Two-weight norm inequalities for maximal functions on homogeneous spaces and boundary estimates

by

SÉRGIO LUÍS ZANI (São Carlos)

Abstract. Let $D$ be an open subset of a homogeneous space $(X, d, \mu)$. Consider the maximal function

$$M_\nu f(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f| \, d\nu,$$

where the supremum is taken over all balls of the form $B = B(a(x), r)$ with $r > t(x) = d(x, \partial D)$, $a(x) \in \partial D$ is such that $d(a(x), x) < \frac{1}{2} t(x)$ and $\nu$ is a nonnegative set function defined for all Borel sets of $X$ satisfying the quasi-monotonicity and doubling properties. We give a necessary and sufficient condition on the weights $w$ and $\nu$ for the weighted norm inequality

$$\left( \int_D [M_\nu f]^{q} w \, d\mu \right)^{1/q} \leq c \left( \int_D [f^{p}]^q \, d\nu \right)^{1/p}$$

(0.1)

to hold when $1 < p < q < \infty$, $\sigma d\nu = r^{1\cdot\sigma - p} d\sigma$, $r d\sigma$ is a doubling weight, and $d\nu$ is a doubling measure, and give a sufficient condition for (0.1) when $1 < p \leq q < \infty$ without assuming that $\sigma$ is a doubling weight but with an extra assumption on $\sigma$. Another characterisation for (0.1) is also provided for $1 < p \leq q < \infty$ and $D$ of the form $Y \times (0, \infty)$, where $Y$ is a homogeneous space with group structure. These results generalize some known theorems in the case when $M_\nu$ is the fractional maximal function in $\mathbb{R}^n_{+}$, that is, when

$$M_\nu f(x, t) = M_\nu f(x, t) = \sup_{r > t} \frac{1}{|B(x, r)|^{1/\gamma}} \int_{B(x, r)} |f| \, d\nu,$$

where $(x, t) \in \mathbb{R}^{n+1}_{+}$, $0 < \gamma < 1$, and $\nu$ is a doubling measure in $\mathbb{R}^n$.

1. Introduction. We consider a homogeneous space $(X, d, \mu)$ where $d : X \times X \to [0, \infty)$ satisfies:

1991 Mathematics Subject Classification: Primary 42B25.

Key words and phrases: norm inequality, weight, maximal function, homogeneous space.

Research partly supported by CNPq.
1. \( d(x, y) = 0 \) if and only if \( x = y \).
2. \( d(x, y) = d(y, x) \) for any \( x, y \) in \( X \).
3. There exists \( A_0 \geq 1 \) such that
\[
(1.1) \quad d(x, y) \leq A_0 (d(x, z) + d(z, y)) \quad \text{for all } x, y, z \text{ in } X.
\]

\( \mu \) is a measure on the Borel subsets of \( X \) with the doubling property: there exists a positive constant \( A_1 \) such that
\[
(1.2) \quad \mu(B(x, r)) \leq A_1 \mu(B(x, 2r)) \quad \text{for all } x \text{ in } X \text{ and } r > 0,
\]
where \( B(x, r) = \{ y \in X : d(x, y) < r \} \) is the ball of radius \( r \) with center \( x \).

As mentioned in [CM], Macias and Segovia (cf. [MS]) have proved that given a quasi-metric \( d \) on \( X \) there exists a quasi-metric \( d' \) on \( X \) satisfying:
1. \( C^{-1}d'(x, y) \leq d(x, y) \leq Cd'(x, y) \) for all \( x, y \) in \( X \), where \( C \) is a positive constant independent of \( x \) and \( y \).
2. The balls with respect to \( d' \), i.e., \( B'(x, r) = \{ y \in X : d'(y, x) < r \} \), \( x \in X, r > 0 \), are open.

Since the order of magnitude of \( d(x, y) \) will be relevant for us, we may assume that the balls defined by \( d \) are open. We will also assume that all annuli \( A(z, r) = B(z, R) \setminus B(z, r) \) in \( X \) are nonempty for all \( 0 < r < R \). We immediately see that there exists \( c > 0 \) such that the diameter \( \delta \) of \( B(z, r) \) satisfies
\[
\delta(B(z, r)) = \sup \{ d(y, z) : y, z \in B(z, r) \} \geq cr
\]
for all \( z \in X \) and \( r > 0 \). In fact, since \( A = A(z, \frac{1}{2}r, r) \) is nonempty, there exists \( y \in A \). Thus, \( r > d(y, z) \geq \frac{1}{2}r \). Therefore, \( \delta(B(z, r)) \geq d(y, z) \geq \frac{1}{2}r \).

Now, for any \( x, y \in B(z, r) \) we have \( d(x, y) \leq A_0 (d(x, z) + d(z, y)) \leq 2A_0 r \). Thus,
\[
\frac{1}{2}r \leq \delta(B(z, r)) \leq 2A_0 r,
\]
and we say that the diameter of \( B(z, r) \) is equivalent to \( r \).

Let \( D \) be an open subset of \( X \) and \( c \geq 1 \). For each \( x \) in \( X \) let \( t(x) = d(x, \partial D) \) and select \( a(x) \) in \( \partial D \) such that \( d(x, a(x)) \leq ct(x) \). Note that if \( c' \geq 1 \) and \( a'(x) \) is in \( \partial D \) and satisfies \( d(x, a'(x)) \leq c't(x) \) then there exists \( K > 1 \) depending only on \( A_0, c \) and \( c' \) such that
\[
(1.3) \quad B(a(x), r) \subset B(a'(x), Kr) \subset B(a(x), K^2 r)
\]
for all \( x \) in \( X \) and \( r > t(x) \).

Denote by \( \mathcal{B} \) the set of all balls \( B = B(z, r) \) with \( z \) in \( \partial D \) and \( r > 0 \). Suppose \( \varphi \) is a nonnegative set function which is defined for all Borel sets of \( X \) and satisfies the following conditions:

\[\text{(1) (quasi-monotonicity)} \quad \text{There exists } c_0 > 0 \text{ such that } \varphi(B') \leq c_0 \varphi(B) \text{ whenever } B' \subset B, B', B \in \mathcal{B}.\]

\[\text{(2) (doubling)} \quad \text{There exists } c_0 > 0 \text{ such that } \varphi(B(z, 2r)) \leq c_0 \varphi(B(z, r)) \text{ for all } z \text{ in } \partial D \text{ and } r > 0.\]

Observe that (1) and (2) imply that for all \( \gamma > 0 \) there exists \( c(\gamma) > 0 \) such that
\[
(3) \quad \varphi(B(z, \gamma r)) \leq c(\gamma) \varphi(B(z, r)) \quad \text{for all } z \text{ in } \partial D \text{ and } r > 0.
\]

For \( \varphi, t(x) \) and \( a(x) \) as above, define
\[
M_{\varphi, a}(x) = \sup_{r > t(x)} \frac{1}{\varphi(B(a(x), r))} \int_B |f| \nu
\]
where \( B(a(x), r) = B(a(x), r) \cap \partial D, B(a(x), r) = \{ y \in X : d(y, a(x)) < r \} \) and \( \nu \) is a doubling measure on \( \partial D \).

Note that if \( c, c' \geq 1 \) and \( a(x), a'(x) \) are as above then it follows from (1.3) and the properties of \( \varphi \) that there exists \( C > 0 \) depending only on \( A_0, c_0, c \) and \( c' \) such that \( C^{-1}M_{\varphi, a'}f \leq M_{\varphi, a}f \leq CM_{\varphi, a'}f \) for all \( f \). For this reason we will fix \( c = 3/2 \) and write \( M_{\varphi, a} \) instead of \( M_{\varphi, a'} \).

Observe that when \( X = Y \times \mathbb{R}, (Y, \nu, \varphi) \) is a homogeneous space, \( D = Y \times (0, \infty) = \gamma_+ \) and \( \varphi(E) = |E \cap \gamma_+|_{1/\gamma} \), where \( | \cdot |_{\gamma} \) denotes the \( \nu \)-measure and \( 0 < \gamma < 1 \), then
\[
(1.5) \quad M_{\varphi}(x, t) = M_{\varphi}(x, t)
\]
\[
= \sup_{r > 1} \frac{1}{|B(x, r)|^{1-\gamma}} \int_{B(x, r)} |f(y)| \nu(dy), \quad (x, t) \in \gamma_+,
\]

where \( B(x, r) = \{ y \in Y : d(y, x, y) < r \} \) and \( r > 0 \), is the fractional maximal function. In [SWZ], Sawyer, Wheeden and Zhao showed that if \( 1 < p < q < \infty \), \( u(x, t) \) and \( v(y) \) are weights in \( \gamma_+ \) and \( Y \), respectively, and \( \sigma(y)v(dy) = v(y)^{1-\gamma} \sigma(dy) \) with \( p' = p/(p-1) \) and \( \nu(dy) \) satisfy the doubling condition and \( \mu \) is a Borel measure in \( \gamma_+ \), then the weighted norm inequality
\[
(1.6) \quad \left( \int_{\gamma_+} M_{\varphi}(x, t)^q u(x, t) \mu(x, t) \right)^{1/q} \leq C \left( \int_{Y} |f(y)|^p \nu(dy) \right)^{1/p}
\]
holds if and only if there exists a positive constant \( C \) such that
\[
(1.7) \quad \left( \int_B u \right)^{1-\gamma} \left( \int_B v \right)^{1/\gamma} \left( \int_B \sigma \right)^{1/p'} \leq C
\]
for all balls \( B \subset Y \), where \( B = B \times [0, \rho(B)) \) and \( \rho(B) \) is the radius of \( B \). The first result we have for \( M_{\varphi} \) is
THEOREM 1.1. Suppose $1 < p < q < \infty$. Let $w$ and $v$ be weights defined on $D$ and $\partial D$, respectively. Suppose that $\sigma dv = v^{1-\frac{1}{p}} dv$ is a doubling measure. Then the weighted norm inequality
\begin{equation}
\left( \int_D \left| M_{\varphi f} \right|^q w \, d\mu \right)^{1/q} \leq C \left( \int_{\partial D} |f|^p v \, dv \right)^{1/p}
\end{equation}
holds for all $f$, with $C$ independent of $f$, if and only if
\begin{equation}
\varphi(0)^{-1} \left( \int_B \sigma \, d\mu \right)^{1/q} \left( \int_B \sigma \, d\nu \right)^{1/p'} \leq C
\end{equation}
for all balls $B = B(x, r)$ with $z \in \partial D$, $r > 0$, $C$ independent of $B$, and $\bar{B} = B \cap D$, $\bar{\partial} = B \cap \partial D$.

