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Two-weight norm inequalities for maximal functions on homogeneous spaces and boundary estimates

by

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Abstract. Let D be an open subset of a homogeneous space (X,d,μ). Consider the maximal function

$$M_\varphi f(x)=\sup\frac{1}{\varphi(B)}\int\limits_{B\cap\partial D}|f|\,d\nu,\qquad x\in D,$$

where the supremum is taken over all balls of the form $B=B(a(x),r)$ with $r>t(x)=d(x,\partial D)$, $a(x)\in\partial D$ is such that $d(a(x),x)<\frac{3}{2}t(x)$ and φ is a nonnegative set function defined for all Borel sets of X satisfying the quasi-monotonicity and doubling properties. We give a necessary and sufficient condition on the weights w and v for the weighted norm inequality

$$\Big(\int\limits_D[M_\varphi(f)]^q w\,d\mu\Big)^{1/q}\le c\Big(\int\limits_{\partial D}|f|^p v\,d\nu\Big)^{1/p}\tag{0.1}$$

to hold when $1<p<q<\infty$, $\sigma d\nu=v^{1-p'}d\nu$ is a doubling weight, and $d\nu$ is a doubling measure, and give a sufficient condition for (0.1) when $1<p\le q<\infty$ without assuming that σ is a doubling weight but with an extra assumption on φ. Another characterization for (0.1) is also provided for $1<p\le q<\infty$ and D of the form $Y\times(0,\infty)$, where Y is a homogeneous space with group structure. These results generalize some known theorems in the case when M_φ is the fractional maximal function in $\mathbb{R}^{n+1}_+$, that is, when

$$M_\varphi f(x,t)=M_\gamma f(x,t)=\sup_{r>t}\frac{1}{\nu(B(x,r))^{1-\gamma}}\int\limits_{B(x,r)}|f|\,d\nu,$$

where $(x,t)\in\mathbb{R}^{n+1}_+$, $0<\gamma<1$, and ν is a doubling measure in $\mathbb{R}^n$.

1. Introduction. We consider a homogeneous space (X,d,μ) where $d:X\times X\to[0,\infty)$ satisfies:

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1. $d(x,y)=0$ if and only if $x=y$.
2. $d(x,y)=d(y,x)$ for any x,y in X.
3. There exists $A_0\ge 1$ such that

$$d(x,y)\le A_0(d(x,z)+d(z,y))\quad\text{for all } x,y,z \text{ in } X. \tag{1.1}$$

μ is a measure on the Borel subsets of X with the *doubling property*: there exists a positive constant A_1 such that

$$\mu(B(x,2r))\le A_1\mu(B(x,r))\quad\text{for all } x \text{ in } X \text{ and } r>0, \tag{1.2}$$

where $B(x,r)=\{y\in X: d(x,y)<r\}$ is the ball of radius r with center x. If $B=B(x,r)$ and $\lambda>0$, we will write λB for $B(x,\lambda r)$.

As mentioned in [C], Macias and Segovia (cf. [MS]) have proved that given a quasi-metric d on X there exists a quasi-metric d' on X satisfying:

1. $C^{-1}d'(x,y)\le d(x,y)\le Cd'(x,y)$ for all x,y in X, where C is a positive constant independent of x and y.

2. The balls with respect to d', i.e., $B'(x,r)=\{y\in X: d'(y,x)<r\}$, $x\in X$, $r>0$, are open.

Since only the order of magnitude of $d(x,y)$ will be relevant for us, we may assume that the balls defined by d are open. We will also assume that all annuli $A(x,r,R)=B(x,R)\setminus B(x,r)$ in X are nonempty for all $0<r<R$. We immediately see that there exists $c>0$ such that the diameter δ of $B(x,r)$ satisfies

$$\delta(B(x,r))=\sup\{d(y,z): y,z\in B(x,r)\}\ge cr$$

for all x in X and $r>0$. In fact, since $A=A(x,\frac12 r,r)$ is nonempty, there exists y in A. Thus, $r>d(x,y)\ge\frac12 r$. Therefore, $\delta(B(x,r))\ge d(x,y)\ge\frac12 r$.

Now, for any z,y in $B(x,r)$ we have $d(z,y)\le A_0(d(z,x)+d(x,y))\le 2A_0r$. Thus,

$$\tfrac12 r\le\delta(B(x,r))\le 2A_0r,$$

and we say that the diameter of $B(x,r)$ is *equivalent* to r.

Let D be an open subset of X and $c\ge 1$. For each x in X let $t(x)=d(x,\partial D)$ and select $a(x)$ in ∂D such that $d(x,a(x))\le ct(x)$. Note that if $c'\ge 1$ and $a'(x)$ is in ∂D and satisfies $d(x,a'(x))\le c't(x)$ then there exists $K>1$ depending only on A_0, c and c' such that

$$B(a(x),r)\subset B(a'(x),Kr)\subset B(a(x),K^2r)\qquad \text{for all } x\in X \text{ and } r>t(x). \tag{1.3}$$

Denote by $\mathfrak{B}$ the set of all balls $B=B(z,r)$ with z in ∂D and $r>0$. Suppose φ is a nonnegative set function which is defined for all Borel sets of X and satisfies the following conditions:

(1) (*quasi-monotonicity*) There exists $c_0>0$ such that $\varphi(B')\le c_0\varphi(B)$ whenever $B'\subset B$, $B',B\in\mathfrak{B}$.

(2) (*doubling*) There exists $c_0>0$ such that $\varphi(B(z,2r))\le c_0\varphi(B(z,r))$ for all z in ∂D and $r>0$.

Observe that (1) and (2) imply that for all $\gamma>0$ there exists $c(\gamma)>0$ such that

(3) $\varphi(B(z,\gamma r))\le c(\gamma)\varphi(B(z,r))$ for all z in ∂D and $r>0$.

For φ, $t(x)$ and $a(x)$ as above, define

$$M_{\varphi,a}f(x)=\sup_{r>t(x)}\frac{1}{\varphi(B(a(x),r))}\int_{\check{B}}|f|\,d\nu \tag{1.4}$$

where $\check{B}(a(x),r)=B(a(x),r)\cap\partial D$, $B(a(x),r)=\{y\in X: d(y,a(x))<r\}$ and ν is a doubling measure on ∂D.

Note that if $c,c'\ge 1$ and $a(x),a'(x)$ are as above then it follows from (1.3) and the properties of φ that there exists $C>0$ depending only on A_0,c_0,c and c' such that $C^{-1}M_{\varphi,a'}f\le M_{\varphi,a}f\le CM_{\varphi,a'}f$ for all f. For this reason we will fix $c=3/2$ and write $M_\varphi f$ instead of $M_{\varphi,a}f$.

Observe that when $X=Y\times\mathbb{R}$, (Y,ν,ϱ) is a homogeneous space, $D=Y\times(0,\infty)=\widehat{Y}_+$ and $\varphi(E)=|E\cap\partial\widehat{Y}_+|_\nu^{1-\gamma}$, where $|\cdot|_\nu$ denotes the ν-measure and $0\le\gamma<1$, then

$$\begin{aligned} M_\varphi f(x,t)&=M_\gamma f(x,t)\\ &=\sup_{r>t}\frac{1}{|B(x,r)|_\nu^{1-\gamma}}\int_{B(x,r)}|f(y)|\,d\nu(y),\qquad (x,t)\in\widehat{Y}_+,\end{aligned} \tag{1.5}$$

where $B(x,r)=\{y\in Y:\varrho(x,y)<r\}$ and $r>0$, is the fractional maximal function. In [SWZ], Sawyer, Wheeden and Zhao showed that if $1<p<q<\infty$, $w(x,t)$ and $v(y)$ are weights in $\widehat{Y}_+$ and Y, respectively, and $\sigma(y)d\nu(y)=v(y)^{1-p'}d\nu(y)$ (with $p'=p/(p-1)$) and $d\nu(y)$ satisfy the doubling condition and μ is a Borel measure in $\widehat{Y}_+$, then the weighted norm inequality

$$\Big(\int_{\widehat{Y}_+}M_\gamma f(x,t)^q w(x,t)\,d\mu(x,t)\Big)^{1/q}\le c\Big(\int_Y|f(y)|^p v(y)\,d\nu(y)\Big)^{1/p} \tag{1.6}$$

holds if and only if there exists a positive constant C such that

$$\Big(\int_B d\nu\Big)^{1-\gamma}\Big(\int_{\widehat{B}}w\,d\mu\Big)^{1/q}\Big(\int_B\sigma\,d\nu\Big)^{1/p'}\le C \tag{1.7}$$

for all balls $B\subset Y$, where $\widehat{B}=B\times[0,r(B))$ and $r(B)$ is the radius of B. The first result we have for M_φ is

THEOREM 1.1. *Suppose* $1 < p < q < \infty$. *Let* w *and* v *be weights defined on* D *and* ∂D, *respectively. Suppose that* $\sigma d\nu = v^{1-p'} d\nu$ *is a doubling measure. Then the weighted norm inequality*

$$(1.8)\qquad \Big(\int_{D} |M_\varphi f|^q w\, d\mu\Big)^{1/q} \le c\Big(\int_{\partial D} |f|^p v\, d\nu\Big)^{1/p}$$

holds for all f, *with* c *independent of* f, *if and only if*

$$(1.9)\qquad \varphi(B)^{-1}\Big(\int_{\widehat{B}} w\, d\mu\Big)^{1/q}\Big(\int_{\breve{B}} \sigma\, d\nu\Big)^{1/p'} \le C$$

for all balls $B = B(z,r)$ *with* z *in* ∂D, $r > 0$, C *independent of* B, *and* $\widehat{B} = B \cap D$, $\breve{B} = B \cap \partial D$.

In the next theorem we present a condition implying that the weighted norm inequality

$$(1.10)\qquad \Big(\int_{D} [M_\varphi f]^q w\, d\mu\Big)^{1/q} \le c\Big(\int_{\partial D} |f|^p v\, d\nu\Big)^{1/p},$$

holds for all f, where $1 < p \le q < \infty$, without imposing the doubling condition on σ but with an additional condition on φ. The theorem is stated in terms of dyadic cubes (see Definition 2.11) centered at points of the boundary of a fixed open subset D of a homogeneous space. For a similar result for off-centered maximal operators, see [W2].

THEOREM 1.2. *Suppose* $1 < p \le q < \infty$ *and let* w *and* v *be weights defined on* D *and* ∂D, *respectively. Let* $\sigma = v^{1-p'}$. *Suppose that there exist* $C > 0$ *and* $0 < \varepsilon \le 1$ *such that*

$$\Big(\frac{|Q'|_\nu}{|Q|_\nu}\Big)^{\varepsilon} \le c\frac{\varphi(Q'^*)}{\varphi(Q^*)},$$

for all dyadic cubes Q' *and* Q *such that* $Q' \subset Q$, *where* Q^* *is the exterior ball associated with* Q. *Suppose also that there exists* $r > 1$ *such that*

$$(1.11)\qquad \varphi(Q^*)|\breve{Q}|_\nu^{1/p'}|3A_0^2\widehat{Q}^*|_w^{1/q}\Big(\frac{1}{|\breve{Q}|_\nu}\int_{\breve{Q}} \sigma^r\, d\nu\Big)^{1/rp'} \le C$$

for all dyadic cubes Q *centered at points of* ∂D, *where* Q^* *is the exterior ball associated with* Q, $\breve{Q} = Q \cap \partial D$ *and* $3A_0^2\widehat{Q}^* = (3A_0^2 Q^*) \cap D$. *Then there exists* $c > 0$ *such that*

$$(1.12)\qquad \Big(\int_{D} |M_\varphi f|^q w\, d\mu\Big)^{1/q} \le c\Big(\int_{\partial D} |f|^p v\, d\nu\Big)^{1/p}$$

for all f.

