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Fixed points of Lipschitzian semigroups in Banach spaces

by

JAROSŁAW GÓRNICKI (Rzeszów)

Abstract. We prove the following theorem: Let $p > 1$ and let E be a real p-uniformly convex Banach space, and C a nonempty bounded closed convex subset of E. If $\mathcal{T} = \{T_s : C \to C : s \in G = [0,\infty)\}$ is a Lipschitzian semigroup such that

$$g = \liminf_{G \ni \alpha \to \infty} \inf_{G \ni \delta \ge 0} \frac{1}{\alpha} \int_0^\alpha \|T_{\beta+\delta}\|^p \, d\beta < 1 + c,$$

where $c > 0$ is some constant, then there exists $x \in C$ such that $T_s x = x$ for all $s \in G$.

1. Introduction. Throughout this paper, E will always stand for a Banach space with norm $\|\cdot\|$. Let C be a nonempty (and, generally, bounded closed convex) subset of E. A mapping $T : C \to C$ is called *Lipschitzian* if there exists a positive number k such that

$$\|Tx - Ty\| \le k\|x - y\| \quad \text{for all } x, y \text{ in } C,$$

and in particular *nonexpansive* in the case $k = 1$. For these mappings we have the well known result (see, for example, [11]):

THEOREM 1 (Browder, Göhde, Kirk, 1965). *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E. If $T : C \to C$ is nonexpansive then it has a fixed point.* ■

On the other hand, it is quite easy to find examples for which fixed point free nonexpansive self-mappings exists. Also, if B is the closed unit ball in ℓ^2, $\varepsilon > 0$, $e_1 = (1, 0, 0, \ldots)$ and S is the right shift operator, then the mapping $T : B \to B$ defined by $Tx = \varepsilon(1 - \|x\|)e_1 + Sx$ is a fixed point free mapping having Lipschitz constant $1 + \varepsilon$. This example shows that Theorem 1 may fail to hold for the class of mappings T having Lipschitz constant $k > 1$, no matter how near to 1 we choose k.

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In 1973, Goebel and Kirk [10] introduced the notion of a *uniformly k-Lipschitzian mapping* ($k \ge 1$) on C:

$$\|T^n x - T^n y\| \le k\|x-y\| \quad \text{for every } x,y \in C \text{ and } n = 1,2,\ldots$$

The fixed point theory for these mappings has a double interest:

(i) uniformly Lipschitzian mappings are a natural generalization of the nonexpansive mappings,

(ii) there is a close connection between these mappings and the stability problem for the fixed point property for nonexpansive mappings (cf. [11]).

It is natural, therefore, to study the existence of fixed points of such mappings.

The first fixed point theorem for uniformly k-Lipschitzian mapping was given by Goebel and Kirk [10] in uniformly convex Banach spaces (cf. [32]). A different and more general approach was proposed by Lifshitz [23] and recently by Domínguez Benavides *et al.* (cf. [4], [5], [6], [14]). In particular, they established the following

THEOREM 2 (Lifshitz, 1975). *Let C be a nonempty bounded closed convex subset of a Hilbert space and suppose $T : C \to C$ is uniformly k-Lipschitzian for $k < \sqrt{2}$. Then T has a fixed point in C.* ■

Lifshitz [23] found an example of a fixed point free uniformly $\pi/2$-Lipschitzian mapping which leaves invariant a bounded closed convex subset of ℓ^2 (cf. [1], [11]). The validity of Theorem 2 for $\sqrt{2} \le k < \pi/2$ remains open.

The existence of a fixed point of a uniformly k-Lipschitzian mapping has been widely investigated by many authors (cf. [1, 3, 4, 5, 6, 16, 23–25, 27–31, 34]). Downing and Ray [7] showed that Lifshitz's Theorem is valid for a uniformly k-Lipschitzian semigroup which is left reversible, and many authors extended this result to more general semigroups (cf. [12, 18–20, 27, 32, 33, 36]).

