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Minimal pairs of bounded closed convex sets

by

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Abstract. The existence of a minimal element in every equivalence class of pairs of bounded closed convex sets in a reflexive locally convex topological vector space is proved. An example of a non-reflexive Banach space with an equivalence class containing no minimal element is presented.

Let $X=(X,\tau)$ be a topological vector space over the field $\mathbb{R}$. Let $\mathcal{B}_\tau(X)$ (resp. $\mathcal{K}_\tau(X)$) be the collection of all bounded closed (resp. compact) convex subsets of X. For $A,B\subset X$, let

$$A+B:=\{a+b\mid a\in A,\ b\in B\}$$

and let $A\dot{+}B$ denote the closure of $A+B$. For $(A,B),(C,D)\in\mathcal{B}_\tau^2(X)$, let $(A,B)\sim(C,D)$ if and only if $A\dot{+}D=B\dot{+}C$. Let $(A,B)\le(C,D)$ if and only if $A\subset C$, $B\subset D$ and $(A,B)\sim(C,D)$. The relation " $\sim$ " is an equivalence relation by the ordered law of cancellation [5] in $\mathcal{B}_\tau^2(X)$ and " $\le$ " is an ordering in the equivalence class $[A,B]$ of any pair (A,B).

The study of minimal pairs of compact convex sets was stimulated by the development of quasidifferential calculus [1]. Any given quasidifferential may be identified with the equivalence class of a pair of compact convex sets (A,B), where A and B are, respectively, a super- and a sub-differential.

The existence of minimal pairs of compact convex sets in all topological vector spaces and the uniqueness up to translates in $\mathbb{R}^2$ were already proved in [2] and [4].

In this paper we extend our investigations to pairs of bounded closed convex sets.

THEOREM. *Let (X,τ) be a reflexive locally convex topological vector space. Every class $[A,B]\in\mathcal{B}_\tau^2(X)/{\sim}$ contains a minimal element (C,D) such that $(C,D)\le(A,B)$.*

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Proof. In the case of finite-dimensional vector spaces, bounded closed sets are compact, and our theorem follows from [3]. Denote by τ^* the weak topology in X. Every bounded closed set $A \in \mathcal{B}_\tau(X)$ is compact in the topology τ^* and, consequently, belongs to $\mathcal{K}_{\tau^*}(X)$. On the other hand, every τ^*-compact set is closed in τ^* and in τ (since $\tau^* \subset \tau$). Take any $(A, B) \in \mathcal{B}^2_\tau(X) \subset \mathcal{K}^2_{\tau^*}(X)$. Then

$$A + B \in \mathcal{K}_{\tau^*}(X) \quad \text{and} \quad A \dot{+} B \in \mathcal{B}_\tau(X).$$

Then $A + B$ is τ-closed, convex and contained in $A \dot{+} B$, which is bounded. Therefore, $A+B \in \mathcal{B}_\tau(X)$ and, in consequence, $A\dot{+}B = A+B$ in all reflexive vector spaces. Hence $[A, B] \subset [A, B]^* \in \mathcal{K}^2_{\tau^*}(X)/\sim$, where $[A, B]^*$ is the equivalence class of pairs of compact sets in the topology τ^* as defined in [3]. According to the Theorem of [3], the equivalence class $[A, B]_{\tau^*}$ contains a minimal element $(C, D) \in \mathcal{K}^2_{\tau^*}(X)$ such that $C \subset A$ and $D \subset B$. Since C, D are τ-closed, convex and contained in bounded sets, it follows that $(C, D) \in \mathcal{B}^2_\tau(X)$. Moreover, $(C, D) \in [A, B] \subset [A, B]^*$. Therefore, (C, D) is a minimal element in $[A, B]$ and, of course, $(C, D) \le (A, B)$. ∎

Let c_0 be the space of all infinite sequences $a = (a_n)$ of real numbers such that $\lim_n a_n = 0$. Let $\|a\| = \sup_n |a_n| = \max_n |a_n|$ be the norm in c_0. The space c_0 is a non-reflexive Banach space.

EXAMPLE. Let U be the unit ball in c_0. Let $A := \{a \in U \mid a_n \ge 0 \text{ for all } n \in \mathbb{N}\}$ and $B := -A$. Then $A + B = U$. Let $A_m := \{a \in A \mid a_1 = \ldots = a_m = 1/2\}$ and $B_m := -A_m$, where $m \in \mathbb{N}$. Then $(A_m, B_m) \in \mathcal{B}^2(c_0)$ for all $n \in \mathbb{N}$. Moreover, $A + B_m = A_m + B$. Thus (A_m, B_m) is a decreasing chain of pairs in $[A, B]$, i.e.

$$(A, B) \ge (A_1, B_1) \ge \ldots \ge (A_m, B_m) \ge \ldots$$

with empty intersection, i.e. $\bigcap_m A_m = \bigcap_m B_m = \emptyset$. In the proof of existence of minimal pairs of compact convex sets, it was essential that the intersection of a decreasing chain of pairs of compact convex sets is non-empty.

