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Estimates for the Poisson kernels and their derivatives on rank one NA groups

by

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Abstract. For rank one solvable Lie groups of the type NA estimates for the Poisson kernels and their derivatives are obtained. The results give estimates on the Poisson kernel and its derivatives in a natural parametrization of the Poisson boundary (minus one point) of a general homogeneous, simply connected manifold of negative curvature.

The class NA of solvable Lie groups has attracted considerable attention in recent years (cf. e.g. [B], [BBE], [V1], [V2]). We say that a Lie group G is of the form NA if it is a semidirect product of a nilpotent group N extended by an Abelian group A. The name NA comes from the most important examples of such groups: the NA part of the Iwasawa decomposition NAK of a semisimple group (non-compact, finite center). The symmetric space NAK/K admits a simply transitive group of isometries of the form NA acting on the left and so can be identified with NA.

Also every proper homogeneous cone Ω in $\mathbb{R}^n$ admits a simply transitive group of linear transformations which is of the form NA (see [Vi]), and every bounded homogeneous domain $D \subset \mathbb{C}^n$ admits a simply transitive group of biholomorphic transformations of the form NA (see [P]).

We say that *a group NA is of rank one* if A is one-dimensional and in the adjoint action of the Lie algebra $\mathcal{A}$ of A on the Lie algebra $\mathcal{N}$ of N the real parts of the eigenvalues of ad_H for $H \in \mathcal{A}$ are positive. NA groups of rank one can be equipped with a left-invariant riemannian metric for which the sectional curvature is negative. In fact, all the homogeneous riemannian manifolds of negative curvature are of this form [He].

All known examples of non-compact riemannian harmonic spaces, also the non-symmetric ones, have the form of a rank one NA group, N being a so-called group of the Heisenberg type [DR].

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Let L be a left-invariant subelliptic operator on a group $S = NA$. Bounded L-harmonic functions have been studied extensively on symmetric spaces, L being the Laplace–Beltrami operator. For a number of results, like the Fatou theorem, the identification of the symmetric space with the NA part of the Iwasawa decomposition has been essential. For general NA groups and general degenerate elliptic operators bounded harmonic functions were investigated in e.g. [R], [D1], [DH1], [DH2]. These functions are all of the form

$$F(s) = \int_{N/N_0} f(s\cdot y)\nu(y)\,dy,$$

where N_0 is an A-invariant subgroup of N, $s\cdot y$ stands for the action of an element $s\in S$ on $S/N_0A = N/N_0$ and ν is a positive, integrable function on N/N_0 which we call the *Poisson kernel*. For general NA groups the following estimates have been proved (cf. [D] and [DH2]).

Let ϱ be an N-invariant distance in N/N_0. We have

(i) There exists $\eta > 0$ such that $\int_{N/N_0}\varrho(y)^\eta\nu(y)\,dy < \infty$.

(ii) For every multi-index I there are constants $c, M > 0$ such that $|\partial^I\nu(y)| \le c(1+\varrho(y))^M$.

(iii) There exist $c,\varepsilon > 0$ such that $\nu(y) \le c(1+\varrho(y))^{-\varepsilon}$.

Properties (i)–(iii) seem to be the best estimates for ν which can be expressed in terms of a norm in N/N_0, for multidimensional A. They are, however, sufficient to prove a satisfactory Fatou type theorem about the almost everywhere convergence of the Poisson integrals of functions in L^p, $1 < p \le \infty$, to their boundary values ([D], [DH1], [DH3], cf. also [So]). If NA is a harmonic space and L is the Laplace–Beltrami operator, then a formula for ν, very similar to the corresponding one for symmetric spaces of rank one, is proved in [DR]. It follows that in this case

$$\nu(x) \simeq c\varrho(x)^{-2Q},$$

where $\|\cdot\|$ is a specific homogeneous gauge on N and Q is the homogeneous dimension of N.

This has been our starting point for a search of better estimates on ν and their derivatives in the case of a general rank one NA group.

In the case of a rank one NA group every degenerate second order elliptic operator can be written in the form

$$Lf(xa) = \Big((a\partial_a)^2 - \gamma a\partial_a + \sum_{i=1}^m \sigma_a(B_i)^2 + \sigma_a(B)\Big)f(xa),$$

where $\sigma_a = e^{\mathrm{ad}_{\log a H}}$, $A = \mathbb{R}H$ and $B, B_1,\ldots,B_m$ are left-invariant vector fields on N.

By an application of the potential theoretic methods discovered by Alano Ancona [A], which he used to describe the minimal positive harmonic functions on riemannian spaces of negative curvature, the following estimate has been proved in [D2], [D3]:

$$c^{-1}(1+|x|)^{-\alpha-Q} \le \nu(x) \le c(1+|x|)^{-\alpha-Q},$$

where Q is the sum of the real parts of the eigenvalues of ad_H acting on $\mathcal{N}$. The proofs are based on a boundary Harnack inequality for positive harmonic functions on NA.

In [D2] and [D3] the geometry of negatively curved manifolds is used to estimate the Poisson kernel for an NA group. In the present paper we go in the opposite direction: all our arguments are based on some group invariance. We do not use either geometry or the potential theoretic methods, the latter not being adaptable for estimating the derivatives of ν. However, our results give estimates on the Poisson kernel and its derivatives in a natural parametrization of the Poisson boundary (minus one point) of a general homogeneous, simply connected manifold of negative curvature.

We obtain the following estimates for the derivatives of ν (see Theorem (5.1)):

$$(*)\qquad |X^I\nu(x)| \le C(1+|x|)^{-Q-\gamma-\|I\|}(\log(2+|x|))^{\|I\|_0},$$

where the norm $|\cdot|$ is as in (0.20), $\|I\|$ is a suitably defined length of the multi-index and $\|I\|_0$ is a certain number depending on I and the nilpotent part of ad_H. $\|I\|_0$ is equal to 0 when the action of H on $\mathcal{N}$ is diagonal.

To prove $(*)$ we are going to revisit a probabilistic method used in [DH1]. The idea goes back to Malliavin [M] (cf. [T]).

We consider the diffusion $s(t) = x(t)\mathbf{a}(t)$ generated by L on NA. For a fixed continuous function $\mathbf{a} : \mathbb{R}^+ \ni t \to \mathbf{a}(t) \in \mathbb{R}^+$ the "horizontal component" $x(t)$ under the condition that the "vertical component" is $\mathbf{a}$, is the diffusion on N generated by the time dependent operator

$$\sum_{i=1}^m \sigma^{\mathbf{a}(t)}(B_i)^2 + \sigma^{\mathbf{a}(t)}(B).$$

Thus given a trajectory $\mathbf{a}$ of the Brownian motion on $\mathbb{R}$ associated with the operator $\partial_t^2 - \gamma\partial_t = (a\partial_a)^2 - \gamma a\partial_a$ with $a = e^t$ we consider

$$L_{\mathbf{a}} = \sum_{j=1}^m \Phi_t(B_j)^2 + \Phi_t(B) - \partial_t,$$

where $\Phi_t = \sigma_*^{\mathbf{a}(t)}$. Let $P_{\mathbf{a}}(s,t,x)$, $0\le s<t<\infty$, $x\in N$, be the fundamental solution of $L_{\mathbf{a}}$. Then $\lim_{t\to\infty} P_{\mathbf{a}}(0,t,x)$ exists and

$$(**)\qquad X^I\nu^\chi(x) = \mathrm{E}_\chi X^I P_{\mathbf{a}}(0,\infty,x),$$

where ν^χ is the harmonic measure corresponding to $\exp\chi H$, i.e.

$$F(\exp\chi H) = \int_N f(y)\nu^\chi(y)\,dy$$

and E_χ is the integral with respect to the Wiener measure on $C(\mathbb{R})$ with $a(0) = \chi$.

The proof of estimate $(*)$ consists basically of three steps. First we prove the existence of $P_{\mathbf{a}}(0,\infty,x)$ and some estimates of its derivatives (see Theorem (4.20)). Next, in Theorem (5.7), we estimate the integral $\mathrm{E}_\chi X^I P_{\mathbf{a}}(0,\infty,x)$. Finally, we prove equality $(**)$ (Lemma (5.11)).

Section 4 is devoted to estimation of $P_{\mathbf{a}}(0,\infty,x)$. The crucial point is the behavior of the trajectory $\mathbf{a}$. Of course $\mathbf{a}(t) \asymp -\gamma t$ as $t\to\infty$. The following three quantities are of interest: $A_d = \int_0^\infty e^{d\mathbf{a}(t)}\,dt$ with an appropriate positive d, $\Lambda = \max_{0<t<\infty}\mathbf{a}(t)$ and $\lambda = \min_{\zeta<t<\zeta+1}\mathbf{a}(t)$, where $\zeta = \min\{t : \mathbf{a}(t) = \Lambda\}$. In Theorem (4.20) we formulate our estimates in terms of A_d, Λ, λ. To prove them we develop a quite general approach to evolutions with continuous coefficients on N. This is described in Section 3.

The next step is to study the integral $\mathrm{E}_\chi X^I P_{\mathbf{a}}(0,\infty,x)$. Although we do not know the joint distribution of A_d, Λ, λ, we are able to define a stopping time τ and estimate the probability of the set $\{\mathbf{a} : A_d^{1/d} \asymp e^{k_1},\ \Lambda(\Theta_\tau\mathbf{a}) \asymp k_2,\ \lambda(\Theta_\tau\mathbf{a}) \asymp k_3\}$ in such a way that it suffices to obtain an appropriate estimate of E_χ. This requires a number of lemmas about the behavior of the Brownian motion with a negative drift. They are given in Section 2. The appropriate estimate is

$$(***)\qquad \mathrm{E}_\chi X^I P_{\mathbf{a}}(0,\infty,x) \le Ce^{\gamma\chi}$$

and it is proved in Lemma (5.7). The rest follows by a homogeneity argument described at the beginning of Section 5. Finally, we show that $(***)$ implies $(**)$ (see Lemma (5.11)).

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Preliminaries. Let $\mathcal{N}$ be a nilpotent Lie algebra. We fix a basis $\{\mathbf{B}_\alpha\}_{\alpha\in\Omega}$ of $\mathcal{N}$. Let

$$\Delta = \sum_{\alpha\in\Omega}\mathbf{B}_\alpha^2 \quad\text{and}\quad \Lambda = I - \Delta,$$
$$\widetilde{\Delta} = \sum_{\alpha\in\Omega}\widetilde{\mathbf{B}}_\alpha^2 \quad\text{and}\quad \widetilde{\Lambda} = I - \widetilde{\Delta},$$

where $\widetilde{X}$ is the *right-invariant* vector field on N corresponding to the element $X\in\mathcal{N}$.

Every element of the enveloping algebra of $\mathcal{N}$ is a sum of elements

$$\mathbf{B}^I = \prod_{\alpha\in\Omega}\mathbf{B}_\alpha^{i_\alpha},$$

where $I = \{i_\alpha\}_{\alpha\in\Omega}$ and i_α are non-negative integers. We write

$$|I| = \sum_{\alpha\in\Omega} i_\alpha.$$

The operators Λ and $\widetilde{\Lambda}$ are positive, essentially self-adjoint on $C_c^\infty(N)\subset L^2(N)$ so the operators Λ^s and $\widetilde{\Lambda}^s$ are well defined for arbitrary $s\in\mathbb{R}$. We have

$$\widetilde{\Lambda}^s\widetilde{\Lambda}^t = \widetilde{\Lambda}^{s+t} \quad\text{for } s,t\in\mathbb{R}.$$

Of course, $X\widetilde{\Lambda}^s = \widetilde{\Lambda}^s X$ for every $X\in\mathcal{N}$ and $s\in\mathbb{R}$.

We introduce the Sobolev norms:

$$\|u\|_s^2 = (\widetilde{\Lambda}^s u, u) = \|\widetilde{\Lambda}^{s/2}u\|_{L^2}^2,\qquad s\in\mathbb{R},\ u\in C_c^\infty(N).$$

By the spectral theorem, $s\le t$ implies $\|u\|_s\le\|u\|_t$.