In 1988, Górnicki and Krüppel [16] indicated some applications of an asymptotic density in fixed point theorems for uniformly k-Lipschitzian mappings and next proved the following (cf. [13, 16, 21]):

THEOREM 3 ([17]). *Let C be a nonempty bounded closed convex subset of a real p-uniformly convex Banach space ($p > 1$). If $T : C \to C$ is a mapping such that*

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \|T^i\|^p < 1 + d,$$

where $d > 0$ is some constant, then T has a fixed point in C (for a Hilbert space $p = 2$ and $d = 1$). ■

The purpose of this paper is to prove some new fixed point theorem for Lipschitzian semigroups in p-uniformly convex Banach spaces.

2. Preliminaries. A family $\mathcal{T} = \{T_s : s \in G = [0,\infty)\}$ of mappings T_s of C into itself is said to be a *Lipschitzian semigroup* on C if it satisfies the following conditions:

(i) $T_{s+h}x = T_s T_h x$ for all $s,h \in G$ and $x \in C$;

(ii) for each $x \in C$, the mapping $s \to T_s x$ from G into C is continuous;

(iii) $\|T_s x - T_s y\| \le \|T_s\| \cdot \|x-y\|$ for all $x,y \in C$ and $s \in G$, where

$$\|T_s\| = \sup\{\|T_s x - T_s y\|/\|x-y\| : x \ne y,\ x,y \in C\} < \infty$$

for every $s \in G$.

Let $p > 1$, $\lambda \in [0,1]$ and $W_p(\lambda) = \lambda(1-\lambda)^p + \lambda^p(1-\lambda)$.

The functional $\|\cdot\|^p$ is said to be *uniformly convex* [34] on the Banach space E if

(*) there exists a positive constant c_p such that for all $\lambda \in [0,1]$ and $x,y \in E$,

$$\|\lambda x + (1-\lambda)y\|^p \le \lambda\|x\|^p + (1-\lambda)\|y\|^p - c_p W_p(\lambda)\|x-y\|^p.$$

Xu [34] proved that $\|\cdot\|^p$ is uniformly convex on E if and only if E is p-*uniformly convex*, i.e. there exists a constant $c > 0$ such that the modulus of convexity of E (see [11]) satisfies $\delta_E(\varepsilon) \ge c\varepsilon^p$ for all $0 \le \varepsilon \le 2$. We note that a Hilbert space $\mathcal{H}$ is 2-uniformly convex (indeed, $\delta_{\mathcal{H}}(\varepsilon) = 1 - \sqrt{1-(\varepsilon/2)^2} \ge (1/8)\varepsilon^2$) and an L^p space ($1 < p < \infty$) is $\max(2,p)$-uniformly convex.

Now, we establish an existence lemma in p-uniformly convex Banach spaces.

LEMMA 1. *Let $p > 1$, let E be a real p-uniformly convex Banach space and C a nonempty closed convex subset of E, and let $\{x_\alpha\}_{\alpha\in G} \subset E$ be bounded. Then there exists a unique point z in C such that*

$$\limsup_{G\ni\alpha\to\infty} \frac{1}{\alpha}\int_0^\alpha \|z - x_\beta\|^p\,d\beta \le \limsup_{G\ni\alpha\to\infty} \frac{1}{\alpha}\int_0^\alpha \|x - x_\beta\|^p\,d\beta - c_p\|x-z\|^p$$

for every x in C, where $c_p > 0$ is the constant given in (*).

Proof. *Continuity.* For any fixed $\beta \in G$, $x,y \in C$, we have

$$\|x - x_\beta\|^p \le (\|y - x_\beta\| + \|x-y\|)^p.$$

Let $a = \|y - x_\beta\|$ and $b = \|x-y\|$. Then by the Mean Value Theorem, $(a+b)^p = a^p + bp\zeta^{p-1}$ for some $\zeta \in (a, a+b)$. Since $\zeta < a + b < 2d$, where

$$d = \max\{\operatorname{diam}(C), \sup\{\|z - x_s\| : z \in C,\ s \in G\}\},$$

we get

$$\|x - x_\beta\|^p \le \|y - x_\beta\|^p + \|x - y\|p(2d)^{p-1}.$$

Hence

$$r(x) \le r(y) + \|x - y\|p(2d)^{p-1},$$

where

$$r(\xi) = \limsup_{G\ni\alpha\to\infty} \frac{1}{\alpha}\int_0^\alpha \|\xi - x_\beta\|^p\, d\beta, \qquad \xi \in C.$$

