46}$$

or equivalently,

$$\frac{1}{1+2\alpha_{\Phi_s}^0}\le P(\ell^{(\Phi_s)})\le\frac{1}{1+2^{s/2}}. \tag{47}$$

Furthermore, if $\Phi\notin\nabla_2(0)$, we have

$$K(\ell^{(\Phi_s)})=2^{1-s/2}, \tag{48}$$

or equivalently,

$$P(\ell^{(\Phi_s)})=\frac{1}{1+2^{s/2}}. \tag{49}$$

Proof. The left side of (46) follows from Lemma 2.2. Now we prove the right side of (46). Since $0<s\le 1$, one has $\ell^{(\Phi_s)}=h^{(\Phi_s)}$ by Lemma 4.2. For any given $\{x_i:i=1,2,\ldots\}\subset S(\ell^{(\Phi_s)})$ satisfying $R\le\|x_i-x_j\|_{(\Phi_s)}, i\ne j$, choose $n\ge 2, c_i=1/n$, $1\le i\le n$. It follows from (44) that

$$\frac{n(n-1)}{n^2}R^{2/(2-s)}\le 2\left(1-\frac1n\right)^{2(1-s)/(2-s)},$$

and so

$$R\le 2^{1-s/2}\left(1-\frac1n\right)^{-s/2}.$$

Letting $n\to\infty$ in the above, we obtain the right side of (46).

If $\Phi\notin\nabla_2(0)$, then $\alpha_\Phi^0=1/2$ in view of Theorem 1.4. Therefore, from (43) we have

$$\alpha_{\Phi_s}^0=(\alpha_\Phi^0)^{1-s}2^{-s/2}=2^{s/2-1}. \tag{50}$$

Finally, (48) is deduced from (46) and (50). ■

EXAMPLE 4.4. Let $\Phi(u)=e^{|u|^r}-1$ with $1<r<\infty$. Then for $0<s\le 1$ we have

$$\Phi_s^{-1}(u)=[\ln(1+u)]^{(1-s)/r}u^{s/2},\quad u\ge 0.$$

Note that $\alpha_\Phi^0=2^{-1/r}$ and

$$\alpha_{\Phi_s}^0=2^{(s-1)/r-s/2}.$$

By (46) and (47), we get

$$2^{(1-s)/r+s/2}\le K(\ell^{(\Phi_s)})\le 2^{1-s/2}$$

and

$$\frac{1}{1+2^{(1-s)(1-1/r)+s/2}} \le P(\ell^{(\Phi_s)}) \le \frac{1}{1+2^{s/2}}.$$

EXAMPLE 4.5. Let $M(u)$ be the inverse of

$$M^{-1}(u) = \begin{cases} 0 & \text{if } u = 0, \\ u^{3/4}(\ln(1/u))^{1/2} & \text{if } 0 < u \le e^{-2}, \\ (2u/e)^{1/2} & \text{if } e^{-2} \le u < \infty. \end{cases}$$

Then

(51) $$K(\ell^{(M)}) = 2^{3/4}, \quad P(\ell^{(M)}) = \frac{1}{1+2^{1/4}}.$$

In fact, let $\Phi(u)$ be the inverse of

$$\Phi^{-1}(u) = \begin{cases} 0 & \text{if } u = 0, \\ u\ln(1/u) & \text{if } 0 < u \le e^{-2}, \\ (2/e)u^{1/2} & \text{if } e^{-2} \le u < \infty. \end{cases}$$

Then $\Phi(u)$ is an N-function, since $\Phi^{-1}(u)$ is strictly concave on $[0,\infty)$ and satisfies

$$\lim_{u\searrow 0} \frac{u}{\Phi^{-1}(u)} = 0, \quad \lim_{u\nearrow\infty} \frac{u}{\Phi^{-1}(u)} = \infty.$$

Note that $M^{-1}(u) = [\Phi_s^{-1}(u)]_{s=1/2}$ and

$$\alpha_\Phi^0 = \beta_\Phi^0 = \lim_{u\to 0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)} = \frac{1}{2}.$$

Therefore, $\Phi \notin \nabla_2(0)$ by Theorem 1.4(ii) and hence, (51) follows from Theorem 4.3.

To find the values of the Kottman and packing constants for a class of reflexive spaces equipped with the Orlicz norm we need the following two lemmas.

LEMMA 4.6. *Let Φ be an N-function and let Φ_s be the inverse of* (43). *Then*

(52) $$\frac{1}{A_{\Phi_s}^0} = \frac{1-s}{A_\Phi^0} + \frac{s}{2}$$

and

(53) $$\frac{1}{B_{\Phi_s}^0} = \frac{1-s}{B_\Phi^0} + \frac{s}{2}.$$

Proof. Note that $\Phi_s'(t) = \phi_s(t)$ a.e on $(0,\infty)$. Putting $t = \Phi_s^{-1}(u)$ for $0 < u < \infty$, we have

(54) $$\frac{\Phi_s(t)}{t\Phi_s'(t)} = \frac{u[\Phi_s^{-1}(u)]'}{\Phi_s^{-1}(u)} = \frac{(1-s)[\Phi^{-1}(u)]^{-s}[\Phi^{-1}(u)]'u^{1+s/2} + [\Phi^{-1}(u)]^{1-s}(s/2)u^{s/2}}{[\Phi^{-1}(u)]^{1-s}u^{s/2}} = \frac{(1-s)u[\Phi^{-1}(u)]'}{\Phi^{-1}(u)} + \frac{s}{2}.$$