In the next theorem we present a condition implying that the weighted norm inequality
\begin{equation}
\left( \int_D \left| M_{\varphi f} \right|^q w \, d\mu \right)^{1/q} \leq C \left( \int_{\partial D} |f|^p v \, dv \right)^{1/p}
\end{equation}
holds for all $f$, where $1 < p \leq q < \infty$, without imposing the doubling condition on $\sigma$ but with an additional condition on $\varphi$. The theorem is stated in terms of dyadic cubes (see Definition 2.11) centered at points of the boundary of a fixed open subset $D$ of a homogeneous space. For a similar result for off-centered maximal operators, see [W2].

THEOREM 1.2. Suppose $1 < p \leq q < \infty$ and let $w$ and $v$ be weights defined on $D$ and $\partial D$, respectively. Let $\sigma = v^{1-\frac{1}{p}}$. Suppose that there exist $C > 0$ and $0 < \varepsilon \leq 1$ such that
\begin{equation}
\varphi(Q) \cdot \left( \frac{Q^*}{Q} \right)^{\varepsilon} \leq C \frac{\varphi(Q^*)}{\varphi(Q)}
\end{equation}
for all dyadic cubes $Q'$ and $Q$ such that $Q' \subset Q$, where $Q^*$ is the exterior ball associated with $Q$. Suppose also that there exists $r > 1$ such that
\begin{equation}
\varphi(Q^*) \left( \frac{|Q^*|}{|Q|} \right)^{1/p'} \left( 3A_2^3 |\partial Q^*| \right)^{1/q} \left( \int_{\partial Q^*} \sigma^r \, dv \right)^{1/qp} \leq C
\end{equation}
for all dyadic cubes $Q$ centered at points of $\partial D$, where $Q^*$ is the exterior ball associated with $Q$, $Q = Q \cap D$ and $3A_2^3 Q^* = (3A_2^3 Q^*) \cap D$. Then there exists $c > 0$ such that
\begin{equation}
\left( \int_D \left| M_{\varphi f} \right|^q w \, d\mu \right)^{1/q} \leq C \left( \int_{\partial D} |f|^p v \, dv \right)^{1/p}
\end{equation}
for all $f$.

The next result is a characterization of weighted norm inequalities of type $(p, q)$ with $1 < p \leq q < \infty$, when $D$ has the form $X \times (0, \infty)$, where $(X, d, \nu)$ is a homogeneous space. We will suppose additional conditions on $X, d$ and $\nu$. Let $(X, d, \nu)$ be a homogeneous space and define $X = X \times \R$ and $X_+ = X \times (0, \infty)$. Assume that $X$ admits a group structure (not necessarily commutative) such that for all $x, y, z \in X$ and all balls $B \subset X$ we have
\begin{align}
d(x + z, y + z) &= d(x, y), \\
d(0, z) &= d(0, -x)
\end{align}
where $0$ is the identity element of $X$,
\begin{align}
\nu(-B + x) &= \nu(-B) \\
\nu(B) &= \nu(-B).
\end{align}
where $-B = \{ x : -x \in B \}$, and

Consider the following maximal function:
\begin{equation}
M_{\varphi f}(x, t) = \sup_{r > 1} \frac{1}{\varphi(B(x, r))} \int_{B(x, r)} |f| \, dv, \quad (x, t) \in \bar{X}_+,
\end{equation}
where $B(x, r) \subset X$ and $\varphi$ has the (quasi-)monotonicity and doubling properties and is defined for all Borel sets of $X$. The following theorem characterizes (1.10) with $M_{\varphi}$ as in (1.17), and is based on the main theorem of [S1].

THEOREM 1.3. Let $(X, d, \nu)$ be a homogeneous space, where $X$ has a group structure and $d$ and $\nu$ satisfy conditions (1.13)–(1.16). Let $w$ and $v$ be weights in $\bar{X}_+$ and $X$, respectively, and suppose that $1 < p \leq q < \infty$. Then
\begin{equation}
\left( \int_{\bar{X}_+} \left| M_{\varphi f} \right|^q w \, d\mu \right)^{1/q} \leq C \left( \int_X |f|^p v \, dv \right)^{1/p}
\end{equation}
for all $f$ if and only if
\begin{equation}
\left( \int_{Q_{x+z}} \left| M_{\varphi(\sigma Q_{x+z})} \right|^q w \, d\mu \right)^{1/q} \leq C \left( \int_{Q_{x+z}} \sigma \, d\nu \right)^{1/p}
\end{equation}
for all dyadic cubes $Q \subset X$ and $z$ in $X$, where $Q = Q \times l(Q)$ and $l(Q)$ is the edge length of $Q$.

The proofs of the above theorems can be found in Section 3.

2. Dyadic balls and dyadic cubes. In this section we will adapt the constructions of dyadic balls and dyadic cubes in homogeneous spaces as presented in [SW2] to obtain corresponding families of balls and cubes whose centers lie on the boundary of a fixed open subset of $X$. For a slightly
different concept of dyadic cubes in homogeneous spaces, see [C]. We will also prove some lemmas that will be used in the next section.

**Lemma 2.1.** Let $(X, d, \mu)$ be a separable homogeneous space, i.e., $X$ has a countable dense subset. Let $D$ be an open subset of $X$ and $\partial D$ be its boundary. Suppose $\{B_\alpha = B(z_\alpha, r_\alpha)\}_{\alpha \in A}$ is a family of balls with $z_\alpha \in \partial D$ and $B_\alpha \subset B(z, r)$ for some $z \in \partial D$ and $r > 0$. Then there exists a countable subcollection $\{B_i\}_{i \in I}$ of these balls such that

1. $B_i \cap B_j = \emptyset$ if $i \neq j$.
2. Given $\alpha \in A$ there exists $i \in I$ such that $B_\alpha \subset (A_0 + 4A_0^2)B_i$.
3. $\mu(\bigcup_{\alpha \in A} B_\alpha) \leq c \sum_{i \in I} \mu(B_i)$ where $c$ depends only on the positive constants $A_0$ and $A_1$, as in (1.1) and (1.2), respectively.

The proof is the same as that of Lemma 3.3 in [SW2] and will be omitted.

As in [SW2], p. 843, we now construct a sequence of balls in $X$ centered at points of $\partial D$.

**Definition 2.2.** Set $\lambda = A_0(2A_0 + 1)$, where $A_0$ is as in (1.1). For each integer $k$ select a sequence $\{x_j^k\}_j$ of points of $\partial D$ such that the sequence of balls $\overline{B}_j^k = B(x_j^k, \lambda^{-k-1})$ is maximal with respect to the property that $\overline{B}_j^k \cap \overline{B}_j^k = \emptyset$ if $i \neq j$. Set $B_j^k = \lambda \overline{B}_j^k$. We refer to $\{B_j^k\}_{k,j}$ as the collection of dyadic balls.

Note that if, for the same collection $\lambda = A_0(2A_0 + 1)$, we had chosen another sequence of points $\{x_j^k\}_j$ of $\partial D$ satisfying the same maximality condition above with the corresponding collection of dyadic balls $\{B_j^k\}_{k,j}$ then for each $k$ and $j$ there would be $i$ such that

$$\lambda^{-1} \overline{B}_i^k \subset B_j^k \subset \lambda B_j^k.$$ 

This readily implies that there exists $c > 1$ such that for all $k$ and $j$, $c^{-1} \varphi(B_j^k) \leq \varphi(B_j^k) \leq c \varphi(B_j^k)$ for some $i$.

For dyadic balls we have:

(2.3) Given $x \in \partial D$ and an integer $k$ there exists $j$ such that $B(x, \lambda^{k-1}) \subset B_j^k$.

(2.4) $\sum_j x_j^k \leq M$ for some $M > 0$ depending only on $A_0$ and $A_1$.

(2.5) $\overline{B}_j^k \cap \overline{B}_j^k = \emptyset$ for $i \neq j$, $k \in \mathbb{Z}$.

The statement (2.3) follows from the maximality of $\{\overline{B}_j^k\}$. Note that if $x \in B_j^k$, $1 \leq j \leq N$, then $\bigcup_{j=1}^N B_j^k \subset B(x, 2A_0 \lambda^k)$. Since the $B_j^k$ are pairwise disjoint in $j$, we obtain

$$N \mu(B(x, 2A_0 \lambda^k)) \leq c_{\Lambda} \sum_{j=1}^N \mu(B_j^k) = c_{\Lambda} \mu\left(\bigcup_{j=1}^N \overline{B}_j^k\right) \leq c_{\Lambda} \mu(B(x, 2A_0 \lambda^k)).$$

Hence, $N \leq c_{\Lambda}$ and this proves (2.4). Finally, (2.5) follows trivially from the definition of $B_j^k$.

**Lemma 2.6.** Suppose $B = \{B_\alpha\}_{\alpha \in A}$ is a family of dyadic balls as above. If $\{B_j\}_{j \in I}$ is a collection of maximal (with respect to inclusion) balls in $B$, then the balls $\{\overline{B}_j\}_{j \in I}$ are pairwise disjoint.

**Proof.** We know that $\overline{B}_j^k \cap \overline{B}_j^k = \emptyset$ if $i \neq j$. Suppose there exists $x \in \overline{B}_j^k \cap \overline{B}_j^k$, where $l < k$. Then $B_j^l \subset \overline{B}_j^k$. In fact, if $x \in B_j^l$ then

$$d(x_j^l, w) \leq A_0[d(x_j^l, x_0) + A_0] \leq A_0 \lambda^{k-l} \leq A_0 \lambda^{k-1} + A_0 \lambda^{k-l} \leq A_0 \lambda^{k-1} + A_0 \lambda^{k-1} = \lambda^k.$$ 

The result now follows from the maximality of the balls. $lacksquare$

The following lemma is a version of Lemma 3.21 of [SW2] that is suitable for boundary estimates.