Let $\varrho(x)$ be a riemannian distance of x from $e\in N$. For the unit ball

$$B(1) = \{x : \varrho(x) < 1\}$$

and every s there exist constants C_s and c_s such that for $f\in C_c^\infty(B)$ we have

$$C_s^{-1}\|\widetilde{\Lambda}^{s/2}f\|_{L^2} \le \|\Lambda^{s/2}f\|_{L^2} \le C_s\|\widetilde{\Lambda}^{s/2}f\|_{L^2},$$
$$c_s^{-1}\|f\|_s \le \|f\|_{H_0^s} \le c_s\|f\|_s,$$

where H_0^s is the ordinary euclidean Sobolev space on B. Accordingly, on each H_0^t the operator $\widetilde{\Lambda}^{s/2}$ is elliptic of order s, as for $\phi\in C_c^\infty(B)$ the commutator $[\widetilde{\Lambda}^{s/2},\phi]$ is of order $s-1$, i.e.

$$\|[\widetilde{\Lambda}^{s/2},\phi]u\|_{L^2(B)} \le c_\phi\|\widetilde{\Lambda}^{(s-1)/2}u\|_{L^2(B)}.$$

Let $\{\nu_t\}_{t>0}$ be the semigroup of positive measures on N generated by $-\widetilde{\Lambda}$. Then

$$\nu_t = e^{-t}p_t,$$

where p_t are the smooth probability measures in the semigroup generated by the elliptic operator $\widetilde{\Delta}$. The following well known estimate for the semigroup generated by the elliptic Laplacian $\widetilde{\Delta}$ goes back to Nelson and Aronson and is not difficult to prove (cf. e.g. [Ro]):

$$p_t(x) \le ct^{-(\dim N)/2}e^{-\beta\varrho(x)^2/t}, \quad 0 < t \le 1, \tag{0.1}$$

for some positive c and β.

Let R be a fixed positive number and let $\phi \in C^\infty(N)$ be such that

$$0 \le \phi(x) \le 1, \quad \operatorname{supp}\phi \subset B(R/2)^c, \quad \phi(x) = 1 \text{ for } x \in B(R)^c. \tag{0.2}$$

The following estimate is well known (cf. e.g. [Ro]); we include an easy proof.

(0.3) Lemma. *For every $r > 0$ and every multi-index I there are positive constants c_I and $\eta > 0$ such that*

$$\int_{B(r/2)^c} |\widetilde{\mathbf{B}}^I p_t(x)|\,dx \le c_I e^{-\eta/t}, \quad 0 \le t \le 1.$$

Proof. Let $r \ge R$. We have

$$\begin{aligned}\Big(\int_{B(r/2)^c} |\widetilde{\mathbf{B}}^I p_t(x)|\,dx\Big)^2 &\le \Big(\int_N |\phi(x)\widetilde{\mathbf{B}}^I p_t(x)|\,dx\Big)^2 \\ &\le c\int_N (\phi(x)\widetilde{\mathbf{B}}^I p_t(x))^2 e^{\varrho(x)}\,dx \\ &\le c\sum_{|J|\le 2|I|}\int_N |\widetilde{\mathbf{B}}^J p_t(x)|\cdot\psi(x)p_t(x)e^{\varrho(x)}\,dx \\ &\le c\sum_{|J|\le 2|I|}\int_N |\widetilde{\mathbf{B}}^J p_t(x)|^2\,dx\cdot\int_N \psi(x)^2 p_t(x)^2 e^{2\varrho(x)}\,dx,\end{aligned} \tag{0.4}$$

where ψ satisfies (0.2). Since

$$\|\widetilde{\mathbf{B}}^J u\|_{L^2} \le c\|\widetilde{\Lambda}^{[|J|/2]+1}u\|_{L^2}$$

(cf. [N]), we have

$$\begin{aligned}\int_N |\widetilde{\mathbf{B}}^J p_t(x)|^2\,dx &\le c\int_N |\widetilde{\Lambda}^{|J|/2+1}p_{t/2} * p_{t/2}(x)|^2\,dx \\ &\le \|\widetilde{\Lambda}^{|J|/2+1}p_{t/2}\|^2_{L^2\to L^2}\|p_{t/2}\|^2_{L^2}.\end{aligned}$$

Hence, by the spectral theorem and (0.1) for some $d = d_I$ and $0 < t \le 1$ we obtain

$$\int_N |\widetilde{\mathbf{B}}^J p_t(x)|^2\,dx \le ct^{-d}.$$

By (0.2), the second factor on the right hand side of (0.4) is estimated by

$$\int_N \psi(x)^2 p_t(x)^2 e^{2\varrho(x)}\,dx \le Ce^{-\eta'/t}$$

for some $\eta' > 0$, which completes the proof.

Consequently, for every multi-index I there are positive constants c_I and η such that

$$\|\widetilde{\mathbf{B}}^I(\phi\nu_t)\|_{L^1} \le c_I e^{-(\eta/t+t)} \quad \text{for all } t > 0. \tag{0.5}$$

In particular,

$$\|\widetilde{\Delta}(\phi\nu_t)\|_{L^1} \le \mathbf{c}e^{-(\eta/t+t)} \quad \text{for all } t > 0. \tag{0.6}$$

For $0 < s < 1$ and $f \in \mathcal{D}(\widetilde{\Delta})$ we have

$$\widetilde{\Lambda}^s f = \mathbf{c}\int_0^\infty t^{-s}(\nu_t * f - f)\,\frac{dt}{t} = \lambda^s * f,$$

where λ^s is the distribution given by

$$\langle\lambda^s, f\rangle = \mathbf{c}\int_0^\infty t^{-s}(\langle\nu_t, f\rangle - f(e))\,\frac{dt}{t}.$$

For a fixed radius R and ϕ as in (0.2) we define

$$\mu^{(s)} = \mathbf{c}\int_0^\infty t^{-s}\phi\nu_t\,\frac{dt}{t} \tag{0.7}$$

for $s < 1$ and we note that by (0.6) the integral is absolutely convergent. It also follows that $\mu^{(s)}$ is a smooth measure. Let

$$\lambda^{(s)} = \lambda^s - \mu^{(s)}, \quad \widetilde{\Lambda}_0^{(s)}f = \lambda^{(s)} * f, \quad M^{(s)}f = \mu^{(s)} * f.$$

Of course, if $\operatorname{supp} f \subset B(R)^c$, then $\langle\lambda^s, f\rangle = \langle\mu^{(s)}, f\rangle$, which implies

$$\operatorname{supp}\lambda^{(s)} \subset \overline{B(R)}.$$

We then have

$$\widetilde{\Lambda}^s = \widetilde{\Lambda}_0^{(s)} + M^{(s)} \quad \text{for } s < 1. \tag{0.8}$$

But since for arbitrary $t \in \mathbb{R}^+$, $\widetilde{\Lambda}^t = \widetilde{\Lambda}^k\widetilde{\Lambda}^s$, where k is a natural number, (0.8) and (0.6) imply

$$\widetilde{\Lambda}^s = \widetilde{\Lambda}_0^{(s)} + M^{(s)} \quad \text{for all } s \in \mathbb{R}, \tag{0.9}$$

where $\widetilde{\Lambda}_0^{(s)}$ is convolution on the left by a compactly supported distribution and $M^{(s)}$ is convolution on the left by a smooth bounded measure such that for every multi-index I,

$$\|\mathbf{B}^I\mu^{(s)}\|_{L^1} \le c_I. \tag{0.10}$$

(0.11) LEMMA. *For all $s_1 > 0$ and $0 < s_2 < 1$, $\lambda^{(s_2)} * \mu^{(s_1)}$ is an integrable function.*

Proof. We have

$$\lambda^{(s_2)} * \mu^{(s_1)} = c\int_0^\infty u^{-s_2-1}\int_0^\infty t^{-s_1-1}[\nu_u * (\phi\nu_t) - (\phi\nu_t)]\,dt\,du.$$

Hence, since $\phi\nu_t \in \mathcal{D}(\widetilde{\Delta})$,

$$\begin{aligned}\|\lambda^{(s_2)} * \mu^{(s_1)}\|_{L^1} &\le c\int_0^\infty u^{-s_2-1}\int_0^\infty t^{-s_1-1}\int_0^u \|\partial_w(\nu_w * (\phi\nu_t))\|_{L^1}\,dw\,dt\,du\\ &\le c\int_0^\infty u^{-s_2-1}t^{-s_1-1}\int_0^u \|\widetilde{\Delta}(\phi\nu_t) * \nu_w\|_{L^1}\,dw\,du\,dt\\ &\le c\int_0^\infty\int_0^\infty u^{-s_2-1}t^{-s_1-1}\min\{u,1\}e^{-(\eta/t+t)}\,du\,dt,\end{aligned}$$

which is finite.

(0.12) COROLLARY. *For a fixed radius $R > 0$ and arbitrary $s_1 > 0$ and $s_2 > 0$ there is a constant $c = c(R, s_1, s_2)$ such that*

$$\|\widetilde{\Lambda}^{s_1}M^{(s_2)}f\|_{L^2} \le c\|f\|_{L^2}.$$

Let $\mathcal{B} = \{B_1,\ldots,B_k\}$ be a generating set of the Lie algebra $\mathcal{N}$. An absolutely continuous curve $\gamma : [0,1] \to N$ is called *$\mathcal{B}$-admissible* if

$$\frac{d}{dt}\gamma(t) = \sum_{j=1}^k \alpha_j(t)B_j.$$

We write

$$|\gamma|_{\mathcal{B}} = \int_0^1 \sqrt{\sum_{j=1}^k \alpha_j(t)^2}\,dt$$

and for $x \in N$,

$$|x|_{\mathcal{B}} = \inf\{|\gamma|_{\mathcal{B}} : \gamma \text{ is } \mathcal{B}\text{-admissible},\ \gamma(0) = e,\ \gamma(1) = x\}.$$

If $\mathcal{B}$ is a generating set, then for every $x \in N$ there is a $\mathcal{B}$-admissible curve γ such that $\gamma(0) = e$ and $\gamma(1) = x$ (cf. e.g. [V3]).

If the generating set is the basis $\{\mathbf{B}_\alpha\}$, then

$$|x|_{\{\mathbf{B}_\alpha\}} = \varrho(x)$$

is the riemannian distance of x from e in N.

It is well known [V3] and not difficult to prove that for a generating set $\mathcal{B}$ contained in a basis $\{\mathcal{B}_\alpha\}$ there is a $\delta \ge 1$ such that

$$|x|_{\mathcal{B}} \le \varrho(x)^{1/\delta} \quad \text{for } \varrho(x) \le 1.$$

Let Φ be an automorphism of the Lie algebra $\mathcal{N}$ and let $\Phi\mathcal{B} = \{\Phi B_1,\ldots,\Phi B_k\}$. In view of the above formula, it is easy to verify that

$$|x|_{\Phi\mathcal{B}} \le \|\Phi^{-1}\|^{1/\delta}\varrho^{1/\delta}, \tag{0.13}$$

where $\|\Phi\|$ is the norm of the linear operator Φ computed with respect to the scalar product defining ϱ.

(0.14) KOHN'S LEMMA. *Let $\mathcal{B} = \{B_1,\ldots,B_k\}$ be a generating set of the Lie algebra $\mathcal{N}$ and let Φ be an automorphism of $\mathcal{N}$. Let $X_1 = \Phi B_1,\ldots, X_k = \Phi B_k$ and*

$$L_\Phi = \sum_{j=1}^k X_j^2.$$

Then there exist c and $\varepsilon > 0$ which depend only on the algebra $\mathcal{N}$ and the set $B_1,\ldots,B_k$ such that

$$\|\Lambda^{\varepsilon/2}u\|_{L^2}^2 \le c(1 + \|\Phi^{-1}\|)^{2/\delta}((1 - L_\Phi)u, u)$$

for every $u \in C_c^\infty(N)$.

Proof. First we show that

$$\|f_h - f\|_{L^2}^2 = \int_N |f(xh) - f(x)|^2\,dx \le |h|_{\mathcal{B}}^2\sum_{j=1}^k \|B_jf\|_{L^2}^2. \tag{0.15}$$

Indeed, let γ be a $\mathcal{B}$-admissible curve such that $\gamma(1) = h$. Then

$$\begin{aligned}&\int |f(xh) - f(x)|^2\,dx \\ &= \int\Big|\int_0^1 \partial_t f(x\gamma(t))\,dt\Big|^2\,dx = \int\Big|\int_0^1\sum_j \alpha_j(t)B_jf(x\gamma(t))\,dt\Big|^2\,dx\\ &\le \int\mathbf{B}\Big(\int_0^1\sqrt{\sum_j \alpha_j(t)^2}\cdot\sqrt{\sum_j |B_jf(x\gamma(t))|^2}\,dt\Big)^2\,dx\end{aligned} \tag{0.16}$$

$$\leq \iint_0^1 \sqrt{\sum_j \alpha_j(t)^2} \cdot \sum_j |B_j f(x\gamma(t))|^2 \, dt \cdot \int_0^1 \sqrt{\sum_j \alpha_j(t)^2} \, dt \, dx$$

$$= |\gamma|_{\mathcal{B}}^2 \sum_j \|B_j f\|_{L^2}^2.$$

Consequently, if we replace $\mathcal{B}$ by the generating set $\Phi\mathcal{B} = \{X_1, \ldots, X_k\}$ we obtain

$$\|f_h - f\|_{L^2}^2 \leq |h|_{\Phi\mathcal{B}}^2 \sum_{j=1}^k \|X_j f\|_{L^2}^2.$$