Changing the roles of x and y, we get

$$|r(x) - r(y)| \le \|x - y\|p(2d)^{p-1}.$$

Uniform convexity. By the inequality $(*)$, we have

$$\begin{aligned}\|(1-\lambda)x + \lambda y - x_\beta\|^p &= \|(1-\lambda)(x - x_\beta) + \lambda(y - x_\beta)\|^p\\ &\le (1-\lambda)\|(x - x_\beta)\|^p + \lambda\|(y - x_\beta)\|^p - c_p W_p(\lambda)\|x - y\|^p,\end{aligned}$$

hence

(1) $$r((1-\lambda)x + \lambda y) \le (1-\lambda)r(x) + \lambda r(y) - c_p W_p(\lambda)\|x - y\|^p.$$

Thus the functional $r(\cdot)$ is uniformly convex [35] on E and by Lemma 2 of [17], there is a unique point $z \in C$ (called the *asymptotic center* of $\{x_\alpha\}_{\alpha\in G}$ in C) such that

$$r(z) = \inf\{r(\xi) : \xi \in C\}.$$

It follows from inequality (1) that

$$r(z + \lambda(x - z)) \le \lambda r(x) + (1-\lambda)r(z) - c_p W_p(\lambda)\|x - y\|^p$$

for x in C and $0 \le \lambda \le 1$. Noticing that $r(z) \le r(z + \lambda(x - z))$ for x in C and $0 \le \lambda \le 1$, we derive that

$$0 \le \lim_{\lambda\to 0_+} \frac{r(z + \lambda(x - z)) - r(z)}{\lambda} \le r(x) - r(z) - c_p\|x - y\|^p$$

and the desired inequality follows. ■

3. Main result. In this section we prove an existence theorem for fixed points of mappings with Lipschitzian iterates by means of techniques of asymptotic centers and inequalities in real Banach spaces.

THEOREM 4. *Let $p > 1$ and let E be a real p-uniformly convex Banach space, and C a nonempty bounded closed convex subset of E. If $\mathcal{T} = \{T_s : C \to C : s \in G = [0,\infty)\}$ is a Lipschitzian semigroup such that*

$$g = \liminf_{G\ni\alpha\to\infty} \inf_{G\ni\delta\ge 0} \frac{1}{\alpha}\int_0^\alpha \|T_{\beta+\delta}\|^p\, d\beta < 1 + c_p,$$

then there exists $x_0 \in C$ such that $T_s x_0 = x_0$ for all $s \in G$.

Proof. Let $\{\alpha_\gamma\} \subset G$ ($\alpha_\gamma \to \infty$ as $\gamma \to \infty$) and $\{\delta_\gamma\} \subset G$ be such that

$$\liminf_{G\ni\alpha\to\infty} \inf_{G\ni\delta\ge 0} \frac{1}{\alpha}\int_0^\alpha \|T_{\beta+\delta}\|^p\, d\beta = \lim_{\gamma\to\infty} \frac{1}{\alpha_\gamma}\int_0^{\alpha_\gamma} \|T_{\beta+\delta_\gamma}\|^p\, d\beta = g < 1 + c_p.$$

By Lemma 1 and induction, we define a sequence $\{x_m\}_{m=1}^\infty$ in C in the following manner:

$$\begin{cases} x_1 \in C & \text{arbitrary}\\ x_{m+1} = z(x_m), & m = 1, 2, \ldots, \end{cases}$$

where $z(x_m)$ is the unique point in C that minimizes the functional

$$r_m(x) = \limsup_{\gamma\to\infty} \frac{1}{\alpha_\gamma}\int_0^{\alpha_\gamma} \|x - T_{\beta+\delta_\gamma} x_m\|^p\, d\beta$$

over x in C. In view of inequality $(*)$, for any $s \in G$, $\beta + \delta_\gamma \in G$ and $0 \le \lambda \le 1$, we have

$$\begin{aligned}\|\lambda x_m + (1-\lambda)T_s x_m - T_{\beta+\delta_\gamma} x_{m-1}\|^p &= \|\lambda(x_m - T_{\beta+\delta_\gamma} x_{m-1}) + (1-\lambda)(T_s x_m - T_{\beta+\delta_\gamma} x_{m-1})\|^p\\ &\le \lambda\|x_m - T_{\beta+\delta_\gamma} x_{m-1}\|^p + (1-\lambda)\|T_s x_m - T_{\beta+\delta_\gamma} x_{m-1}\|^p\\ &\quad - c_p W_p(\lambda)\|x_m - T_s x_m\|^p\end{aligned}$$