The next result is a characterization of weighted norm inequalities of type (p,q) with $1 < p \le q < \infty$, when D has the form $X \times (0,\infty)$, where (X,d,ν) is a homogeneous space. We will suppose additional conditions on X, d and ν. Let (X,d,ν) be a homogeneous space and define $\widehat{X} = X \times \mathbb{R}$ and $\widehat{X}_+ = X \times (0,\infty)$. Assume that X admits a group structure (not necessarily commutative) such that for all $x, y, z \in X$ and all balls $B \subset X$ we have

$$(1.13)\qquad d(x+z, y+z) = d(x,y),$$

$$(1.14)\qquad d(0,x) = d(0,-x)$$

where 0 is the identity element of X,

$$(1.15)\qquad \nu(-B+x) = \nu(-B)$$

where $-B = \{x : -x \in B\}$, and

$$(1.16)\qquad \nu(B) = \nu(-B).$$

Consider the following maximal function:

$$(1.17)\qquad M_\varphi f(x,t) = \sup_{r>t} \frac{1}{\varphi(B(x,r))}\int_{B(x,r)} |f|\, d\nu, \qquad (x,t) \in \widehat{X}_+,$$

where $B(x,r) \subset X$ and φ has the (quasi-)monotonicity and doubling properties and is defined for all Borel sets of X. The following theorem characterizes (1.10) with M_φ as in (1.17), and is based on the main theorem of [S1].

THEOREM 1.3. *Let* (X,d,ν) *be a homogeneous space, where* X *has a group structure and* d *and* ν *satisfy conditions* (1.13)–(1.16). *Let* w *and* v *be weights in* $\widehat{X}_+$ *and* X, *respectively, and suppose that* $1 < p \le q < \infty$. *Then*

$$(1.18)\qquad \Big(\int_{\widehat{X}_+} [M_\varphi f]^q w\, d\mu\Big)^{1/q} \le C\Big(\int_{X} |f|^p v\, d\nu\Big)^{1/p}$$

for all f *if and only if*

$$(1.19)\qquad \Big(\int_{\widehat{Q}+z} [M_\varphi(\sigma\chi_{Q+z})]^q w\, d\mu\Big)^{1/q} \le C\Big(\int_{Q+z} \sigma\, d\nu\Big)^{1/p}$$

for all dyadic cubes $Q \subset X$ *and* z *in* X, *where* $\widehat{Q} = Q \times l(Q)$ *and* $l(Q)$ *is the edgelength of* Q.

The proofs of the above theorems can be found in Section 3.

2. Dyadic balls and dyadic cubes. In this section we will adapt the constructions of dyadic balls and dyadic cubes in homogeneous spaces as presented in [SW2] to obtain corresponding families of balls and cubes whose centers lie on the boundary of a fixed open subset of X. For a slightly different concept of dyadic cubes in homogeneous spaces, see [C]. We will also prove some lemmas that will be used in the next section.

LEMMA 2.1. *Let* (X,d,μ) *be a separable homogeneous space, i.e.,* X *has a countable dense subset. Let* D *be an open subset of* X *and* ∂D *be its boundary. Suppose* $\{B_\alpha = B(z_\alpha, r_\alpha)\}_{\alpha\in A}$ *is a family of balls with* $z_\alpha\in\partial D$ *and* $B_\alpha\subset B(z,r)$ *for some* $z\in\partial D$ *and* $r>0$. *Then there exists a countable subcollection* $\{B_i\}_{i\in I}$ *of these balls such that*

1. $B_i\cap B_j=\emptyset$ *if* $i\neq j$.
2. *Given* $\alpha\in A$ *there exists* $i\in I$ *such that* $B_\alpha\subset(A_0+4A_0^2)B_i$.
3. $\mu(\bigcup_{\alpha\in A}B_\alpha)\le c\sum_{i\in I}\mu(B_i)$ *where* c *depends only on the positive constants* A_0 *and* A_1, *as in* (1.1) *and* (1.2), *respectively.*

The proof is the same as that of Lemma 3.3 in [SW2] and will be omitted.

As in [SW2], p. 843, we now construct a sequence of balls in X centered at points of ∂D.

DEFINITION 2.2. Set $\lambda=A_0(2A_0+1)$, where A_0 is as in (1.1). For each integer k select a sequence $\{z_j^k\}_j$ of points of ∂D such that the sequence of balls $\widetilde{B}_j^k = B(z_j^k,\lambda^{k-1})$ is maximal with respect to the property that $\widetilde{B}_i^k\cap\widetilde{B}_j^k=\emptyset$ if $i\neq j$. Set $B_j^k=\lambda\widetilde{B}_j^k$. We refer to $\{B_j^k\}_{k,j}$ as *the collection of dyadic balls*.

Note that if, for the same $\lambda=A_0(2A_0+1)$, we had chosen another sequence of points $\{z_j'^k\}_j$ of ∂D satisfying the same maximality condition above with the corresponding collection of dyadic balls $\{B_j'^k\}_{k,j}$ then for each k and j there would be i such that

$$\lambda^{-1}B_j'^k\subset B_i^k\subset\lambda B_j'^k.$$

This readily implies that there exists $c>1$ such that for all k and j, $c^{-1}\varphi(B_j'^k)\le\varphi(B_i^k)\le c\varphi(B_j'^k)$ for some i.

For dyadic balls we have:

(2.3) Given z in ∂D and an integer k there exists j such that $B(z,\lambda^{k-1})\subset B_j^k$.

(2.4) $\sum_j\chi_{B_j^k}\le M$ for some $M>0$ depending only on A_0 and A_1.

(2.5) $\widetilde{B}_i^k\cap\widetilde{B}_j^k=\emptyset$ for $i\neq j$, $k\in\mathbb{Z}$.

The statement (2.3) follows from the maximality of $\{\widetilde{B}_j^k\}$. Note that if $x\in B_j^k$, $1\le j\le N$, then $\bigcup_j B_j^k\subset B(x,2A_0\lambda^k)$. Since the B_j^k are pairwise disjoint in j, we obtain

$$N\mu(B(x,2A_0\lambda^k))\le c_\lambda\sum_{j=1}^{N}\mu(\widetilde{B}_j^k)=c_\lambda\mu\Big(\bigcup_{j=1}^{N}\widetilde{B}_j^k\Big)\le c_\lambda\mu(B(x,2A_0\lambda^k)).$$

Hence, $N\le c_\lambda$ and this proves (2.4). Finally, (2.5) follows trivially from the definition of B_j^k.

LEMMA 2.6. *Suppose* $\mathcal{B}=\{B_\alpha\}_{\alpha\in A}$ *is a family of dyadic balls as above. If* $\{B_j\}_{j\in J}$ *is a collection of maximal (with respect to inclusion) balls in* $\mathcal{B}$, *then the balls* $\{\widetilde{B}_j\}_{j\in J}$ *are pairwise disjoint.*

Proof. We know that $\widetilde{B}_i^k\cap\widetilde{B}_j^k=\emptyset$ if $i\neq j$. Suppose there exists $x\in\widetilde{B}_j^k\cap\widetilde{B}_i^l$, where $l<k$. Then $B_i^l\subset B_j^k$. In fact, if $w\in B_i^l$ then

$$\begin{aligned}d(z_j^k,w)&\le A_0[d(z_j^k,x)+A_0[d(x,z_i^l)+d(z_i^l,w)]]\\&\le A_0[\lambda^{k-1}+A_0(\lambda^{k-1}+\lambda^k)]\le A_0[\lambda^{k-1}+2A_0\lambda^{k-1}]=\lambda^k.\end{aligned}$$

The result now follows from the maximality of the balls. ■

The following lemma is a version of Lemma 3.21 of [SW2] that is suitable for boundary estimates.

LEMMA 2.7. *Suppose* (X,d) *is a separable quasi-metric space and* $D\subset X$ *is an open subset. Then there exists* $\lambda>1$ *depending only on* A_0 *such that for every* $m\in\mathbb{Z}$, *there are points* $z_j^k\in\partial D$ *and Borel sets* E_j^k, $1\le j<n_k$, $k\ge m$, *where* $n_k\in\mathbb{N}\cup\{\infty\}$, *such that*

(2.8) $B(z_j^k,\lambda^k)\cap\partial D\subset E_j^k\subset B(z_j^k,\lambda^{k+1}),\quad 1\le j<n_k,\ k\ge m,$

(2.9) $\partial D=\bigcup_j E_j^k\cap\partial D$ *for all* $k\ge m$, $\quad E_j^k\cap E_i^k=\emptyset$ *if* $i\neq j$,

and given i,j,k,l *with* $m\le k<l$, *we have either*

(2.10) $E_j^k\subset E_i^l$ *or* $E_j^k\cap E_i^l=\emptyset$.

DEFINITION 2.11. For each $m\in\mathbb{Z}$ let $\mathcal{D}_m=\{E_j^k : k\ge m,\ 1\le j<n_k\}$. We refer to the elements of $\mathcal{D}_m$ as *dyadic cubes*.

DEFINITION 2.12. Let $Q=E_j^k$ in $\mathcal{D}_m$ and write $\overline{Q}=B(z_j^k,\lambda^k)\cap\partial D$ and $Q^*=B(z_j^k,\lambda^{k+1})$. We will refer to $\overline{Q}$ and Q^* as the *inner* and *outer balls*, respectively, *associated with* Q. The diameter of Q will be called the *edgelength* of Q and will be denoted by $l(Q)$.

Note that since any annulus is nonempty, $\lambda^k\le$ edgelength of $Q\le\lambda^{k+1}$.

Proof of Lemma 2.7. We will only define the sets E_j^k and the rest of the proof will be omitted for it is essentially the same as that of Lemma 3.21 in [SW2].

Set $\lambda=8A_0^5$. For each $k\in\mathbb{Z}$ choose a sequence $\{z_j^k\}_{1\le j<n_k}$ of points on ∂D maximal with respect to the property that the balls $\{B(z_j^k,3A_0^2\lambda^k)\}$, $1\le j<n_k$, are pairwise disjoint. Notice that since X is separable, the cardinality of $\{z_j^k\}$ is at most countable.

Fix $m\in\mathbb{Z}$ and define

$$E_1^m=B(z_1^m,6A_0^3\lambda^m)-\bigcup_{i\neq 1}B(z_i^m,\lambda^m)$$

and for $j>1$,

$$E_j^m=B(z_j^m,6A_0^3\lambda^m)-\bigcup_{i\neq j}B(z_i^m,\lambda^m)-\bigcup_{i<j}E_i^m.$$

Observe that $E_j^m\cap E_i^m=\emptyset$ if $i\neq j$, $B(z_j^m,\lambda^m)\subset E_j^m\subset B(z_j^m,\lambda^{m+1})$ and $\partial D=\bigcup_j E_j^m\cap\partial D$.