We write

$$\|\Lambda^{\varepsilon/2} f\|_{L^2} \leq \int_0^\infty \|t^{-\varepsilon/2}(f * \nu_t - f)\|_{L^2} \frac{dt}{t}$$

$$= \int_{t\leq 1} t^{-\varepsilon/2} \|f * \nu_t - f\|_{L^2} \frac{dt}{t} + \int_{t>1} t^{-\varepsilon/2} \|f * \nu_t - f\|_{L^2} \frac{dt}{t}.$$

But

$$\int_{t>1} t^{-\varepsilon/2} \|f * \nu_t - f\|_{L^2} \frac{dt}{t} \leq c\|f\|_{L^2}$$

with c depending on ε, and

$$\int_{t\leq 1} t^{-\varepsilon/2} \|f * \nu_t - f\| \frac{dt}{t}$$

$$\leq \int \int_{t\leq 1} t^{-\varepsilon/2} \|f_h - f\|_{L^2} \nu_t(h) \, dh \frac{dt}{t}$$

$$\leq c \int \int_{t\leq 1} t^{-\varepsilon/2} \|f_h - f\|_{L^2} t^{-(\dim N)/2} e^{-\varrho(h)^2/t} \, dh \frac{dt}{t}$$

$$\leq c \int_{\varrho(h)\leq 1} \varrho(h)^{-\dim N - \varepsilon} \|f_h - f\|_{L^2} \, dh$$

$$+ c\|f\|_{L^2} \int_{\varrho(h)>1} \int_{t\leq 1} t^{(\dim N)/2} e^{-\varrho(x)^2/t} \, dh \frac{dt}{t}$$

$$\leq c \int_{\varrho(h)\leq 1} \varrho(h)^{-\dim N - \varepsilon + 1/\delta} \, dh \, \|\Phi^{-1}\|^{1/\delta} \Big(\sum_j \|X_j f\|_{L^2}^2 \Big)^{1/2} + c\|f\|_{L^2}$$

$$\leq c(1 + \|\Phi^{-1}\|^{1/\delta}) \Big(\Big(\sum_j \|X_j f\|_{L^2}^2 \Big)^{1/2} + \|f\|_{L^2} \Big)$$

for $\varepsilon < 1/\delta$.

Let D be a derivation of $\mathcal{N}$. The automorphism e^{tD} of $\mathcal{N}$ defines an automorphism σ^t of N by

$$\sigma^t(\exp X) = \exp[e^{tD} X].$$

Clearly, $\sigma_*^t = e^{tD}$.

Let $\mathcal{N}^{\mathbb{C}}$ be the complexification of $\mathcal{N}$. We define

$$\mathcal{N}_\lambda^{\mathbb{C}} = \{X \in \mathcal{N}^{\mathbb{C}} : \exists_{k>0} \ (D - \lambda I)^k x = 0\}.$$

Then

$$\mathcal{N} = \bigoplus_{\Im\lambda \geq 0} V_\lambda, \tag{0.17}$$

where

$$V_\lambda = V_{\overline{\lambda}} = (\mathcal{N}_\lambda^{\mathbb{C}} \oplus \mathcal{N}_{\overline{\lambda}}^{\mathbb{C}}) \cap \mathcal{N} \quad \text{if } \Im\lambda \neq 0,$$
$$V_\lambda = \mathcal{N}_\lambda^{\mathbb{C}} \cap \mathcal{N} \quad \text{if } \Im\lambda = 0.$$

We specify: since ad_H is a derivation of the Lie algebra $\mathcal{N}$ we have

$$X \in V_{\lambda_1} \text{ and } Y \in V_{\lambda_2} \ \Rightarrow \ [X, Y] \in V_{\lambda_1 + \lambda_2}. \tag{0.18}$$

Of course, $V_{\lambda_1} \neq 0$ and $V_{\lambda_2} \neq 0$ does not imply $V_{\lambda_1+\lambda_2} \neq 0$.

From now on we assume

$$\text{If } V_\lambda \neq 0, \text{ then } \Re\lambda > 0. \tag{0.19}$$

Under this assumption, (0.17) is a gradation of $\mathcal{N}$. Let us order λ's so that

$$0 < \Re\lambda_1 \leq \ldots \leq \Re\lambda_k.$$

Let $(\cdot,\cdot)$ be an arbitrary fixed inner product in $\mathcal{N}$. We define

$$\langle X, Y \rangle = \int_0^\infty (\sigma_*^{-t} X, \sigma_*^{-t} Y) \, dt, \qquad \|X\| = \sqrt{\langle X, X \rangle}.$$

Let

$$|X| = (\inf\{e^t > 0 : \|\sigma_*^t X\| \geq 1\})^{-1}. \tag{0.20}$$

We write

$$|\exp X| = |X|.$$

Since for $X \neq 0$,

$$\lim_{t\to-\infty} \|\sigma_*^t X\| = 0 \quad \text{and} \quad \lim_{t\to\infty} \|\sigma_*^t X\| = \infty$$

and the function $t \to \|\sigma_*^t X\|$ is increasing, it follows that

(0.21) for every $Y \neq 0$ there is precisely one t such that

$$Y = \sigma_*^t X, \quad |X| = 1, \quad |Y| = e^t.$$

First we fix an inner product $\langle \cdot, \cdot \rangle$ in $\mathcal{N}$ in such a way that the spaces $V_{\lambda_1}, \ldots, V_{\lambda_k}$ are mutually orthogonal, and an orthonormal basis $\mathcal{X}_1, \ldots, \mathcal{X}_n$

accordingly. The enveloping algebra of $\mathcal{N}$ is identified with the polynomials in $\mathcal{X}_1, \ldots, \mathcal{X}_n$. We introduce a bilinear inner product in the enveloping algebra of $\mathcal{N}$ by

$$\langle X_1 \otimes \ldots \otimes X_r, Y_1 \otimes \ldots \otimes Y_r \rangle = \prod_{j=1}^{r} \langle X_j, Y_j \rangle.$$

We denote by V_j^r the symmetric tensor product of r copies of V_{λ_j}, which, since (0.17) is a gradation of $\mathcal{N}$, is identified with a linear subspace of the enveloping algebra of $\mathcal{N}$. For a sequence $I = (i_1, \ldots, i_k)$ of non-negative integers let

$$X^I = X_1^{(i_1)} \ldots X_k^{(i_k)}, \quad \text{where } X_j^{(i_j)} \in V_j^{i_j}. \tag{0.22}$$

We have

$$\sigma_*^t X^I = \sigma_*^t X_1^{(i_1)} \ldots \sigma_*^t X_k^{(i_k)}.$$

Since

$$\|\sigma_*^t\|_{V_\lambda \to V_\lambda} = \|\sigma_*^{-t}\|_{V_\lambda \to V_\lambda} \le c e^{t\Re\lambda}(1 + |t|)^{\dim V_\lambda - 1},$$

we have

$$\begin{aligned} \|\sigma_*^t X^I\| &\le \prod_{j=1}^{k} \|\sigma_*^t\|_{V_j^{i_j} \to V_j^{i_j}} \|X_j^{(i_j)}\| \\ &= \exp\Big(\sum_{j=}^{k} i_j[d_j t + D_j \log(1 + |t|)]\Big) \cdot \prod_{j=1}^{k} \|X_j^{(i_j)}\|, \end{aligned} \tag{0.23}$$

where $d_j = \Re\lambda_j$ and $D_j = \dim V_{\lambda_j} - 1$.

1. Poisson kernel. Let $\mathcal{S}$ be a solvable Lie algebra of a rank one solvable Lie group. It is the sum $\mathcal{S} = \mathcal{N} \oplus \mathcal{A}$ of its nilpotent ideal $\mathcal{N}$ and a one-dimensional algebra $\mathcal{A} = \mathbb{R}$. We assume that

(1.0) there exists $H \in \mathcal{A}$ such that the real parts of all the eigenvalues of $\mathrm{ad}_H : \mathcal{N} \to \mathcal{N}$ are positive.

Let N, A, S be the connected and simply connected Lie groups whose Lie algebras are $\mathcal{N}, \mathcal{A}, \mathcal{S}$, respectively. Then $S = NA$ is a semidirect product of N and $A = \mathbb{R}^+$. We consider a second order degenerate elliptic left-invariant operator

$$L = \sum_{i,j=0}^{m} \alpha_{ij} Y_i Y_j + Y,$$

on S such that $Y_0 = H$ and $Y_1(e), \ldots, Y_m(e) \in \mathcal{N}$. It follows from elementary linear algebra that for $\alpha_{00} \ne 0$, L can be written in the form

$$L = \alpha_{00}(H + Y_0')^2 + \sum_{j=1}^{m} Y_j'^2 + Y', \tag{1.1}$$

where $Y_0', \ldots, Y_m'$ are left-invariant vector fields on S such that $Y_0'(e), \ldots, Y_m'(e) \in \mathcal{N}$. We may assume $\alpha_{00} = 1$.

The decomposition of S into a semidirect product of the maximal nilpotent normal subgroup N and $A = \mathbb{R}^+$ is not unique, i.e. there is no canonical choice of A. We put $A = \exp \mathcal{A}'$ with $\mathcal{A}' = \mathrm{lin}(H + Y_0)$. Clearly the real parts of the eigenvalues of ad_{H+Y_0} are again strictly positive.

Decomposing $s \in S$ as $s = xa$, $x \in N$, $a = \exp[(\log a)(H + Y_0)]$, we have

$$S = N \exp \mathcal{A}' = NA$$

and for some γ,

$$L = (a\partial_a)^2 - \gamma a\partial_a + \sum_{i=1}^{m} \Phi_a(B_i)^2 + \Phi_a(B), \tag{1.2}$$

where $\Phi_a = \mathrm{Ad}_{\exp(\log a(H+Y_0))}$ and $B, B_1, \ldots, B_m$ are left-invariant vector fields on N. If $\gamma \le 0$ then all the bounded harmonic functions are constant. This is a consequence of a result in [BR] (cf. [DH1]). Thus for the rest of the paper we assume $\gamma > 0$.

Let $\mathcal{H}$ be the space of bounded harmonic functions. Functions $F \in \mathcal{H}$ and $f \in L^\infty(N)$ are in a one-one correspondence established by the Poisson integral

$$F(s) = \int_N f(s \cdot x)\nu(x)\, dx,$$

where $x \to s \cdot x$ denotes the action of S on $N = S/A$.

ν is a smooth, bounded positive function with $\int_N \nu(x)\, dx = 1$. This correspondence, i.e. the existence of ν, follows from a theorem due to A. Raugi [R] (Theorems 8.4 and 9.2) applied to the semigroup $\{\mu_t\}_{t>0}$ of probability measures with the infinitesimal generator L (see [D1]). Smoothness, boundedness and positivity of ν are consequences of the Harnack inequality (for the details cf. Theorem 3.15 of [D1], where the above properties are proved in the case of a diagonal action of A, but the proofs clearly generalize.)

Since [R] deals with the Poisson integrals for a general Lie group and functions harmonic with respect to a probability measure, the proof of the existence of the Poisson kernel in that generality is long and technical. In our case it is much easier. We include here the main steps (of the classical proof [R]) deriving as a by-product the formula (1.12) which we need later.

(1.3) THEOREM. *There is a smooth, bounded, positive function ν with $\int_N \nu(x)\,dx = 1$ such that the integral*

$$F(s) = \int_N f(s\cdot x)\nu(x)\,dx, \qquad f \in L^\infty(N), \tag{1.4}$$

gives a one-one correspondence between $f \in L^\infty(N)$ and L-harmonic functions F on S.

Proof. We fix $t > 0$ and consider the random walk $S_n^t(\omega)$ starting at e with law $\mu = \check{\mu}_t$, where $\check{\mu}(A) = \mu(A^{-1})$. Let $Y_n = x_n(\omega)\,a_n(\omega)$, $x_n(\omega) \in N$, $a_n(\omega) \in A$, $n \ge 1$, be a random variable with values in S and distribution law μ. Then

$$\begin{aligned} S_n^t(\omega) &= x_1(\omega)a_1(\omega)\,\ldots\,x_n(\omega)a_n(\omega) \\ &= x_1(\omega)x_2(\omega)^{a_1(\omega)}\,\ldots\,x_n(\omega)^{a_1(\omega)\ldots a_{n-1}(\omega)}\cdot a_1(\omega)\,\ldots\,a_n(\omega), \end{aligned}$$

where $x(\omega)^{a(\omega)} = a(\omega)x(\omega)a(\omega)^{-1}$. To shorten the notation we omit ω. We have

$$\pi(S_n^t) = x_1\,x_2^{a_1}\,\ldots\,x_n^{a_1\ldots a_{n-1}},$$

where $\pi : S \to N$ is given by $\pi(xa) = x$, $x \in N$, $a \in A$. We are going to prove that

$$\lim_{n\to\infty} \pi(S_n^t(\omega)) = Z(\omega)$$

exists a.e. and the distribution law of Z is precisely the Poisson kernel $P(x)\,dx$ we are looking for.