and

$$\begin{aligned}\limsup_{\gamma\to\infty} \frac{1}{\alpha_\gamma}\int_0^{\alpha_\gamma} \|\lambda x_m + (1-\lambda)T_s x_m - T_{\beta+\delta_\gamma} x_{m-1}\|^p\, d\beta &\le \lambda \limsup_{\gamma\to\infty} \frac{1}{\alpha_\gamma}\int_0^{\alpha_\gamma} \|x_m - T_{\beta+\delta_\gamma} x_{m-1}\|^p\, d\beta\\ &\quad + (1-\lambda)\limsup_{\gamma\to\infty} \frac{1}{\alpha_\gamma}\int_0^{\alpha_\gamma} \|T_s x_m - T_{\beta+\delta_\gamma} x_{m-1}\|^p\, d\beta\\ &\quad - c_p W_p(\lambda)\|x_m - T_s x_m\|^p.\end{aligned}$$

Hence

$$\begin{aligned}
c_pW_p(\lambda)\|x_m - T_sx_m\|^p \\
&\le \lambda \limsup_{\gamma\to\infty} \frac{1}{\alpha_\gamma}\int_0^{\alpha_\gamma} \|x_m - T_{\beta+\delta_\gamma}x_{m-1}\|^p\,d\beta \\
&\quad + (1-\lambda)\limsup_{\gamma\to\infty} \frac{1}{\alpha_\gamma}\int_0^{\alpha_\gamma} \|T_sx_m - T_{\beta+\delta_\gamma}x_{m-1}\|^p\,d\beta \\
&\quad - \limsup_{\gamma\to\infty} \frac{1}{\alpha_\gamma}\int_0^{\alpha_\gamma} \|\lambda x_m + (1-\lambda)T_sx_m - T_{\beta+\delta_\gamma}x_{m-1}\|^p\,d\beta \\
&\le (1-\lambda)\limsup_{\gamma\to\infty} \frac{1}{\alpha_\gamma}\int_0^{\alpha_\gamma} \|T_sx_m - T_{\beta+\delta_\gamma}x_{m-1}\|^p\,d\beta \\
&\quad - (1-\lambda)\limsup_{\gamma\to\infty} \frac{1}{\alpha_\gamma}\int_0^{\alpha_\gamma} \|x_m - T_{\beta+\delta_\gamma}x_{m-1}\|^p\,d\beta.
\end{aligned}$$

Dividing by $1-\lambda$ and taking $\lambda = 1$, we get

$$\begin{aligned}
c_p\|x_m - T_sx_m\|^p &\le \limsup_{\gamma\to\infty} \frac{1}{\alpha_\gamma}\int_0^{\alpha_\gamma} \|T_sx_m - T_{\beta+\delta_\gamma}x_{m-1}\|^p\,d\beta \\
&\quad - \limsup_{\gamma\to\infty} \frac{1}{\alpha_\gamma}\int_0^{\alpha_\gamma} \|x_m - T_{\beta+\delta_\gamma}x_{m-1}\|^p\,d\beta \\
&\le \limsup_{\gamma\to\infty}\Big\{\frac{1}{\alpha_\gamma}\int_0^{s} \|T_sx_m - T_{\beta+\delta_\gamma}x_{m-1}\|^p\,d\beta \\
&\quad + \|T_s\|^p\frac{1}{\alpha_\gamma}\int_s^{\alpha_\gamma} \|x_m - T_{(\beta-s)+\delta_\gamma}x_{m-1}\|^p\,d\beta\Big\} \\
&\quad - \limsup_{\gamma\to\infty} \frac{1}{\alpha_\gamma}\int_0^{\alpha_\gamma} \|x_m - T_{\beta+\delta_\gamma}x_{m-1}\|^p\,d\beta \\
&\le \limsup_{\gamma\to\infty}\Big\{\frac{1}{\alpha_\gamma}\int_0^{s} \|T_sx_m - T_{\beta+\delta_\gamma}x_{m-1}\|^p\,d\beta \\
&\quad + \|T_s\|^p\Big(\frac{1}{\alpha_\gamma}\int_0^{\alpha_\gamma} \|x_m - T_{\beta+\delta_\gamma}x_{m-1}\|^p\,d\beta \\
&\quad - \frac{1}{\alpha_\gamma}\int_{\alpha_\gamma-s}^{\alpha_\gamma} \|x_m - T_{\beta+\delta_\gamma}x_{m-1}\|^p\,d\beta\Big)\Big\}
\end{aligned}$$