**Lemma 2.7.** Suppose $(X, d)$ is a separable quasi-metric space and $D \subset X$ is an open subset. Then there exists $\lambda > 1$ depending only on $A_0$ such that for every $m \in \mathbb{Z}$, there are points $x_j^k \in \partial D$ and Borel sets $E_j^k$, $1 \leq j \leq n_k$, $k \geq m$, where $n_k \in \mathbb{N} \cup \{\infty\}$, such that

(2.8) $B(x_j^k, \lambda^k) \cap \partial D \subset E_j^k \subset B(x_j^k, \lambda^{k+1})$, $1 \leq j \leq n_k$, $k \geq m$,

(2.9) $\partial D = \bigcup_{k \geq m} E_j^k \subset \partial D$ for all $k \geq m$, $E_j^k \cap E_l^s = \emptyset$ if $i \neq j$,

and given $i, j, k, l$ with $m \leq k < l$, we have either

(2.10) $E_j^k \subset E_l^s$ or $E_j^k \cap E_l^s = \emptyset$.

**Definition 2.11.** For each $m \in \mathbb{Z}$ let $D_m = \{E_j^k : k \geq m, 1 \leq j \leq n_k\}$. We refer to the elements of $D_m$ as dyadic cubes.

**Definition 2.12.** Let $Q = E_j^k$ in $D_m$ and write $\overline{Q} = B(x_j^k, \lambda^k) \cap \partial D$ and $Q^* = B(x_j^k, \lambda^{k+1})$. We will refer to $Q$ and $Q^*$ as the inner and outer balls, respectively, associated with $Q$. The diameter of $Q$ will be called the edgelength of $Q$ and will be denoted by $l(Q)$.

Note that since any annulus is nonempty, $\lambda^k \leq$ edgelength of $Q \leq \lambda^{k+1}$.

**Proof of Lemma 2.7.** We will only define the sets $E_j^k$ and the rest of the proof will be omitted for it is essentially the same as that of Lemma 3.21 in [SW2].

Set $\lambda = 8A_0^3$. For each $k \in \mathbb{Z}$ choose a sequence $\{x_j^k\}_{1 \leq j \leq n_k}$ of points on $\partial D$ maximal with respect to the property that the balls $\{B(x_j^k, 3A_0 \lambda^k)\}$, $1 \leq j \leq n_k$, are pairwise disjoint. Notice that since $X$ is separable, the cardinality of $\{x_j^k\}$ is at most countable.
Fix \( m \in \mathbb{Z} \) and define
\[
E_1^m = B(z_1^m, 6A_0^3 \lambda^m) - \bigcup_{i \neq 1} B(z_i^m, \lambda^m)
\]
and for \( j > 1 \),
\[
E_j^m = B(z_j^m, 6A_0^3 \lambda^m) - \bigcup_{i,j} B(z_i^m, \lambda^m) - \bigcup_{i < j} E_i^m.
\]
Observe that \( E_i^m \cap E_i^m = \emptyset \) if \( i \neq j \), \( B(z_j^m, \lambda^m) \subset E_j^m \subset B(z_j^m, \lambda^{m+1}) \) and \( \partial D = \bigcup_j E_j^m \cap \partial D \).

We now define \( \{E^k_j\} \) for \( k \geq m \). Suppose \( \{E_j^k\}_{1 \leq j \leq n_k} \) has been defined for \( k = m, m+1, \ldots, k \), for some \( k \geq m \) so as to satisfy
\[
B(z_j^l, \lambda^l) \cap \partial D \subset E_j^k \subset B(z_j^l, \lambda^{l+1}), \quad m \leq l < k,
\]
and \( m \leq l_1 < l_2 \leq k \) and any \( i, j \) we have either
\[
E_j^k \cap E_i^l = \emptyset \quad \text{or} \quad E_i^l \subset E_j^k.
\]
For any \( R > 0 \) and \( 1 \leq j < n_{k+1} \), let
\[
\tilde{B}(z_j^{k+1}, R) = \bigcup\{E_i^k : E_i^k \cap B(z_j^{k+1}, R) \neq \emptyset\}.
\]
Define
\[
E_i^{k+1} = \tilde{B}(z_i^{k+1}, 6A_0^3 \lambda^{k+1}) - \bigcup_{i \neq j} \tilde{B}(z_j^{k+1}, \lambda^{k+1})
\]
and for \( j > 1 \),
\[
E_j^{k+1} = \tilde{B}(z_j^{k+1}, 6A_0^3 \lambda^{k+1}) - \bigcup_{i,j} \tilde{B}(z_i^{k+1}, \lambda^{k+1}) - \bigcup_{i < j} E_i^{k+1}.
\]

The next lemma, which is an application of the Marcinkiewicz interpolation theorem, is analogous to Lemma 3.15 of [SW2].

**Lemma 2.16.** Let \( D \subset X \) be an open subset and \( \partial D \) its boundary. Suppose \( a(B) \) is a nonnegative set function, defined for all balls \( B \in \mathcal{B} \), that satisfies
\[
a(B(z, 2r)) \leq ca(B(z, r)) \quad \text{for all} \quad z \in \partial D \quad \text{and} \quad r > 0
\]
and
\[
\sum_{B \in \Omega} a(B) \leq ca(B_0)
\]
whenever \( \Omega \) is a collection of pairwise disjoint balls \( B = B(z, r) \) in \( \mathcal{B} \) contained in a ball \( B_0 = B(x_0, r_0) \) of \( \mathcal{B} \). In addition, assume that \( u(z) \geq 0 \) on \( \partial D \), \( \beta \geq 1 \) and \( \Gamma \) is a countable collection of dyadic balls centered in \( \partial D \) such that
\[
\int_B u \, d\nu \leq ca(B) \quad \text{for all} \quad B \in \Gamma,
\]
where \( \tilde{B} = B \cap \partial D \) and \( d\nu \) is a doubling measure on \( \partial D \), i.e.,
\[
|B(z, 2r) \cap \partial D| \leq c|B(z, r) \cap \partial D| \quad \text{for all} \quad z \in \partial D \quad \text{and} \quad r > 0.
\]
Furthermore, assume that
\[
\sum_{B \in \Gamma} a(B)^\beta \leq ca(B_0)^\beta \quad \text{for all} \quad B_0 \in \Gamma.
\]

Then
\[
\left( \sum_{B \in \Gamma} a(B)^\beta \left( \frac{1}{a(B)} \int_B f \, d\nu \right)^t \right)^{1/t} \leq c_{s, \beta} \left( \int_{\partial D} f^s \, d\nu \right)^{1/s}
\]
for all \( f \geq 0 \) on \( \partial D \) and \( t = s\beta, 1 < s < \infty \).

**Proof.** Consider the map
\[
f \mapsto \left\{ \frac{1}{a(B)} \int_B f \, d\nu \right\}_{B \in \Gamma}.
\]
For \( f \in L^\infty(\partial D, d\nu) \), we have
\[
\left| \frac{1}{a(B)} \int_B f \, d\nu \right| \leq ||f||_{\infty} \frac{1}{a(B)} \int_B f \, d\nu \leq c||f||_{\infty} \quad \text{by (2.19)}.
\]
Now, suppose \( f : \partial D \rightarrow [0, \infty) \) is bounded with support contained in \( B_1 \cap \partial D \), where \( B_1 = B(z_1, r_1) \) with \( z_1 \in \partial D \). Let \( \Gamma' \) be a finite subset of \( \Gamma \). For \( \lambda > 0 \), let \( \{Q_j\}_{j \in \Gamma'} \) be the maximal dyadic balls \( B \) in \( \Gamma' \) such that
\[
a(B)^\beta \leq \frac{1}{a(B)} \int_B \right| f \, d\nu > \lambda
\]
Note that \( B \cap B_1 \neq \emptyset \) for any ball \( B \) that satisfies (2.22) since the support of \( f \) is contained in \( B_1 \cap \partial D \). Thus,
\[
\sum_{B \in \Gamma'} \sum_{B \in Q_j} a(B)^\beta \leq c \sum_{j \in \Gamma'} \sum_{B \in Q_j} a(Q_j)^\beta \quad \text{by (2.20)}
\]
(2.24) \[ \leq c \left( \sum_{j \in J'} a(Q_j) \right)^\beta \] since \( \beta \geq 1 \)

(2.25) \[ \leq c \left( \sum_{j \in J'} a(Q_j) \right)^\beta \] by (2.17)

where \( Q_j \) denotes the ball concentric with \( Q_j \) and with radius \( r(Q_j)/(A_0(2A_0 + 1)) \). Recall that the \( Q_j \) are pairwise disjoint (see Lemma 2.6).

Now, since \( J' \) is finite, \( \bigcup_{j \in J'} \bar{Q}_j \subset B' \) for some ball \( B' \) centered at \( \partial D \). Therefore, by Lemma 2.1, there exists a pairwise disjoint subcollection \( \{Q_i\}_{i \in I'} \) of the dyadic balls \( \{Q_j\}_{j \in J'} \) such that every \( Q_j \) is contained in some \( Q_i, i \in I' \), where \( Q_i \) is the ball concentric with \( Q_i \) and with radius \( A_0(4A_0 + 1)r(Q_i) \). Thus,

\[ \sum_{j \in J'} a(Q_j) \leq \sum_{i \in I'} \sum_{Q_i \subset Q_j} a(Q_j) \leq c \sum_{i \in I'} a(Q_i)^\beta \] by (2.18) and Lemma 2.6

\[ \leq c \sum_{i \in I'} a(Q_i) \] by (2.17)

\[ \leq \frac{c}{\lambda} \sum_{i \in I'} \int_{Q_i} f u \, dv \leq \frac{c}{\lambda} \int_{\partial D} f u \, dv, \]

since \( \{Q_i\}_{i \in I'} \) is a collection of pairwise disjoint balls. Thus, since \( I' \) was an arbitrary finite subset of \( I \) and the constant \( c \) above depends only on \( A_0 \) and \( A_1 \), we conclude that the map

\[ f \mapsto \left\{ \frac{1}{a(B)} \int_B f u \, dv \right\}_{B \in I'} \]

is both of weak-type \((\infty, \infty)\) and weak-type \((1, \beta)\), i.e., it takes \( L_{\infty}(\partial D, u \, dv) \) to weak \( L^\beta(I, a(B)) \). Therefore, applying the Marcinkiewicz interpolation theorem we obtain (2.21).