Let ϱ be a riemannian left-invariant distance on N. It is enough to show that

$$\begin{aligned} &\limsup_{n\to\infty} \varrho(x_{n+1}^{a_1\ldots a_n})^{1/n} \\ &\quad = \limsup_{n\to\infty} \varrho((x_1x_2^{a_1}\,\ldots\,x_n^{a_1\ldots a_{n-1}})^{-1}\,x_1x_2^{a_1}\,\ldots\,x_{n+1}^{a_0\ldots a_n})^{1/n} < 1 \quad \text{a.e.} \end{aligned} \tag{1.5}$$

We have

$$\varrho(x_{n+1}^{a_1\ldots a_n}) \le \|\mathrm{Ad}_{a_1\ldots a_n}\|\varrho(x_{n+1}),$$

where $\|\mathrm{Ad}_a\|$ is the norm of the linear transformation Ad_a for the scalar product corresponding to ϱ. So, if we prove

$$\lim_{n\to\infty} \|\mathrm{Ad}_{a_1\ldots a_n}\|^{1/n} < 1 \quad \text{a.e.} \tag{1.6}$$

and

$$\limsup_{n\to\infty} \varrho(x_{n+1})^{1/n} \le 1 \quad \text{a.e.,} \tag{1.7}$$

(1.5) will follow. For (1.7) we refer to [R] (p. 69). It follows from the fact that $\varrho(\pi(s))$ is μ-integrable. To prove (1.6) we use the following simple fact.

LEMMA [R]. *Let $\{Q_n\}_{n\ge1}$ be a sequence of upper triangular $d\times d$ matrices. If the diagonal elements q_n^i, $i = 1,\ldots,d$, satisfy*

$$\lim_{n\to\infty} |q_1^i\cdots q_n^i|^{1/n} = q^i > 0 \quad \text{and} \quad \limsup_{n\to\infty} \|Q_n\|^{1/n} \le 1,$$

then $\lim_{n\to\infty} \|Q_1\ldots Q_n\|^{1/n}$ exists and is equal to $\sup_{\{i=1,\ldots,d\}} q^i$.

Therefore what we have to verify is

$$0 < \lim_{n\to\infty} e^{(1/n)\Re\lambda_j \log(a_0\ldots a_n)} < 1 \quad \text{a.e.} \tag{1.8}$$

for all λ_j, and that

$$\limsup_{n\to\infty} \|\mathrm{Ad}_{a_n}\|^{1/n} \le 1 \quad \text{a.e.} \tag{1.9}$$

Just as (1.7) formula (1.9) follows from the fact that $\|\mathrm{Ad}_a\|$ is μ-integrable. (1.8) is equivalent to

$$-\infty < \lim_{n\to\infty} \frac{1}{n}\Re\lambda_j \log(a_1\,\ldots\,a_n) < 0 \quad \text{a.e.} \tag{1.10}$$

$\log a_1, \log a_2, \ldots$ is a sequence of identically distributed independent random variables with values in $\mathbb{R}$. By the strong law of large numbers,

$$\lim_{n\to\infty} \frac{1}{n}\log(a_1\,\ldots\,a_n) = \int_S \log a(s)\,d\mu(s).$$

But

$$\int_S \log a(s)\;d\mu(s) = \int_A \log a\,d\pi_A\mu(a), \tag{1.11}$$

where $\pi_A(xa) = a$ is the canonical homomorphism $\pi_A : S \to S/N = A$. Since $\pi_A\mu_t$ is the gaussian semigroup with the infinitesimal generator $(a\partial_a)^2 - \gamma a\partial_a$, the integral in (1.11) is easily computable and when $\gamma > 0$,

$$\int_A \log a\,d\pi_A\mu(a) < 0.$$

But $\Re\lambda_j > 0$ so (1.10) follows. Now, as in [R], [D1] notice that

$$\check{\mu}_1 * \nu_n = \nu_{n+1},$$

where ν_n is the distribution law of $\pi(S_n(\omega))$. Therefore if

$$\nu = \lim_{n\to\infty} \nu_n \tag{1.12}$$

then $\check{\mu}_t * \nu = \nu$, which shows that the Poisson integrals against ν are L-harmonic. For the regularity properties of ν see [D1] (Theorem 3.15) and for the one-one correspondence see [R] (Theorem 8.4 for the μ-harmonic case or e.g. [DH1] (Theorem 3.8) in the case of a differential operator).

2. Probabilistic lemmas. Let $b(\cdot)$ be the Brownian motion on $\mathbb{R}$ starting from χ and normalized so that

$$\mathrm{E}_\chi f(b(t)) = \int f(\chi + y)\frac{1}{\sqrt{4\pi t}} e^{-y^2/(4t)}\, dy.$$

Let $\mathcal{F}_t$ be the σ-field generated by $(b(s)\in[a,b])$, $s<t$, $a,b\in\mathbb{R}$. Let

$$\mathbf{a}(t) = b(t) - \gamma t \quad\text{and}\quad \alpha(t) = e^{\mathbf{a}(t)}. \tag{2.0}$$

Let $d>0$. For $0\le s<t\le\infty$ we define the following Brownian random variables:

$$A_d(s,t) = \int_s^t \alpha(u)^d\, du, \qquad A_d = A_d(0,\infty),$$
$$\Lambda(s,t) = \max_{s\le u\le t} \mathbf{a}(u), \qquad \Lambda = \Lambda(0,\infty),$$
$$\lambda(s,t) = \min_{s\le u\le t} \mathbf{a}(u).$$

In [U] Urbanik has shown how his theory of analytic stochastic processes yields the theorem which follows. We present a direct proof which seems to be simpler.

(2.1) THEOREM. *Let $\chi\in\mathbb{R}$ be the starting point of the Brownian motion $b(\cdot)$. We have*

$$\mathrm{E}_\chi f(A_d) = c_{d,\gamma} e^{\gamma\chi}\int_0^\infty f(\sigma)\sigma^{-\gamma/d}\exp\left(-\frac{e^{d\chi}}{d^2\sigma}\right)\frac{d\sigma}{\sigma}. \tag{2.2}$$

Proof. By scaling the Brownian motion and changing the variable, we see that it suffices to prove (2.2) for $d=2$.

First we notice that

$$\left(\partial_r^2 - \frac{\kappa-1}{r}\partial_r\right)(r^\kappa f(r)) = r^\kappa\left(\partial_r^2 + \frac{\kappa+1}{r}\partial_r\right)f(r). \tag{2.3}$$

Let

$$W_t(r) = t^{-(\kappa+3)/2}e^{-r^2/(4t)}.$$

Then

$$\left(\partial_r^2 + \frac{\kappa+2}{r}\partial_r\right)W_t(r) = \partial_t W_t(r).$$

This, by (2.3), implies that for $w\in C^\infty(\mathbb{R}^+)$ which together with all its derivatives is bounded and vanishes fast enough at ∞, and for

$$u(r) = r^{\kappa+1}\int_0^\infty W_t(r)w(t)\, dt,$$

we have

$$\left(\partial_r^2 - \frac{\kappa}{r}\partial_r\right)u(r) = r^{\kappa+1}\int_0^\infty \partial_t W_t(r)w(t)\, dt = -r^{\kappa+1}\int_0^\infty W_t(r)w'(t)\, dt.$$

Since also

$$u(r) = \int_0^\infty t^{-(\kappa+1)/2}e^{-1/(4t)}w(r^2t)\frac{dt}{t},$$

we have

$$\lim_{r\to 0} u(r) = w(0)\int_0^\infty t^{-(\kappa+1)/2}e^{-1/(4t)}\frac{dt}{t}.$$

Let

$$u_\lambda(r) = r^{\kappa+1}\int_0^\infty W_t(r)e^{-\lambda t}\, dt.$$

Then

$$\left(\partial_r^2 - \frac{\kappa}{r}\partial_r\right)u_\lambda(r) = \lambda u_\lambda(r)$$

or

$$((r\partial_r)^2 - (\kappa+1)(r\partial_r))u_\lambda(r) = \lambda r^2 u_\lambda(r).$$

We substitute

$$v_\lambda(x) = cu_\lambda(e^x), \quad\text{where}\quad c^{-1} = \int_0^\infty t^{-(\kappa+1)/2}e^{-1/(4t)}\frac{dt}{t}.$$

Hence

$$(\partial_x^2 - (\kappa+1)\partial_x - \lambda e^{2x})v_\lambda(x) = 0 \tag{2.4}$$

and

$$0\le v_\lambda\le 1, \qquad \lim_{x\to-\infty} v_\lambda(x) = 1, \qquad \lim_{x\to\infty} v_\lambda(x) = 0.$$

Let $\{T_t\}_{t>0}$ be the semigroup of operators on $C_\infty(\mathbb{R})$ generated by the Schrödinger operator

$$H_\lambda = \partial_x^2 - \gamma\partial_x - \lambda e^{2x} \quad\text{and}\quad \gamma = \kappa+1.$$

By the Feynman–Kac formula, we have

$$T_t f(x) = \mathrm{E}_x \exp\left[-\lambda\int_0^t e^{2(b(s)-\gamma s)}\, ds\right] f(b(t)-\gamma t).$$

We put $f\equiv 1$ and $t=\infty$. Then the function

$$\phi_\lambda(x) = \mathrm{E}_x \exp\left[-\lambda\int_0^\infty e^{2(b(t)-\gamma s)}\, ds\right]$$

is harmonic with respect to H_λ and

$$0 \le \phi_\lambda \le 1, \qquad \lim_{x\to-\infty} \phi_\lambda(x) = 1, \qquad \lim_{x\to\infty} \phi_\lambda(x) = 0.$$

Therefore $v_\lambda(x) = \phi_\lambda(x)$, which completes the proof of Theorem (2.1).

(2.5) COROLLARY. *For every $l \in \mathbb{R}$ and T such that $\gamma T > |l|$,*

$$\text{(2.6)}\qquad \mathrm{P}_0\Big[\int_T^\infty e^{d(b(t)-\gamma t)}\,dt > e^{dl}\Big] \le \Big(c_{d,\gamma}\frac{d}{\gamma}+1\Big)\pi^{-1/2}\frac{2T^{3/2}\gamma}{\gamma^2T^2-l^2}e^{-(\gamma T+l)^2/(4T)},$$

$c_{d,\gamma}$ being the same constant as in (2.2).

Proof. By the Markov property and (2.2) we have

$$\begin{aligned}
\mathrm{P}_0\Big[\int_T^\infty e^{d(b(t)-\gamma t)}\,dt > e^{dl}\Big]
&= \int_{-\infty}^\infty \frac{1}{\sqrt{4\pi T}}e^{-(x+\gamma T)^2/(4T)}\mathrm{P}_x\Big[\int_0^\infty e^{d(b(t)-\gamma t)}\,dt > e^{dl}\Big]\,dx\\
&= c_{d,\gamma}\int_{-\infty}^\infty \frac{1}{\sqrt{4\pi T}}e^{-(x+\gamma T)^2/(4T)}e^{\gamma x}\int_{e^{dl}}^\infty \sigma^{-\gamma/d}\exp\Big(-\frac{e^{dx}}{d^2\sigma}\Big)\frac{d\sigma}{\sigma}\\
&= c_{d,\gamma}\int_{-\infty}^\infty \frac{1}{\sqrt{4\pi T}}e^{-(x+\gamma T)^2/(4T)}\int_{e^{d(l-x)}}^\infty \sigma^{-\gamma/d}\exp\Big(-\frac{1}{d^2\sigma}\Big)\frac{d\sigma}{\sigma}.
\end{aligned}$$

But

$$c_{d,\gamma}\int_\xi^\infty \sigma^{-\gamma/d}\exp\Big(-\frac{1}{d^2\sigma}\Big)\frac{d\sigma}{\sigma} \le \begin{cases} c_{d,\gamma}(d/\gamma)/\xi^{-\gamma/d} & \text{if } \xi > 1,\\ 1 & \text{if } \xi \le 1.\end{cases}$$

Therefore

$$\mathrm{P}_0\Big[\int_T^\infty e^{d(b(t)-\gamma t)}\,dt > e^{dl}\Big] \le I_1 + I_2,$$

where

$$I_1 = \int_l^\infty \frac{1}{\sqrt{4\pi T}}e^{-(x+\gamma T)^2/(4T)}\,dx$$

$$I_2 = \int_{-\infty}^l \frac{1}{\sqrt{4\pi T}}e^{-(x+\gamma T)^2/(4T)}c_{d,\gamma}\frac{d}{\gamma}e^{-\gamma(l-x)}\,dx.$$

Applying to I_1 the inequality

$$\text{(2.7)}\qquad \int_a^\infty e^{-y^2/4}\,dy \le \frac{2}{a}e^{-a^2/4}, \qquad a > 0,$$

we obtain

$$\text{(2.8)}\qquad \int_{(l+\gamma T)/\sqrt{T}}^\infty (4\pi)^{-1/2}e^{-x^2/4}\,dx \le \frac{\sqrt{T}}{\sqrt{\pi}(l+\gamma T)}e^{-(l+\gamma T)^2/(4T)}.$$

Analogously for I_2, we have

$$\begin{aligned}
\text{(2.9)}\qquad I_2 &= c_{d,\gamma}\frac{d}{\gamma}e^{-\gamma l}\int_{-\infty}^l \frac{1}{\sqrt{4\pi T}}e^{-(x-\gamma T)^2/(4T)}\,dx\\
&= c_{d,\gamma}\frac{d}{\gamma}e^{-\gamma l}\int_{(-l+\gamma T)/\sqrt{T}}^\infty \frac{1}{\sqrt{4\pi}}e^{-x^2/4}\,dx\\
&\le c_{d,\gamma}\frac{d}{\gamma}e^{-\gamma l}\frac{\sqrt{T}}{\sqrt{\pi}(-l+\gamma T)}e^{-(-l+\gamma T)^2/(4T)}\\
&= c_{d,\gamma}\frac{d}{\gamma}\frac{\sqrt{T}}{\sqrt{\pi}(-l+\gamma T)}e^{-(l+\gamma T)^2/(4T)}.
\end{aligned}$$

(2.8) and (2.9) together give (2.6).

