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# Hardy spaces associated with some Schrödinger operators

by

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**Abstract.** For a Schrödinger operator $A = -\Delta + V$, where $V$ is a nonnegative polynomial, we define a Hardy $H^1_A$ space associated with $A$. An atomic characterization of $H^1_A$ is shown.

**1. Introduction.** Let $A$ be a Schrödinger operator on $\mathbb{R}^d$ which has the form

(1.1) $$A = -\Delta + V,$$

where $V(x) = \sum_{\beta\le\alpha} a_\beta x^\beta$ is a nonnegative nonzero polynomial on $\mathbb{R}^d$, $\alpha = (\alpha_1, \ldots, \alpha_d)$.

These operators have attracted attention of a number of authors (cf. [Fe], [HN], [Z]). Recent results of J. Zhong [Z] deal with the Riesz transforms $R_j = \frac{\partial}{\partial x_j} A^{-1/2}$. Among other things it is proved in [Z] that $H^1(\mathbb{R}^d)$ is mapped by $R_j$ into $L^1(\mathbb{R}^d)$. In general, however, this does not characterize $H^1(\mathbb{R}^d)$, i.e. the norm $\|f\|_{L^1} + \sum_{j=1}^d \|R_j f\|_{L^1}$ is not equivalent to the $H^1(\mathbb{R}^d)$ norm.

The operator $A$, however, gives rise to a perhaps more natural notion of the space $H^1_A$ which is the following. Let $\{T_t\}_{t>0}$ be the semigroup of operators generated by $-A$ (e.g. on $L^2(\mathbb{R}^d)$), $T_t(x,y)$ being their kernels. We notice that, since $V$ is nonnegative, we have

(1.2) $$0 \le T_t(x,y) \le \widetilde{T}_t(x,y) = (4\pi t)^{-d/2} \exp(-|x-y|^2/(4t)).$$

Let

(1.3) $$\mathcal{M}f(x) = \sup_{t>0} |T_t f(x)|.$$

By (1.2), $\mathcal{M}$ is of weak type (1,1). Therefore we may say that a function $f$ is in the Hardy space $H^1_A$ associated with $A$ if

(1.4) $$\|f\|_{H^1_A} = \|\mathcal{M}f\|_{L^1} < \infty.$$

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As we shall see later $H^1(\mathbb{R}^d)$ is a *proper* subspace of $H^1_A$.

The aim of this paper is to study the space $H^1_A$ in more detail. Our main theorem concerns an atomic characterization of $H^1_A$.

Since the operator $A$ is neither translation nor dilation invariant, the position and size of the support of atoms play an important role. Indeed, our notion of atom is the same as that of the classical $(1,\infty)$-atom, except that the mean zero condition is required only if the diameter of the support of the atom is small and its center and size are related to the level sets of the potential and its derivatives. Therefore the use of Goldberg's theory of local Hardy spaces with a localization properly scaled is natural here (cf. Section 3 for the details).

We make the utmost use of the idea which relates the operators $-\Delta+V$ to operators $\Pi_P$, where $\Pi$ is a unitary representation of a nilpotent Lie group and $P$ is a specific left-invariant homogeneous operator on the group. The results of P. Głowacki [G] and W. Hebisch [He] are crucial in this context. These enable us to derive appropriate estimates for the kernels $T_t(x,y)$ of the semigroup; the details will appear in a separate paper where some other applications will be presented. The estimates will be used to show that in time and space variables our kernels behave "locally" as the appropriately localized Weierstrass heat kernels and globally, for large time, they are small.

As one might expect, the space $H^1_A$ can be characterized by means of suitable Littlewood–Paley square functions. Also our $H^1_A$ space is natural in the sense that the Zhong Riesz transforms characterize it. Indeed, the norm $\|f\|_{H^1_A}$ is equivalent to the norm $\sum_{j=1}^d \|(\partial/\partial x_j)A^{-1/2}f\|_{L^1}+\|f\|_{L^1}$.

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## 2. Decomposition of $\mathbb{R}^d$

For our potential $V(x)=\sum_{\beta\le\alpha}a_\beta x^\beta$ and for $n=1,2,3,\ldots$ we define the sets $\mathcal{A}_n$ by

$$\mathcal{A}_n=\{x\in\mathbb{R}^d: 2^{n/2}\le\sup_{\beta\le\alpha}\{2^{-n(|\beta|+1)/2}|D^\beta V(x)|\}\le 2^{(2+|\alpha|+n)/2}\}. \tag{2.1}$$

We also set

$$\mathcal{A}_0=\{x\in\mathbb{R}^d:\sup_{\beta\le\alpha}\{|D^\beta V(x)|\}\le 2^{(2+|\alpha|)/2}\}. \tag{2.2}$$

Let $\mathcal{B}_0=\mathcal{A}_0$, and $\mathcal{B}_n=\mathcal{A}_n-\bigcup_{k=0}^{n-1}\mathcal{A}_k$. We have $\mathbb{R}^d=\bigcup_{n=0}^\infty\mathcal{B}_n$. We will denote by $B(x,r)$ the ball in $\mathbb{R}^d$ with center $x$ and radius $r$.