We will need the following analogue of Lemma 2.10 in [SW2]:

**Lemma 2.23.** Let \( u(x) \geq 0 \) on \( \partial D \) and for each \( m \in \mathbb{Z} \), let \( \{Q_i\}_{i \in I} \) be a countable collection of dyadic cubes in \( \mathcal{D}_m \). Suppose that there exists a sequence \( \{a_i\}_{i \in I} \) of positive numbers and \( \beta \geq 1 \) satisfying

(2.27) \[ \int_{Q_i} u \, dv \leq \alpha a_i, \]

(2.28) \[ \sum_{j : Q_j \subset Q_i} a_j^\beta \leq \alpha a_i^\beta, \]

where \( \alpha > 0 \) is independent of \( i \) and \( m \). Then

\[ \left( \sum_{i \in I} \frac{f u}{a_i} \int_{Q_i} f u \, dv \right)^{1/\beta} \leq c \left( \int_{\partial D} f u \, dv \right)^{1/p} \]

for all \( f \geq 0 \) on \( \partial D \) where \( 1 < p < \infty, \beta = p \) and \( c \) is independent of \( m \).

**Proof.** The map \( f \mapsto \{f u/a_i\}_{Q_i} \) takes \( L_p^\infty(\partial D) \) into \( L_p^\infty(I) \) by condition (2.27) and \( L_p^\infty(\partial D) \) into weak \( L_p^\infty(I) \) by condition (2.28), as we now show. Suppose \( f \) is bounded with compact support in \( \partial D \). Let \( t > 0 \) and let \( \{Q_j\}_{j \in J} \) be the maximal dyadic cubes from the collection \( \{Q_i\}_{i \in I} \) such that \( \{f u/a_i\}_{Q_i} \geq t \) (we may assume the collection \( \{Q_i\}_{i \in I} \) is finite). Then

(2.29) \[ \sum_{i \in I} \left( \frac{1}{a_i} \int_{Q_i} f u \, dv \right)^{1/\beta} \leq \sum_{j \in J} \sum_{Q_i \subset Q_j} a_j^\beta \]

(2.30) \[ \leq \frac{c}{\lambda} \sum_{j \in J} \int_{Q_j} f u \, dv \]

(2.31) \[ \leq c \left( \sum_{j \in J} a_j^\beta \right) \]

(2.32) \[ \leq \frac{1}{\lambda} \sum_{j \in J} \int_{Q_j} f u \, dv \]

(2.33) \[ \leq \frac{1}{\lambda} \int_{\partial D} f u \, dv, \]

since the maximal dyadic cubes are disjoint. The Marcinkiewicz interpolation theorem now completes the proof of Lemma 2.26.

The next lemma can be found in [W1] and its proof is omitted.

**Lemma 2.34 (Reverse doubling).** Suppose \( \mu \) is a doubling measure on a homogeneous space \( X \). Then \( \mu \) satisfies the reverse doubling condition: there exist \( \alpha, \beta > 1 \) such that \( |B(x, r\tau)|_{\mu} \geq \beta |B(x, r)|_{\mu} \) for all \( x \in X \) and \( r > 0 \).

We will need the concept of dyadic cubes in \( X \). The construction of these sets can be found in Lemma 3.21 of [SW2]. We will need the following lemma of [SW2] (Lemma 3.21 there):

**Lemma 2.35.** Suppose \( (X,d) \) is a separable quasi-metric space. Let \( \lambda = \mathcal{A}_{B_{\infty}} \). Then for every \( m \in \mathbb{Z} \) there are points \( x_j^m \) in \( X \) and Borel sets \( E_j^m \), \( 1 \leq j < n_k, k \geq m \), where \( n_k \in \mathbb{N} \cup \{\infty\} \), such that

(2.36) \[ \mu(\lambda, \lambda^k) \subset \mathcal{D}_\mathcal{E} \subset E_j^m \subset B(x_j^m, \lambda^{k+1}), \quad 1 \leq j < n_k, k \geq m, \]
(2.37) \( X = \bigcup_{j} E_j^k \) for all \( k \geq m \), \( E_j^k \cap E_i^l = \emptyset \) if \( i \neq j \),

and given \( i, j, k, l \) with \( m \leq k < l \), we have either

\[ E_j^k \subseteq E_i^l \quad \text{or} \quad E_i^l \cap E_j^k = \emptyset. \]

Recall that \( D_m = \{ E_j^k : k \geq m, 1 \leq j \leq n_k \} \). If \( Q = E_j^k \in D_m \), let \( l(Q) \)

be the edgelength of \( Q \) and define \( \hat{Q} = Q \times (0, l(Q)) \).

We will need the following lemma (cf. Lemma 3.20 of [SW1]).

**Lemma 2.38.** Let \( \mu(z) > 0 \) on \( X \). For each \( m \in \mathbb{Z} \), suppose that \( \{Q_i\} \) is a countable collection of dyadic cubes in \( D_m \), and for each \( z \in X \), \( \{a_i(z)\}_{i \in I} \) and \( \{b_i(z)\}_{i \in I} \) are positive numbers satisfying

\[
\int_{Q_i + z} u \, d\mu \leq a_i(z) \quad \text{for all } i \in I \text{ and } z \in X,
\]

\[
\sum_{j : Q_j \subset Q_i} b_j(z) \leq c_i(z) \quad \text{for all } i \in I \text{ and } z \in X,
\]

with \( c \) independent of \( m \). Then

\[
\left( \sum_{i \in I} b_i(z) \left( \frac{1}{a_i(z)} \int_{Q_i + z} g \, d\mu \right) \right)^{\frac{1}{q}} \leq c \left( \frac{1}{X} \int g^q \, d\mu \right)^{\frac{1}{q}}
\]

for all \( g \geq 0 \) on \( X \) and \( z \in X \), where \( 1 < q < \infty \) and \( c_q \) is independent of \( z, m \) and \( g \).

**Proof.** The map \( g \mapsto \{(1/a_i(z)) \int_{Q_i + z} g \, d\mu\} \) takes \( L^q(X) \) into \( L^{q'}(I) \) by condition (2.39) and \( L^q(X) \) into weak \( L^{q'}(I) \) by condition (2.40), as we now show. If \( g \) is bounded with compact support in \( X \) and \( t > 0 \), let \( \{Q_j\}_{j \in J} \) be the maximal dyadic cubes from the collection \( \{Q_j\}_{i \in I} \) such that \( \{(1/a_i(z)) \int_{Q_i + z} g \, d\mu\} > t \) (we may assume the collection \( \{Q_j\}_{j \in J} \) is finite). Then

\[
\sum_{j : (1/a_j(z)) \int_{Q_j + z} g \, d\mu > t} b_j(z) \leq \sum_{j \in J} a_j(z) \quad \text{by (2.40)}
\]

\[
\geq c \sum_{j \in J} a_j(z)
\]

\[
\leq c \sum_{j \in J} \int_Q g \, d\mu \quad \text{by (2.40)}
\]

\[
\leq c \int_{Q} g \, d\mu.
\]

(2.41)

(2.42)

(2.43)

(2.44)

since the maximal dyadic cubes are disjoint. The Marcinkiewicz interpolation theorem now completes the proof of Lemma 2.38. \( \blacksquare \)

3. **Proofs of the main theorems.** In this section we will prove the theorems introduced in Section 1.

**Proof of Theorem 1.1.** First, we prove that (1.8) implies (1.9). Given \( z \in B \) and \( r > 0 \), let \( B = B(z, r) \) and set \( f(y) = \sigma(y)X_B(y) \). Thus,

\[
|\int_D M^q_g(\sigma_XB)w \, d\mu|^{\frac{1}{q}} \leq c \left( \int_{D} |\sigma_XBv| w \, d\nu \right)^{\frac{1}{p}}
\]

\[
= c \left( \int_{B} |\sigma v| w \, d\nu \right)^{\frac{1}{p}} = c|\hat{B}|^{\frac{1}{p}}.
\]

For \( x \in X \) let \( a(x) \) and \( t(x) \) be as in the definition of \( M_\varphi \). We claim that \( B = B(z, r) \subseteq B(a(x), 3A_0^2[r + d(z, x)]) \). Indeed, if \( y \in B \) then

\[
d(y, a(x)) \leq A_0^2[d(y, z) + d(z, x) + d(x, a(x))]
\]

\[
\leq A_0^2[r + d(z, x) + \frac{3}{2}d(z, x)] \leq 3A_0^2[r + d(z, x)],
\]

which proves our claim. Obviously \( 3A_0^2[r + d(z, x)] \geq t(x) \). Hence,

\[
M_\varphi(\sigma_XB)(x) \geq \frac{1}{\varphi(B(a(x), 3A_0^2[r + d(z, x)]))} \int_{B(a(x), 3A_0^2[r + d(z, x)])} |\sigma_XBv| \, dv
\]

(3.2)

\[
|\hat{B}|^{\frac{1}{p}} \leq \varphi(B(a(x), 3A_0^2[r + d(z, x)]))^{\frac{1}{p}}
\]

for any \( x \in X \) and \( r > 0 \).

We claim that

\[
B(a(x), 3A_0^2[r + d(z, x)]) \subseteq B(z, 18A_0^2r)
\]

for \( x \in B = B(z, r) \). In fact, for \( y \in B(a(x), 3A_0^2[r + d(z, x)]) \), we have

\[
d(y, z) \leq A_0^2[d(y, a(x)) + d(a(x), x) + d(z, x)]
\]

\[
\leq A_0^2[3A_0^2[r + d(z, x)] + t(x) + d(z, x)]
\]

\[
\leq 9A_0^2[r + d(z, x)] \quad \text{since } t(x) = d(z, \partial B) \leq d(z, x)
\]

\[
\leq 18A_0^2r,
\]

as claimed. Thus, \( \varphi(B(a(x), 3A_0^2[r + d(z, x)]) \leq c\varphi(B(z, r)) \). Therefore, it follows from (3.2) that

\[
M_\varphi(\sigma_XB)(x) \geq \frac{|\hat{B}|^{\frac{1}{p}}}{\varphi(B(z, r))}
\]

for all \( x \in B \).

Combining (3.1) and (3.3) we obtain

\[
\varphi(B)^{-\frac{1}{p}}|\hat{B}|^{\frac{1}{p}} \leq |\hat{B}|^{\frac{1}{p}}
\]

(3.4)

\[
= \varphi(B)^{-\frac{1}{p}}|\hat{B}|^{\frac{1}{p}} \leq |\hat{B}|^{\frac{1}{p}}.
\]

Thus, if \( |\hat{B}| \) is neither 0 nor \( \infty \), we obtain (1.9) from (3.4). If \( |\hat{B}| = 0 \) then (1.9) is obvious. Suppose \( |\hat{B}| = \infty \). For any \( \epsilon > 0 \) we have \( |v(y) + \epsilon|^{1-p} \leq |v(y)|^{1-p} \), so
\( \epsilon^{1-p'} < \infty \) and, therefore, \(|\hat{B}|_{x_{\epsilon}} \leq \epsilon^{1-p'} |\hat{B}|_v \) where \( \sigma_v(y) = [v(y) + \epsilon^{1-p'}] \). Hence, since (1.8) implies (1.8) with \( u \) replaced by \( v + \epsilon \), we obtain (1.9) with \( v \) replaced by \( v + \epsilon \), i.e.,
\[
\varphi(B)^{-1} |\hat{B}|_w^{1/q} |\hat{B}|_v^{1/p'} \leq C.
\]

Letting \( \epsilon \) tend to 0, we conclude that \( w = 0 \) a.e. (\( \mu \)) since \( |\hat{B}|_w = \infty \). Thus, (1.9) follows immediately.