(2.10) COROLLARY. *Let $\chi \in \mathbb{R}$ and η, d_1, d_2, d_3 be positive numbers. Then there is $C = C(\eta, d_1, d_2, d_3, \chi)$ such that for every $t \ge 1$,*

$$\mathrm{E}_\chi \exp\Big[-\frac{\eta}{A_{d_1}(t,\infty)^{d_2} + A_{d_1}(t,\infty)^{d_3}}\Big] \le Ce^{-\gamma^2 t/4}.$$

Proof. We proceed as in the proof of Corollary (2.5). Adjusting η we may assume $\chi = 0$. By the Markov property and (2.2),

$$\begin{aligned}
\mathrm{E}_0 \exp\Big(-\frac{\eta}{A_{d_1}(t,\infty)^{d_2} + A_{d_1}(t,\infty)^{d_3}}\Big)
&= \mathrm{E}_0\mathrm{E}_{\mathbf{a}(t)}\exp\Big(-\frac{\eta}{A_{d_1}^{d_2} + A_{d_1}^{d_3}}\Big)\\
&= c_{d,\gamma}\int_{-\infty}^\infty \frac{1}{\sqrt{4\pi t}}e^{-(x+\gamma t)^2/(4t)}e^{\gamma x}\\
&\quad\times\int_0^\infty \sigma^{-\gamma/d_1}\exp\Big(-\frac{\eta}{\sigma^{d_2}+\sigma^{d_3}}\Big)\exp\Big(-\frac{e^{d_1 x}}{d_1^2\sigma}\Big)\frac{d\sigma}{\sigma}\,dx = I.
\end{aligned}$$

Now

$$I \le C(I_1 + I_2),$$

where

$$I_1 = \int_{-\infty}^{0} \frac{1}{\sqrt{4\pi t}} e^{-[(x+\gamma t)^2/(4t)]+\gamma x}\,dx = \int_{-\infty}^{0} \frac{1}{\sqrt{4\pi t}} e^{-(x-\gamma t)^2/(4t)}\,dx,$$

$$I_2 = \int_{0}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-(x+\gamma t)^2/(4t)}\,dx.$$

Indeed, if $x \le 0$, we estimate

$$\int_0^\infty \sigma^{-\gamma/d_1} \exp\left(-\frac{\eta}{\sigma^{d_2}+\sigma^{d_3}}\right)\exp\left(-\frac{e^{d_1 x}}{d_1^2\sigma}\right)\frac{d\sigma}{\sigma}\,dx \le \int_0^\infty \sigma^{-\gamma d_1} \exp\left(-\frac{\eta}{\sigma^{d_2}+\sigma^{d_3}}\right)\frac{d\sigma}{\sigma}\,dx < \infty,$$

and if $x \ge 0$, then

$$e^{\gamma x}\int_0^\infty \sigma^{-\gamma/d_1} \exp\left(-\frac{\eta}{\sigma^{d_2}+\sigma^{d_3}}\right)\exp\left(-\frac{e^{d_1 x}}{d_1^2\sigma}\right)\frac{d\sigma}{\sigma}\,dx \le e^{\gamma x}\int_0^\infty \sigma^{-\gamma/d_1}\exp\left(-\frac{e^{d_1 x}}{d_1^2\sigma}\right)\frac{d\sigma}{\sigma}\,dx = 1.$$

The conclusion now follows from inequality (2.7).

Let

(2.11) $$\zeta = \min\{t : \mathbf{a}(t) = \Lambda\}.$$

(2.12) LEMMA. *There is a $c > 0$ such that*

(2.13) $$\mathrm{P}_0\{k \le \zeta \le k+1\} \le ce^{-\gamma^2 k/8}.$$

Proof. Since $\Lambda \ge 0$ P_0-almost surely,

$$\begin{aligned}\mathrm{P}_0\{k \le \zeta \le k+1\} &\le \mathrm{P}_0\{\max_{k\le t\le k+1} b(t) \ge \gamma k\} \\ &\le 2\mathrm{P}_0\{b(k+1) \ge \gamma k\} \\ &= 2\frac{1}{\sqrt{4\pi(k+1)}}\int_{\gamma k}^\infty e^{-x^2/(4(k+1))}\,dx,\end{aligned}$$

which gives (2.13).

Given $s > 0$, we have

(2.14) $$\mathrm{P}_0\{\Lambda(s,s+1) \ge n + \lambda(s,s+1)\} \le e^{-(n-\gamma)^2/4}.$$

(2.15) LEMMA. *For every D there is a constant c_D such that for every $n > 0$,*

(2.16) $$\mathrm{P}_0\{\Lambda(\zeta,\zeta+1) - \lambda(\zeta,\zeta+1) \ge n\} \le c_D e^{-nD}.$$

Proof. We have

$$\begin{aligned}&\mathrm{P}_0\{\Lambda(\zeta,\zeta+1) - \lambda(\zeta,\zeta+1) \ge n\} \\ &\quad\le \sum_k \mathrm{P}_0\{\Lambda(\zeta,\zeta+1) - \lambda(\zeta,\zeta+1) \ge n \wedge k \le \zeta \le k+1\} \\ &\quad\le \sum_k \mathrm{P}_0\{\Lambda(k,k+2) - \lambda(k,k+2) \ge n\}^{1/2}\mathrm{P}_0\{k \le \zeta \le k+1\}^{1/2}.\end{aligned}$$

Now (2.14) implies (2.16).

We write

$$B \sim e^k \Leftrightarrow e^k \le B \le e^{k+1}.$$

(2.17) THEOREM. *Given d, D there is c_D such that*

(2.18) $$\mathrm{P}_0\{A_d^{1/d} e^{-\Lambda} \sim e^n\} \le c_D e^{-D|n|}.$$

Proof. We have to prove that for $n \ge 1$,

(2.19) $$\mathrm{P}_0\{A_d^{1/d} \ge e^n e^{\Lambda}\} \le c_D e^{-Dn}$$

and

(2.20) $$\mathrm{P}_0\{e^{\Lambda} \ge e^{n-1} A_d^{1/d}\} \le c_D e^{-Dn}.$$

For (2.19) we notice that $\Lambda \ge 0$ P_0-almost surely. Therefore, since

$$e^{d\Lambda} \ge A_d(0,n^2)/n^2,$$

we have

$$\mathrm{P}_0\{A_d^{1/d} \ge e^n e^{\Lambda}\} \le \mathrm{P}_0\{A_d(n^2,\infty) \ge e^{dn}/2\} + \mathrm{P}_0\left\{A_d(0,n^2) \ge \frac{1}{2}\frac{e^{dn}}{n^2}A_d(0,n^2)\right\}.$$

The second set in the above inequality is empty for $n \ge 3$. For the first one we can apply Corollary (2.5) because $\gamma n^2 > n$ if n is large enough. Hence

$$\mathrm{P}_0\{A_d(n^2,\infty) \ge e^{dn}\} \le c_D e^{-Dn}.$$

To prove (2.20) we write

$$\begin{aligned}&\mathrm{P}_0[e^{\Lambda} \ge e^{n-1}A_d^{1/d}] \\ &\quad\le \sum_k \mathrm{P}_0[e^{\Lambda(k,k+1)} \ge e^{n-1}A_d^{1/d}(k,k+1) \wedge k \le \zeta \le k+1] \\ &\quad\le \sum_{k\ge 0} \mathrm{P}_0[e^{\Lambda(k,k+1)} \ge e^{n-1}A_d^{1/d}(k,k+1)]^{1/2}\mathrm{P}_0[k \le \zeta \le k+1]^{1/2}.\end{aligned}$$

But

$$\begin{aligned}&\mathrm{P}_0[e^{\Lambda(k,k+1)} \ge e^{n-1}A_d^{1/d}(k,k+1)] \\ &\quad\le \mathrm{P}_0[\Lambda(k,k+1) \ge n-1+\lambda(k,k+1)] \le e^{-(n-1)^2}.\end{aligned}$$

For a stopping time τ we denote the shift transformation $C(0,\infty) \to C(0,\infty)$ by Θ_τ, i.e. $(\Theta_\tau\omega)(s) = \omega(\tau+s)$.

(2.21) THEOREM. *Given d_1, k_1 let τ be the stopping time defined by*

$$A_{d_1}(0,\tau) = \tfrac{1}{2}e^{d_1k_1}$$

and

$$\Omega_{k_1,k_2,k_3} = \{A_{d_1}^{1/d_1} \sim e^{k_1} \wedge e^{\Lambda(\Theta_\tau \mathbf{a})} \sim e^{k_2} \wedge e^{\lambda(\zeta(\Theta_\tau\mathbf{a}),\zeta(\Theta_\tau\mathbf{a})+1)} \sim e^{k_3}\}.$$

Then

$$\mathrm{P}_\chi(\Omega_{k_1,k_2,k_3}) \le c_D e^{\gamma\chi} e^{-\gamma k_1} e^{-D|k_1-k_2|-D|k_2-k_3|}. \tag{2.22}$$

Proof. If $A_{d_1}(0,\tau) = \frac{1}{2}e^{d_1k_1}$ and $A_{d_1} \sim e^{dk_1}$, then $A_{d_1}(\tau,\infty) \ge \frac{1}{2}e^{d_1k_1}$. Hence

$$\begin{aligned}\mathrm{P}_\chi(\Omega_{k_1,k_2,k_3}) &\le \mathrm{P}_\chi\{\tau<\infty \wedge 2^{-1/d_1}e^{k_1-k_2-1}e^{\Lambda(\Theta_\tau\mathbf{a})} \le A_{d_1}(\tau,\infty)^{1/d_1} \\ &\qquad \wedge \Lambda(\Theta_\tau\mathbf{a}) - \lambda(\zeta(\Theta_\tau\mathbf{a}),\zeta(\Theta_\tau\mathbf{a})+1) \ge k_2-k_3-1\} \\ &= \mathrm{E}_\chi 1_{\{\tau<\infty\}}\Theta_\tau 1_{\Omega_0},\end{aligned}$$

where

$$\Omega_0 = \{2^{-1/d_1}e^{k_1-k_2-1}e^{\Lambda} \le A_{d_1}^{1/d_1} \wedge \Lambda - \lambda(\zeta,\zeta+1) \ge k_2-k_3-1\}.$$

Moreover, $\mathrm{P}_\xi(\Omega_0) = \mathrm{P}_0(\Omega_0)$ for every ξ. Therefore by the strong Markov property

$$\mathrm{E}_\chi 1_{\{\tau<\infty\}}\Theta_\tau 1_{\Omega_0} = \mathrm{P}_\chi\{\tau<\infty\}\mathrm{P}_0(\Omega_0)$$

and so

$$\mathrm{P}_\chi(\Omega_{k_1,k_2,k_3}) \le \mathrm{P}_\chi\{A_{d_1}^{1/d_1} \ge e^{k_1}/2\}\mathrm{P}_0(\Omega_0).$$

Now (2.22) follows from (2.2), Lemma (2.15) and Theorem (2.17).