We now define $\{E_j^k\}_j$ for $k\geq m$. Suppose $\{E_j^l\}_{1\leq j<n_l}$ has been defined for $l=m,m+1,\dots,k$ for some $k\geq m$ so as to satisfy

$$B(z_j^l,\lambda^l)\cap\partial D\subset E_j^l\subset B(z_j^l,\lambda^{l+1}),\quad m\leq l\leq k,\ 1\leq j<n_l, \tag{2.13}$$

$$\partial D=\bigcup_j E_j^l\cap\partial D,\qquad E_j^l\cap E_i^l=\emptyset\quad\text{if } i\neq j, \tag{2.14}$$

and for $m\leq l_1<l_2\leq k$ and any i,j we have either

$$E_j^{l_2}\cap E_i^{l_1}=\emptyset\quad\text{or}\quad E_i^{l_1}\subset E_j^{l_2}. \tag{2.15}$$

For any $R>0$ and $1\leq j<n_{k+1}$, let

$$\widetilde{B}(z_j^{k+1},R)=\bigcup\{E_i^k: E_i^k\cap B(z_i^{k+1},R)\neq\emptyset\}.$$

Define

$$E_1^{k+1}=\widetilde{B}(z_1^{k+1},6A_0^3\lambda^{k+1})-\bigcup_{i\neq 1}\widetilde{B}(z_i^{k+1},\lambda^{k+1})$$

and for $j>1$,

$$E_j^{k+1}=\widetilde{B}(z_j^{k+1},6A_0^3\lambda^{k+1})-\bigcup_{i\neq j}\widetilde{B}(z_i^{k+1},\lambda^{k+1})-\bigcup_{i<j}E_i^{k+1}.$$

The next lemma, which is an application of the Marcinkiewicz interpolation theorem, is analogous to Lemma 3.15 of [SW2].

LEMMA 2.16. *Let $D\subset X$ be an open subset and ∂D its boundary. Suppose $a(B)$ is a nonnegative set function, defined for all balls B in $\mathfrak{B}$, that satisfies*

$$a(B(z,2r))\leq ca(B(z,r))\qquad\text{for all } z\in\partial D \text{ and } r>0 \tag{2.17}$$

and

$$\sum_{B\in\Omega}a(B)\leq ca(B_0) \tag{2.18}$$

whenever Ω is a collection of pairwise disjoint balls $B=B(z,r)$ in $\mathfrak{B}$ contained in a ball $B_0=B(z_0,r_0)$ of $\mathfrak{B}$. In addition, assume that $u(z)\geq 0$ on ∂D, $\beta\geq 1$ and Γ is a countable collection of dyadic balls centered in ∂D such that

$$\int_{\check B}u\,d\nu\leq ca(B)\qquad\text{for all } B\in\Gamma, \tag{2.19}$$

where $\check B=B\cap\partial D$ and $d\nu$ is a doubling measure on ∂D, i.e.,

$$|B(z,2r)\cap\partial D|_\nu\leq c|B(z,r)\cap\partial D|_\nu\qquad\text{for all } z\in\partial D \text{ and } r>0.$$

Furthermore, assume that

$$\sum_{\substack{B\in\Gamma\\ B\subset B_0}}a(B)^\beta\leq ca(B_0)^\beta\qquad\text{for all } B_0\in\Gamma. \tag{2.20}$$

Then

$$\Big(\sum_{B\in\Gamma}a(B)^\beta\Big(\frac{1}{a(B)}\int_{\check B}fu\,d\nu\Big)^t\Big)^{1/t}\leq c_{s,t}\Big(\int_{\partial D}f^s u\,d\nu\Big)^{1/s} \tag{2.21}$$

for all $f\geq 0$ on ∂D and $t=s\beta$, $1<s<\infty$.

Proof. Consider the map

$$f\mapsto\Big\{\frac{1}{a(B)}\int_{\check B}fu\,d\nu\Big\}_{B\in\Gamma}.$$

For $f\in L^\infty(\partial D,ud\nu)$, we have

$$\Big|\frac{1}{a(B)}\int_{\check B}fu\,d\nu\Big|\leq\|f\|_\infty\frac{1}{a(B)}\int_{\check B}u\,d\nu\leq c\|f\|_\infty\qquad\text{by (2.19)}.$$

Now, suppose $f:\partial D\to[0,\infty)$ is bounded with support contained in $B_1\cap\partial D$, where $B_1=B(z_1,r_1)$, $z_1\in\partial D$. Let Γ' be a finite subset of Γ. For $\lambda>0$, let $\{Q_j\}_{j\in\mathcal{J}'}$ be the maximal dyadic balls B in Γ' such that

$$\frac{1}{a(B)}\int_{\check B}fu\,d\nu>\lambda. \tag{2.22}$$

Note that $B\cap B_1\neq\emptyset$ for any ball B that satisfies (2.22) since the support of f is contained in $B_1\cap\partial D$. Thus,

$$\begin{aligned}\sum_{\substack{B\in\Gamma'\\ (1/a(B))\int_{\check B}fu\,d\nu>\lambda}}a(B)^\beta&\leq\sum_{j\in\mathcal{J}'}\sum_{B\subset Q_j}a(B)^\beta\\&\leq c\sum_{j\in\mathcal{J}'}a(Q_j)^\beta\qquad\text{by (2.20)}\end{aligned} \tag{2.23}$$

$$(2.24)\qquad \le c\Big(\sum_{j\in\mathcal{J}'} a(Q_j)\Big)^{\beta} \qquad \text{since } \beta\ge 1$$

$$(2.25)\qquad \le c\Big(\sum_{j\in\mathcal{J}'} a(\widetilde{Q}_j)\Big)^{\beta} \qquad \text{by (2.17)}$$

where $\widetilde{Q}_j$ denotes the ball concentric with Q_j and with radius $r(Q_j)/(A_0(2A_0+1))$. Recall that the $\widetilde{Q}_j$ are pairwise disjoint (see Lemma 2.6).

Now, since $\mathcal{J}'$ is finite, $\bigcup_{i\in\mathcal{J}'}\widetilde{Q}_j\subset B'$ for some ball B' centered at ∂D. Therefore, by Lemma 2.1, there exists a pairwise disjoint subcollection $\{Q_i\}_{i\in\mathcal{I}'}$ of the dyadic balls $\{Q_j\}_{j\in\mathcal{J}'}$ such that every Q_j is contained in some Q_i^*, $i\in\mathcal{I}'$, where Q_i^* is the ball concentric with Q_i and with radius $A_0(4A_0+1)r(Q_i)$. Thus,

$$\sum_{j\in\mathcal{J}'} a(\widetilde{Q}_j)\le\sum_{i\in\mathcal{I}'}\sum_{\substack{j\in\mathcal{J}'\\ Q_j\subset Q_i^*}} a(\widetilde{Q}_j)\le c\sum_{i\in\mathcal{I}'} a(Q_i^*) \qquad \text{by (2.18) and Lemma 2.6}$$
$$\le c\sum_{i\in\mathcal{I}'} a(Q_i) \qquad \text{by (2.17)}$$
$$\le \frac{c}{\lambda}\sum_{i\in\mathcal{I}'}\int_{\check{Q}_i} fu\,d\nu\le\frac{c}{\lambda}\int_{\partial D} fu\,d\nu,$$

since $\{Q_i\}_{i\in\mathcal{I}'}$ is a collection of pairwise disjoint balls. Thus, since Γ' was an arbitrary finite subset of Γ and the constant c above depends only on A_0 and A_1, we conclude that the map

$$f\mapsto\Big\{\frac{1}{a(B)}\int_{\check{B}} fu\,d\nu\Big\}_{B\in\Gamma}$$

is both of weak-type (∞,∞) and weak-type $(1,\beta)$, i.e., it takes $L^1(\partial D, ud\nu)$ to weak $l^{\beta}(\Gamma, a(B)^{\beta})$. Therefore, applying the Marcinkiewicz interpolation theorem we obtain (2.21). ∎

We will need the following analogue of Lemma 2.10 in [SW2]:

LEMMA 2.26. *Let $u(x)\ge 0$ on ∂D and for each $m\in\mathbb{Z}$, let $\{Q_i\}_i$ be a countable collection of dyadic cubes in $\mathcal{D}_m$. Suppose that there exists a sequence $\{a_i\}_{i\in I}$ of positive numbers and $\beta\ge 1$ satisfying*

$$(2.27)\qquad \int_{\check{Q}_i} u\,d\nu\le ca_i,$$

$$(2.28)\qquad \sum_{j:Q_j\subset Q_i} a_j^{\beta}\le ca_i^{\beta},$$

where $c>0$ is independent of i and m. Then

$$\Big(\sum_{i\in I} a_i^{\beta}\Big(\frac{1}{a_i}\int_{\check{Q}_i} fu\,d\nu\Big)^{q}\Big)^{1/q}\le c\Big(\int_{\partial D} f^p u\,d\nu\Big)^{1/p}$$

for all $f\ge 0$ on ∂D where $1<p<\infty$, $q=\beta p$ and c is independent of m.

Proof. The map $f\mapsto\{(1/a_i)\int_{\check{Q}_i} fu\,d\nu\}$ takes $L_u^{\infty}(\partial D)$ into $l^{\infty}_{\{a_i^{\beta}\}}(I)$ by condition (2.27) and $L_u^1(\partial D)$ into weak $l^1_{\{a_i^{\beta}\}}(I)$ by condition (2.28), as we now show. Suppose f is bounded with compact support in ∂D. Let $t>0$ and let $\{Q_j\}_{j\in J}$ be the maximal dyadic cubes from the collection $\{Q_i\}_{i\in I}$ such that $(1/a_i)\int_{\check{Q}_i}|f|u\,d\nu>t$ (we may assume the collection $\{Q_i\}_{i\in I}$ is finite). Then

$$(2.29)\qquad \sum_{i:|(1/a_i)\int_{\check{Q}_i} fu\,d\nu|>t} a_i^{\beta}\le\sum_{j\in J}\sum_{i:Q_i\subset Q_j} a_i^{\beta}$$

$$(2.30)\qquad \le c\sum_{j\in J} a_j^{\beta} \qquad \text{by (2.28)}$$

$$(2.31)\qquad \le c\Big(\sum_{j\in J} a_j\Big)^{\beta} \qquad \text{since } \beta\ge 1$$

$$(2.32)\qquad \le c\Big(\frac{1}{t}\sum_{j}\int_{\check{Q}_j}|f|u\,d\nu\Big)^{\beta}$$

$$(2.33)\qquad \le c\Big(\frac{1}{t}\int_{\partial D}|f|u\,d\nu\Big)^{\beta}$$

since the maximal dyadic cubes are disjoint. The Marcinkiewicz interpolation theorem now completes the proof of Lemma 2.26. ∎

The next lemma can be found in [W1] and its proof is omitted.

LEMMA 2.34 (Reverse doubling). *Suppose μ is a doubling measure on a homogeneous space X. Then μ satisfies the* reverse doubling condition: *there exist $\alpha,\beta>1$ such that $|B(x,\alpha r)|_{\mu}\ge\beta|B(x,r)|_{\mu}$ for all $x\in X$ and $r>0$.*

We will need the concept of dyadic cubes in X. The construction of these sets can be found in Lemma 3.21 of [SW2]. We will need the following lemma of [SW2] (Lemma 3.21 there).

LEMMA 2.35. *Suppose (X,d) is a separable quasi-metric space. Let $\lambda=8A_0^5$. Then for every $m\in\mathbb{Z}$, there are points z_j^k in X and Borel sets E_j^k, $1\le j<n_k$, $k\ge m$, where $n_k\in\mathbb{N}\cup\{\infty\}$, such that*

$$(2.36)\qquad B(z_j^k,\lambda^k)\subset E_j^k\subset B(z_j^k,\lambda^{k+1}),\qquad 1\le j<n_k,\ k\ge m,$$

$$X = \bigcup_j E_j^k \quad \textit{for all } k \ge m, \qquad E_j^k \cap E_i^k = \emptyset \quad \textit{if } i \ne j, \tag{2.37}$$

and given i, j, k, l *with* $m \le k < l$, *we have either*

$$E_j^k \subset E_i^l \quad \textit{or} \quad E_j^k \cap E_i^l = \emptyset.$$

Recall that $\mathcal{D}_m = \{E_j^k : k \ge m,\ 1 \le j < n_k\}$. If $Q = E_j^k \in \mathcal{D}_m$, let $l(Q)$ be the edgelength of Q and define $\widehat{Q} = Q \times (0, l(Q))$.