The following lemmas follow from the definition of $\mathcal{B}_n$ and the Taylor formula.



Lemma 2.3. *There is a constant $C$ such that for every $R>2$ and every $n$, if $x\in\mathcal{B}_n$ then*

$$\#\{k: B(x,2^{-n/2}R)\cap\mathcal{B}_k\ne\emptyset\}\le C\log_2 R.$$

Lemma 2.4. *There is a constant $C$ and a collection of balls $B_{(n,k)}=B(x_{(n,k)},2^{-n/2})$, $n=0,1,2,\ldots$, $k=1,2,\ldots$, such that $x_{(n,k)}\in\mathcal{B}_n$, $\mathcal{B}_n\subset\bigcup_k B_{(n,k)}$, and $\#\{(n',k'): B(x_{(n,k)},R2^{-n/2})\cap B(x_{(n',k')},R2^{-n'/2})\ne\emptyset\}\le R^C$ for every $(n,k)$ and $R\ge2$.*

As a consequence of Lemma 2.4, we obtain

Lemma 2.5. *There are nonnegative functions $\psi_{(n,k)}$ such that*

$$\psi_{(n,k)}\in C_c^\infty(B(x_{(n,k)},2^{1-n/2})), \tag{2.6}$$

$$\sum_{(n,k)}\psi_{(n,k)}(x)=1, \tag{2.7}$$

$$\|\nabla\psi_{(n,k)}\|_{L^\infty}\le C2^{n/2}. \tag{2.8}$$

## 3. Atoms

We say that a function $a$ is an *atom for the Hardy space $H^1_A$* associated with a ball $B(x_0,r)$ if

$$\operatorname{supp} a\subset B(x_0,r), \tag{3.1}$$

$$\|a\|_{L^\infty}\le(\operatorname{vol}B(x_0,r))^{-1}, \tag{3.2}$$

$$\text{if } x_0\in\mathcal{B}_n,\ \text{then } r\le 2^{-n/2}, \tag{3.3}$$

$$\text{if } x_0\in\mathcal{B}_n \text{ and } r\le 2^{-1-n/2},\ \text{then } \int a(x)\,dx=0. \tag{3.4}$$

The atomic norm in the space $H^1_A$ is defined by

$$\|f\|_{\mathrm{a}}=\inf\Big\{\sum_{j=0}^\infty|c_j|\Big\}, \tag{3.5}$$

where the infimum is taken over all decompositions $f=\sum_{j=1}^\infty c_ja_j$, with $a_j$ being $H^1_A$ atoms and $c_j$ being scalars.

Our aim is

Theorem 3.6. *There is a constant $C>0$ such that*

$$C^{-1}\|f\|_{H^1_A}\le\|f\|_{\mathrm{a}}\le C\|f\|_{H^1_A} \tag{3.7}$$

*for every $f\in L^1(\mathbb{R}^d)$.*

We now state some results from the theory of local Hardy spaces (cf. [Go]) we shall need later.

Let $\{\widetilde{T}_t\}_{t>0}$ be the semigroup of linear operators generated by the Laplacian $\Delta$ on $\mathbb{R}^d$. For $n \in \mathbb{Z}$ we define the local maximal function $\widetilde{\mathcal{M}}_n$ setting

$$\widetilde{\mathcal{M}}_n f(x) = \sup_{0<t\le 2^{-n}} |\widetilde{T}_t f(x)|. \tag{3.8}$$

We say that a function $f$ is in the *local Hardy space* $\mathbf{h}_n^1$ if

$$\|f\|_{\mathbf{h}_n^1} = \|\widetilde{\mathcal{M}}_n f\|_{L^1} < \infty.$$

A function $\widetilde{a}$ is an *atom for the local Hardy space* $\mathbf{h}_n^1$ if

$$\operatorname{supp} \widetilde{a} \subset B(x,r), \quad r \le 2^{-n/2}, \tag{3.9}$$

$$\|\widetilde{a}\|_{L^\infty} \le (\operatorname{vol} B(x,r))^{-1}, \tag{3.10}$$

$$\text{if } r \le 2^{-1-n/2}, \text{ then } \int \widetilde{a}(x)\,dx = 0. \tag{3.11}$$

The atomic norm in $\mathbf{h}_n^1$ is defined by

$$\|f\|_{\mathbf{h}_{\mathrm{a},n}^1} = \inf\Big\{\sum_j |c_j|\Big\}, \tag{3.12}$$

where the infimum is taken over all decompositions $f = \sum c_j \widetilde{a}_j$, where $\widetilde{a}_j$ are $\mathbf{h}_n^1$ atoms.