$$\begin{aligned}
&\quad - \limsup_{\gamma\to\infty} \frac{1}{\alpha_\gamma}\int_0^{\alpha_\gamma} \|x_m - T_{\beta+\delta_\gamma}x_{m-1}\|^p\,d\beta \\
&\le (\|T_s\|^p - 1)\limsup_{\gamma\to\infty} \frac{1}{\alpha_\gamma}\int_0^{\alpha_\gamma} \|x_m - T_{\beta+\delta_\gamma}x_{m-1}\|^p\,d\beta \\
&\le (\|T_s\|^p - 1)r_{m-1}(x_m) \le (\|T_s\|^p - 1)r_{m-1}(x_{m-1}),
\end{aligned}$$

since

$$\frac{1}{\alpha_\gamma}\int_0^{s} \|T_sx_m - T_{\beta+\delta_\gamma}x_{m-1}\|^p\,d\beta \to 0 \qquad \text{as } \gamma\to\infty,$$

$$\frac{1}{\alpha_\gamma}\int_{\alpha_\gamma-s}^{\alpha_\gamma} \|x_m - T_{\beta+\delta_\gamma}x_{m-1}\|^p\,d\beta \to 0 \qquad \text{as } \gamma\to\infty,$$

$$r_{m-1}(x_m) \le r_{m-1}(x_{m-1}).$$

Therefore for any $s\in G$, we obtain the estimate

$$c_p\|x_m - T_sx_m\|^p \le (\|T_s\|^p - 1)r_{m-1}(x_{m-1})$$

and as a consequence

$$\begin{aligned}
c_p \limsup_{\gamma\to\infty} \frac{1}{\alpha_\gamma}\int_0^{\alpha_\gamma} \|x_m - T_{\beta+\delta_\gamma}x_m\|^p\,d\beta \\
\le \Big(\lim_{\gamma\to\infty} \frac{1}{\alpha_\gamma}\int_0^{\alpha_\gamma} \|T_{\beta+\delta_\gamma}\|^p\,d\beta - 1\Big)r_{m-1}(x_{m-1})
\end{aligned}$$

and

$$r_m(x_m) \le Br_{m-1}(x_{m-1})$$

where

$$B = \frac{1}{c_p}\Big(\lim_{\gamma\to\infty} \frac{1}{\alpha_\gamma}\int_0^{\alpha_\gamma} \|T_{\beta+\delta_\gamma}\|^p\,d\beta - 1\Big) < 1.$$

In a similar way, we get

$$r_m(x_m) \le B^{m-1}r_1(x_1), \qquad m = 1,2,\ldots$$

Now we show the convergence of the sequence $\{x_m\}_{m=1}^{\infty}$. For a fixed $s\in G$, by Jensen's inequality (cf. [22, p. 183]), we have

$$\|x_{m+1} - x_m\|^p \le 2^{p-1}(\|x_{m+1} - T_sx_m\|^p + \|T_sx_m - x_m\|^p),$$

and hence

$$\|x_{m+1}-x_m\|^p \le 2^{p-1}\Big(\limsup_{\gamma\to\infty}\frac{1}{\alpha_\gamma}\int_0^{\alpha_\gamma}\|x_{m+1}-T_{\beta+\delta_\gamma}x_m\|^p\,d\beta + \limsup_{\gamma\to\infty}\frac{1}{\alpha_\gamma}\int_0^{\alpha_\gamma}\|T_{\beta+\delta_\gamma}x_m-x_m\|^p\,d\beta\Big)$$
$$\le 2^{p-1}(r_m(x_{m+1})+r_m(x_m)) \le 2^p r_m(x_m) \le 2^p B^{m-1} r_1(x_1)\to 0$$