3. Estimates of the evolution kernels I. Let N be a nilpotent Lie group with the Lie algebra $\mathcal{N}$. Assume that we have a continuous family θ_t, $t>0$, of automorphisms of $\mathcal{N}$ and a generating set $\{B_1,\ldots,B_m\}$ of $\mathcal{N}$. We consider the operator

$$L = \sum_{j=1}^m \theta_t(B_j)^2 + \theta_t(B) - \partial_t \tag{3.1}$$

and its fundamental solution

$$P(s,t,x) = P(s,t)(x), \quad 0 \le s < t < \infty,\ x \in N.$$

P is a non-negative function on $N \times \{(s,t) : 0 \le s < t\}$ such that

$$\int_N P(s,t,x)\,dx = 1$$

and for $s<u<t$,

$$P(s,t) = P(s,u) * P(u,t). \tag{3.2}$$

Moreover, if $\phi \in C_c^\infty(N)$, then

$$F(x,t) = \phi * P(s,t)(x)$$

is the solution of the Dirichlet problem on $N \times (s,\infty)$ with the boundary data ϕ, i.e.

$$LF = 0 \quad \text{in } N\times(s,\infty), \qquad \lim_{t\to 0} F(x,t) = \phi(x).$$

For existence of P see e.g. [DH2], where the proofs from ([S], [SV]) are adapted to group-invariant operators of the form (3.1). For $s<t$ let

$$\xi(s,t) = \sup_{s\le u\le t} \|\theta_u\|_{\mathcal{N}\to\mathcal{N}}, \tag{3.3}$$

$$\eta(s,t) = \sup_{s\le u\le t} \|\theta_u^{-1}\|_{\mathcal{N}\to\mathcal{N}}. \tag{3.4}$$

In this section we give an estimate of the L^∞-norm of $X^IP(s,t)$ in terms of $\xi(s,t)$ and $\eta(s,t)$. It can be easily derived from the arguments in the proof of Theorem 4.3 in [DH2]. However, since the role of $\eta(s,t)$ has not been made explicit enough there, we think that we should include here the main steps of the proof estimating the constants important for the rest of this paper.

(3.5) THEOREM. *For every $I=(i_1,\ldots,i_k)$ and $c>0$, there are constants $C, M>0$ such that*

$$\|X^IP(s,t)\|_{L^\infty} \le C(1+\xi(s,t)+\eta(s,t))^M \quad \textit{if } c \le t-s \textit{ and } \|X^I\| \le 1.$$

Proof. Let $U = U_0\times(s,t)$, where U_0 is an open set in N with compact closure and let

$$L_0 = \sum_{j=1}^m \theta_t(B_j)^2$$

(considered as an operator on U). Let $\phi \in C_c^\infty(U)$. Proceeding as in the proof of Theorem 5.3 in [DH2], for a constant $C=C(\phi,U)$, we obtain

$$|\langle L_0(\phi u),\phi u\rangle| \le C(1+\xi(s,t))^2\|u\|^2_{L^2(U)}, \tag{3.6}$$

whenever $Lu=0$ in U. Now the Kohn Lemma (0.14) implies that

$$\begin{aligned}\|\phi u\|_\varepsilon^2 &\le C(1+\eta(s,t))^2(1+\xi(s,t))^2\|u\|^2_{L^2(U)} \\ &\le C(1+\eta(s,t)+\xi(s,t))^4\|u\|^2_{L^2(U)}\end{aligned} \tag{3.7}$$

for a $C=C(\phi,U,\varepsilon)$. Iterating (3.7) we are going to prove by induction that for every positive integer n there is $C=C(\phi,U,\varepsilon,n)$ such that

$$\|\phi u\|_{n\varepsilon} \le C(1+\eta(s,t)+\xi(s,t))^{2n}\|u\|^2_{L^2(U)}. \tag{3.8}$$

Our inductive hypothesis is the following:

H. *Let u be a harmonic function on U, i.e. $Lu(x,r)=0$ for $(x,r)\in U$, and let $\phi\in C_c^\infty(U)$ be such that for a positive R, $B(3R)\operatorname{supp}\phi\subset U$. Then there exists $C=C(n,R)$ such that*

$$\|\phi u\|_{n\varepsilon}\le C(1+\eta(s,t)+\xi(s,t))^{2n}\|u\|_{L^2(U)}.$$

We write

$$\|\widetilde{\Lambda}^{(n+1)\varepsilon/2}\phi u\|_{L^2}\le\|\widetilde{\Lambda}^{n\varepsilon/2}\widetilde{\Lambda}_0^{(\varepsilon/2)}\phi u\|_{L^2}+\|\widetilde{\Lambda}^{n\varepsilon/2}M^{(\varepsilon/2)}\phi u\|_{L^2},$$

where $\widetilde{\Lambda}_0^{(\varepsilon/2)}$ and $M^{(\varepsilon/2)}$ are as in (0.9). Let $\psi\in C_c^\infty(U)$ be a function such that $\psi=1$ on $B(R)\operatorname{supp}\phi$ and $B(R)\operatorname{supp}\psi\subset U$. Hence, since $\widetilde{\Lambda}_0^{(\varepsilon/2)}\psi u$ is harmonic on $B(R)\operatorname{supp}\phi$, by the inductive hypothesis and Corollary (0.12), for a $c=c(\varepsilon,n,r)$ we have

$$\begin{aligned}\|\widetilde{\Lambda}^{(n+1)\varepsilon/2}\phi u\|_{L^2}&\le\|\widetilde{\Lambda}^{n\varepsilon/2}\widetilde{\Lambda}_0^{(\varepsilon/2)}\phi\psi u\|_{L^2}+c\|u\|_{L^2}\\&\le\|\widetilde{\Lambda}^{n\varepsilon/2}\phi\widetilde{\Lambda}_0^{(\varepsilon/2)}\psi u\|_{L^2}\\&\quad+\|\widetilde{\Lambda}^{n\varepsilon/2}[\widetilde{\Lambda}_0^{(\varepsilon/2)},\phi]\psi u\|_{L^2}+c\|u\|_{L^2}.\end{aligned}$$

Since $\widetilde{\Lambda}_0^{(\varepsilon/2)}\psi u$ is harmonic on $\operatorname{supp}\phi$ and

$$\|\widetilde{\Lambda}^{n\varepsilon/2}[\widetilde{\Lambda}_0^{(\varepsilon/2)},\phi]v\|_{L^2}\le C_{\phi,\varepsilon}\|\widetilde{\Lambda}^{n\varepsilon/2}v\|_{L^2}$$

for $v\in C_c^\infty(U)$, by the inductive hypothesis, we have

$$\begin{aligned}\|\widetilde{\Lambda}^{n\varepsilon/2}\phi\widetilde{\Lambda}_0^{(\varepsilon/2)}\psi u\|_{L^2}&\le c(1+\eta(s,t)+\xi(s,t))^{2n}\|\widetilde{\Lambda}_0^{(\varepsilon/2)}\psi u\|_{L^2}\\&\le c(1+\eta(s,t)+\xi(s,t))^{2(n+1)}\|u\|_{L^2}\end{aligned}$$

and

$$\|\widetilde{\Lambda}^{n\varepsilon/2}[\widetilde{\Lambda}_0^{(\varepsilon/2)},\phi]\psi u\|_{L^2}\le c\|\widetilde{\Lambda}^{n\varepsilon/2}\psi u\|_{L^2}\le(1+\eta(s,t)+\xi(s,t))^{2n}\|u\|_{L^2}.$$

Thus (3.8) is proved.

Next applying the Sobolev inequality with respect to the x variable together with (3.8) we see that for every $U'\subset\overline{U'}\subset U$ and every I as in (0.22) there are $C=C(U,U',I)$ and $M=M(I)$ such that

$$\sup_{(x,r)\in U'}|X^Iu(x,r)|\le C(1+\xi(s,t)+\eta(s,t))^M\|u\|_{L^2}\quad\text{ if }\|X^I\|\le 1.\tag{3.9}$$

Assume now that $c_1\le t-s\le c_2$ and take U' of the form $U'=U_0'\times(s',t')$, where $\overline{U_0'}\subset U_0$, $s<s'<t'<t$ and

$$u(x,r)=f*P(w,r)(x),\qquad w<s'<r<t'.$$

Then, in view of (3.9), there is C such that

$$\begin{aligned}|X^I(f*P(w,r))(e)|&\le C(1+\xi(s,t)+\eta(s,t))^M\Big(\int_s^t\int_{U_0}|f*P(w,r)(x)|^2\,dx\,dt\Big)^{1/2}\\&\le C(1+\xi(s,t)+\eta(s,t))^M\|f\|_{L^2}.\end{aligned}\tag{3.10}$$

Therefore $P(w,r)\in C^\infty(N)$ and

$$\|X^IP(w,r)\|_{L^2}\le C(1+\xi(s,t)+\eta(s,t))^M.$$

Applying (3.10) to $w<s'<r_1<(t+s)/2<r_2<t'$, we obtain

$$\begin{aligned}\|X^IP(w,r_2)\|_{L^\infty}&\le\|P(w,r_1)\|_{L^2}\|X^IP(r_1,r_2)\|_{L^2}\\&\le C(1+\xi(s,t)+\eta(s,t))^M\end{aligned}\tag{3.11}$$

for, possibly, another M. Of course, the upper bound c_2 for $t-s$ can be easily removed because of (3.2).

4. Estimates of the evolution kernels II. Let H be as in (1.0) and

$$\mathbf{a}(t)=b(t)-\gamma t$$

be as in (2.0). Assume that $B,B_1,\ldots,B_m\in\mathcal{N}$ and $B_1,\ldots,B_m$ generate $\mathcal{N}$ as a Lie algebra. Let σ^r be the automorphism of N defined by

$$\sigma^r(x)=\exp(rH)x\exp(-rH).$$

For a fixed t let

$$L_{\mathbf{a}}(t)=\sum_{j=1}^m\Phi_t(B_j)^2+\Phi_t(B)=\sum_{j=1}^mX_j^2+X,$$

where $\Phi_t=\sigma_*^{\mathbf{a}(t)}$. For a fixed t, $L_{\mathbf{a}(t)}$ is a left-invariant operator on N.

We consider the operator

$$L_{\mathbf{a}}=\sum_{j=1}^m\Phi_t(B_j)^2+\Phi_t(B)-\partial_t\quad\text{ on }N\times\mathbb{R}^+,\tag{4.1}$$

i.e. $L_{\mathbf{a}}f(x,t)=L_{\mathbf{a}(t)}f(x,t)-\partial_tf(x,t)$. Let $P_{\mathbf{a}}(s,t,x)$, $0\le s<t<\infty$, $x\in N$, be the fundamental solution of $L_{\mathbf{a}}$. We "dilate" $L_{\mathbf{a}}$ in an appropriate way. Namely, let

$$L_{\mathbf{a}}^r=\sum_{j=1}^m\sigma_*^{-r}\Phi_t(B_j)^2+\sigma_*^{-r}\Phi_t(B)-\partial_t,\tag{4.2}$$

where $r>0$ will be chosen later. Then, clearly,

$$L_{\mathbf{a}(t)}^r(f\circ\sigma^r)=(L_{\mathbf{a}(t)}f)\circ\sigma^r.\tag{4.3}$$

Let $P^r_{\mathbf{a}}(s,t,x)$, $0\le s<t<\infty$, $x\in N$, be the fundamental solution of $L^r_{\mathbf{a}}$. In view of (4.3),

$$P^r_{\mathbf{a}}(s,t,x)=|\det\sigma^r_*|P_{\mathbf{a}}(s,t,\sigma^r(x)) \tag{4.4}$$

and

$$|\det\sigma^r_*|=e^{rQ}\quad\text{where}\quad Q=\sum_j\Re\lambda_j.$$

First we estimate $P^r_{\mathbf{a}}$. We put

$$\theta_t=\sigma_*^{-r}\Phi_t. \tag{4.5}$$

By (0.23) we have

(4.6) Lemma. *There exist $C,d_1,d_2>0$ such that for every $u\in\mathbb{R}$,*

$$\|\sigma^u_*\|_{\mathcal{N}\to\mathcal{N}}\le C(e^{d_1u}+e^{d_2u}). \tag{4.7}$$

Now we choose r in (4.2). Let d_1,d_2 be as above. Let

$$\begin{aligned}A=A(T_1,T_2)&=A_{d_1}(T_1,T_2)^{1/d_1}+A_{d_2}(T_1,T_2)^{1/d_2}\\&\quad+A_{2d_1}(T_1,T_2)^{1/(2d_1)}+A_{2d_2}(T_1,T_2)^{1/(2d_2)}.\end{aligned} \tag{4.8}$$

We put

$$r=r(T_1,T_2)=\log A.$$

Of course, r depends on the trajectory $\mathbf{a}$; also, our T_1 and T_2 will vary and consequently so will r. In view of Lemma (4.6) we now have

(4.9) Lemma. *There are $c,d_1,d_2>0$ such that*

$$\|\theta_t\|_{\mathcal{N}\to\mathcal{N}}\le C((e^{\mathbf{a}(t)}A(T_1,T_2)^{-1})^{d_1}+(e^{-\mathbf{a}(t)}A(T_1,T_2))^{d_2}),$$
$$\|\theta_t^{-1}\|_{\mathcal{N}\to\mathcal{N}}\le C((e^{\mathbf{a}(t)}A(T_1,T_2)^{-1})^{-d_1}+(e^{-\mathbf{a}(t)}A(T_1,T_2))^{-d_2}).$$

We start with the following integral estimate for $P^r_{\mathbf{a}}$.

(4.10) Theorem. *There is C such that for every $\mathbf{a}$, every T_1, T_2 and every $T_1\le s<t\le T_2$,*

$$\int P^r_{\mathbf{a}}(s,t,x)e^{\varrho(x)}\,dx\le C, \tag{4.11}$$

where $r=r(T_1,T_2)$.