Conversely, (1.9) implies (1.8). To see this, let
\[
m^{q,v}_w f(x) = \sup_{\text{dyadic ball } B} \frac{1}{\varphi(B)} \int_B |f| \, dv, \quad x \in X,
\]
where a dyadic ball is a ball as in Definition 2.2. We claim that there exists \( c > 0 \) such that
\[
M^q_v f(x) \leq cm^{q,v}_w f(x)
\]
for all \( f \) and \( x \in X \).

Given \( B = B(a(x), r) \) with \( r > t(x) \), let \( k \in \mathbb{Z} \) be such that \( \lambda^{k-1} < \frac{3}{2} r \leq \lambda^k \) (as in Lemma 2.6). Thus, there exists a dyadic ball \( B^{k+1}_j \) of radius \( \lambda^{k+1} \) such that \( B \subset B(a(x), \frac{3}{2} r) \subset B(a(x), \lambda^k) \subset B^{k+1}_j \). Clearly, \( \varphi(B) \leq c \varphi(B^{k+1}) \). We show that \( \varphi(B^{k+1}) \leq c \varphi(B) \). Indeed, if \( y \in B^{k+1}_j = B(x^{k+1}_j, \lambda^{k+1}) \) then
\[
d(y, a(x)) \leq A_0 [d(y, x^{k+1}_j) + d(x^{k+1}_j, a(x))] \leq 2A_0 \lambda^{k+1}.
\]
Thus, \( B^{k+1}_j \supset 3A_0 \lambda^2 B \), and, hence, \( \varphi(B^{k+1}) \leq c \varphi(B) \). Therefore,
\[
\frac{1}{\varphi(B)} \int_B |f| \, dv \leq \frac{1}{\varphi(B^{k+1})} \int_{B^{k+1}_j} |f| \, dv.
\]
Clearly, \( x \) and \( a(x) \) lie in \( B^{k+1}_j \). Hence, our claim follows from (3.6).

Due to (3.5) it is enough to prove (1.8) with \( M^q_w \) replaced by \( m^{q,v}_w \). We may assume \( f \geq 0 \). Let \( D_k = \{ x \in D : m^{q,v}_w(f)(x) > 2^k \} \), \( k \in \mathbb{Z} \). Thus,
\[
\int_D [m^{q,v}_w(f)]^q w \, d\mu = \sum_k \int_{D_k \setminus D_{k+1}} [m^{q,v}_w(f)]^q w \, d\mu \leq 2^k \int_{D_k \setminus D_{k+1}} w \, d\mu.
\]
It follows from the definition of \( m^{q,v}_w \) that if \( x \in D_k \) then there exists a dyadic ball \( B \) with \( x, a(x) \in B \) satisfying
\[
\frac{1}{\varphi(B)} \int_B f \, dv > 2^k.
\]
Let \( \{ B_{k,j} \} \) be the collection of maximal dyadic balls with respect to inclusion which satisfy (3.7). Observe that \( D_k \subset \bigcup_j B_{k,j} \). In fact, if \( x \in D_k \) then
there exists a dyadic ball \( B \) with \( x, a(x) \in B \) satisfying (3.7). Hence, by the maximality of the \( B_{k,j} \), there exists \( j \) such that \( x \in B \subset B_{k,j} \).

Thus,
\[
\sum_k 2^k \int_{D_k \setminus D_{k+1}} w \, d\mu \leq \sum_{k,j} \int_{B_{k,j}} \left[ \frac{1}{\varphi(B_{k,j})} \int_{B_{k,j}} f \, d\mu \right]^q w \, d\mu
\]
\[
= \sum_{k,j} a_{k,j} \int_{B_{k,j}} f \, d\mu
\]
where
\[
a_{k,j} = \int_{B_{k,j}} \sigma \, dv \quad \text{and} \quad b_{k,j} = \left[ \frac{a_{k,j}}{\varphi(B_{k,j})} \right]^q \int_{B_{k,j}} w \, d\mu.
\]
From (1.9) it follows that
\[
b_{k,j} = a_{k,j}^{\frac{q}{p}} \varphi(B_{k,j})^{-q} |B_{k,j}|_\sigma \leq c a_{k,j}^{\frac{q}{p}} |B_{k,j}|_{\sigma}^{q/p} = c a_{k,j}^{q/p}.
\]

We claim that there exists \( c > 0 \) such that
\[
\sum_{B_{k,j} \subset B_{s,t}} a_{k,j}^{q/p} \leq c a_{s,t}^{q/p} \quad \text{for all } s, t.
\]
Indeed, with \( \lambda = A_0(2A_0 + 1) \), we have
\[
\sum_{B_{k,j} \subset B_{s,t}} a_{k,j}^{q/p} \leq \sum_{B_{k,j} \subset B_{s,t}} |B_{k,j}|_{\sigma}^{q/p} \leq \sum_{l=0}^{\infty} \sum_{\text{dyadic ball } \subset B_{s,t}} |B_{k,j}|_{\sigma}^{q/p} \leq c \sum_{l=0}^{\infty} \sum_{\text{dyadic ball } \subset B_{s,t}} (\delta |\hat{B}_{s,t}|) \sigma^{1/q-1} |B_{k,j}|_{\sigma}
\]
for some \( \delta < 1 \) since \( \sigma dv \) is doubling and, therefore, the reverse doubling condition (see Lemma 2.34) is satisfied. Hence
\[
\sum_{B_{k,j} \subset B_{s,t}} a_{k,j}^{q/p} \leq c \sum_{l=0}^{\infty} (\delta |\hat{B}_{s,t}|) \sigma^{1/q-1} M |\hat{B}_{s,t}|_{\sigma} \quad \text{by (2.4)}
\]
\[
= c M |\hat{B}_{s,t}|_{\sigma} \sum_{l=0}^{\infty} \delta^{q/(q-1)} \sigma = c |\hat{B}_{s,t}|_{\sigma}^{q/p} = c a_{s,t}^{q/p}
\]
since \( 0 < \delta < 1 \) and \( q > p \), and this finishes the proof of our claim.

We now obtain (1.8) from Lemma 2.16 with \( a(B_{k,j}) = a_{k,j}, \beta = q/p, \gamma = \{B_{k,j}\} \), \( t = q, s = p, u = \sigma \), and \( \Gamma = \{B_{k,j}\} \) since \( \sigma dv \) is a doubling measure. This concludes the proof of Theorem 1.1.
Proof of Theorem 1.2. Let \( \mathcal{B} = \{ B = B(z, r) : z \in \partial D, r > 0 \} \). Given \( x \in D \) and \( r > t(x) \), let \( B = B(a(x), r) \) and \( B_0 = B(a(x), \frac{2}{3}r) \). Then, clearly, \( B \subset B_0 \) and \( x \in B_0 \) since \( d(x, a(x)) \leq \frac{2}{3}t(x) < \frac{2}{3}r \). Thus, since \( \varphi \) is doubling we have

\[
\frac{1}{\varphi(B)} \int_B |f| \, dv \leq \frac{c}{\varphi(B_0)} \int_{B_0} |f| \, dv \leq c \sup_{B \in \mathcal{B}} \int_{a(x) \in B} |f| \, dv = c m_{\varphi, f}(x), \quad \text{say},
\]

and hence,

\[
M_{\varphi} f \leq c m_{\varphi, f}.
\]

For each \( m \in \mathbb{Z} \), define

\[
m_{\varphi, m, f}(x) = \sup_{B \in \mathcal{B}} \int_B |f| \, dv,
\]

where \( \lambda = 8.45^m \) as in Lemma 2.7.

It will be convenient to majorize \( m_{\varphi, m, f} \) by a suitable dyadic operator defined in terms of the dyadic cubes \( Q \in \mathcal{D}_m \) that were introduced in Lemma 2.7. Given \( B = B(z, r) \in \mathcal{B}, r > \lambda^m \), select \( k > m \) such that \( \lambda^k \leq r < \lambda^{k+1} \). Suppose \( B \cap Q = \emptyset, Q = E_j \in \mathcal{D}_m \) and \( B = B(z, r) \) as above. Then, if \( y \in B \) and \( x \in B \cap Q \), we have

\[
d(y, z_j) \leq A_0 |d(y, z) + d(z, x) + d(x, z_j)| \leq A_0 \lambda^{k+1} + \lambda^{k+1} + \lambda^{k+1}
\]

since \( Q \subset B(z_j, \lambda^{k+1}) \) (see Lemma 2.7).