Let m be the space of all bounded sequences of real numbers with the norm $\|a\| = \sup_n |a_n|$. Let $c = \{a \in m \mid \lim_n a_n \text{ exists in } \mathbb{R}\}$. Of course, $c_0 \subset c \subset m$.

THEOREM 2. *Let $X = c_0, c$, or m. There exists a class $[A, B] \in \mathcal{B}^2(X)/\sim$ that contains no minimal elements.*

Proof. Let A and B be the sets defined in the Example. Let $p : X \to \mathbb{R}$ be defined by $p(a) := a_1$. For $E \in \mathcal{B}(X)$ and $\alpha \in p(E)$ let

$$E_\alpha := \{a \in E \mid p(a) = \alpha\}.$$

Take any $(C, D) \in [A, B]$. We shall prove that (C, D) is not a minimal element.

Case 1: $\operatorname{card} p(C) > 1$. Since $p(C)$ is an interval, $\operatorname{int} p(C)$ is non-empty. Fix $\alpha \in \operatorname{int} p(C)$. Let $c \in C_\alpha$, $b \in B_0$ (according to the definition of E_α, $B_0 = \{a \in B \mid p(a) = 0\}$) and $\varepsilon > 0$. There exist $c', c'' \in C$ such that $\|c' - c\|$, $\|c'' - c\| < \varepsilon/4$ and $p(c') < \alpha < p(c'')$. Let $\delta = \min(p(c'') - \alpha, \alpha - p(c'))$. Let $e = (1, 0, \ldots, 0, \ldots) \in X$. Since $B \dot{+} C = A \dot{+} D$, we have $(b - e) + c' \in B + C \subset A \dot{+} D$. Then there exist $a' \in A$ and $d' \in D$ such that

$$\|b - e + c' - a' - d'\| < \delta.$$

Hence

$$\delta > |p(b - e + c' - a' - d')| = |0 - 1 + p(c') - p(a') - p(d')|.$$

Then

$$p(d') < p(c') - p(a') - 1 + \delta \le \alpha - \delta - 0 - 1 + \delta = \alpha - 1.$$

Similarly, $b + c'' \in B + C \subset A \dot{+} D$, and there exist $a'' \in A$ and $d'' \in D$ such that

$$\|b + c'' - a'' - d''\| < \delta.$$

Then $|0 + p(c'') - p(a'') - p(d'')| < \delta$, and

$$p(d'') > p(c'') - p(a'') - \delta \ge \alpha + \delta - 1 - \delta + \alpha - 1.$$

Since $p(d') < \alpha - 1 < p(d'')$, there exist $\beta, \gamma > 0$ with $\beta + \gamma = 1$ such that $\beta p(d') + \gamma p(d'') = \alpha - 1$. Let $d''' = \beta d' + \gamma d''$, $a''' = \beta a' + \gamma a''$ and $c''' = \beta c' + \gamma c''$. Since the sets A, C, D are convex we have $a''' \in A$, $c''' \in C$ and $d''' \in D$. The inequalities $\|b-e+c'-a'-d'\| < \delta$ and $\|b+c''-a''-d''\| < \delta$ imply that $\|b-\beta e+c'''-a'''-d'''\| < \delta$. Hence $|0-\beta+p(c''')-p(a''')-\alpha+1| < \delta$, and

$$|\beta + p(a''') - 1| < |p(c''') - \alpha| + \delta \le \varepsilon/4 + \delta < \varepsilon/2.$$

Now notice that

$$\begin{aligned}
&\|b + c - a''' - (1 - p(a'''))e - d'''\| \\
&\quad = \|b + c - c''' + c''' - \beta e + \beta e - a''' - (1 - p(a'''))e - d'''\| \\
&\quad \le \|b - \beta e + c''' - a''' - a''' - d'''\| + \|c - c'''\| + |\beta - 1 + p(a''')| \\
&\quad < \delta + \varepsilon/4 + \varepsilon/2 < \varepsilon.
\end{aligned}$$

We have $a''' + (1 - p(a'''))e \in A_1$ and $d''' \in D_{\alpha-1}$. Since ε may be arbitrarily small, it follows that $b+c \in A_1 \dot{+} D_{\alpha-1}$. We have just proved that $B_0 + C_\alpha \subset A_1 \dot{+} D_{\alpha-1}$. Since $\alpha - 1 \in \operatorname{int} p(D)$, we can prove in the same way that $A_1 + D_{\alpha-1} \subset B_0 \dot{+} C_\alpha$. Therefore, $(C_\alpha, D_{\alpha-1}) \sim (A_1, B_0) \sim (A, B) \sim (C, D)$. Thus $(C_\alpha, D_{\alpha-1}) \le (C, D)$ and $(C_\alpha, D_{\alpha-1}) \ne (C, D)$, and the pair (C, D) is not minimal.