We will need the following lemma (cf. Lemma 3.20 of [SW1]).

LEMMA 2.38. *Let* $u(x) \ge 0$ *on* X. *For each* $m \in \mathbb{Z}$, *suppose that* $\{Q_i\}_i$ *is a countable collection of dyadic cubes in* $\mathcal{D}_m$, *and for each* $z \in X$, $\{a_i(z)\}_{i\in I}$ *and* $\{b_i(z)\}_{i\in I}$ *are positive numbers satisfying*

$$\int\limits_{Q_i+z} u\, d\mu \le c a_i(z) \quad \textit{for all } i \in I \textit{ and } z \in X, \tag{2.39}$$

$$\sum_{j: Q_j \subset Q_i} b_j(z) \le c a_i(z) \quad \textit{for all } i \in I \textit{ and } z \in X, \tag{2.40}$$

with c *independent of* m. *Then*

$$\Big(\sum_{i\in I} b_i(z) \Big(\frac{1}{a_i(z)} \int\limits_{Q_i+z} g u\, d\mu\Big)^q\Big)^{1/q} \le c_q \Big(\int\limits_X g^q u\, d\mu\Big)^{1/q}$$

for all $g \ge 0$ *on* X *and* $z \in X$, *where* $1 < q < \infty$ *and* c_q *is independent of* z, m *and* g.

Proof. The map $g \mapsto \{(1/a_i(z)) \int_{Q_i+z} g u\, d\mu\}$ takes $L_u^\infty(X)$ into $l^\infty_{\{b_i(z)\}}(I)$ by condition (2.39) and $L_u^1(X)$ into weak $l^1_{\{b_i(z)\}}(I)$ by condition (2.40), as we now show. If g is bounded with compact support in X and $t > 0$, let $\{Q_j\}_{j\in J}$ be the maximal dyadic cubes from the collection $\{Q_i\}_{i\in I}$ such that $(1/a_i(z)) \int_{Q_i+z} |g| u\, d\mu > t$ (we may assume the collection $\{Q_i\}_{i\in I}$ is finite). Then

$$\sum_{i : |(1/a_i(z)) \int_{Q_i+z} g u\, d\mu| > t} b_i(z) \le \sum_{j\in J} \sum_{i: Q_i \subset Q_j} b_i(z) \tag{2.41}$$

$$\le c \sum_{j \in J} a_j(z) \quad \text{by (2.40)} \tag{2.42}$$

$$\le \frac{c}{t} \sum_j \int\limits_{Q_j+z} |g| u\, d\mu \tag{2.43}$$

$$\le \frac{c}{t} \int\limits_X |g| u\, d\mu \tag{2.44}$$

since the maximal dyadic cubes are disjoint. The Marcinkiewicz interpolation theorem now completes the proof of Lemma 2.38. ∎

3. Proofs of the main theorems.

In this section we will prove the theorems introduced in Section 1.

Proof of Theorem 1.1. First, we prove that (1.8) implies (1.9). Given z in ∂D and $r > 0$, let $B = B(z,r)$ and set $f(y) = \sigma(y)\chi_{\check{B}}(y)$. Thus,

$$\begin{aligned} \Big(\int\limits_D M_\varphi^q(\sigma\chi_{\check{B}}) w\, d\mu\Big)^{1/q} &\le c\Big(\int\limits_{\partial D} \sigma^p \chi_{\check{B}} v\, d\nu\Big)^{1/p} \\ &= c\Big(\int\limits_{\check{B}} \sigma\, d\nu\Big)^{1/p} = c|\check{B}|_\sigma^{1/p}. \end{aligned} \tag{3.1}$$

For $x \in X$ let $a(x)$ and $t(x)$ be as in the definition of M_φ. We claim that $B = B(z,r) \subset B(a(x), 3A_0^2[r + d(z,x)])$. Indeed, if $y \in B$ then

$$\begin{aligned} d(y, a(x)) &\le A_0^2[d(y,z) + d(z,x) + d(x, a(x))] \\ &\le A_0^2\big[r + d(z,x) + \tfrac{3}{2} d(z,x)\big] \le 3A_0^2[r + d(z,x)], \end{aligned}$$

which proves our claim. Obviously $3A_0^2[r + d(z,x)] \ge t(x)$. Hence,

$$\begin{aligned} M_\varphi(\sigma\chi_{\check{B}})(x) &\ge \frac{1}{\varphi(B(a(x), 3A_0^2[r + d(z,x)]))} \int\limits_{\check{B}(a(x), 3A_0^2[r+d(z,x)])} \sigma\chi_B\, d\nu \\ &= \frac{|\check{B}|_\sigma}{\varphi(B(a(x), 3A_0^2[r + d(z,x)]))} \end{aligned} \tag{3.2}$$

for any $x \in X$ and $r > 0$.

We claim that

$$B(a(x), 3A_0^2[r + d(z,x)]) \subset B(z, 18A_0^4 r)$$

for x in $B = B(z,r)$. In fact, for $y \in B(a(x), 3A_0^2[r + d(z,x)])$, we have

$$\begin{aligned} d(y,z) &\le A_0^2[d(y, a(x)) + d(a(x), x) + d(x,z)] \\ &\le A_0^2[3A_0^2[r + d(z,x)] + t(x) + d(x,z)] \\ &\le 9A_0^4[r + d(x,z)] \quad \text{since } t(x) = d(x, \partial D) \le d(x,z) \\ &\le 18A_0^4 r, \end{aligned}$$

as claimed. Thus, $\varphi(B(a(x), 3A_0^2[r + d(z,x)])) \le c\varphi(B(z,r))$. Therefore, it follows from (3.2) that

$$M_\varphi(\sigma\chi_{\check{B}})(x) \ge c\frac{|\check{B}|_\sigma}{\varphi(B(z,r))} \quad \text{for all } x \text{ in } B. \tag{3.3}$$

Combining (3.1) and (3.3) we obtain

$$\varphi(B)^{-q} |\check{B}|_\sigma^q |\widehat{B}|_w \le |\check{B}|_\sigma^{q/p}. \tag{3.4}$$

Thus, if $|\check{B}|_\sigma$ is neither 0 nor ∞, we obtain (1.9) from (3.4). If $|\check{B}|_\sigma = 0$ then (1.9) is obvious. Suppose $|\check{B}|_\sigma = \infty$. For any $\varepsilon > 0$ we have $[v(y) + \varepsilon]^{1-p'} \le$

$\varepsilon^{1-p'} < \infty$ and, therefore, $|\check{B}|_{\sigma_\varepsilon} \le \varepsilon^{1-p'}|\check{B}|_\nu$ where $\sigma_\varepsilon(y) = [v(y)+\varepsilon]^{1-p'}$. Hence, since (1.8) implies (1.8) with v replaced by $v+\varepsilon$, we obtain (1.9) with v replaced by $v+\varepsilon$, i.e.,

$$\varphi(B)^{-1}|\widehat{B}|_w^{1/q}|\check{B}|_{\sigma_\varepsilon}^{1/p'} \le C.$$

Letting ε tend to 0, we conclude that $w = 0$ a.e. (μ) since $|\check{B}|_\sigma = \infty$. Thus, (1.9) follows immediately.

Conversely, (1.9) implies (1.8). To see this, let

$$m_\varphi^{dy} f(x) = \sup_{\substack{\text{dyadic ball } B\\ x,a(x)\in B}} \frac{1}{\varphi(B)} \int_{\check{B}} |f|\,d\nu, \qquad x \in X,$$

where a dyadic ball is a ball as in Definition 2.2. We claim that there exists $c > 0$ such that

$$M_\varphi f(x) \le c m_\varphi^{dy} f(x) \tag{3.5}$$

for all f and $x \in X$.

Given $B = B(a(x), r)$ with $r > t(x)$, let k in $\mathbb{Z}$ be such that $\lambda^{k-1} < \frac{3}{2}r \le \lambda^k$ (with λ as in Lemma 2.6). Thus, there exists a dyadic ball B_j^{k+1} of radius λ^{k+1} such that $B \subset B\big(a(x), \frac{3}{2}r\big) \subset B(a(x), \lambda^k) \subset B_j^{k+1}$. Clearly, $\varphi(B) \le c_0\varphi(B_j^{k+1})$. We show that $\varphi(B_j^{k+1}) \le c\varphi(B)$. Indeed, if $y \in B_j^{k+1} = B(z_j^{k+1}, \lambda^{k+1})$ then

$$d(y, a(x)) \le A_0[d(y, z_j^{k+1}) + d(z_j^{k+1}, a(x))] \le 2A_0\lambda^{k+1}.$$

Thus, $B_j^{k+1} \subset 3A_0\lambda^2 B$, and, hence, $\varphi(B_j^{k+1}) \le c\varphi(B)$. Therefore,

$$\frac{1}{\varphi(B)}\int_{\check{B}} |f|\,d\nu \le c\frac{1}{\varphi(B_j^{k+1})}\int_{\check{B}_j^{k+1}} |f|\,d\nu. \tag{3.6}$$

Clearly, x and $a(x)$ lie in B_j^{k+1}. Hence, our claim follows from (3.6).

Due to (3.5) it is enough to prove (1.8) with M_φ replaced by m_φ^{dy}. We may assume $f \ge 0$. Let $D_k = \{x \in D : m_\varphi^{dy}(f\sigma)(x) > 2^k\}$, $k \in \mathbb{Z}$. Thus,

$$\int_D [m_\varphi^{dy}(f\sigma)]^q w\,d\mu = \sum_k \int_{D_k\setminus D_{k+1}} [m_\varphi^{dy}(f\sigma)]^q w\,d\mu \approx \sum_k 2^{kq}\int_{D_k\setminus D_{k+1}} w\,d\mu.$$

It follows from the definition of m_φ^{dy} that if $x \in D_k$ then there exists a dyadic ball B with $x, a(x) \in B$ satisfying

$$\frac{1}{\varphi(B)}\int_{\check{B}} f\,d\nu > 2^k. \tag{3.7}$$

Let $\{B_{k,j}\}$ be the collection of maximal dyadic balls with respect to inclusion which satisfy (3.7). Observe that $D_k \subset \bigcup_j \widehat{B}_{k,j}$. In fact, if $x \in D_k$ then there exists a dyadic ball B with $x, a(x) \in B$ satisfying (3.7). Hence, by the maximality of the $B_{k,j}$, there exists j such that $x \in B \subset B_{k,j}$.