Thus,

\[
B(z, r) \subset B(z_j, 3A_0 \lambda^{k+1}).
\]

Also, if \( u \in Q \), we have

\[
d(u, z) \leq A_0 |d(u, z_j) + d(z_j, z) + d(z, x)| \leq A_0 \lambda^{k+1} + \lambda^{k+1} + \lambda^{k+1} = 3A_0 \lambda^{k+1}.
\]

Thus,

\[
Q \subset E_j \subset B(z_j, 3A_0 \lambda^{k+1}).
\]

Now, if \( E_j \not\subset \emptyset \), \( \alpha = 1, \ldots, N \), we obtain

\[
N |\hat{B}(z, r)| \leq c \sum_{\alpha=1}^N |\hat{B}(z_{\alpha}, 3A_0 \lambda^{k+1})| \quad \text{by (3.8)}
\]

(3.10)

\[
\leq c \sum_{\alpha=1}^N |\hat{B}(z_{\alpha}, \lambda^k)| \quad \text{since \( \nu \) is doubling}
\]

(3.11)

\[
\leq c \sum_{\alpha=1}^N |\hat{B}(z_{\alpha}, \lambda^k)| \quad \text{by Lemma 2.7}
\]

(3.12)

\[
= c \sum_{\alpha=1}^N |\hat{B}(z_{\alpha}, \lambda^k)| \quad \text{since the cubes are disjoint}
\]

(3.13)

\[
\leq c |\hat{B}(z, 3A_0 \lambda^{k+1})| \quad \text{by (3.9)}
\]

(3.14)

\[
\leq c |\hat{B}(z, r)| \quad \text{since \( \nu \) is doubling}
\]

(3.15)

Therefore, \( N \leq c \). Since \( \partial D = \bigcup_j E_j \) for all \( k \geq m \) by Lemma 2.7, any ball \( B \in \mathcal{B} \) with \( r(B) > \lambda^m \) is contained in a finite union of dyadic cubes \( \{ E_j \}_{j=1}^N \), say, and the number of such dyadic cubes is bounded by a constant that depends only on the doubling constant of \( \nu \). Thus, for any \( B \in \mathcal{B} \) with \( r(B) > \lambda^m \) and \( \lambda^k \leq r < \lambda^{k+1} \), we have \( \hat{B} \subset \bigcup_{\alpha=1}^N \hat{E}_{\alpha} \) with \( \hat{B} \cap \hat{E}_{\alpha} \neq \emptyset \) for some \( N \leq c. \) Hence, for some \( 1 \leq \alpha_0 \leq N \),

(3.16)

\[
\frac{1}{\varphi(B)} \int_B |f| \sigma \, dv \leq \frac{1}{\varphi(B)} \int_{\bigcup_{\alpha=1}^N \hat{E}_{\alpha}} |f| \sigma \, dv \leq c \sum_{\alpha=1}^N \frac{1}{\varphi(B)} \int_{\hat{E}_{\alpha}} |f| \sigma \, dv
\]

\[
\leq c \frac{1}{\varphi(B)} \int_{\hat{E}_{\alpha_0}} |f| \sigma \, dv.
\]

(3.17)

It follows from (3.9) with \( j \) replaced by \( j_{\alpha_0} \) that if \( B \) and \( E_0 \) are as above then \( E_{\alpha_0} \subset B(z, 3A_0 \lambda^{k+1}) \) and, therefore, \( \varphi(E_{\alpha_0}) \leq c \varphi(B) \). Since \( \varphi \) is doubling, \( E_{\alpha_0} \) is the outer ball associated with \( E_{\alpha_0} \). Hence, it follows from (3.16) that

(3.18)

\[
\frac{1}{\varphi(B)} \int_B |f| \sigma \, dv \leq c \frac{1}{\varphi(E_{\alpha_0})} \int_{E_{\alpha_0}} |f| \sigma \, dv.
\]

Since \( B \cap E_{\alpha_0} \neq \emptyset \), it follows from (3.8) that

\[
B \subset B(z_j, 3A_0 \lambda^{k+1}) = 3A_0 B(z_j, \lambda^{k+1}) = 3A_0 E_{j_0}.
\]

Thus, if \( B \) is as above with \( x \in \partial D \) and \( \sigma(x) \in \partial B \) then (3.17) implies that

\[
\frac{1}{\varphi(B)} \int_B |f| \sigma \, dv \leq c \sup_{Q \in \mathcal{D}_m : x \in 3A_0 Q} \frac{1}{\varphi(Q)} \int_Q |f| \sigma \, dv = c m_{\varphi, m, f}(\sigma)(x), \quad \text{say}.
\]

Therefore, we have

(3.19)

\[
m_{\varphi, m, f}(\sigma)(x) \leq c m_{\varphi, m, f}(\sigma)(x), \quad x \in D.
\]

For each \( k \in \mathbb{Z} \) let \( \Omega_{k,m} = \{ x \in D : m_{\varphi, m, f}(\sigma)(x) > a k \} \), where \( a > 1 \) will be chosen later. Thus, \( x \in \Omega_{k,m} \) if and only if there exists \( Q \in \mathcal{D}_m \) such that \( x \in 3A_0 Q^* \) and
Let \( \{Q^*_j\}_j \) be the maximal (with respect to inclusion) dyadic cubes in \( D_m \) which satisfy (3.19). We claim that \( \Omega_{k,m} = \bigcup_{j} 3A_2 Q^*_j \). Indeed, since any cube \( Q^*_j \) satisfies (3.19), for any \( x \in 3A_2 Q^*_j \) we have
\[
\mathcal{M}_{\omega}^2(f) \lesssim \frac{1}{\omega(Q^*_j)} \int_{Q^*_j} |f| \sigma \, dv > a^k,
\]
which implies that \( x \in \Omega_{k,m} \). On the other hand, if \( x \in \Omega_{k,m} \) then there exists \( Q \in D_m \) such that \( x \in 3A_2 Q \) and (3.19) holds. Since the \( Q^*_j \) are maximal with respect to inclusion, there exists \( j_0 \) such that \( Q \subset Q^*_j \) and, therefore, \( x \in 3A_2 Q \subset 3A_2 Q^*_j \), that is, \( x \in \bigcup_{j} 3A_2 Q^*_j \); this proves our claim.

Now
\[
(3.20) \quad \| \mathcal{M}_{\omega}^2(f) \|_{L^\infty(D)}^2 \leq a^9 \sum_{k,j} a^{kq} \left( \int_{\Omega_{k,m} \setminus \Omega_{k+1,m}} w \, d\mu \right) \leq a^9 \sum_{k,j} a^{kq} \left( \int_{\Omega_{k,m} \setminus \Omega_{k+1,m}} w \, d\mu \right) \leq a^9 \sum_{k,j} a^{kq} \left( \int_{\Omega_{k,m} \setminus \Omega_{k+1,m}} w \, d\mu \right)
\]
where \( F_{\omega,k,m} = 3A_2 Q^*_j \setminus \Omega_{k+1,m} \)
\[
\leq a^9 \sum_{k,j} \left[ \varphi(Q^*_j)^{-1} \int_{Q^*_j} |f| \sigma \, dv \right]^q \left( \int_{F_{\omega,k,m}} w \right) \quad \text{by (3.19) with } Q = Q^*_j
\]
\[
= a^9 \sum_{k,j} \left[ \frac{1}{A(Q^*_j)} \int_{Q^*_j} |f| \sigma \, dv \right]^q \quad \text{where } A(Q) = |Q| / \sigma(Q \sigma \, dv)^{1/r}
\]
\[
\leq a^9 \sum_{k,j} \left[ 3A_2 Q^*_j \right] \varphi(Q^*_j)^{-1} A(Q^*_j) \left[ \frac{1}{A(Q^*_j)} \int_{Q^*_j} |f| \sigma \, dv \right]^q
\]
where \( A(Q) = (Q \sigma \, dv)^{1/r} \).

We claim that
\[
\| \mathcal{M}_{\omega}^2(f) \|_{L^\infty(D)}^2 \leq c \sum_{k,j} A(Q^*_j)^{q/p} \left[ \frac{1}{A(Q^*_j)} \int_{Q^*_j} |f| \sigma \, dv \right]^q.
\]
We claim that
\[
\sum_{k,j} A(Q^*_j)^{q/p} \leq c A(Q^*_j)^{q/p}
\]
with \( c > 0 \) independent of \( l \) and \( i \). First, we show that \( Q^*_j \subset Q^*_l \) implies \( l \leq k \). If \( Q^*_j \subset Q^*_l \), then it follows from the maximality of the \( Q^*_j \) that
\[
a' < \varphi(Q^*_j)^{-1} \int_{Q^*_j} |f| \sigma \, dv \leq a^k
\]
which yields \( l \leq k \), since \( a > 1 \). Now, suppose that \( Q^*_j = Q^*_l \), and for any \( Q \in D_m \) let \( Q^* \) denote the smallest dyadic cube in \( D_m \) such that \( Q^* \supseteq Q \). Thus, since \( \nu \) is doubling, we have
\[
a' < \varphi(Q^*_l)^{-1} \int_{Q^*_l} |f| \sigma \, dv = \varphi(Q^*_j)^{-1} \int_{Q^*_j} |f| \sigma \, dv.
\]
\[
\leq c \varphi(Q_j^{(n)})^{-1} \int_{Q_j^{(n)}} |f| \sigma \, d\nu \leq c a^k \quad \text{since } Q_j^k \text{ is maximal}
\]
\[
\leq a^{k+1} \quad \text{if we select } a \geq c.
\]
Therefore, \( l \leq k \).

Observe that the same argument employed in (3.24) can be used to obtain
\[
A_{Q_j^k}^{q/p} \leq \left( \sum_{k,j:Q_j^k \subset Q_j^l} A_{Q_j^l}^{q/p} \right)^{q/p}.
\]
Thus, since \( q \geq p \), we have
\[
A_{Q_j^k}^{q/p} \leq \left( \sum_{k,j:Q_j^k \subset Q_j^l} A_{Q_j^l}^{q/p} \right)^{q/p}.
\]
Now, for a fixed \( k \geq l \), we have
\[
\sum_{j:Q_j^k \subset Q_j^l} A_{Q_j^k}^{q/p} \leq \sum_{j:Q_j^k \subset Q_j^l} \left( \int_{Q_j^k} \sigma \, d\nu \right)^{1/r} \leq \left( \sum_{j:Q_j^k \subset Q_j^l} \left( \int_{Q_j^k} \sigma \, d\nu \right)^{1/r} \right)^{q/p}.
\]
by Hölder's inequality
\[
\leq \left( \sum_{j:Q_j^k \subset Q_j^l} \left( \int_{Q_j^k} \sigma \, d\nu \right)^{1/r} \right)^{q/p},
\]
since the \( Q_j^k \) are pairwise disjoint in \( j \). Using the first inequality of (3.25) and the fact that \( \|Q_j^k \|_v / \|Q_j^l \|_v \leq \varphi(Q_j^k) / \varphi(Q_j^l) \), we obtain
\[
\sum_{j:Q_j^k \subset Q_j^l} A_{Q_j^k}^{q/p} \leq c \varphi(Q_j^l)^{1/r} \left( \sum_{j:Q_j^k \subset Q_j^l} a^{-k} \left( \int_{Q_j^k} |f| \sigma \, d\nu \right)^{1/s} \right)^{1/s} \quad \text{since } 1/s \geq 1
\]
\[
\leq c \varphi(Q_j^l)^{1/r} \left( \sum_{j:Q_j^k \subset Q_j^l} a^{-k} \left( \int_{Q_j^k} |f| \sigma \, d\nu \right)^{1/s} \right)^{1/s} \quad \text{since the } Q_j^k \text{ are pairwise disjoint in } j
\]
by the second inequality of (3.25) with \( k = l \). Thus, combining (3.26)–(3.28), we obtain
\[
\sum_{k,j:Q_j^k \subset Q_j^l} A_{Q_j^k}^{q/p} \leq \left( \sum_{k=1}^{\infty} \left( \int_{Q_j^k} |f| \sigma \, d\nu \right)^{1/r} \right)^{q/p} \leq \left( \sum_{k=1}^{\infty} a^{(l+1-k)/(1-r)} |Q_j^k|^{1/r} \right)^{q/p} \leq a^{q/(1-r)} \left( \sum_{k=0}^{\infty} a^{-k/(1-r)} A_{Q_j^k}^{q/p} \right)^{q/p} = c A_{Q_j^l}^{q/p},
\]
where \( c = a^{q/(1-r)}(\sum_{k=0}^{\infty} a^{-k/(1-r)} q/p \right)^{q/p}, \) and this yields (3.22).