THEOREM 3.13 [Go]. *The norms* $\|\ \|_{\mathbf{h}_n^1}$ *and* $\|\ \|_{\mathbf{h}_{\mathrm{a},n}^1}$ *are equivalent with constants independent of* $n \in \mathbb{Z}$. *Moreover, if* $f \in \mathbf{h}_n^1$ *and* $\operatorname{supp} f \subset B(x, 2^{1-n/2})$, *then there are* $\mathbf{h}_n^1$ *atoms* $\widetilde{a}_j$ *such that* $\operatorname{supp} \widetilde{a}_j \subset B(x, 2^{2-n/2})$ *and*

$$f = \sum_j c_j \widetilde{a}_j, \quad \sum_j |c_j| \le C\|f\|_{\mathbf{h}_n^1} \tag{3.14}$$

*with a constant* $C$ *independent of* $n$.

## 4. Estimates of kernels.

Let

$$Af = \int_0^\infty \lambda\, dE_A(\lambda) f$$

be the spectral resolution of the operator $A$. The results in [He] combined with [DHJ] (see [D] for details) imply that there is a nilpotent Lie group $G$, a unitary representation $\Pi$, and a regular symmetric kernel $P$ of order 2 such that $\Pi_P = A$. The construction of the Lie group and the representation $\Pi$ allow us to show that for a $C^\infty$ function $\varphi$ such that

$$\varphi \in C_c^\infty([1/2,2]), \quad |\varphi(\lambda)| > c > 0 \quad \text{for } \lambda \in [3/4, 7/4] \tag{4.1}$$

the operators

$$Q_\mu f = \varphi(2^{-\mu}A) f = \int_0^\infty \varphi(2^{-\mu}\lambda)\, dE_A(\lambda) f \tag{4.2}$$



are expressed by

$$Q_\mu f(x) = \int_{\mathbb{R}^d} Q_\mu(x,y) f(y)\,dy, \tag{4.3}$$

where

$$Q_\mu(x,y) = 2^{d\mu/2} F(2^{\mu/2}(y-x), 2^{-\mu}V(x), \ldots, 2^{-\mu(|\beta|+2)/2} D^\beta V(x), \ldots) \tag{4.4}$$

with $F$ being in the Schwartz space of functions on $\mathbb{R}^d \times \mathbb{R}^D$, $D = (\alpha_1+1) \times (\alpha_2+1)\ldots(\alpha_d+1)$. Moreover, the Schwartz class functions $\widetilde{F}_{2^{-\mu}}(x) = 2^{d\mu/2} F(-2^{\mu/2}x, 0)$ are the convolution kernels of the operators

$$\widetilde{Q}_\mu f = \varphi(-2^{-\mu}\Delta) f. \tag{4.5}$$

Denote by $T_t(x,y)$ the kernels of the operators $T_t$. It was proved in [D, Proposition 3.17] that for every $b > 0$ there is a constant $C_b$ such that

$$0 \le T_t(x,y) \le C_b t^{-d/2}(1 + t^{-1/2}|y-x|)^{-b} \times \prod_{\beta\le\alpha} (1 + |t^{(|\beta|+2)/2} D^\beta V(x)|)^{-b}. \tag{4.6}$$

Proposition 3.13 of [D] asserts that the kernels $K_s(x,y)$ of the operators $AT_s = -\frac{d}{ds}T_s$ are given by

$$K_s(x,y) = s^{-1}s^{-d/2}\Xi(s^{-1/2}(y-x), sV(x), \ldots, s^{(|\beta|+2)/2} D^\beta V(x), \ldots), \tag{4.7}$$

where

$$|\Xi(x,\xi)| \le C(1+|x|)^{-d-2}, \tag{4.8}$$

and

$$|\Xi(x,\xi) - \Xi(x,0)| \le C(1+|x|)^{-d-1}|\xi|^\varepsilon \quad \text{with some } \varepsilon > 0. \tag{4.9}$$

Similarly

$$\widetilde{K}_s(x) = s^{-1}s^{-d/2}\Xi(s^{-1/2}(-x), 0, \ldots, 0) \tag{4.10}$$

is the convolution kernel of the operator $-\Delta\widetilde{T}_s$.