Thus,

$$\sum_k 2^{kq}\int_{D_k\setminus D_{k+1}} w\,d\mu \le \sum_{k,j}\bigg[\frac{1}{\varphi(B_{k,j})}\int_{\check{B}_{k,j}} f\sigma\,d\nu\bigg]^q \int_{\widehat{B}_{k,j}} w\,d\mu$$
$$= \sum_{k,j} b_{k,j}\bigg[\frac{1}{a_{k,j}}\int_{\check{B}_{k,j}} f\sigma\,d\nu\bigg]^q$$

where

$$a_{k,j} = \int_{\check{B}_{k,j}} \sigma\,d\nu \quad\text{and}\quad b_{k,j} = \bigg[\frac{a_{k,j}}{\varphi(B_{k,j})}\bigg]^q \int_{\widehat{B}_{k,j}} w\,d\mu.$$

From (1.9) it follows that

$$b_{k,j} = a_{k,j}^q\varphi(B_{k,j})^{-q}|\widehat{B}_{k,j}|_w \le ca_{k,j}^q|\check{B}_{k,j}|_\sigma^{-q/p'} = ca_{k,j}^{q/p}.$$

We claim that there exists $c > 0$ such that

$$\sum_{B_{k,j}\subset B_{s,t}} a_{k,j}^{q/p} \le ca_{s,t}^{q/p} \quad \text{for all } s,t.$$

Indeed, with $\lambda = A_0(2A_0+1)$, we have

$$\sum_{B_{k,j}\subset B_{s,t}} a_{k,j}^{q/p} = \sum_{B_{k,j}\subset B_{s,t}} |\check{B}_{k,j}|_\sigma^{q/p} \le \sum_{l=0}^\infty \sum_{\substack{\text{dyadic ball}\subset B_{s,t}\\ r(B)/r(B_{s,t})=\lambda^{-l}}} |\check{B}_{k,j}|_\sigma^{q/p-1}|\check{B}_{k,j}|_\sigma$$
$$\le c\sum_{l=0}^\infty \sum_{\substack{\text{dyadic ball}\subset B_{s,t}\\ r(B)/r(B_{s,t})=\lambda^{-l}}} (\delta^l|\check{B}_{s,t}|_\sigma)^{q/p-1}|\check{B}_{k,j}|_\sigma$$

for some $\delta < 1$ since $\sigma d\nu$ is doubling and, therefore, the reverse doubling condition (see Lemma 2.34) is satisfied. Hence

$$\sum_{B_{k,j}\subset B_{s,t}} a_{k,j}^{q/p} \le c\sum_{l=0}^\infty (\delta^l|\check{B}_{s,t}|_\sigma)^{q/p-1}M|\check{B}_{s,t}|_\sigma \quad \text{by (2.4)}$$
$$= cM|\check{B}_{s,t}|_\sigma^{q/p}\sum_{l=0}^\infty \delta^{l(q/p-1)} = c|\check{B}_{s,t}|_\sigma^{q/p} = ca_{s,t}^{q/p}$$

since $0 < \delta < 1$ and $q > p$, and this finishes the proof of our claim.

We now obtain (1.8) from Lemma 2.16 with $a(B_{k,j}) = a_{k,j}$, $\beta = q/p$, $t = q$, $s = p$, $u = \sigma$ and $\Gamma = \{B_{k,j}\}$ since $\sigma d\nu$ is a doubling measure. This concludes the proof of Theorem 1.1. ■

Proof of Theorem 1.2. Let $\mathfrak{B}=\{B=B(z,r) : z\in\partial D,\ r>0\}$. Given $x\in D$ and $r>t(x)$, let $B=B(a(x),r)$ and $B_0=B\big(a(x),\frac{3}{2}r\big)$. Then, clearly, $B\subset B_0$ and $x\in B_0$ since $d(x,a(x))\le\frac{3}{2}t(x)<\frac{3}{2}r$. Thus, since φ is doubling we have

$$\frac{1}{\varphi(B)}\int\limits_{\check B}|f|\,d\nu\le\frac{c}{\varphi(B_0)}\int\limits_{\check B_0}|f|\,d\nu\le c\sup_{\substack{B\in\mathfrak{B}\\ x,a(x)\in B}}\int\limits_{\check B}|f|\,d\nu=c\mathfrak{m}_{\varphi}f(x),\qquad\text{say,}$$

and hence,

$$M_{\varphi}f\le c\mathfrak{m}_{\varphi}f.$$

For each $m\in\mathbb{Z}$, define

$$\mathfrak{m}_{\varphi,m}f(x)=\sup_{\substack{B\in\mathfrak{B}\\ x,a(x)\in B,\,r(B)>\lambda^m}}\int\limits_{\check B}|f|\,d\nu,$$

where $\lambda=8A_0^5$ as in Lemma 2.7.

It will be convenient to majorize $\mathfrak{m}_{\varphi,m}f$ by a suitable dyadic operator defined in terms of the dyadic cubes $Q\in\mathcal{D}_m$ that were introduced in Lemma 2.7. Given $B=B(z,r)\in\mathfrak{B}$, $r>\lambda^m$, select $k>m$ such that $\lambda^k\le r<\lambda^{k+1}$. Suppose $B\cap Q\ne\emptyset$, $Q=E_j^k\in\mathcal{D}_m$ and $B=B(z,r)$ is as above. Then, if $y\in B$ and $x\in B\cap Q$, we have

$$\begin{aligned}d(y,z_j^k)&\le A_0^2[d(y,z)+d(z,x)+d(x,z_j^k)]\\&\le A_0^2[\lambda^{k+1}+\lambda^{k+1}+\lambda^{k+1}]\\&\qquad\qquad\text{since } Q\subset B(z_j^k,\lambda^{k+1})\ \text{(see Lemma 2.7)}\\&=3A_0^2\lambda^{k+1}.\end{aligned}$$

Thus,

$$B(z,r)\subset B(z_j^k,3A_0^2\lambda^{k+1}).\tag{3.8}$$

Also, if $u\in Q$, we have

$$\begin{aligned}d(u,z)&\le A_0^2[d(u,z_j^k)+d(z_j^k,x)+d(x,z)]\\&\le A_0^2[\lambda^{k+1}+\lambda^{k+1}+\lambda^{k+1}]=3A_0^2\lambda^{k+1}.\end{aligned}$$

Thus,

$$Q=E_j^k\subset B(z,3A_0^2\lambda^{k+1}).\tag{3.9}$$

Now, if $E_{j_\alpha}^k\cap B\ne\emptyset$, $\alpha=1,\dots,N$, we obtain

$$N|\check B(z,r)|_\nu\le\sum_{\alpha=1}^N|\check B(z_{j_\alpha}^k,3A_0^2\lambda^{k+1})|_\nu\qquad\text{by (3.8)}\tag{3.10}$$

$$\le c\sum_{\alpha=1}^N|\check B(z_{j_\alpha}^k,\lambda^k)|_\nu\qquad\text{since }\nu\text{ is doubling}\tag{3.11}$$

$$\le c\sum_{\alpha=1}^N|\check E_{j_\alpha}^k|_\nu\qquad\text{by Lemma 2.7}\tag{3.12}$$

$$=c\Big|\bigcup_{\alpha=1}^N\check E_{j_\alpha}^k\Big|_\nu\qquad\text{since the cubes are disjoint}\tag{3.13}$$

$$\le c|\check B(z,3A_0^2\lambda^{k+1})|_\nu\qquad\text{by (3.9)}\tag{3.14}$$

$$\le c|\check B(z,r)|_\nu\qquad\text{since }\nu\text{ is doubling.}\tag{3.15}$$

Therefore, $N\le c$. Since $\partial D=\bigcup_j\check E_j^k$ for all $k\ge m$ by Lemma 2.7, any ball $B\in\mathfrak{B}$ with $r(B)>\lambda^m$ is contained in a finite union of dyadic cubes $\{E_{j_\alpha}^k\}_{1\le\alpha\le N}$, say, and the number of such dyadic cubes is bounded by a constant that depends only on the doubling constant of ν. Thus, for any $B\in\mathfrak{B}$ with $r(B)>\lambda^m$ and $\lambda^k\le r<\lambda^{k+1}$, we have $\check B\subset\bigcup_{\alpha=1}^N\check E_{j_\alpha}^k$ with $B\cap E_{j_\alpha}^k\ne\emptyset$ for some $N\le c$. Hence, for some $1\le\alpha_0\le N$,

$$\begin{aligned}\frac{1}{\varphi(B)}\int\limits_{\check B}|f|\sigma\,d\nu&\le\frac{1}{\varphi(B)}\int\limits_{\bigcup_{\alpha=1}^N\check E_{j_\alpha}^k}|f|\sigma\,d\nu=\sum_{\alpha=1}^N\frac{1}{\varphi(B)}\int\limits_{\check E_{j_\alpha}^k}|f|\sigma\,d\nu\\&\le\frac{N}{\varphi(B)}\int\limits_{\check E_{j_{\alpha_0}}^k}|f|\sigma\,d\nu\le\frac{c}{\varphi(B)}\int\limits_{\check E_{j_{\alpha_0}}^k}|f|\sigma\,d\nu.\end{aligned}\tag{3.16}$$

It follows from (3.9) with j replaced by j_{α_0} that if B and k are as above then $E_{\alpha_0}^k\subset B(z,3A_0^2\lambda^{k+1})$ and, therefore, $\varphi(E_{\alpha_0}^{k*})\le c\varphi(B(z,r))=c\varphi(B)$, since φ is doubling, where $E_{\alpha_0}^{k*}$ is the outer ball associated with $E_{\alpha_0}^k$. Hence, it follows from (3.16) that

$$\frac{1}{\varphi(B)}\int\limits_{\check B}|f|\sigma\,d\nu\le\frac{c}{\varphi(E_{\alpha_0}^{k*})}\int\limits_{E_{\alpha_0}^k}|f|\sigma\,d\nu.\tag{3.17}$$

Since $B\cap E_{\alpha_0}^k\ne\emptyset$, it follows from (3.8) that

$$B\subset B(z_j^k,3A_0^2\lambda^{k+1})=3A_0^2B(z_j^k,\lambda^{k+1})=3A_0^2E_{j_\alpha}^{k*}.$$

Thus, if B is as above with $x\in\widehat B$ and $a(x)\in\check B$ then (3.17) implies that

$$\frac{1}{\varphi(B)}\int\limits_{\check B}|f|\sigma\,d\nu\le c\sup_{Q\in\mathcal{D}_m:x\in3A_0^2\widehat Q^*}\frac{1}{\varphi(Q^*)}\int\limits_{\check Q}|f|\sigma\,d\nu=c\mathfrak{M}_{\varphi,m}^{dy}(f\sigma)(x),\qquad\text{say.}$$

Therefore, we have

$$\mathfrak{m}_{\varphi,m}(f\sigma)(x)\le c\mathfrak{M}_{\varphi,m}^{dy}(f\sigma)(x),\qquad x\in D.\tag{3.18}$$

For each $k\in\mathbb{Z}$ let $\Omega_{k,m}=\{x\in D:\mathfrak{M}_{\varphi,m}^{dy}(f\sigma)(x)>a^k\}$, where $a>1$ will be chosen later. Thus, $x\in\Omega_{k,m}$ if and only if there exists $Q\in\mathcal{D}_m$ such that $x\in3A_0^2\widehat Q^*$ and

$$\frac{1}{\varphi(Q^*)}\int_{\check{Q}}|f|\sigma\,d\nu > a^k. \tag{3.19}$$

Let $\{Q_j^k\}_j$ be the maximal (with respect to inclusion) dyadic cubes (in $\mathcal{D}_m$) which satisfy (3.19). We claim that $\Omega_{k,m} = \bigcup_j 3A_0^2\widehat{Q}_j^{k*}$. Indeed, since any cube Q_j^k satisfies (3.19), for any $x \in 3A_0^2\widehat{Q}_j^{k*}$ we have

$$\mathfrak{M}_{\varphi,m}^{dy}(f\sigma)(x) \geq \frac{1}{\varphi(Q_j^k)}\int_{\check{Q}_j^k}|f|\sigma\,d\nu > a^k,$$

which implies that $x \in \Omega_{k,m}$. On the other hand, if $x \in \Omega_{k,m}$ then there exists $Q \in \mathcal{D}_m$ such that $x \in 3A_0^2\widehat{Q}$ and (3.19) holds. Since the Q_j^k are maximal with respect to inclusion, there exists j_0 such that $Q \subset Q_{j_0}^k$ and, therefore, $x \in 3A_0^2\widehat{Q} \subset 3A_0^2\widehat{Q}_j^{k*}$, that is, $x \in \bigcup_j 3A_0^2\widehat{Q}_j^{k*}$; this proves our claim.