Thus, if we take \( \beta = q/p \) and \( I \) to be the set of indices \( k, j \), and \( a_i = A_{Q_j^k} \) and \( u = \sigma \), then from (3.21) and Lemma 2.26 we obtain
\[
\left\| m_{\varphi, m} f \sigma \right\|_{L^q(B)} \leq c \sum_{k,j} A_{Q_j^k}^{q/p} \left[ \frac{1}{A_{Q_j^k}^{r}} \left( \int_{Q_j^k} |f| \sigma \, d\nu \right)^{r/p} \right]^q \leq c \left( \int_{\partial D} f^p u^q \, d\nu \right)^{q/p},
\]
provided we verify (2.27) and (2.28). Note that (2.27) follows from Hölder's inequality with \( c = 1 \). Condition (2.28) holds due to (2.22). It follows from (3.18) that
\[
\left\| \varphi_{m, m} f \sigma \right\|_{L^q(B)} \leq c \left( \int_{\partial D} f^p u \, d\nu \right)^{q/p},
\]
where
\[
m_{\varphi, m} f \sigma(x) = \sup_{B \ni x} \left( \int_{\partial B} f \sigma \, d\nu \right) \quad x \in D.
\]
Therefore, letting \( m \) tend to \(-\infty\) in (3.30), we obtain
\[
\left\| m f \sigma \right\|_{L^q(B)} \leq c \left( \int_{\partial D} f^p \sigma \, d\nu \right)^{q/p}.
\]
Replacing \( f \) by \( f^{1-1} \) in (3.31) and using the fact that \( M_{\varphi, f} \leq c m_{\varphi, f} \) (see the beginning of the proof of Theorem 1.2), we obtain (1.12). This concludes the proof of Theorem 1.2.
In order to prove Theorem 1.3 it will be convenient to define the following dyadic maximal function: for \( m \in \mathbb{Z} \) and \( z \in \mathbb{X} \) put
\[
\mathfrak{m}^{dy}_{\varphi, m, z} f(x, t) = \sup_{Q \in D_j} \frac{1}{\varphi(Q + z)} \int_{Q + z} |f| \, d\nu, \quad (x, t) \in \mathbb{X}_+, 
\]
and for \( k > m \), let
\[
M^k_{\varphi, m} f(x, t) = \sup_{\lambda^k \geq \rho \geq \max \{ \lambda^m, t \}} \frac{1}{\varphi(B(x, r))} \int_{B(x, r)} |f| \, d\nu. \tag{3.33}
\]

We have

**Lemma 3.32.** Let \((X, d, \mu)\) be a homogeneous space with group structure satisfying conditions (1.13)–(1.16). Suppose \( 1 \leq p < \infty \) and \( w(x, t) \geq 0 \) in \( \mathbb{X}_+ = X \times (0, \infty) \). Then there exists \( C > 0 \) such that
\[
\left( \int_{\mathbb{X}_+} (M^k_{\varphi, m} f(x, t))^p w(x, t) \, d\mu(x, t) \right)^{1/p} \leq C \sup_{x \in X} \left( \int_{\mathbb{X}_+} (\mathfrak{m}^{dy}_{\varphi, m, z} f(x, t))^p w(x, t) \, d\mu(x, t) \right)^{1/p}
\]
for all \( f \geq 0 \) with \( C \) independent of \( m \).

**Proof.** Fix \( k > m \) and \((x, t) \in \mathbb{X}_+\). Suppose \( B(x, r) \) is a ball in \( X \) satisfying
\[
\frac{1}{\varphi(B(x, r))} \int_{B(x, r)} |f| \, d\nu > \frac{1}{2} M^k_{\varphi, m} f(x, t).
\]
Now, select \( m \leq l \leq k \) such that \( \lambda^{l-1} \leq r \leq \lambda^l \). Let \( B_k \) be the ball in \( X \) of radius \( \lambda^k \) about the identity element of \( X \). Define \( \Omega = \{ z \in B_{k+3} : \exists Q \in T_m \) with \( \lambda^k \leq l(Q) \leq \lambda^{l+1} \) and \( B(x, r) \subset Q + z \} \). Thus, if \( Q = E_j^{l+1} \) and \( z \in \Omega \) then \( Q + z \subset B(2A_0 \lambda^k r) \) and, therefore, \( \varphi(Q + z) \leq c \varphi(B(x, r)) \). Hence,
\[
\mathfrak{m}^{dy}_{\varphi, m, z} f(x, t) \geq \frac{1}{\varphi(Q + z)} \int_{Q + z} |f| \, d\nu \\
\geq \frac{c}{\varphi(B(x, r))} \int_{B(x, r)} |f| \, d\nu \geq c M^k_{\varphi, m} f(x, t).
\]
Thus,
\[
(3.33) \quad \int_{B_{k+3}} \mathfrak{m}^{dy}_{\varphi, m, z} f(x, t) \, d\nu(x) \geq c M^k_{\varphi, m} f(x, t) \nu(B_{k+3}), \tag{3.34}
\]
provided we show \( \nu(B_{k+3}) \leq c \nu(\Omega) \).

Let \( \Gamma = \{ j : E_j^{l+1} \cap B(x, \lambda^{k+2}) \neq \emptyset \} \), where \( l \) is as in the definition of \( \Omega \). Now suppose \( z \) is in \( -B(z_j^{l+1}, \lambda^l) + x \) where \( -B = \{ w \in X : -w \in B \} \). We show that \( B(x, r) \subset E_j^{l+1} + z \). First note that
\[
x - z \in B(z_j^{l+1}, \lambda^l) \subset B(z_j^{l+1}, \lambda^{k+1}) \subset E_j^{l+1},
\]
whence \( x \in E_j^{l+1} + z \). Now, let \( y \in B(x, r) \). Since \( x - z \in B(z_j^{l+1}, \lambda^l) \) and \( d(y - z, x - z) = d(y, x) \leq \lambda^l \) by (1.13), we have
\[
d(y - z, z_j^{l+1}) \leq A_0 [d(y - z, x - z) + d(x - z, z_j^{l+1})] < A_0 [\lambda^l + \lambda^l] < \lambda^{l+1}.
\]
Thus, \( y - z \in B(z_j^{l+1}, \lambda^{k+1}) \subset E_j^{l+1} \) and, therefore, \( y \in E_j^{l+1} + z \). Now if \( j \in \Gamma \) then \( -B(z_j^{l+1}, \lambda^l) + x \subset B_{k+3} \). Indeed, if \( w \in B(z_j^{l+1}, \lambda^l) \) and \( u \in E_j^{l+1} \cap B(x, \lambda^{k+2}) \) (which exists since \( j \in \Gamma \)), then
\[
d(u, w) \leq A_0 [d(u, z_j^{l+1}) + d(z_j^{l+1}, w)] < A_0 [\lambda^{l+2} + \lambda^l] < 2A_0 \lambda^{k+2},
\]
which implies
\[
d(x, w) \leq A_0 [d(x, u) + d(u, w)] < A_0 [\lambda^{k+2} + 2A_0 \lambda^{k+2}] < 3A_0 \lambda^{k+2},
\]
which in turn yields
\[
d(0, w) \leq A_0 [d(0, x) + d(x, w)] < A_0 [\lambda^3 + 3A_0^2 \lambda^{k+2}] < 4A_0^2 \lambda^{k+2},
\]
since \( x \in B_{k+3} \). Thus,
\[
d(0, -w + x) \leq A_0 [d(0, x) + d(x, -w + x)] < A_0 [\lambda^k + d(0, w)]
\]
since \( x \in B_{k+3} \) and by (1.13) and (1.14)
\[
< A_0 [\lambda^k + 4A_0 \lambda^{k+2}] < 5A_0^2 \lambda^{k+3} < \lambda^{k+4},
\]
as desired. Now, since the sets \( \{ -B(z_j^{l+1}, \lambda^l) + x \}_{j \in \Gamma} \) are pairwise disjoint, we have
\[
(3.35) \quad \nu(\Omega) \geq \sum_{j \in \Gamma} \nu(-B(z_j^{l+1}, \lambda^l) + x) \quad \text{by (1.15) and (1.16)}
\]
\[
(3.36) \quad \geq c \sum_{j \in \Gamma} \nu(E_j^{l+1}) \quad \text{since } \nu \text{ is doubling}
\]
\[
(3.37) \quad = \nu \left( \bigcup_{j \in \Gamma} E_j^{l+1} \right) \geq \nu(B(x, \lambda^{k+2})) \quad \text{by (2.37)}
\]
\[
(3.38) \quad \geq c \nu(B(x, \lambda^{k+3})) \quad \text{since } \mu \text{ is doubling}
\]
\[
(3.39) \quad = c \nu(B(0, \lambda^{k+3}) + x) \quad \text{by (1.13)}
\]
\[
(3.40) \quad = c \nu(B(0, \lambda^{k+3}) + B_{k+3}) \quad \text{by (1.15)}
\]
We are now able to finish the proof of the lemma. By (3.34) and Minkowski's inequality, we get
\[
\left[ \int_{B_k \times (0, n(B_k))} \left( \sum_{x \in B_k} \left( \sum_{t \in B_k} (M_{\varphi, m} f(x, t))^p w(x, t) d\mu(x, t) \right)^{1/p} \right) \right]^{1/p} \leq c \sup_{z \in B_k} \left[ \int_{X_+} \left( \sum_{x \in B_k} \left( \sum_{t \in B_k} (M_{\varphi, m} f(x, t))^p w(x, t) d\mu(x, t) \right)^{1/p} \right) \right]^{1/p} d\nu(z) \leq c \sup_{z \in B_k} \left[ \int_{X_+} \left( \sum_{x \in B_k} \left( \sum_{t \in B_k} (M_{\varphi, m} f(x, t))^p w(x, t) d\mu(x, t) \right)^{1/p} \right) \right]^{1/p},
\]
and letting \( k \to \infty \) finishes the proof of Lemma 3.32. ■