## 5. Some lemmas

LEMMA 5.1. *There is a constant* $C$ *such that for every nonnegative integer* $n$,

$$\Big\| \sup_{0<t\le 2^{-n}} |\widetilde{T}_t(\psi_{(n,k)}f)(x) - T_t(\psi_{(n,k)}f)(x)| \Big\|_{L^1(dx)} \le C\|\psi_{(n,k)}f\|_{L^1}, \tag{5.2}$$

*where* $\psi_{(n,k)}$ *are the functions from Lemma 2.5.*

Proof. Let $0 < t \le 2^{-n}$. Then

$$|\widetilde{T}_t(\psi_{(n,k)}f)(x) - T_t(\psi_{(n,k)}f)(x)| = \Big|\int_0^t \frac{d}{ds}(\widetilde{T}_s - T_s)(\psi_{(n,k)}f)(x)\,ds\Big| \le \int_0^{2^{-n}} |(\Delta\widetilde{T}_s + AT_s)(\psi_{(n,k)}f)(x)|\,ds.$$

By (4.7) and (4.10), the functions $R_s(x,y) = K_s(x,y) - \widetilde{K}_s(x-y)$ are the kernels of the operators $\Delta\widetilde{T}_s + AT_s$.

Obviously,

$$\|(\Delta\widetilde{T}_s + AT_s)(\psi_{(n,k)}f)\|_{L^1} \le D_{s,n,k}\|\psi_{(n,k)}f\|_{L^1}, \tag{5.3}$$

where $D_{s,n,k} = \sup_y \int |R_s(x,y)|\chi_{B^*_{(n,k)}}(y)\,dx$, and $B^*_{(n,k)} = B(x_{(n,k)}, 2^{1-n/2})$.

If $x \in B^{**}_{(n,k)} = B(x_{(n,k)}, 2^{2-n/2})$, then $2^{-n(|\beta|+1)/2}|D^\beta V(x)| \le C2^{n/2}$, and, consequently, $|s^{(|\beta|+2)/2}D^\beta V(x)| \le C2^n s$. From (4.9) we conclude that

$$\sup_y \int_{x\in B^{**}_{(n,k)}} |R_s(x,y)|\chi_{B^*_{(n,k)}}(y)\,dx \le C2^{n\varepsilon}s^{\varepsilon-1}. \tag{5.4}$$

If $x \notin B^{**}_{(n,k)}$, then $|x-y| \ge 2^{-n/2}$ for $y \in B^*_{(n,k)}$. Since $|R_s(x,y)| \le s^{-1}s^{-d/2}C(1+|s^{-1/2}(y-x)|)^{-d-2}$ (cf. (4.8)), we obtain

$$\sup_y \int_{x\notin B^{**}_{(n,k)}} |R_s(x,y)|\chi_{B^*_{(n,k)}}(y)\,dx \le s^{-1}s^{-d/2}C\int_{|x|>2^{-n/2}} (1+|s^{-1/2}x|)^{-d-2}\,dx \le C2^n.$$

Therefore $D_{s,n,k} \le C(2^n + 2^{\varepsilon n}s^{\varepsilon-1})$. Finally, the left-hand side of (5.2) is estimated by

$$\|\psi_{(n,k)}f\|_{L^1}\int_0^{2^{-n}} D_{s,n,k}\,ds \le C\|\psi_{(n,k)}f\|_{L^1},$$

which completes the proof of the lemma.

Let $\omega^{[N]}(x) = (1+|x|)^{-N}$. From Lemma 2.4 we deduce that for $N$ sufficiently large,

$$\sum_{(n,k)} \omega^{[N]}(2^{n/2}(x - x_{(n,k)})) \in L^\infty \quad \text{as a function of } x. \tag{5.6}$$

We define the maximal function $\mathcal{M}_{(n,k)}$ by

$$\mathcal{M}_{(n,k)}f(x) = \sup_{0<t\le 2^{-n}} |T_t(\psi_{(n,k)}f)(x) - \psi_{(n,k)}(x)T_tf(x)|.$$



LEMMA 5.7. *For every $N > 0$ there is a constant $C$ such that*

$$\|\mathcal{M}_{(n,k)}f\|_{L^1} \le C\|f(x)\omega^{[N]}(2^{n/2}(x - x_{(n,k)}))\|_{L^1(dx)}. \tag{5.8}$$

Proof. We have

$$[\psi_{(n,k)}, T_t]f(x) = \sum_{(n',k')} T_{t,(n,k),(n',k')}f(x),$$

where $T_{t,(n,k),(n',k')}f(x) = \int f(y)T_t(x,y)(\psi_{(n,k)}(x) - \psi_{(n,k)}(y))\psi_{(n',k')}(y)\,dy$. Let

$$\mathcal{M}_{(n,k),(n',k')}f(x) = \sup_{0<t\le 2^{-n}} |T_{t,(n,k),(n',k')}f(x)|.$$

Set $J_{(n,k)} = \{(n',k') : |x_{(n',k')} - x_{(n,k)}| \le C'2^{-n/2}\}$, and $I_{(n,k)} = \{(n',k') : |x_{(n',k')} - x_{(n,k)}| > C'2^{-n/2}\}$.