Now

$$\begin{aligned}
&\|\mathfrak{M}_{\varphi,m}^{dy}(f\sigma)\|_{L_w^q(D)}^q \\
&\quad= \int_D[\mathfrak{M}_{\varphi,m}^{dy}(f\sigma)]^q w\,d\mu \leq a^q\sum_k a^{kq}\int_{\Omega_{k,m}\setminus\Omega_{k+1,m}} w\,d\mu \\
&\quad\leq a^q\sum_{k,j}a^{kq}\int_{3A_0^2\widehat{Q}_j^{k*}\setminus\Omega_{k+1,m}} w\,d\mu = a^q\sum_{k,j}a^{kq}|F_{j,m}^k|_w \\
&\qquad\qquad\text{where } F_{j,m}^k = 3A_0^2\widehat{Q}_j^{k*}\setminus\Omega_{k+1,m} \\
&\quad\leq a^q\sum_{k,j}\Big[\varphi(Q_j^{k*})^{-1}\int_{\check{Q}_j^k}|f|\sigma\,d\nu\Big]^q|F_{j,m}^k|_w \quad \text{by (3.19) with } Q = Q_j^k \\
&\quad= a^q\sum_{k,j}|F_{j,m}^k|_w[\varphi(Q_j^{k*})^{-1}\mathcal{A}(Q_j^k)]^q\Big[\frac{1}{\mathcal{A}(Q_j^k)}\int_{\check{Q}_j^k}|f|\sigma\,d\nu\Big]^q \\
&\qquad\qquad\text{where } \mathcal{A}(Q) = |\check{Q}|_\nu^{1/r'}\big(\textstyle\int_{\check{Q}}\sigma^r\,d\nu\big)^{1/r} \\
&\quad\leq a^q\sum_{k,j}|3A_0^2\widehat{Q}_j^{k*}|_w[\varphi(Q_j^{k*})^{-1}\mathcal{A}(Q_j^k)]^q\Big[\frac{1}{\mathcal{A}(Q_j^k)}\int_{\check{Q}_j^k}|f|\sigma\,d\nu\Big]^q \\
&\qquad\qquad\text{since } F_{j,m}^k \subset 3A_0^2\widehat{Q}_j^{k*} \\
&\quad= a^q\sum_{k,j}|3A_0^2\widehat{Q}_j^{k*}|_w\varphi(Q_j^{k*})^{-q}|\check{Q}_j^k|_\nu^q[|\check{Q}_j^k|_\nu^{-1}\mathcal{A}(Q_j^k)]^q \\
&\qquad\times\Big[\frac{1}{\mathcal{A}(Q_j^k)}\int_{\check{Q}_j^k}|f|\sigma\,d\nu\Big]^q.
\end{aligned} \tag{3.20}$$

We claim that

$$|3A_0^2\widehat{Q}_j^{k*}|_w\varphi(Q_j^{k*})^{-q}|\check{Q}_j^k|_\nu^q[|\check{Q}_j^k|_\nu^{-1}\mathcal{A}(Q_j^k)]^q \leq c\mathcal{A}(Q_j^k)^{q/p} \quad \text{for all } k, j.$$

Indeed,

$$\begin{aligned}
&|3A_0^2\widehat{Q}_j^{k*}|_w\varphi(Q_j^{k*})^{-q}|\check{Q}_j^k|_\nu^q[|\check{Q}_j^k|_\nu^{-1}\mathcal{A}(Q_j^k)]^q \\
&\quad\leq c|\check{Q}_j^k|_\nu^{q/p}\Big(\frac{1}{|\check{Q}_j^k|_\nu}\int_{\check{Q}_j^k}\sigma^r\,d\nu\Big)^{-q/(rp')}[|\check{Q}_j^k|_\nu^{-1}\mathcal{A}(Q_j^k)]^q \quad \text{by (1.11)} \\
&\quad= c|\check{Q}_j^k|_\nu^{-q/p'}\Big(\frac{1}{|\check{Q}_j^k|_\nu}\int_{\check{Q}_j^k}\sigma^r\,d\nu\Big)^{-q/(rp')}\mathcal{A}(Q_j^k)^q \\
&\quad= c\Big(\frac{|\check{Q}_j^k|_\nu^r}{|\check{Q}_j^k|_\nu}\int_{\check{Q}_j^k}\sigma^r\,d\nu\Big)^{-q/(rp')}\mathcal{A}(Q_j^k)^q \\
&\quad= c\Big(|\check{Q}_j^k|_\nu^{(r-1)/r}\Big(\int_{\check{Q}_j^k}\sigma^r\,d\nu\Big)^{1/r}\Big)^{-q/p'}\mathcal{A}(Q_j^k)^q \\
&\quad= c\mathcal{A}(Q_j^k)^{-q/p'}\mathcal{A}(Q_j^k)^q = c\mathcal{A}(Q_j^k)^{q/p},
\end{aligned}$$

which proves our claim. Thus, it follows from (3.20) that

$$\|\mathfrak{M}_{\varphi,m}^{dy}(f\sigma)\|_{L_w^q(D)}^q \leq c\sum_{k,j}\mathcal{A}(Q_j^k)^{q/p}\Big[\frac{1}{\mathcal{A}(Q_j^k)}\int_{\check{Q}_j^k}|f|\sigma\,d\nu\Big]^q. \tag{3.21}$$

We claim that

$$\sum_{k,j:Q_j^k\subset Q_i^l}\mathcal{A}(Q_j^k)^{q/p} \leq c\mathcal{A}(Q_i^l)^{q/p} \tag{3.22}$$

with $c > 0$ independent of l and i. First, we show that $Q_j^k \subset Q_i^l$ implies $l \leq k$. If $Q_j^k \subsetneq Q_i^l$ then it follows from the maximality of the Q_j^k that

$$a^l < \varphi(Q_i^{l*})^{-1}\int_{\check{Q}_i^l}|f|\sigma\,d\nu \leq a^k, \tag{3.23}$$

which yields $l < k$, since $a > 1$. Now, suppose that $Q_j^k = Q_i^l$, and for any $Q \in \mathcal{D}_m$ let $Q^\star$ denote the smallest dyadic cube in $\mathcal{D}_m$ such that $Q^\star \supsetneq Q$. Thus, since ν is doubling, we have

$$a^l < \varphi(Q_i^{l*})^{-1}\int_{\check{Q}_i^l}|f|\sigma\,d\nu = \varphi(Q_j^{k*})^{-1}\int_{\check{Q}_j^k}|f|\sigma\,d\nu \tag{3.24}$$

$$\le c\varphi(Q_j^{k**})^{-1}\int_{\check{Q}_j^{k*}}|f|\sigma\,d\nu \le ca^k \quad \text{since } Q_j^k \text{ is maximal}$$

$$\le a^{k+1} \quad \text{if we select } a \ge c.$$

Therefore, $l \le k$.

Observe that the same argument employed in (3.24) can be used to obtain

$$a^k < \varphi(Q_j^k)^{-1}\int_{\check{Q}_j^k}|f|\sigma\,d\nu \le a^{k+1} \quad \text{for all } j,k. \tag{3.25}$$

Thus, since $q \ge p$, we have

$$\sum_{k,j:Q_j^k\subset Q_i^l}\mathcal{A}(Q_j^k)^{q/p} \le \Big(\sum_{k,j:Q_j^k\subset Q_i^l}\mathcal{A}(Q_j^k)\Big)^{q/p} = \Big(\sum_{k=l}^{\infty}\sum_{j:Q_j^k\subset Q_i^l}\mathcal{A}(Q_j^k)\Big)^{q/p}. \tag{3.26}$$

Now, for a fixed $k \ge l$, we have

$$\begin{aligned}\sum_{j:Q_j^k\subset Q_i^l}\mathcal{A}(Q_j^k) &= \sum_{j:Q_j^k\subset Q_i^l}|\check{Q}_j^k|_\nu^{1/r'}\Big(\int_{\check{Q}_j^k}\sigma^r\,d\nu\Big)^{1/r}\\ &\le \Big(\sum_{j:Q_j^k\subset Q_i^l}|\check{Q}_j^k|_\nu\Big)^{1/r'}\Big(\sum_{j:Q_j^k\subset Q_i^l}\int_{\check{Q}_j^k}\sigma^r\,d\nu\Big)^{1/r}\\ &\qquad\text{by Hölder's inequality}\\ &\le \Big(\sum_{j:Q_j^k\subset Q_i^l}|\check{Q}_j^k|_\nu\Big)^{1/r'}\Big(\int_{\check{Q}_i^l}\sigma^r\,d\nu\Big)^{1/r}\end{aligned} \tag{3.27}$$

since the Q_j^k are pairwise disjoint in j. Using the first inequality of (3.25) and the fact that $(|Q'|_\nu/|Q|_\nu)^\varepsilon \le c\varphi(Q'^*)/\varphi(Q^*)$, we obtain

$$\begin{aligned}\sum_{j:Q_j^k\subset Q_i^l}|\check{Q}_j^k|_\nu &\le c\frac{|\check{Q}_i^l|_\nu}{\varphi(Q_i^{l*})^{1/\varepsilon}}\sum_{j:Q_j^k\subset Q_i^l}\Big(a^{-k}\int_{\check{Q}_j^k}|f|\sigma\,d\nu\Big)^{1/\varepsilon}\\ &\le c\frac{|\check{Q}_i^l|_\nu}{\varphi(Q_i^{l*})^{1/\varepsilon}}\Big(\sum_{j:Q_j^k\subset Q_i^l}a^{-k}\int_{\check{Q}_j^k}|f|\sigma\,d\nu\Big)^{1/\varepsilon} \qquad \text{since } 1/\varepsilon \ge 1\\ &\le c\frac{|\check{Q}_i^l|_\nu}{\varphi(Q_i^{l*})^{1/\varepsilon}}\Big(a^{-k}\int_{\check{Q}_i^l}|f|\sigma\,d\nu\Big)^{1/\varepsilon}\\ &\qquad\text{since the } Q_j^k \text{ are pairwise disjoint in } j\\ &\le c\frac{|\check{Q}_i^l|_\nu}{\varphi(Q_i^{l*})^{1/\varepsilon}}(a^{l+1-k}\varphi(Q_i^{l*}))^{1/\varepsilon} = a^{(l+1-k)/\varepsilon}|\check{Q}_i^l|_\nu\end{aligned} \tag{3.28}$$

by the second inequality of (3.25) with $k = l$. Thus, combining (3.26)–(3.28), we obtain

$$\begin{aligned}\sum_{k,j:Q_j^k\subset Q_i^l}\mathcal{A}(Q_j^k)^{q/p} &\le \Big(\sum_{k=l}^{\infty}\Big(a^{(l+1-k)/(1-\gamma)}|\check{Q}_i^l|_\nu\Big)^{1/r'}\Big(\int_{\check{Q}_i^l}\sigma^r\,d\nu\Big)^{1/r}\Big)^{q/p}\\ &= \Big(\sum_{k=l}^{\infty}a^{(l+1-k)/((1-\gamma)r')}|\check{Q}_i^l|_\nu^{1/r'}\Big(\int_{\check{Q}_i^l}\sigma^r\,d\nu\Big)^{1/r}\Big)^{q/p}\\ &= a^{q/((1-\gamma)pr')}\Big(\sum_{k=0}^{\infty}a^{-k/((1-\gamma)r')}\mathcal{A}(Q_i^l)\Big)^{q/p} = c\mathcal{A}(Q_i^l)^{q/p},\end{aligned}$$

where $c = a^{q/((1-\gamma)pr')}(\sum_{k=0}^{\infty}a^{-k/((1-\gamma)r')})^{q/p}$, and this yields (3.22).