Proof. As in [H] we take a non-negative, not identically zero function $\phi\in C_c(N)$ and we notice that

$$c^{-1}\varrho*\phi(x)\le\varrho(x)\le c\varrho*\phi(x)$$

for a $c=c(\phi)$. Moreover, $|X_j\varrho*\phi(x)|$ and $|X_iX_j\varrho*\phi(x)|$ are bounded functions on N. For fixed s we write

$$\langle P^r_{\mathbf{a}}(s,t),e^{\varrho}\rangle\le C\langle P^r_{\mathbf{a}}(s,t),e^{\varrho*\phi}\rangle$$

and

$$\begin{aligned}\frac{d}{dt}\langle P^r_{\mathbf{a}}(s,t),e^{\varrho*\phi}\rangle&=\Big\langle\frac{d}{dt}P^r_{\mathbf{a}}(s,t),e^{\varrho*\phi}\Big\rangle=\langle L^r_{\mathbf{a}(t)}P^r_{\mathbf{a}}(s,t),e^{\varrho*\phi}\rangle\\&=\langle P^r_{\mathbf{a}}(s,t),(L^r_{\mathbf{a}(t)})^*e^{\varrho\phi}\rangle\le c(\|\theta_t\|+\|\theta_t\|^2)\langle P^r_{\mathbf{a}}(s,t),e^{\varrho*\phi}\rangle.\end{aligned}$$

Therefore there are constants $c_1,c_2>0$ such that

$$\langle P^r_{\mathbf{a}}(s,t),e^{\varrho*\phi}\rangle\le c_1\exp\Big(c_2\int_s^t(\|\theta_u\|+\|\theta_u\|^2)\,du\Big).$$

Now (4.11) follows from (4.8) and Lemma (4.9).

(4.12) Corollary. *Given a neighborhood U of e in N, there are $C,\eta>0$ such that for every T_1,T_2, every $\mathbf{a}$ and every $T_1\le s<t\le T_2$,*

$$\int_{U^c}P_{\mathbf{a}}(s,t,x)\,dx\le C\exp\Big(-\frac{\eta}{A^{d_1}+A^{d_2}}\Big), \tag{4.13}$$

where $A=A(T_1,T_2)$.

Proof. By (4.4),

$$\int_{U^c}P_{\mathbf{a}}(s,t,x)\,dx=\int_{U^c}e^{-rQ}P^r_{\mathbf{a}}(s,t,\sigma^{-r}(x))\,dx=\int_{\sigma^{-r}(U^c)}P^r_{\mathbf{a}}(s,t,x)\,dx.$$

Since σ^{-r} is an automorphism of N, by (4.7) we have

$$\varrho(x)\le\|\sigma^r_*\|\varrho(\sigma^{-r}(x))\le C(A^{d_1}+A^{d_2})\varrho(\sigma^{-r}(x)). \tag{4.14}$$

Therefore

$$\varrho(y)\ge\frac{\eta}{A^{d_1}+A^{d_2}} \tag{4.15}$$

for $y\in\sigma^{-r}(U^c)$. So, in view of (4.11),

$$\int_{\sigma^{-r}(U^c)}P^r_{\mathbf{a}}(s,t,y)\exp\Big(-\frac{\eta}{A^{d_1}+A^{d_2}}\Big)\,dy\le C$$

and the conclusion follows.

We need the following simple lemma with almost the same proof as Lemma (0.3).

(4.16) Lemma. *For a constant $C=C(I,\varrho)$ we have*

$$\|X^If\|_{L^2(e^{\varrho})}\le C\sum_{|J|\le|2I|}\|X^Jf\|^{1/2}_{L^\infty}\|f\|^{1/2}_{L^1(e^{\varrho})}.$$

Indeed,

$$\|X^I f\|^2_{L^2(e^\varrho)} \le \int_N (X^I f(x))\cdot(X^I f(x))e^{\varrho(x)}\,dx \le \sum_{|J|\le|2I|} c_J \int_N |X^J f(x)\cdot f(x)e^{\varrho(x)}|\,dx \le \sum_{|J|\le|2I|} c_J\|X^J f\|_{L^\infty}\|f\|_{L^1(e^\varrho)}.$$

Let now $T_1 = 0$, $T_2 = \infty$, $r = r(0,\infty)$ and θ_t be defined by (4.5).

(4.17) LEMMA. *Let $\beta : \Omega \to \mathbb{R}^+$ be any function which to every trajectory $\mathbf{a}$ assigns a non-negative number $\beta(\mathbf{a})$. For every multi-index $I = (i_1,\ldots,i_k)$, there are $C, M > 0$ such that for every trajectory $\mathbf{a}$, every $x\in N$ and every $t > \beta(\mathbf{a})+1$, if $\|X^I\|\le 1$ then*

$$|X^I P^r_{\mathbf{a}}(0,t,x)e^{\varrho(x)}| \le C(1+\xi(\beta,\beta+1)+\eta(\beta,\beta+1))^M. \tag{4.18}$$

Proof. Let $\widetilde{X}^I$ be the right-invariant differential operator corresponding to the *left-invariant* operator X^I. Since by Theorem (4.10),

$$\|P^r_{\mathbf{a}}(s,t)\|_{L^1(e^\varrho)} \le C$$

for every $\mathbf{a}$, every $t > \beta(\mathbf{a})+1$ and every $0\le s<t$, we have

$$|\widetilde{X}^I P^r_{\mathbf{a}}(0,t,x)e^{\varrho(x)}| \le C\|\widetilde{X}^I P^r_{\mathbf{a}}(0,\beta+1,\cdot)e^{\varrho(\cdot)}\|_{L^\infty}\|P^r_{\mathbf{a}}(\beta+1,t,\cdot)\|_{L^1(e^\varrho)} \le C\|X^I P^r_{\mathbf{a}}(0,\beta+1,\cdot)e^{2\varrho(\cdot)}\|_{L^\infty}.$$

But again, in view of (2.2) and Theorem (4.10),

$$\begin{aligned}\|X^I P^r_{\mathbf{a}}(0,\beta+1,\cdot)e^{2\varrho(\cdot)}\|_{L^\infty} &\le C\|P^r_{\mathbf{a}}(0,s,\cdot)\|_{L^1(e^{2\varrho})}\|X^I P^r_{\mathbf{a}}(\beta,\beta+1,\cdot)e^{2\varrho(\cdot)}\|_{L^\infty}\\ &\le C\|X^I P^r_{\mathbf{a}}(\beta,\beta+1,\cdot)e^{2\varrho(\cdot)}\|_{L^\infty}\\ &\le C\|P^r_{\mathbf{a}}(\beta,\beta+1/2,\cdot)\|_{L^2(e^{2\varrho})}\|X^I P^r_{\mathbf{a}}(\beta+1/2,\beta+1,\cdot)\|_{L^2(e^{2\varrho})}.\end{aligned}$$

Now by Lemma (4.16) and Theorem (4.10),

$$\|P^r_{\mathbf{a}}(\beta,\beta+1/2,\cdot)\|_{L^2(e^{2\varrho})}\|X^I P^r_{\mathbf{a}}(\beta+1/2,\beta+1,\cdot)\|_{L^2(e^{2\varrho})} \le \|P^r_{\mathbf{a}}(\beta,\beta+1/2,\cdot)\|^{1/2}_{L^\infty}\sum_{|J|\le 2|I|}\|X^J P^r_{\mathbf{a}}(\beta+1/2,\beta+1,\cdot)\|^{1/2}_{L^\infty},$$

which, together with Theorem (3.5), implies (4.18).

Now we are going to prove the existence of the limit

$$\lim_{t\to\infty} P_{\mathbf{a}}(0,t,x) = P_{\mathbf{a}}(0,\infty,x) \tag{4.19}$$

and to estimate $P_{\mathbf{a}}(0,\infty)$ and its derivatives.

(4.20) THEOREM. *For every $I = (i_1,\ldots,i_k)$ there are constants C_1, C_2, M_1, M_2 such that for every $\mathbf{a}$,*

$$\begin{aligned}|X^I P_{\mathbf{a}}(0,\infty,x)| \le C_1 e^{-Q\log A}(A^{-d_1}+A^{-d_2})^{M_1}(1+(e^{\Lambda(\beta,\beta+1)}A^{-1})^{d_1}\\ +(e^{\Lambda(\beta,\beta+1)}A^{-1})^{d_2}+(e^{-\lambda(\beta,\beta+1)}A)^{d_1}\\ +(e^{-\lambda(\beta,\beta+1)}A)^{d_2})^{M_2}e^{-\varrho(x)/(C_2(A^{d_1}+A^{d_2}))},\end{aligned}$$

where $A = A(0,\infty)$ and $\|X^I\|\le 1$.

Proof. Let $\widetilde{X}^I$ be the right-invariant differential operator corresponding to X^I. Then

$$\omega(s,t,x) = |\widetilde{X}^I P_{\mathbf{a}}(0,s)(x) - \widetilde{X}^I P_{\mathbf{a}}(0,t)(x)| \le \int_N |\widetilde{X}^I P_{\mathbf{a}}(0,s,x) - \widetilde{X}^I P_{\mathbf{a}}(0,s,xy^{-1})|P_{\mathbf{a}}(s,t,y)\,dy.$$

If $\mathbf{a}$ is fixed, in view of (4.4) and (4.18), $\widetilde{X}^I P_{\mathbf{a}}(0,s)$ is bounded independently of $s>\beta(\mathbf{a})+1$. Given a compact set K, there is $U_{\mathbf{a}}$ such that for every $s>\beta(\mathbf{a})+1$, every $y\in U_{\mathbf{a}}$ and every $x\in K$,

$$|\widetilde{X}^I P_{\mathbf{a}}(0,s,x) - \widetilde{X}^I P_{\mathbf{a}}(0,s,xy^{-1})| \le \varepsilon.$$

Therefore

$$\omega(s,t,x) \le \varepsilon + 2\|\widetilde{X}^I P_{\mathbf{a}}(0,s)\|_{L^\infty}\int_{U^c_{\mathbf{a}}} P_{\mathbf{a}}(s,t,y)\,dy.$$

But in view of Corollary (4.12) applied to $T_1 = s$, $T_2=\infty$,

$$\int_{U^c} P_{\mathbf{a}}(s,t,y)\,dy \le C\exp\left(-\frac{\eta(\mathbf{a})}{A(s,\infty)^{d_1}+A(s,\infty)^{d_2}}\right),$$

and so

$$\omega(s,t,x) \le \varepsilon + 2\|\widetilde{X}^I P_{\mathbf{a}}(0,s)\|_{L^\infty}\exp\left(-\frac{\eta(\mathbf{a})}{A(s,\infty)^{d_1}+A(s,\infty)^{d_2}}\right).$$

Consequently, $\limsup_{s,t\to\infty}\omega(s,t,x) \le \varepsilon$ for $x\in K$. This proves the existence of $X^I P_{\mathbf{a}}(0,\infty,x)$. Also we see that $X^I P_{\mathbf{a}}(0,t,x)$ converges to $X^I P_{\mathbf{a}}(0,\infty,x)$ as $t\to\infty$, uniformly on compact sets. Therefore, in view of Lemma (4.17), for $A = A(0,\infty)$ we have

$$\begin{aligned}|X^I P_{\mathbf{a}}(0,\infty,x)| &= \lim_{t\to\infty}|X^I P_{\mathbf{a}}(0,t,x)|\\ &\le \lim_{t\to\infty} e^{-Q\log A}|\sigma^r_* X^I P^r_{\mathbf{a}}(0,t,\sigma^{-r}(x))|\\ &\le Ce^{-Q\log A}\|\sigma^{-r}_*\|^{M_1}\\ &\quad\times(1+\xi(\beta,\beta+1)+\eta(\beta,\beta+1))^{M_2}e^{-\varrho(\sigma^{-r}(x))}.\end{aligned} \tag{4.21}$$

Therefore

$$|X^I P_{\mathbf{a}}(0,\infty,x)| \le C_1 e^{-Q\log A}(A^{-d_1}+A^{-d_2})^{M_1} \times (1+\xi(\beta,\beta+1)+\eta(\beta,\beta+1))^M e^{-\varrho(x)/(C_2(A^{d_1}+A^{d_2}))}$$

and the conclusion follows by Lemma (4.9).

5. Estimation of the Poisson kernel. The main goal of this section is to obtain pointwise estimates for the Poisson kernel and its derivatives in terms of the norm defined by (0.20).