Note that the number of elements in $J_{(n,k)}$ is bounded by a constant independent of $(n,k)$. Moreover, taking $C'$ sufficiently large we see that $B^{**}_{(n,k)} \cap B^{**}_{(n',k')} = \emptyset$ for $(n',k') \in I_{(n,k)}$. Thus, by (4.6), we get

$$\|\mathcal{M}_{(n,k),(n',k')}f\|_{L^1} \le \begin{cases} C\|f\|_{L^1(B^*_{(n',k')})} & \text{if } (n',k') \in J_{(n,k)},\\ C_N 2^{-Nn/2}|x_{(n,k)} - x_{(n',k')}|^{-N}\|f\|_{L^1(B^*_{(n',k')})} & \text{if } (n',k') \in I_{(n,k)}. \end{cases}$$

Applying the above estimates, we have

$$\begin{aligned}\|\mathcal{M}_{(n,k)}f\|_{L^1} &\le \sum_{(n',k')} \|\mathcal{M}_{(n,k),(n',k')}f\|_{L^1}\\ &\le C\sum_{(n',k')\in J_{(n,k)}} \|f\|_{L^1(B(x_{(n,k)},C2^{-n/2}))}\\ &\quad + C\sum_{(n',k')\in I_{(n,k)}} C_N 2^{-Nn/2}|x_{(n,k)} - x_{(n',k')}|^{-N}\|f\|_{L^1(B^*_{(n',k')})}\\ &\le C\|f(x)\omega^{[N]}(2^{n/2}(x - x_{(n,k)}))\|_{L^1(dx)}.\end{aligned}$$

## 6. Proof of Theorem 3.6.

Let

$$\mathcal{M}_nf(x) = \sup_{0<t\le 2^{-n}} |T_tf(x)|.$$

Proof of Theorem 3.6. We first assume that $f \in H^1_A$. By Lemmas 5.1 and 5.7,

$$\begin{aligned}\|\widetilde{\mathcal{M}}_n(\psi_{(n,k)}f)\|_{L^1} &\le C(\|\mathcal{M}_n(\psi_{(n,k)}f)\|_{L^1} + \|\psi_{(n,k)}f\|)\\ &\le C(\|\psi_{(n,k)}(\mathcal{M}_nf)\|_{L^1}\\ &\quad + \|f(x)\omega^{[N]}(2^{n/2}(x - x_{(n,k)}))\|_{L^1(dx)}).\end{aligned}$$

From (5.6) we conclude that

$$\sum_{(n,k)} \|\widetilde{\mathcal{M}}_n(\psi_{(n,k)}f)\|_{L^1} \le C(\|\mathcal{M}f\|_{L^1} + \|f\|_{L^1}). \tag{6.1}$$

An application of Theorem 3.13 gives

$$\psi_{(n,k)}f = \sum_j c_j^{(n,k)} a_j^{(n,k)}, \quad \text{where } a_j^{(n,k)} \text{ are } H^1_A \text{ atoms,} \tag{6.2}$$

and

$$\sum_j |c_j^{(n,k)}| \le C\|\widetilde{\mathcal{M}}_n(\psi_{(n,k)}f)\|_{L^1}. \tag{6.3}$$

Finally, using (6.1) and (6.3), we obtain the required $H^1_A$ atomic decomposition

$$f = \sum_{(n,k)}\sum_j c_j^{(n,k)} a_j^{(n,k)} \quad \text{and} \quad \sum_{(n,k)}\sum_j |c_j^{(n,k)}| \le C\|\mathcal{M}f\|_{L^1}, \tag{6.4}$$

and the inequality $\|f\|_{\mathrm{a}} \le C\|f\|_{H^1_A}$ is proved.

In order to prove the reverse inequality it suffices to show that there exists a constant $C > 0$ such that

$$\|\mathcal{M}a\|_{L^1} \le C \tag{6.5}$$

for every $H^1_A$ atom $a$.