as $m\to\infty$. This shows that $\{x_m\}_{m=1}^\infty$ is norm Cauchy and strong convergent. Let $x_0=\lim_{m\to\infty}x_m$. Then for $\beta+\delta_\gamma\in G$, by Jensen's inequality, we have

$$\|x_0-T_{\beta+\delta_\gamma}x_0\|^p \le (\|x_0-x_m\|+\|x_m-T_{\beta+\delta_\gamma}x_m\|+\|T_{\beta+\delta_\gamma}x_m-T_{\beta+\delta_\gamma}x_0\|)^p$$
$$\le 3^{p-1}(\|x_0-x_m\|^p+\|x_m-T_{\beta+\delta_\gamma}x_m\|^p+\|T_{\beta+\delta_\gamma}x_m-T_{\beta+\delta_\gamma}x_0\|^p).$$

Thus

$$\limsup_{\gamma\to\infty}\frac{1}{\alpha_\gamma}\int_0^{\alpha_\gamma}\|x_0-T_{\beta+\delta_\gamma}x_0\|^p\,d\beta \le 3^{p-1}\Big(\|x_0-x_m\|^p+\limsup_{\gamma\to\infty}\frac{1}{\alpha_\gamma}\int_0^{\alpha_\gamma}\|x_m-T_{\beta+\delta_\gamma}x_m\|^p\,d\beta + \|x_m-x_0\|^p\lim_{\gamma\to\infty}\frac{1}{\alpha_\gamma}\int_0^{\alpha_\gamma}\|T_{\beta+\delta_\gamma}\|^p\,d\beta\Big)$$
$$\le 3^{p-1}((1+g)\|x_m-x_0\|^p+B^{m-1}r_1(x_1))\to 0$$

as $m\to\infty$. Therefore

$$r(x_0)=\limsup_{\gamma\to\infty}\frac{1}{\alpha_\gamma}\int_0^{\alpha_\gamma}\|x_0-T_{\beta+\delta_\gamma}x_0\|^p\,d\beta=0.$$

This implies that for any $s\in G$ and for any $\varepsilon>0$ there exists $s_\varepsilon\in G$ such that

$$\|x_0-T_{s_\varepsilon}x_0\|<\varepsilon \quad\text{and}\quad \|x_0-T_{s+s_\varepsilon}x_0\|<\varepsilon. \tag{2}$$

Otherwise, we have

$$\exists_{s\in G}\,\exists_{\varepsilon>0}\,\forall_{s_\varepsilon\in G}\quad (\|x_0-T_{s_\varepsilon}x_0\|>\varepsilon\vee\|x_0-T_{s+s_\varepsilon}x_0\|>\varepsilon)$$

and

$$r(x_0)=\limsup_{\gamma\to\infty}\frac{1}{\alpha_\gamma}\int_0^{\alpha_\gamma}\|x_0-T_{\beta+\delta_\gamma}x_0\|^p\,d\beta \ge \liminf_{\gamma\to\infty}\frac{1}{\alpha_\gamma}\int_0^{\alpha_\gamma}\|x_0-T_{\beta+\delta_\gamma}x_0\|^p\,d\beta \ge \liminf_{\alpha\to\infty}\frac{1}{\alpha}\int_0^{\alpha}\|x_0-T_{\beta+\delta_\gamma}x_0\|^p\,d\beta\ge\varepsilon^p.$$

Let $s\in G$ and $\varepsilon>0$. Then by (2),

$$\|x_0-T_sx_0\|\le\|x_0-T_{s+s_\varepsilon}x_0\|+\|T_{s+s_\varepsilon}x_0-T_sx_0\|$$
$$\le\|x_0-T_{s+s_\varepsilon}x_0\|+\|T_s\|\cdot\|T_{s_\varepsilon}x_0-x_0\|\le\varepsilon+\|T_s\|\varepsilon\to 0$$

as $\varepsilon\to 0_+$. This completes the proof. ∎

4. The corollaries. In this section we give applications of the established inequalities analogous to $(*)$ in Banach spaces. We begin with the following.