Thus, if we take $\beta = q/p$ and I to be the set of indices k,j, and $a_i = \mathcal{A}(Q_i)$ and $u = \sigma$, then from (3.21) and Lemma 2.26 we obtain

$$\begin{aligned}\|\mathfrak{M}^{dy}_{\varphi,m}(f\sigma)\|^q_{L^q_w(D)} &\le c\sum_{k,j}\mathcal{A}(Q_j^k)^{q/p}\Big[\frac{1}{\mathcal{A}(Q_j^k)}\int_{\check{Q}_j^k}|f|\sigma\,d\nu\Big]^q\\ &\le c\Big(\int_{\partial D}f^p u\,d\nu\Big)^{q/p},\end{aligned} \tag{3.29}$$

provided we verify (2.27) and (2.28). Note that (2.27) follows from Hölder's inequality with $c = 1$. Condition (2.28) holds due to (3.22).

It follows from (3.18) that

$$\|\mathfrak{m}_{\varphi,m}(f\sigma)\|^q_{L^q_w(D)} \le c\Big(\int_{\partial D}f^p\sigma\,d\nu\Big)^{q/p}, \tag{3.30}$$

where

$$\mathfrak{m}_{\varphi,m}(f\sigma)(x) = \sup_{\substack{B\in\mathfrak{B}\\ x,a(x)\in B,\, r(B)>\lambda^m}}\int_{\check{B}}|f|\sigma\,d\nu, \qquad x\in D.$$

Therefore, letting m tend to $-\infty$ in (3.30), we obtain

$$\|\mathfrak{m}_{\varphi}(f\sigma)\|^q_{L^q_w(D)} \le c\Big(\int_{\partial D}f^p\sigma\,d\nu\Big)^{q/p}. \tag{3.31}$$

Replacing f by $f\sigma^{-1}$ in (3.31) and using the fact that $M_\varphi f \le c\mathfrak{m}_\varphi f$ (see the beginning of the proof of Theorem 1.2), we obtain (1.12). This concludes the proof of Theorem 1.2. ∎

In order to prove Theorem 1.3 it will be convenient to define the following dyadic maximal function: for $m \in \mathbb{Z}$ and $z \in X$ put

$$\mathfrak{M}^{dy}_{\varphi,m,z} f(x,t) = \sup_{\substack{Q \in \mathcal{D}_m \\ (x,t) \in \widehat{Q}+z}} \frac{1}{\varphi(Q+z)} \int\limits_{Q+z} |f| \, d\nu, \qquad (x,t) \in \widehat{X}_+,$$

and for $k > m$, let

$$M^k_{\varphi,m} f(x,t) = \sup_{\lambda^k \geq r > \max\{\lambda^m, t\}} \frac{1}{\varphi(B(x,r))} \int\limits_{B(x,r)} |f| \, d\nu.$$

We have

LEMMA 3.32. *Let (X,d,μ) be a homogeneous space with group structure satisfying conditions* (1.13)–(1.16). *Suppose $1 \leq p < \infty$ and $w(x,t) \geq 0$ in $\widehat{X}_+ = X \times (0,\infty)$. Then there exists $C > 0$ such that*

$$\Big(\int\limits_{\widehat{X}_+} (M_{\varphi,m} f(x,t))^p w(x,t) \, d\mu(x,t) \Big)^{1/p} \leq C \sup_{z \in X} \Big(\int\limits_{\widehat{X}_+} (\mathfrak{M}^{dy}_{\varphi,m,z} f(x,t))^p w(x,t) \, d\mu(x,t) \Big)^{1/p}$$

for all $f \geq 0$ with C independent of m.

Proof. Fix $k > m$ and $(x,t) \in \widehat{X}_+$. Suppose $B(x,r)$ is a ball in X satisfying

$$\frac{1}{\varphi(B(x,r))} \int\limits_{B(x,r)} |f| \, d\nu > \frac{1}{2} M^k_{\varphi,m} f(x,t).$$

Now, select $m < l \leq k$ such that $\lambda^{l-1} < r \leq \lambda^l$. Let B_k be the ball in X of radius λ^k about the identity element of X. Define $\Omega = \{z \in B_{k+3} : \exists Q \in \mathcal{D}_m$ with $\lambda^k \leq l(Q) \leq \lambda^{l+1}$ and $B(x,r) \subset Q + z\}$. Thus, if $Q = E^{l+1}_i$ and $z \in \Omega$ then $Q + z \subset B(x, 2A_0\lambda^3 r)$ and, therefore, $\varphi(Q+z) \leq c\varphi(B(x,r))$. Hence,

$$\begin{aligned} \mathfrak{M}^{dy}_{\varphi,m,z} f(x,t) &\geq \frac{1}{\varphi(Q+z)} \int\limits_{Q+z} |f| \, d\nu \\ &\geq \frac{c}{\varphi(B(x,r))} \int\limits_{B(x,r)} |f| \, d\nu \geq c M^k_{\varphi,m} f(x,t). \end{aligned}$$

Thus,

$$\int\limits_{B_{k+3}} \mathfrak{M}^{dy}_{\varphi,m,z} f(x,t) \, d\nu(z) \geq c M^k_{\varphi,m} f(x,t) \nu(\Omega) \tag{3.33}$$

$$\geq c M^k_{\varphi,m} f(x,t) \nu(B_{k+3}), \tag{3.34}$$

provided we show $\nu(B_{k+3}) \leq c\nu(\Omega)$.

Let $\Gamma = \{j : E^{l+1}_j \cap B(x, \lambda^{k+2}) \neq \emptyset\}$, where l is as in the definition of Ω. Now suppose z is in $-B(z^{l+1}_j, \lambda^l) + x$ where $-B = \{w \in X : -w \in B\}$. We show that $B(x,r) \subset E^{l+1}_j + z$. First note that

$$x - z \in B(z^{l+1}_j, \lambda^l) \subset B(z^{l+1}_j, \lambda^{l+1}) \subset E^{l+1}_j,$$

whence $x \in E^{l+1}_j + z$. Now, let $y \in B(x,r)$. Since $x - z \in B(z^{l+1}_j, \lambda^l)$ and $d(y-z, x-z) = d(y,x) \leq \lambda^l$ by (1.13), we have

$$d(y-z, z^{l+1}_j) \leq A_0[d(y-z, x-z) + d(x-z, z^{l+1}_j)] < A_0[\lambda^l + \lambda^l] < \lambda^{l+1}.$$

Thus, $y - z \in B(z^{l+1}_j, \lambda^{l+1}) \subset E^{l+1}_j$ and, therefore, $y \in E^{l+1}_j + z$. Now if $j \in \Gamma$ then $-B(z^{l+1}_j, \lambda^l) + x \subset B_{k+3}$. Indeed, if $w \in B(z^{l+1}_j, \lambda^l)$ and $u \in E^{l+1}_j \cap B(x, \lambda^{k+2})$ (which exists since $j \in \Gamma$), then

$$d(u,w) \leq A_0[d(u, z^{l+1}_j) + d(z^{l+1}_j, w)] < A_0[\lambda^{l+2} + \lambda^l] < 2A_0\lambda^{k+2},$$

which implies

$$d(x,w) \leq A_0[d(x,u) + d(u,w)] < A_0[\lambda^{k+2} + 2A_0\lambda^{k+2}] < 3A_0^2\lambda^{k+2},$$

which in turn yields

$$d(0,w) \leq A_0[d(0,x) + d(x,w)] < A_0[\lambda^k + 3A_0^2\lambda^{k+2}] < 4A_0^3\lambda^{k+2},$$

since $x \in B_k$. Thus,

$$\begin{aligned} d(0, -w+x) &\leq A_0[d(0,x) + d(x, -w+x)] < A_0[\lambda^k + d(0,w)] \\ &\qquad\qquad \text{since } x \in B_k \text{ and by (1.13) and (1.14)} \\ &< A_0[\lambda^k + 4A_0^3\lambda^{k+2}] < 5A_0^4\lambda^{k+2} < \lambda^{k+3}, \end{aligned}$$

as desired. Now, since the sets $\{-B(z^{l+1}_j, \lambda^l) + x\}_{j \in \Gamma}$ are pairwise disjoint, we have

$$\nu(\Omega) \geq \sum_{j \in \Gamma} \nu(-B(z^{l+1}_j, \lambda^l) + x) \qquad \text{by (1.15) and (1.16)} \tag{3.35}$$

$$\geq c \sum_{j \in \Gamma} \nu(E^{l+1}_j) \qquad \text{since } \nu \text{ is doubling} \tag{3.36}$$

$$= c\nu\Big(\bigcup_{j \in \Gamma} E^{l+1}_j \Big) \geq c\nu(B(x, \lambda^{k+2})) \qquad \text{by (2.37)} \tag{3.37}$$

$$\geq c\nu(B(x, \lambda^{k+3})) \qquad \text{since } \mu \text{ is doubling} \tag{3.38}$$

$$= c\nu(B(0, \lambda^{k+3}) + x) \qquad \text{by (1.13)} \tag{3.39}$$

$$= c\nu(B(0, \lambda^{k+3})) = c\nu(B_{k+3}) \qquad \text{by (1.15).} \tag{3.40}$$

We are now able to finish the proof of the lemma. By (3.34) and Minkowski's inequality, we get

$$\Big[\int\limits_{B_k\times(0,r(B_k))} (M^k_{\varphi,m}f(x,t))^p w(x,t)\,d\mu(x,t)\Big]^{1/p}$$

$$\le c\frac{1}{\mu(B_{k+3})}\int\limits_{B_{k+3}}\Big[\int\limits_{\widehat{X}_+}(\mathfrak{M}^{dy}_{\varphi,m,z}f(x,t))^p w(x,t)\,d\mu(x,t)\Big]^{1/p}d\nu(z)$$

$$\le c\sup_{z\in B_{k+3}}\Big[\int\limits_{\widehat{X}_+}(\mathfrak{M}^{dy}_{\varphi,m,z}f(x,t))^p w(x,t)\,d\mu(x,t)\Big]^{1/p},$$

and letting $k\to\infty$ finishes the proof of Lemma 3.32. ∎

We now prove a version of Theorem 1.3 with $M_\varphi f$ replaced by $\mathfrak{M}^{dy}_{\varphi,m,z}f$. We have

THEOREM 3.1. *Let (X,d,ν) be a homogeneous space, where X has a group structure and d and ν satisfy conditions (1.13)–(1.16). Let w and v be weights in $\widehat{X}_+$ and X, respectively, and suppose that $1<p\le q<\infty$. Then*

$$\Big(\int\limits_{\widehat{X}_+}[\mathfrak{M}^{dy}_{\varphi,m,z}f]^q w\,d\mu\Big)^{1/q}\le C\Big(\int\limits_X|f|^p v\,d\nu\Big)^{1/p} \tag{3.41}$$

for all f in $L^p_{vd\nu}$, m in $\mathbb{Z}$ and z in X if and only if

$$\Big(\int\limits_{\widehat{Q}+z}[\mathfrak{M}^{dy}_{\varphi,m,z}(\sigma\chi_{Q+z})]^q w\,d\mu\Big)^{1/q}\le C\Big(\int\limits_{Q+z}\sigma\,d\nu\Big)^{1/p} \tag{3.42}$$

for all dyadic cubes Q in $\mathcal{D}_m$, m in $\mathbb{Z}$ and z in X, where $\widehat{Q}=Q\times l(Q)$.

Proof. Let $f(x)=\sigma\chi_{Q+z}(x)$ where Q is a dyadic cube in $\mathcal{D}_m$, $m\in\mathbb{Z}$ and $z\in X$. Now, if (3.41) holds we obtain

$$\Big(\int\limits_{\widehat{Q}+z}[\mathfrak{M}^{dy}_{\varphi,m,z}(\sigma\chi_{Q+z})]^q w\,d\mu\Big)^{1/q}$$

$$\le\Big(\int\limits_{\widehat{X}_+}[\mathfrak{M}^{dy}_{\varphi,m,z}(\sigma\chi_{Q+z})]^q w\,d\mu\Big)^{1/q}$$

$$\le C\Big(\int\limits_X|\sigma\chi_{Q+z}|^p v\,d\nu\Big)^{1/p}=\Big(\int\limits_{Q+z}\sigma\,d\nu\Big)^{1/p},$$

which is (3.42).