Let $a$ be an $H^1_A$ atom associated with a ball $B(x_0, r)$. Let $n$ be such that $x_0 \in \mathcal{B}_n$. By definition, $r \le 2^{-n/2}$. By Lemma 5.1, $\|\mathcal{M}_n a\|_{L^1} \le C\|\widetilde{\mathcal{M}}_n a\|_{L^1}$. Theorem 3.13 asserts that the right-hand side of this inequality is bounded by a constant $C$ independent of $a$. It remains to show that

$$\|\sup_{t>2^{-n}} |T_t a(x)|\|_{L^1(dx)} \le C, \tag{6.6}$$

with a constant $C$ independent of $a$.

Let $P^t_m$ be the operator defined by

$$P^t_m f(x) = \int |f(y)| t^{-d/2}\chi_{B(0,m)}(t^{-1/2}(y-x)) \times \Big(\prod_{\beta\le\alpha} \chi_{[-m,m]}(t^{(|\beta|+2)/2} D^\beta V(x))\Big)\, dy.$$

According to (4.6), we have

$$|T_t a(x)| \le \sum_{m\ge 2} b_m P^t_m a(x), \tag{6.7}$$

where $b_m < C_p(1+m)^{-p}$ for every positive $p$. We shall use the following lemma.



LEMMA 6.8. *There is a constant $C_1 \ge 1$ independent of $n$ such that for every $f \in L^1(\mathbb{R}^d)$ such that* $\operatorname{supp} f \subset B(x_n, 2^{1-n/2})$, $x_n \in \mathcal{B}_n$, *and for every $m \ge 2$, we have*

$$P^t_m f = 0 \quad \textit{for } t > m^{C_1} 2^{-n}. \tag{6.9}$$

The proof of Lemma 6.8 will be presented below. Using (6.7) and Lemma 6.8, we get

$$\begin{aligned}\|\sup_{t>2^{-n}} |T_t a(x)|\,\|_{L^1(dx)} &= \|\sup_{2^{-n}<t\le 2^{-n}m^{C_1}} |T_t a(x)|\|_{L^1(dx)} \\ &\le \sum_{m\ge 2} b_m \|\sup_{2^{-n}<t\le 2^{-n}m^{C_1}} |P^t_m a(x)|\|_{L^1(dx)} \\ &\le C\sum_{m\ge 2} b_m m^{dC_1}\|a\|_{L^1(dx)} \le C.\end{aligned}$$

Proof of Lemma 6.8. Let $f \in L^1(\mathbb{R}^d)$, $\operatorname{supp} f \subset B(x_n, 2^{1-n/2})$, $x_n \in \mathcal{B}_n$. Then, by Lemma 2.3, $\operatorname{supp} f \subset \bigcup_{k=n-C_2}^{n+C_2} \mathcal{B}_k$. Assume that $P^t_m f \not\equiv 0$. Then there are $x \in \mathbb{R}^d$ and $y \in \operatorname{supp} f$ such that

$$t^{(|\beta|+2)/2} D^\beta V(x) \in B(0,m) \text{ for all } \beta \le \alpha,$$
$$t^{-1/2}(y-x) \in B(0,m).$$

Since $D^\gamma V(y) = \sum_{\beta\le\alpha} \frac{1}{\beta!} D^{\beta+\gamma} V(x)(y-x)^\beta$ for every $\gamma \le \alpha$,

$$|D^\gamma V(y)| \le Cm^{|\alpha|+1} t^{-(|\gamma|+2)/2}.$$

On the other hand, $|D^\gamma V(y)| \ge 2^{(n-C_2+(n-C_2)(|\gamma|+1))/2}$, thus

$$2^{(n-C_2+(n-C_2)(|\gamma|+1))/2} \le Cm^{|\alpha|+1} t^{-(|\gamma|+2)/2}.$$

This implies $t \le m^{C_1} 2^{-n}$, which completes the proof of the lemma.

**7. Characterization of $H^1_A$ by square functions and Riesz transforms.** For a $C^\infty$ function $\varphi$ satisfying (4.1) we define the square function

$$Sf(x) = \Big(\sum_{\mu\in\mathbb{Z}} |Q_\mu f(x)|^2\Big)^{1/2}, \tag{7.1}$$

where $Q_\mu = \varphi(2^{-\mu}A)$. Our purpose is

THEOREM 7.2. *There is a constant $C > 0$ such that*

$$C^{-1}\|f\|_{H^1_A} \le \|Sf\|_{L^1} + \|f\|_{L^1} \le C\|f\|_{H^1_A}. \tag{7.3}$$

For an integer $n$ we define the truncated square functions

$$S_n f(x) = \Big(\sum_{\mu\ge n} |Q_\mu f(x)|^2\Big)^{1/2}, \quad \widetilde{S}_n f(x) = \Big(\sum_{\mu\ge n} |\widetilde{Q}_\mu f(x)|^2\Big)^{1/2}, \tag{7.4}$$

where $\widetilde{Q}_\mu = \varphi(-2^{-\mu}\Delta)$.