LEMMA 2. (a) *In a Hilbert space $\mathcal{H}$ we have the identity*

$$\|\lambda x+(1-\lambda)y\|^2=\lambda\|x\|^2+(1-\lambda)\|y\|^2-\lambda(1-\lambda)\|x-y\|^2$$

for all x,y in $\mathcal{H}$ and $\lambda\in[0,1]$.

(b) *If $1<p\le 2$, then for all x,y in L^p and $\lambda\in[0,1]$, we have*

$$\|\lambda x+(1-\lambda)y\|^2\le\lambda\|x\|^2+(1-\lambda)\|y\|^2-\lambda(1-\lambda)(p-1)\|x-y\|^2$$

(see [25], [31]).

(c) *Assume $2<p<\infty$ and t_p is the unique zero of the function $g(x)=-x^{p-1}+(p-1)x+p-2$ in $(1,\infty)$. Let*

$$c_p=(p-1)(1+t_p)^{2-p}=\frac{1+t_p^{p-1}}{(1+t_p)^{p-1}}. \tag{3}$$

Then

$$\|\lambda x+(1-\lambda)y\|^p\le\lambda\|x\|^p+(1-\lambda)\|y\|^p-c_pW_p(\lambda)\|x-y\|^p$$

for all x,y in L^p and $\lambda\in[0,1]$ (see [24], [25], [34]). ∎

Remark. All constants appearing in the inequalities of Lemma 2 (e.g., $p-1$ and c_p) are the best possible [24, 25].

COROLLARY 1 ([15]). *Let C be a nonempty bounded closed convex subset of a Hilbert space. If $\mathcal{T}=\{T_s:C\to C:s\in G\}$ is a Lipschitzian semigroup* *such that*

$$g = \liminf_{G\ni\alpha\to\infty} \inf_{G\ni\delta\ge 0} \frac{1}{\alpha}\int_0^\alpha \|T_{\beta+\delta}\|^2\,d\beta < 2,$$

then there exists $x_0 \in C$ *such that* $T_s x_0 = x_0$ *for all* $s \in G$. ■

COROLLARY 2. *Let* C *be a nonempty bounded closed convex subset of real* L^p $(1 < p \le 2)$. *If* $\mathcal{T} = \{T_s : C \to C : s \in G\}$ *is a Lipschitzian semigroup such that*

$$g = \liminf_{G\ni\alpha\to\infty} \inf_{G\ni\delta\ge 0} \frac{1}{\alpha}\int_0^\alpha \|T_{\beta+\delta}\|^2\,d\beta < p,$$

then there exists $x_0 \in C$ *such that* $T_s x_0 = x_0$ *for all* $s \in G$. ■

COROLLARY 3. *Let* C *be a nonempty bounded closed convex subset of real* L^p $(2 < p < \infty)$. *If* $\mathcal{T} = \{T_s : C \to C : s \in G\}$ *is a Lipschitzian semigroup such that*

$$g = \liminf_{G\ni\alpha\to\infty} \inf_{G\ni\delta\ge 0} \frac{1}{\alpha}\int_0^\alpha \|T_{\beta+\delta}\|^p\,d\beta < 1 + c_p,$$

where $c_p > 0$ *is the constant given by* (3), *then there exists* $x_0 \in C$ *such that* $T_s x_0 = x_0$ *for all* $s \in G$. ■

Using the result of Prus–Smarzewski [28, 30] and Xu [34] we can obtain from Theorem 4 a fixed point theorem, for example, for Hardy and Sobolev spaces.

Let $H^p, 1 < p < \infty$, denote the Hardy space [9] of all functions x analytic in the unit disc $|z| < 1$ of the complex plane and such that

$$\|x\| = \lim_{r\to 1_-}\left(\frac{1}{2\pi}\int_0^{2\pi} |x(re^{i\Theta})|^p\,d\Theta\right)^{1/p} < \infty.$$

Now, let Ω be an open subset of $\mathbb{R}^n$. Denote by $W^{k,p}(\Omega)$, with $k \ge 0$, $1 < p < \infty$, the Sobolev space [2, p. 149] of distributions x such that $D^\alpha x \in L^p(\Omega)$ for all $|\alpha| = \alpha_1 + \ldots + \alpha_n \le k$ equipped with the norm

$$\|x\| = \Big(\sum_{|\alpha|\le k}\int_\Omega |D^\alpha x(\omega)|^p\,d\omega\Big)^{1/p}.$$