Conversely, assume that (3.42) holds. Define

$$\mathfrak{M}^{dy,R}_{\varphi,m,z}f(x,t)=\sup_{\substack{Q\in\mathcal{D}_m\\(x,t)\in\widehat{Q}+z\\\operatorname{diam}(Q^*)\le R}}\frac{1}{\varphi(Q^*+z)}\int\limits_{Q+z}|f|\,d\nu.$$

Let $f\in L^p_{vd\nu}$ and for each k in $\mathbb{Z}$ and $R>0$ define

$$\Omega^R_k=\{(x,t)\in\widehat{X}_+:\mathfrak{M}^{dy,R}_{\varphi,m,z}(f\sigma)(x,t)>2^k\}.$$

For a fixed m in $\mathbb{Z}$ let

$$\Omega^R_{k,m}=\{(x,t)\in\Omega^R_k:\exists Q\in\mathcal{D}_m,(x,t)\in\widehat{Q}+z,\ Q^*+z\subset\Omega^R_k\}.$$

Let $\{Q^k_j\}_{j\in J^m_k}$ denote the dyadic cubes maximal among those $Q\in\mathcal{D}_m$ with the property that Q^* is contained in Ω^R_k. Then

$$\Omega^R_{k,m}=\bigcup_{j\in J^m_k}(\widehat{Q}^k_j+z)\quad\text{for } k \text{ in } \mathbb{Z}, \tag{3.43}$$

$$Q^k_i\cap Q^k_j=\emptyset\quad\text{for } i\ne j, k \text{ in } \mathbb{Z}, \tag{3.44}$$

$$\frac{1}{\varphi(Q^{k^*}_j+z)}\int\limits_{Q^k_j+z}|f|\sigma\,d\nu>2^k\quad\text{for } j \text{ in } J^m_k,\ k \text{ in } \mathbb{Z}. \tag{3.45}$$

We have

$$\varphi(Q^{k^*}_j+z)\le 2^{-k}\int\limits_{Q^k_j+z}|f|\sigma\,d\nu\quad\text{by (3.45)}$$

$$\le 2^{-k}\Big(\int\limits_{Q^k_j+z}|f|^p\sigma\,d\nu\Big)^{1/p}\Big(\int\limits_{Q^k_j+z}\sigma\,d\nu\Big)^{1/p'}\quad\text{by Hölder's inequality.}$$

Let $\widetilde{Q}^k_j$ be such that $\widetilde{Q}^k_j+z=(\widehat{Q}^k_j+z)\setminus\Omega^R_{k+1,m}$. We have

$$\int\limits_{\widehat{X}_+}[\mathfrak{M}^{dy,R}_{\varphi,m,z}(f\sigma)]^q w\,d\mu$$

$$\le 2^q\sum_{k\in\mathbb{Z}}2^{kq}\int\limits_{\Omega^R_{k,m}\setminus\Omega^R_{k+1,m}}w\,d\mu$$

$$\le 2^q\sum_{\substack{k\in\mathbb{Z}\\j\in J^m_k}}|\widetilde{Q}^k_j+z|_{wd\mu}\Big(\frac{1}{\varphi(Q^{k^*}_j+z)}\int\limits_{Q^k_j+z}|f|\sigma\,d\nu\Big)^q\quad\text{by (3.43) and (3.45)}$$

$$= 2^q \sum_{\substack{k\in\mathbb{Z}\\ j\in J_k^m}} |\widetilde{Q}_j^k + z|_{w\,d\mu}\Big(\frac{1}{\varphi({Q_j^k}^* + z)} \int\limits_{Q_j^k+z} \sigma\,d\nu\Big)^q$$

$$\times \Big(\frac{1}{|Q_j^k + z|_{\sigma d\nu}} \int\limits_{Q_j^k+z} |f|\sigma\,d\nu\Big)^q$$

$$\le 2^q \sum_{\substack{k\in\mathbb{Z}\\ j\in J_k^m}} \Big(\int\limits_{\widetilde{Q}_j^k+z} (\mathfrak{M}^{dy}_{\varphi,m,z}(\sigma\chi_{Q_j^k+z}))^q w\,d\mu\Big)\Big(\frac{1}{|Q_j^k + z|_{\sigma d\nu}} \int\limits_{Q_j^k+z} |f|\sigma\,d\nu\Big)^q$$

$$= 2^q \sum_{\substack{k\in\mathbb{Z}\\ j\in J_k^m}} b_j^k(z)\Big(\frac{1}{a_j^k(z)} \int\limits_{Q_j^k+z} |f|\sigma\,d\nu\Big)^q,$$

where

$$b_j^k(z) = \int\limits_{\widetilde{Q}_j^k+z} (\mathfrak{M}^{dy}_{\varphi,m,z}(\sigma\chi_{Q_j^k+z}))^q w\,d\mu \quad\text{and}\quad a_j^k(z) = |Q_j^k + z|_{\sigma d\nu}.$$

We show that

$$\sum_{\substack{k\in\mathbb{Z},\, j\in J_k^m\\ Q_j^k\subset Q_0}} b_j^k(z) \le c a_0(z)^{q/p}$$

for all Q_0 in $\mathcal{D}_m$, z in X and m in $\mathbb{Z}$, where $a_0(z) = |Q_0 + z|_{\sigma d\nu}$.

If Q_0 is as above then

$$\sum_{\substack{k\in\mathbb{Z},\, j\in J_k^m\\ Q_j^k\subset Q_0}} b_j^k(z) = \sum_{\substack{k\in\mathbb{Z},\, j\in J_k^m\\ Q_j^k\subset Q_0}} \int\limits_{\widetilde{Q}_j^k+z} (\mathfrak{M}^{dy}_{\varphi,m,z}(\sigma\chi_{Q_j^k+z}))^q w\,d\mu$$

$$\le \int\limits_{\widehat{Q}_0+z} (\mathfrak{M}^{dy}_{\varphi,m,z}(\sigma\chi_{Q_0+z}))^q w\,d\mu \quad \text{since the } \widetilde{Q}_j^k\text{'s are disjoint in } k \text{ and } j$$

$$\le c\Big(\int\limits_{Q_0+z} \sigma\,d\nu\Big)^{q/p} \quad \text{by (3.42)}$$

$$= c a_0(z)^{q/p}.$$

Using Lemma 2.38 we obtain

$$\Big(\int\limits_{\widehat{X}_+} [\mathfrak{M}^{dy,R}_{\varphi,m,z} f]^q w\,d\mu\Big)^{1/q} \le C\Big(\int\limits_X |f|^p v\,d\nu\Big)^{1/p}$$

and letting R tend to infinity we have (3.41). ■

Now we are able to finish the proof of Theorem 1.3.

Proof of Theorem 1.3. If in (1.18) we set $f = \sigma\chi_{Q+z}$ where $Q \in \bigcup_m \mathcal{D}_m$ and $z \in X$, we obtain (1.19).

Now suppose that (1.19) holds. It is enough to show (1.18) with M_φ replaced by $M_{\varphi,m}$ and c independent of m, since we obtain the desired result by letting m tend to $-\infty$. By Lemma 3.32 it is enough to show (1.8) with M_φ replaced by $\mathfrak{M}^{dy}_{\varphi,m,z}$ and c independent of m and z. By Theorem 3.1 it is enough to show that

$$\Big(\int\limits_{\widehat{Q}+z} [\mathfrak{M}^{dy}_{\varphi,m,z}(\sigma\chi_{Q+z})]^q w\,d\mu\Big)^{1/q} \le C\Big(\int\limits_{Q+z} \sigma\,d\nu\Big)^{1/p},$$

for all $Q \in \bigcup_m \mathcal{D}_m$ and z in X.

Let Q_0 be a dyadic cube in $\mathcal{D}_m$ and let $\overline{Q}$ and $\widehat{Q}_0^*$ be the inner and outer balls, respectively, associated with Q_0. Let $z \in X$. If $x - z \in Q_0$ then

$$\frac{1}{\varphi(Q_0 + z)} \int\limits_{Q_0+z} \sigma\,d\nu \le \frac{c}{\varphi(\overline{Q}_0 + z)} \int\limits_{Q_0+z} \sigma\,d\nu$$

$$\le \frac{c}{\varphi(Q_0^* + z)} \int\limits_{Q_0^*+z} \chi_{Q_0+z}\sigma\,d\nu \le c M_{\varphi,m}(\chi_{Q_0+z}\sigma)(x,t),$$

since $x - z \in Q_0$ and the radius of Q_0^* is larger than t. Hence,

$$\mathfrak{M}^{dy}_{\varphi,m,z}(\sigma\chi_{Q_0+z})(x,t) \le c M_{\varphi,m}(\chi_{Q_0+z}\sigma)(x,t)$$

for all (x,t) in $\widehat{Q}_0 + z$ and, therefore,

$$\Big(\int\limits_{\widehat{Q}_0+z} [\mathfrak{M}^{dy}_{\varphi,m,z}(\sigma\chi_{Q_0+z})]^q w\,d\mu\Big)^{1/q}$$

$$\le c\Big(\int\limits_{\widehat{Q}_0+z} [M_{\varphi,m}(\sigma\chi_{Q_0+z})]^q w\,d\mu\Big)^{1/q} \le \Big(\int\limits_{Q_0+z} \sigma\,d\nu\Big)^{1/p},$$

by (1.19). This finishes the proof of Theorem (1.3). ■

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Minimal pairs of bounded closed convex sets

by

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Abstract. The existence of a minimal element in every equivalence class of pairs of bounded closed convex sets in a reflexive locally convex topological vector space is proved. An example of a non-reflexive Banach space with an equivalence class containing no minimal element is presented.

Let $X = (X, \tau)$ be a topological vector space over the field $\mathbb{R}$. Let $\mathcal{B}_\tau(X)$ (resp. $\mathcal{K}_\tau(X)$) be the collection of all bounded closed (resp. compact) convex subsets of X. For $A, B \subset X$, let

$$A + B := \{a + b \mid a \in A,\ b \in B\}$$

and let $A \dotplus B$ denote the closure of $A + B$. For $(A, B), (C, D) \in \mathcal{B}^2_\tau(X)$, let $(A, B) \sim (C, D)$ if and only if $A \dotplus D = B \dotplus C$. Let $(A, B) \le (C, D)$ if and only if $A \subset C$, $B \subset D$ and $(A, B) \sim (C, D)$. The relation “ $\sim$ ” is an equivalence relation by the ordered law of cancellation [5] in $\mathcal{B}^2_\tau(X)$ and “ $\le$ ” is an ordering in the equivalence class $[A, B]$ of any pair (A, B).

The study of minimal pairs of compact convex sets was stimulated by the development of quasidifferential calculus [1]. Any given quasidifferential may be identified with the equivalence class of a pair of compact convex sets (A, B), where A and B are, respectively, a super- and a sub-differential.

The existence of minimal pairs of compact convex sets in all topological vector spaces and the uniqueness up to translates in $\mathbb{R}^2$ were already proved in [2] and [4].

In this paper we extend our investigations to pairs of bounded closed convex sets.

THEOREM. *Let (X, τ) be a reflexive locally convex topological vector space. Every class $[A, B] \in \mathcal{B}^2_\tau(X)/\!\sim$ contains a minimal element (C, D) such that $(C, D) \le (A, B)$.*

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