Theorem 7.5 below is a simple consequence of the results of Goldberg [Go].

THEOREM 7.5. *There is a constant $C > 0$ independent of $n$ such that*

$$C^{-1}\|f\|_{\mathbf{h}_n^1} \le \|\widetilde{S}_n f\|_{L^1} + \|f\|_{L^1} \le C\|f\|_{\mathbf{h}_n^1}. \tag{7.6}$$

The two lemmas below can be proved by the same means we used in Section 5.

LEMMA 7.7. *There is a constant $C > 0$ such that for every integer $n$,*

$$\|\widetilde{Q}_\mu(\psi_{(n,k)}f) - Q_\mu(\psi_{(n,k)}f)\|_{L^1} \le C2^{(n-\mu)/2}\|\psi_{(n,k)}f\|_{L^1}.$$

LEMMA 7.8. *For every $N > 0$ there is a constant $C >$ independent of $n$ such that*

$$\|S_n(\psi_{(n,k)}f) - \psi_{(n,k)}(S_n f)\|_{L^1} \le C\|f(x)\omega^{[N]}(2^{n/2}(x - x_{(n,k)}))\|_{L^1(dx)}.$$

A simple application of Lemma 7.7 leads to

COROLLARY 7.9. *There is a constant $C$ such that for every $n$ and $k$,*

$$\|\widetilde{S}_n(\psi_{(n,k)}f)\| \le C(\|S_n(\psi_{(n,k)}f)\|_{L^1} + \|\psi_{(n,k)}f\|_{L^1}), \tag{7.10}$$

$$\|S_n(\psi_{(n,k)}f)\| \le C(\|\widetilde{S}_n(\psi_{(n,k)}f)\|_{L^1} + \|\psi_{(n,k)}f\|_{L^1}). \tag{7.11}$$

Proof of Theorem 7.2. Assume that $f \in H_A^1$. Since the infimum of the spectrum of $A$ is strictly positive, $Q_\mu = 0$ for $\mu \le B$. Hence

$$\|Sf\|_{L^1} \le \sum_{(n,k)} \|S_n(\psi_{(n,k)}f)\|_{L^1} + \sum_{(n,k)} \|S_n^0(\psi_{(n,k)}f)\|_{L^1}, \tag{7.12}$$

where $S_n^0(\psi_{(n,k)}f)(x) = (\sum_{B\le\mu<n} |Q_\mu(\psi_{(n,k)}f)(x)|^2)^{1/2}$. By (7.11), Theorem 7.5, and Lemma 5.1,

$$\begin{aligned}\|S_n(\psi_{(n,k)}f)\|_{L^1} &\le C(\|\widetilde{S}_n(\psi_{(n,k)}f)\|_{L^1} + \|\psi_{(n,k)}f\|_{L^1})\\ &\le C\|\widetilde{\mathcal{M}}_n(\psi_{(n,k)}f)\|_{L^1} \le C\|\mathcal{M}_n(\psi_{(n,k)}f)\|_{L^1}.\end{aligned}$$

According to Lemma 5.7, we obtain

$$\begin{aligned}&\|S_n(\psi_{(n,k)}f)\|_{L^1}\\ &\quad\le C\|\psi_{(n,k)}\mathcal{M}f\|_{L^1} + \|f(x)\omega^{[N]}(2^{n/2}(x - x_{(n,k)}))\|_{L^1(dx)}.\end{aligned} \tag{7.13}$$

As a consequence of Lemma 6.8 we have

$$\|S_n^0(\psi_{(n,k)}f)\|_{L^1} \le C\|\psi_{(n,k)}f\|_{L^1}. \tag{7.14}$$

Therefore, by (7.13), (7.14), and (5.6), we get $\|Sf\|_{L^1} + \|f\|_{L^1} \le C\|f\|_{H_A^1}$.