(5.1) THEOREM. (a) *There are constants C_1, C_2 such that*

$$C_1(1+|x|)^{-Q-\gamma} \le \nu(x) \le C_2(1+|x|)^{-Q-\gamma}, \quad x\in N.$$

(b) *For $I=(i_1,\dots,i_k)$ and all $X^I = X_1^{(i_1)}\dots X_k^{(i_k)}$, where $X_j^{(i_j)} \in V_j^{i_j}$, with $\|X^I\|\le 1$, there are constants C such that*

$$|X^I\nu(x)| \le C(1+|x|)^{-Q-\gamma-\|I\|}[\log(2+|x|)]^{\|I\|_0},$$

where

$$\|I\| = \sum_{j=1}^k i_j d_j, \qquad d_j = \Re\lambda_j,$$

$$\|I\|_0 = \sum_{j=1}^k i_j D_j, \quad D_j = \dim V_{\lambda_j} - 1.$$

Proof. Let ν^χ be the measure defined by

$$F(\exp\chi H) = \int_N f(x)\,\nu^\chi(x)\,dx.$$

Then, in view of (1.4),

$$\nu^\chi = |\det\sigma_*^{-\chi}|\nu(\sigma^{-\chi}(x)). \tag{5.2}$$

The crucial estimates are:

$$|X^I\nu^\chi(x)| \le Ce^{\gamma\chi} \quad \text{for } |x|=1, \tag{5.3}$$

$$\nu^\chi(x) \ge C_1 e^{\gamma\chi} \quad \text{for } |x|=1. \tag{5.4}$$

Then the conclusion is obtained via a homogeneity argument. Indeed, for y in N by (0.21) there is exactly one χ such that $e^{-\chi}=|y|$ and $y=\sigma^{-\chi}(x)$ with $|x|=1$.

We then apply (0.23) to obtain for $\|X^I\|\le 1$,

$$|X^I\nu(y)| = |\det\sigma_*^\chi|\cdot|\sigma_*^\chi X^I\nu^\chi(y)| \le |\det\sigma_*^\chi|M(\chi)\sup_{\|Y^I\|\le 1}|Y^I\nu^\chi(y)| \le C|\det\sigma_*^\chi|M(\chi)e^{\chi\gamma},$$

where

$$M(\chi) = \exp\Big(\sum_{j=1}^k i_j[d_j\chi + D_j\log(2+|\chi|)]\Big),$$

with $d_j=\Re\lambda_j$ and $D_j=\dim V_{\lambda_j}-1$. Hence, substituting $-\chi=\log|y|$ we obtain (b).

For the lower bound for ν we proceed precisely in the same way using (5.4) and the fact that $\nu>0$.

(5.3) for $I=0$ and (5.4) were proved in [D2] for ad_H diagonal. The proof relied heavily on boundary Harnack inequalities due to Ancona [A], [D2]. It turns out that the space $S=NA$ considered here fits in the framework of Ancona theory precisely in the same way as it did for the diagonal action of ad_H (cf. [D3]). All the proofs given in [D2] adapt easily to our situation and lead to (5.3) for $I=0$ and (5.4). Since Ancona's method is based heavily on potential theory nothing of that works for the derivatives. Therefore, to estimate them we use the evolution $P_{\mathbf{a}}(s,t,x)$. We prove that

$$X^I\nu^\chi(x) = \mathrm{E}_\chi X^I P_{\mathbf{a}}(0,\infty,x) \tag{5.5}$$

and estimate the right side of (5.5) for $|x|\ge 1$. This is done in the rest of this chapter.

Let $d_3=2d_1$, $d_4=2d_2$, $\mathbf{k}=(k_1,k_2,k_3)$, τ be the stopping time defined in Theorem (2.21), and ξ in (2.11). Let

$$\Omega_{\mathbf{k},p} = \{\mathbf{a} : A_{d_p}^{1/d_p} \sim e^{k_1} \wedge A_{d_j}^{1/d_j} \le e^{k_1} \text{ for } j\ne p \wedge \Lambda(\Theta_\tau\mathbf{a}) \sim k_2 \wedge \lambda(\zeta(\Theta_\tau\mathbf{a}),\zeta(\Theta_\tau\mathbf{a})+1)\sim k_3\}$$

for $p=1,2,3,4$. Then by (2.22),

$$\mathrm{P}_\chi(\Omega_{\mathbf{k},p}) \le c_D e^{\gamma\chi}e^{-\gamma k_1}e^{-D|k_1-k_2|-D|k_1-k_3|}. \tag{5.6}$$

(5.7) LEMMA. *Given $r>0$ there is $C=C(r)$ such that for $\varrho(x)\ge r>0$,*

$$\mathrm{E}_\chi|X^I P_{\mathbf{a}}(0,\infty,x)| \le Ce^{\gamma\chi}.$$

Proof. Let $\mathbf{a}\in\Omega_{\mathbf{k},p}$. Then $e^{k_1}\le A\le 4e^{k_1}$. Putting $\sigma(\mathbf{a}) = \zeta(\Theta_\tau\mathbf{a})$, in view of (4.21) we have

$$|X^I P_{\mathbf{a}}(0,\infty,x)| \le C_1 e^{-k_1 Q}(e^{-k_1 d_1}+e^{-k_1 d_2})^{|I|} \times (1+e^{(k_2-k_1)d_1}+e^{(k_2-k_1)d_2}+e^{(k_1-k_3)d_1}+e^{(k_1-k_3)d_2})^M \times \exp\Big(-\frac{\varrho(x)}{C_2(e^{k_1 d_1}+e^{k_1 d_2})}\Big)$$

for every $\mathbf{a}\in\Omega_{\mathbf{k}}$. Let $d=\max(d_1,d_2)$ and $d_0=\min(d_1,d_2)$. Clearly if $k_1\ge 0$, then

$$(5.8)\qquad |X^I P_{\mathbf{a}}(0,\infty,x)| \le C_1 e^{-k_1 Q} e^{Md|k_2-k_1|} e^{Md|k_1-k_3|},$$

and if $k_1 \le 0$, then

$$(5.9)\qquad |X^I P_{\mathbf{a}}(0,\infty,x)| \le C_1 e^{-k_1 Q - k_1 d|I|} e^{Md|k_2-k_1|} e^{Md|k_1-k_3|} \exp\left(-\frac{\varrho(x)}{C_2 e^{k_1 d_0}}\right).$$

Now we are able to estimate

$$\mathrm{E}_\chi |X^I P_{\mathbf{a}}(0,\infty,x)| = \sum_{\mathbf{k},p} \mathrm{E}_\chi 1_{\Omega_{\mathbf{k},p}} |X^I P_{\mathbf{a}}(0,\infty,x)|$$

for x such that $\varrho(x) \ge r$. In view of (5.13), (5.15), (5.16) we have

$$\sum_{\mathbf{k},p} \mathrm{E}_\chi 1_{\Omega_{\mathbf{k},p}} |X^I P_{\mathbf{a}}(0,\infty,x)| \le c_D \sum_{k_1\ge 0, k_2,k_3,p} e^{\gamma\chi} e^{-k_1 Q - \gamma k_1} e^{(Md-D)|k_2-k_1| + (Md-D)|k_1-k_3|}$$
$$+ c_D \sum_{k_1\le 0, k_2,k_3,p} e^{\gamma\chi} e^{-k_1 Q - k_1 d|I| - \gamma k_1} \times e^{(Md-D)|k_2-k_1| + (Md-D)|k_1-k_3|} \exp\left(-\frac{r}{c_2 e^{k_1 d_0}}\right).$$

Hence the conclusion follows.

Let

$$\check{\mu}_t^\chi(U) = \check{\mu}_t(\sigma - \chi U).$$

We have (cf. [T])

$$\pi_N(\check{\mu}_t^\chi)(x) = \int P_{\mathbf{a}}(0,t,x)\, dW_\chi(d\mathbf{a}) = \mathrm{E}_\chi P_{\mathbf{a}}(0,t,x).$$

On the other hand, $\pi_N(\check{\mu}_t^\chi)$ tends *weakly to ν^χ as $t\to\infty$. Indeed,

$$\int_N f(x)\, d\pi_N(\check{\mu}_t^\chi)(x) = \int_S f\circ\pi_N(s)\, d\check{\mu}_t^\chi(s) = \int_S f(\pi_N(e^\chi s))\, d\check{\mu}_t(s)$$
$$= \int_S f(e^\chi \pi_N(s))\, d\check{\mu}_t(s) = \int_N f(e^\chi x)\, d\nu_t(x)$$
$$\to \int_N f(e^\chi x)\, d\nu(x) = \int_N f(x)\, d\nu_t^\chi(x).$$

Therefore,

$$(5.10)\qquad \langle f, X^I \nu^\chi\rangle = \lim_{t\to\infty} \langle X^I f, \pi_N(\check{\mu}_t^\chi)\rangle = \lim_{t\to\infty} \langle X^I f, \mathrm{E}_\chi P_{\mathbf{a}}(0,t)\rangle.$$

In order to prove (5.5) we must pass with t to infinity, i.e. replace $P_{\mathbf{a}}(0,t)$ by $P_{\mathbf{a}}(0,\infty)$ in (5.10). This is included in the following lemma.

(5.11) LEMMA. *For every* $f \in C_c^\infty(N\backslash\{e\})$,

$$\lim_{t\to\infty} \langle X^I f, \mathrm{E}_\chi P_{\mathbf{a}}(0,t)\rangle = \langle f, \mathrm{E}_\chi X^I P_{\mathbf{a}}(0,\infty)\rangle,$$

Proof. First we prove that for $f \in C_c^\infty$,

$$\lim_{t\to\infty} \langle X^I f, \mathrm{E}_\chi P_{\mathbf{a}}(0,t)\rangle = \mathrm{E}_\chi \langle f, X^I P_{\mathbf{a}}(0,\infty)\rangle.$$

Clearly $\langle X^I f, \mathrm{E}_\chi P_{\mathbf{a}}(0,t)\rangle = \mathrm{E}_\chi \langle X^I f, P_{\mathbf{a}}(0,t)\rangle$.

Since $P_{\mathbf{a}}(0,t)$ tends to $P_{\mathbf{a}}(0,\infty)$ uniformly on compact sets (which is shown in the proof of Theorem (4.20)), we have

$$\lim_{t\to\infty} \langle X^I f, P_{\mathbf{a}}(0,t)\rangle = \langle X^I f, P_{\mathbf{a}}(0,\infty)\rangle.$$

But

$$|\langle X^I f, P_{\mathbf{a}}(0,t)\rangle| \le \|X^I f\|_{L^\infty}.$$

Hence (5.12) follows by the Lebesgue bounded convergence theorem.

Since by Lemma (5.7),

$$\langle |f|, \mathrm{E}_\chi |X^I P_{\mathbf{a}}(0,\infty)|\rangle \le C e^{\gamma\chi} |f|_{L^1} < \infty,$$

we have

$$\mathrm{E}_\chi \langle f, X^I P_{\mathbf{a}}(0,\infty)\rangle = \langle f, \mathrm{E}_\chi X^I P_{\mathbf{a}}(0,\infty)\rangle$$

and the conclusion of Lemma (5.11) follows.

Now, (5.10) and Lemma (5.11) imply (5.5), which together with Lemma (5.14) leads to the estimate (5.3).

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Hardy spaces associated with some Schrödinger operators

by

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Abstract. For a Schrödinger operator $A = -\Delta + V$, where V is a nonnegative polynomial, we define a Hardy H^1_A space associated with A. An atomic characterization of H^1_A is shown.

1. Introduction. Let A be a Schrödinger operator on $\mathbb{R}^d$ which has the form

$$A = -\Delta + V, \tag{1.1}$$

where $V(x) = \sum_{\beta\le\alpha} a_\beta x^\beta$ is a nonnegative nonzero polynomial on $\mathbb{R}^d$, $\alpha = (\alpha_1, \ldots, \alpha_d)$.

These operators have attracted attention of a number of authors (cf. [Fe], [HN], [Z]). Recent results of J. Zhong [Z] deal with the Riesz transforms $R_j = \frac{\partial}{\partial x_j} A^{-1/2}$. Among other things it is proved in [Z] that $H^1(\mathbb{R}^d)$ is mapped by R_j into $L^1(\mathbb{R}^d)$. In general, however, this does not characterize $H^1(\mathbb{R}^d)$, i.e. the norm $\|f\|_{L^1} + \sum_{j=1}^d \|R_j f\|_{L^1}$ is not equivalent to the $H^1(\mathbb{R}^d)$ norm.

The operator A, however, gives rise to a perhaps more natural notion of the space H^1_A which is the following. Let $\{T_t\}_{t>0}$ be the semigroup of operators generated by $-A$ (e.g. on $L^2(\mathbb{R}^d)$), $T_t(x,y)$ being their kernels. We notice that, since V is nonnegative, we have

$$0 \le T_t(x,y) \le \widetilde{T}_t(x,y) = (4\pi t)^{-d/2} \exp(-|x-y|^2/(4t)). \tag{1.2}$$

Let

$$\mathcal{M}f(x) = \sup_{t>0} |T_t f(x)|. \tag{1.3}$$

By (1.2), $\mathcal{M}$ is of weak type (1,1). Therefore we may say that a function f is in the Hardy space H^1_A associated with A if

$$\|f\|_{H^1_A} = \|\mathcal{M}f\|_{L^1} < \infty. \tag{1.4}$$

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