Case 2: $\operatorname{card} p_n(C) > 1$ for some $n \in \mathbb{N}$ where $p_n : X \to \mathbb{R}$ is defined by $p_n := a_n$. The proof is the same as in Case 1.

Case 3: $\operatorname{card} p_n = 1$ for all $n \in \mathbb{N}$. Then $\operatorname{card} C = 1$. Let $C = \{c = (c_n)\}$. Hence

$$[0,1] + p_n(D) \subset p_n(A \dotplus D) = p_n(B \dotplus C) = p_n(B + c) = [c_n - 1, c_n].$$

Thus $p_n(D) = \{c_n - 1\}$. Then for any $d \in D$ we have $d_n = c_n - 1$, and so for any $a \in A$ and $b \in B$ we obtain

$$\|a + d - b - c\| \geq \limsup_n |a_n + d_n - b_n - c_n| = \limsup_n |0 + c_n - 1 - 0 - c_n| = 1.$$

Since a, b, c, d are arbitrary elements of A, B, C and D we get $A \dotplus D \neq B \dotplus C$. Therefore, this case is impossible. ■

Remark. Let A and B be the subsets of l^1 defined in the same way as A and B in c_0 (see Example). A and B belong to $\mathcal{B}(l^1)$. Let $(C, D) \in \mathcal{B}^2(l^1), (C, D) \leq (A, B)$. Let $e_i = (0, \ldots, 0, 1, 0, \ldots)$, $i = 1, 2, \ldots$ Let $p_i : l^1 \to \mathbb{R}^2$, $p_i(a_n) = (a_1, a_i)$ for $i = 2, 3, \ldots$ Notice that

$$(\overline{p_i(C)}, \overline{p_i(D)}) \leq (p_i(A), p_i(B))$$

and that $(p_i(A), p_i(B))$ is a pair of closed triangles in $\mathbb{R}^2$. Since $p_i(B) = -p_i(A)$, the pair $(p_i(A), p_i(B))$ is a pair of convex compact sets in $\mathbb{R}^2$. Then

$$\overline{p_i(C)} = p_i(A) \quad \text{and} \quad \overline{p_i(D)} = p_i(B).$$

Since $(0,1) \in \overline{p_i(C)}$ and $a = e_i$ is the only element of A such that $p_i(a) = (0,1)$, and C is a closed set contained in A, we conclude that $e_i \in C$. In a similar way $e_1 \in C$.

Now, let $q : l^1 \to \mathbb{R}$, $q(a_n) = \sum_{n=1}^{\infty} a_n$. Again

$$(q(C), q(D)) \leq (q(A), q(B)) = ([0,1], [-1,0]).$$

We know that $1 = q(e_i) \in C$ and $-1 \in q(D)$. Then $0 \in q(C)$. Since $a = 0 = (0, \ldots, 0, \ldots)$ is the only element of A such that $q(a) = 0$, we get $0 \in C$. Since $A = \operatorname{conv}(\{0\} \cup \{e_i \mid i = 1, 2, \ldots\})$, it follows that $C = A$. Similarly $D = B$. Therefore, (A, B) is a minimal element of $[A, B]$.

Notice that $A_i = \emptyset$ for $i = 3, 4, \ldots$ (see Example).

The following theorem is a simple generalization of a result of [6]:

THEOREM 3. *Let $X = (X, \tau)$ be an infinite-dimensional topological vector space. Let $\mathcal{M}$ be the collection of all minimal pairs in $\mathcal{B}^2_\tau(X)$. Let $\mathcal{NM} := \mathcal{B}^2_\tau(X) \setminus \mathcal{M}$. Then $\operatorname{card} \mathcal{M} = \operatorname{card} \mathcal{NM}$.*

In conclusion, we ask the following question: Does there exist in l^1 or, generally, in every non-reflexive locally convex topological vector space, an equivalence class $[A, B] \in \mathcal{B}^2_\tau(X)/\sim$ containing no minimal elements?

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