Let $(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$, $\alpha \in \Lambda$, be a sequence of positive measure spaces, where Λ is finite or countable. Given a sequence of linear subspaces X_α in $L^p(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$, we denote by $L_{q,p}$, $1 < p < \infty$, $q = \max(2,p)$, the linear space (see [26]) of all sequences

$$x = \{x_\alpha \in X_\alpha : \alpha \in \Lambda\}$$

equipped with the norm

$$\|x\| = \Big[\sum_{\alpha\in\Lambda}(\|x_\alpha\|_{p,\alpha})^q\Big]^{1/q},$$

where $\|\cdot\|_{p,\alpha}$ denotes the norm in $L^p(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$.

Finally, let $L^p = L^p(S_1, \Sigma_1, \mu_1)$ and $L^q = L^q(S_2, \Sigma_2, \mu_2)$, where $1 < p < \infty$, $q = \max(2,p)$ and (S_i, Σ_i, μ_i) are positive measure spaces. Denote by $L_q(L_p)$ the Banach space [8, III.2.10] of all measurable L^p-valued functions x on S_2 such that

$$\|x\| = \Big(\int_{S_2} (\|x(s)\|_p)^q\,\mu_2(ds)\Big)^{1/q}.$$

These spaces are q-uniformly convex with $q = \max(2,p)$ (see [28, 30]) and their norms satisfy

$$\|\lambda x + (1-\lambda)y\|^q \le \lambda\|x\|^q + (1-\lambda)\|y\|^q - dW_q(\lambda)\|x-y\|^q$$

with a constant

$$d = d_p = \frac{p-1}{8} \quad \text{for } 1 < p \le 2,$$

$$d = d_p = \frac{1}{p\cdot 2^p} \quad \text{for } 2 < p < \infty.$$

Hence Theorem 4 yields the following:

COROLLARY 4. *Let* C *be a nonempty bounded closed convex subset of the space* X, *where* $X = H^p$ *or* $X = W^{k,p}(\Omega)$ *or* $X = L_{q,p}$ *or* $X = L_q(L_p)$ *and* $1 < p < \infty$, $q = \max(2,p)$, $k \ge 0$. *If* $\mathcal{T} = \{T_s : C \to C : s \in G\}$ *is a Lipschitzian semigroup such that*

$$g = \liminf_{G\ni\alpha\to\infty} \inf_{G\ni\delta\ge 0} \frac{1}{\alpha}\int_0^\alpha \|T_{\beta+\delta}\|^q\,d\beta < 1 + d,$$

then there exists $x_0 \in C$ *such that* $T_s x_0 = x_0$ *for all* $s \in G$. ■

5. Final remarks. 1. If $G = \mathbb{N}_0$, the set of nonnegative integers, then

$$\frac{1}{\alpha}\int_0^\alpha \Phi(t)\,dt = \frac{1}{n+1}\sum_{k=0}^n \Phi(k),$$

where $n = [\alpha]$, $\alpha \in [0,\infty)$. In this setting, Theorem 4 is the natural generalization of the corresponding result of [17].

2. Recently, Krüppel [21] established the following inequality in all uniformly convex Banach spaces: for all $\|x\| \le 1$, $\|y\| \le 1$, $0 \le \lambda \le 1$,

$$\|\lambda x + (1-\lambda)y\|^p \le \lambda\|x\|^p + (1-\lambda)\|y\|^p - 2\sum_{n=0}^\infty \phi(2^n\lambda)\delta_p\Big(\frac{\|x-y\|}{2^n}\Big),$$

where $\phi : [0,\infty) \to [0,1/2]$ is the periodic function with period 1 such that $\phi(t) = \min\{t, 1-t\}$, $0 \le t \le 1$, and

$$\delta_p(\varepsilon) = \inf\left\{\frac{1}{2}(\|x\|^p + \|y\|^p) - \left\|\frac{1}{2}(x+y)\right\|^p : \|x\| \le 1,\ \|y\| \le 1,\ \|x-y\| \ge \varepsilon\right\}$$

is the p-modulus of convexity of E, $1 < p < \infty$. By means of that inequality and the method described in [21] one can prove Theorem 4 in uniformly convex Banach spaces.

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