Our proof of the reverse inequality is similar. Assume that $\|Sf\|_{L^1} + \|f\|_{L^1} < \infty$. Trivially,

$$\|\mathcal{M}f\|_{L^1} \le \sum_{(n,k)} \|\mathcal{M}_n(\psi_{(n,k)}f)\|_{L^1} + \sum_{(n,k)} \|\mathcal{M}_n^\infty(\psi_{(n,k)}f)\|_{L^1}, \tag{7.15}$$



where $\mathcal{M}_n^\infty(\psi_{(n,k)}f)(x) = \sup_{t>2^{-n}} |T_t(\psi_{(n,k)}f)(x)|$. We conclude from Lemma 6.8 that

$$\|\mathcal{M}_n^\infty(\psi_{(n,k)}f)\|_{L^1} \le C\|(\psi_{(n,k)}f)\|_{L^1}. \tag{7.16}$$

Using Lemma 5.1 and Theorem 7.5, we obtain

$$\|\mathcal{M}_n(\psi_{(n,k)}f)\|_{L^1} \le C\|\widetilde{\mathcal{M}}_n(\psi_{(n,k)}f)\|_{L^1} \le C(\|\widetilde{S}_n(\psi_{(n,k)}f)\|_{L^1} + \|\psi_{(n,k)}f\|_{L^1}).$$

Lemma 7.8 and (7.10) yield

$$\begin{aligned}\|\mathcal{M}_n(\psi_{(n,k)}f)\|_{L^1} &\le C(\|S_n(\psi_{(n,k)}f)\|_{L^1} + \|\psi_{(n,k)}f\|_{L^1})\\ &\le C(\|\psi_{(n,k)}Sf\|_{L^1} + \|f(x)\omega^{[N]}(2^{n/2}(x - x_{(n,k)}))\|_{L^1(dx)} + \|\psi_{(n,k)}f\|_{L^1}).\end{aligned}$$

Finally, by (5.6), we have $\|f\|_{H_A^1} \le C(\|Sf\|_{L^1} + \|f\|_{L^1})$, which ends the proof of Theorem 7.2.

Let us define the Riesz transforms $R_j$, $j = 1, \ldots, d$, associated with the operator $A$ setting

$$R_j f = \frac{\partial}{\partial x_j} A^{-1/2} f = \frac{\partial}{\partial x_j} \int_0^\infty \lambda^{-1/2}\, dE_A(\lambda) f. \tag{7.17}$$

Similar arguments can be used to prove the following characterization of the space $H_A^1$:

THEOREM 7.18. *There is a constant $C > 0$ such that*

$$C^{-1}\|f\|_{H_A^1} \le \|f\|_{L^1} + \sum_{j=1}^d \|R_j f\|_{L^1} \le C\|f\|_{H_A^1}.$$

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# Perfect sets of finite class without the extension property

by

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**Abstract.** We prove that generalized Cantor sets of class $\alpha, \alpha \neq 2$, have the extension property iff $\alpha < 2$. Thus belonging of a compact set $K$ to some finite class $\alpha$ cannot be a characterization for the existence of an extension operator. The result has some interconnection with potential theory.

**1. Introduction.** Let $K$ be a compact set in $\mathbb{R}^m$. Then $\mathcal{E}(K)$ is the space of Whitney jets with the topology defined by the norms (in what follows we will consider only the one-dimensional case)

$$\|f\|_q = |f|_q + \sup\left\{\frac{|(R_y^q f)^{(k)}(x)|}{|x-y|^{q-k}} : x, y \in K,\ x \neq y,\ k = 0, 1, \ldots, q\right\},$$

$q = 0, 1, \ldots,$ where $|f|_q = \sup\{|f^{(k)}(x)| : x \in K,\ k \le q\}$ and $R_y^q f(x) = f(x) - T_y^q f(x)$ is the Taylor remainder. We say that $K$ *has the extension property* if there exists a linear continuous extension operator $L : \mathcal{E}(K) \to C^\infty(\mathbb{R}^m)$. The problem of finding such an operator was investigated by many authors (see e.g. [2], [9], [11], [12], [14]–[17]). In [16] Tidten applied Vogt's condition for a splitting of exact sequences of Fréchet spaces and gave a topological characterization of the extension property (see Th. 1 below). In order to give a corresponding geometric description Tidten introduced in [17] the following property: a compact set $K \subset \mathbb{R}$ is a *perfect set of class* $\alpha$ ($\alpha \ge 1$) if there are constants $C \ge 1$ and $\delta > 0$ such that for any $y \in K$ one can find a sequence $(x_j)_{j=1}^\infty \subset K$ such that $|y - x_j| \downarrow 0, |y - x_1| \ge \delta$ and $C|y - x_{j+1}| \ge |y - x_j|^\alpha$ for any $j \in \mathbb{N}$. In this case we will write $K \in (\alpha)$. It was proved in [17] that

(i) $K \in (1) \Rightarrow$
(ii) $K$ has the extension property $\Rightarrow$
(iii) $K \in (\alpha)$ for some $\alpha \ge 1$.

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