We now prove a version of Theorem 1.3 with \( M_{\varphi} f \) replaced by \( M_{\varphi, m} f \). We have

**Theorem 3.1.** Let \((X, d, \nu)\) be a homogeneous space, where \( X \) has a group structure and \( d \) and \( \nu \) satisfy conditions (1.13)–(1.16). Let \( w \) and \( v \) be weights in \( X_+ \) and \( X \), respectively, and suppose that \( 1 < p \leq q < \infty \). Then
\[
\left( \int_{X_+} \left( \sum_{x \in B_k} \left( \sum_{t \in B_k} (M_{\varphi, m} f(x, t))^p w(x, t) d\mu(x, t) \right)^{1/p} \right) \right]^{1/q} \leq C \left( \int_{X} |f|^{1/p} \nu dv \right)^{1/p}
\]
for all \( f \) in \( L^p_{v} \), \( m \) in \( \mathbb{Z} \) and \( z \) in \( X \) if and only if
\[
\left( \int_{Q+x} \left( \sum_{x \in B_k} \left( \sum_{t \in B_k} (M_{\varphi, m} f(x, t))^p w(x, t) d\mu(x, t) \right)^{1/p} \right) \right]^{1/q} \leq C \left( \int_{Q+x} \sigma dv \right)^{1/p}
\]
for all dyadic cubes \( Q \) in \( D, m \) in \( \mathbb{Z} \) and \( z \) in \( X \), where \( \bar{Q} = Q \times I(Q) \).

**Proof.** Let \( f(x) = \sigma \chi_{Q+x}(x) \) where \( Q \) is a dyadic cube in \( D, m \) in \( \mathbb{Z} \) and \( z \in X \). Now, if (3.41) holds we obtain
\[
\left( \int_{Q+x} \left( \sum_{x \in B_k} \left( \sum_{t \in B_k} (M_{\varphi, m} \sigma \chi_{Q+x}(x))^p w(x, t) d\mu(x, t) \right)^{1/p} \right) \right]^{1/q} \leq \left( \int_{X_+} \left( \sum_{x \in B_k} \left( \sum_{t \in B_k} (M_{\varphi, m} \sigma \chi_{Q+x}(x))^p w(x, t) d\mu(x, t) \right)^{1/p} \right) \right]^{1/q} \leq C \left( \int_{Q+x} |\sigma \chi_{Q+x}|^p dv \right)^{1/p} \left( \int_{Q+x} \sigma dv \right)^{1/p},
\]
which is (3.42).

Conversely, assume that (3.42) holds. Define
\[
\mathfrak{M}_{d_{\varphi, m}} f(x, t) = \sup_{Q \in D_m} \frac{1}{\varphi(Q^* + z)} \left\{ |f| dv \right\}_{Q^* + z}
\]
Let \( f \in L^p_{v} \), and for each \( k \) in \( \mathbb{Z} \) and \( R > 0 \) define
\[
\Omega_k^R = \{(x, t) \in X_+: \mathfrak{M}_{d_{\varphi, m}} f(x, t) > 2^k\}
\]
For a fixed \( m \) in \( \mathbb{Z} \) let
\[
\Omega_k^{R, m} = \{(x, t) \in \Omega_k^R : \exists Q \in D, (x, t) \in Q + z, Q^* + z \subset \Omega_k^R\}
\]
Let \( \{Q_j^k\}_{j \in J_k^m} \) denote the dyadic cubes maximal among those \( Q \in D_m \) with the property that \( Q^* \) is contained in \( \Omega_k^R \). Then
\[
\Omega_k^{R, m} = \bigcup_{j \in J_k^m} (Q_j^k + z) \quad \text{for } k \in \mathbb{Z},
\]
\[
Q_j^k \cap Q_j^k = \emptyset \quad \text{for } i \neq j, k \in \mathbb{Z},
\]
\[
\frac{1}{\varphi(Q_j^k + z)} \int_{Q_j^k + z} |f| \sigma dv > 2^k \quad \text{for } j \in J_k^m, k \in \mathbb{Z}.
\]
We have
\[
\varphi(Q_j^k + z) \leq 2^{-k} \int_{Q_j^k + z} |f| \sigma dv \quad \text{by (3.45)}
\]
\[
\leq 2^{-k} \left( \int_{Q_j^k + z} |f|^{p} \sigma dv \right)^{1/p} \left( \int_{Q_j^k + z} \sigma dv \right)^{1/p} \quad \text{by Hölder's inequality.}
\]
Let \( \hat{Q}_j^k \) be such that \( \hat{Q}_j^k + z = (Q_j^k + z) \setminus \Omega_k^{R, m} \). We have
\[
\int_{Q_j^k + z} |f| \sigma dv \leq 2^{-k} \sum_{j \in J_k^m} w d\mu
\]
\[
\leq 2^k \sum_{j \in J_k^m} \int_{\hat{Q}_j^k + z} w d\mu
\]
\[
\leq 2^k \sum_{j \in J_k^m} \left( \int_{\hat{Q}_j^k + z} w d\mu \right) \left( \frac{1}{\varphi(Q_j^k + z)} \int_{Q_j^k + z} |f| \sigma dv \right)^q \quad \text{by (3.43) and (3.45)}
\]
Now we are able to finish the proof of Theorem 1.3.

Proof of Theorem 1.3. If in (1.18) we set \( f = \sigma \chi_{Q+z} \) where \( Q \in \mathcal{D}_m \) and \( z \in X \), we obtain (1.19).

Now suppose that (1.19) holds. It is enough to show (1.18) with \( M_{\sigma} \) replaced by \( M_{\sigma, m} \) and \( c \) independent of \( m \), since we obtain the desired result by letting \( m \) tend to \( -\infty \). By Lemma 3.32 it is enough to show (1.8) with \( M_{\sigma} \) replaced by \( M_{\sigma, m, z} \) and \( c \) independent of \( m \) and \( z \). By Theorem 3.1 it is enough to show that

\[
\left( \int_{Q+z} \left[ \frac{M_{\sigma, m, z}(\sigma \chi_{Q+z})}{Q+z} \right]^q w \, d\mu \right)^{1/q} \leq C \left( \int_{Q+z} \sigma \, d\nu \right)^{1/p},
\]

for all \( Q \in \mathcal{D}_m \) and \( z \in X \).

Let \( Q_0 \) be a dyadic cube in \( \mathcal{D}_m \) and let \( \bar{Q} \) and \( \bar{Q}'_0 \) be the inner and outer balls, respectively, associated with \( Q_0 \). Let \( z \in X \). If \( x - z \in Q_0 \) then

\[
\frac{1}{\varphi(Q_0 + z)} \int_{Q_0 + z} \sigma \, d\nu \leq \frac{c}{\varphi(Q_0 + z)} \int_{Q_0 + z} \sigma \, d\nu \leq c \sigma_{Q_0 + z} \sigma \, d\nu \leq c M_{\sigma, m}(\chi_{Q_0 + z})(x, t),
\]

since \( x - z \in Q_0 \) and the radius of \( Q_0 \) is larger than \( t \). Hence,

\[
\frac{1}{\varphi(Q_0 + z)} \int_{Q_0 + z} \sigma \, d\nu \leq c \sigma_{Q_0 + z} \sigma \, d\nu \leq c M_{\sigma, m}(\chi_{Q_0 + z})(x, t),
\]

for all \((x, t)\) in \( \bar{Q}_0 + z \) and, therefore,

\[
\left( \int_{Q_0 + z} \left[ \frac{M_{\sigma, m, z}(\sigma \chi_{Q_0+z})}{Q_0+z} \right]^q w \, d\mu \right)^{1/q} \leq c \left( \int_{Q_0 + z} \sigma \, d\nu \right)^{1/p} \leq \left( \int_{Q_0 + z} \sigma \, d\nu \right)^{1/p},
\]

by (1.19). This finishes the proof of Theorem (1.3). □

References

[C] M. Christ, A \( T(b) \) theorem with remarks on analytic capacity and the Cauchy integral, Colloq. Math. 60/61 (1990), 601–628.


Minimal pairs of bounded closed convex sets

by

J. GRZYBOWSKI and R. URBAŃSKI (Poznań)

Abstract. The existence of a minimal element in every equivalence class of pairs of bounded closed convex sets in a reflexive locally convex topological vector space is proved. An example of a non-reflexive Banach space with an equivalence class containing no minimal element is presented.

Let $X = (X, \tau)$ be a topological vector space over the field $\mathbb{R}$. Let $\mathcal{B}_r(X)$ (resp. $\mathcal{K}_r(X)$) be the collection of all bounded closed (resp. compact) convex subsets of $X$. For $A, B \subset X$, let

$$A + B := \{a + b \mid a \in A, b \in B\}$$

and let $A + B$ denote the closure of $A + B$. For $(A, B), (C, D) \in \mathcal{B}_r(X)$, let $(A, B) \sim (C, D)$ if and only if $A + D = B + C$. Let $(A, B) \leq (C, D)$ if and only if $A \subset C, B \subset D$ and $(A, B) \sim (C, D)$. The relation $\sim$ is an equivalence relation by the ordered law of cancellation [5] in $\mathcal{B}_r(X)$ and $\leq$ is an ordering in the equivalence class $[A, B]$ of any pair $(A, B)$.

The study of minimal pairs of compact convex sets was stimulated by the development of quasidifferential calculus [1]. Any given quasidifferential may be identified with the equivalence class of a pair of compact convex sets $(A, B)$, where $A$ and $B$ are, respectively, a super- and a sub-differential.

The existence of minimal pairs of compact convex sets in all topological vector spaces and the uniqueness up to translates in $\mathbb{R}^2$ were already proved in [2] and [4].

In this paper we extend our investigations to pairs of bounded closed convex sets.

**Theorem.** Let $X = (X, \tau)$ be a reflexive locally convex topological vector space. Every class $[A, B] \in \mathcal{B}_r(X)/\sim$ contains a minimal element $(C, D)$ such that $(C, D) \leq (A, B)$.

1991 Mathematics Subject Classification: 52A07, 26A27.

Key words and phrases: convex analysis, pairs of convex sets.