Therefore,

$$\frac{1}{A_{\Phi_s}^0} = \limsup_{t\to 0} \frac{\Phi_s(t)}{t\Phi_s'(t)} = (1-s)\limsup_{u\to 0} \frac{u[\Phi^{-1}(u)]'}{\Phi^{-1}(u)} + \frac{s}{2} = \frac{1-s}{A_\Phi^0} + \frac{s}{2}.$$

Similarly, one can prove (53) by using (54). ■

LEMMA 4.7. *Let Φ, Ψ be a pair of complementary N-functions. Suppose that*

(55) $$C_\Phi^0 = \lim_{t\to 0} \frac{t\phi(t)}{\Phi(t)}$$

exists. Then

(i) $\gamma_\Phi^0 = \lim_{u\to 0} \Phi^{-1}(u)/\Phi^{-1}(2u)$ *exists, and* $\gamma_\Phi^0 = 2^{-1/C_\Phi^0}$;

(ii) $C_\Psi^0 = \lim_{t\to 0} t\psi(t)/\Psi(t)$ *exists, and* $\gamma_\Psi^0 = \lim_{v\to 0} \Psi^{-1}(v)/\Psi^{-1}(2v) = 2^{-1/C_\Psi^0}$; *and*

(iii) $2\gamma_\Phi^0\gamma_\Psi^0 = 1$.

Proof. The assertions follow directly from (8) and (7). ■

The second main result of this paper is as follows.

THEOREM 4.8. *Let Φ be an N-function and let $\Phi_s(u)$ be the inverse of* (43). *If $0 < s \le 1$, then*

(56) $$2\beta_{\Psi_s^+}^0 \le K(\ell^{\Phi_s}) \le 2^{1-s/2},$$

or equivalently,

(57) $$\frac{1}{1+1/\beta_{\Psi_s^+}^0} \le P(\ell^{\Phi_s}) \le \frac{1}{1+2^{s/2}},$$

where Ψ_s^+ is the complementary N-function to Φ_s.

Furthermore, if $\Phi \notin \nabla_2(0)$ and C_Φ^0 exists, then

(58) $$K(\ell^{\Phi_s}) = 2^{1-s/2},$$

or equivalently,

(59) $$P(\ell^{\Phi_s}) = \frac{1}{1+2^{s/2}}.$$

Proof. The left side of (56) follows from Lemma 3.2. By using (45), we can obtain the right side of (56). Now we prove the second part of the theorem. Note that $\Phi \notin \nabla_2(0)$ and C^0_Φ exists if and only if $C^0_\Phi = 1$. It follows from Lemmas 4.6 and 4.7 that

$$(60) \qquad 2\beta^0_{\Psi_s^+} = 1/\alpha^0_{\Phi_s} = 2^{1/C^0_{\Phi_s}} = 2^{(1-s)/C^0_\Phi + s/2} = 2^{1-s/2}.$$

Finally, (58) is a consequence of (56) and (60). ∎

Example 4.9. Let Φ_s, $0 < s \le 1$, be the inverse of

$$\Phi_s^{-1}(u) = \{\exp[\ln(1+u)]^{2/3} - 1\}^{1-s} u^{s/2}, \quad u \ge 0.$$

Then

$$(61) \qquad 2^{(4-s)/6} \le K(\ell^{\Phi_s}) \le 2^{1-s/2}$$

and

$$(62) \qquad \frac{1}{1+2^{(2+s)/6}} \le P(\ell^{\Phi_s}) \le \frac{1}{1+2^{s/2}}.$$

In fact, letting

$$\Phi(u) = (1+|u|)^{\sqrt{\ln(1+|u|)}} - 1$$

(see [8, p. 34]), we have $\Phi_s^{-1}(u) = [\Phi^{-1}(u)]^{1-s}[\Phi_0^{-1}(u)]^s$. As $\lim_{t\to 0} t\phi(t)/\Phi(t) = 3/2$, by Lemmas 4.7 and 4.6 we get

$$2\beta^0_{\Psi_s^+} = 1/\alpha^0_{\Phi_s} = 2^{1/C^0_{\Phi_s}} = 2^{(1-s)/C^0_\Phi + s/2} = 2^{(4-s)/6}.$$

Thus, (61) and (62) follow from (56), (57) and the above equality.

Example 4.10. Let $M(u)$ and $\Phi(u)$ be as in Example 4.5. Then $K(\ell^M) = K(\ell^{(M)})$ and $P(\ell^M) = P(\ell^{(M)})$.

In fact, putting $t = \Phi^{-1}(u)$ we have

$$C^0_\Phi = \lim_{t\to 0} \frac{t\Phi'(t)}{\Phi(t)} = \lim_{u\to 0} \frac{\Phi^{-1}(u)}{u[\Phi^{-1}(u)]'} = 1.$$

By the second part of Theorem 4.8, the desired result follows.

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Department of Mathematics
University of California
Riverside, California 92521
U.S.A.
E-mail: rao@math.ucr.edu

Department of Mathematics
Suzhou University
215006 Suzhou
P.R. China

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