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Hankel multipliers and transplantation operators

by

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Dedicated to Professor Satoru Igari on the occasion of his 60th birthday

Abstract. Connections between Hankel transforms of different order for L^p-functions are examined. Well known are the results of Guy [Guy] and Schindler [Sch]. Further relations result from projection formulae for Bessel functions of different order. Consequences for Hankel multipliers are exhibited and implications for radial Fourier multipliers on Euclidean spaces of different dimensions indicated.

1. Introduction. It is well known that harmonic analysis of radial functions on the Euclidean space $\mathbb{R}^n$, $n \geq 1$, reduces to studying appropriate function spaces on the half-line equipped with the measure $x^{n-1}dx$. The Fourier transform is then replaced by the modified Hankel transform of order $(n-2)/2$. The aim of this paper is to show, among other things, that also studying the non-modified Hankel transform of an arbitrary order $\nu \geq -1/2$ within an appropriate weighted setting leads to corresponding results for the Fourier transform of radial functions. This is seen, for instance, in Section 2 where we discuss multiplier results for the modified Hankel transform. It occurs that they are closely related to two transference theorems of Rubio de Francia for the Fourier transform on Euclidean spaces.

Given $\nu \geq -1/2$ and f, an integrable function on $\mathbb{R}_+ = (0,\infty)$, its (non-modified) *Hankel transform* is defined by

$$\mathcal{H}_\nu f(x) = \int_0^\infty (xy)^{1/2} J_\nu(xy) f(y)\, dy, \quad x > 0. \tag{1.1}$$

Here $J_\nu(x)$ denotes the Bessel function of the first kind of order ν [Sz, (1.17.1)]. For $\nu = -1/2$ or $\nu = 1/2$ one recovers the cosine and sine trans-

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forms on the half-line. The *modified Hankel transform* is given by

$$H_\nu f(x) = \int_0^\infty \frac{J_\nu(xy)}{(xy)^\nu} f(y) y^{2\nu+1}\, dy, \qquad x > 0. \tag{1.2}$$

Due to the estimates on the Bessel function

$$J_\nu(x) = O(x^\nu), \qquad J_\nu(x) = O(x^{-1/2}), \tag{1.3}$$

valid for $x \to 0$ and $x \to \infty$ respectively, $H_\nu f$ is well defined for every function f in $L^p(\mathbb{R}_+, x^{2\nu+1}dx)$, $1 \le p < 4(\nu+1)/(2\nu+3)$. Clearly both transforms are related to each other by

$$\mathcal{H}_\nu f(x) = x^{\nu+1/2} H_\nu((\cdot)^{-(\nu+1/2)} f(\cdot))(x) \tag{1.4}$$

whenever f is an integrable function on $\mathbb{R}_+$, and $\int_0^\infty |f(x)| x^{\nu+1/2}\, dx < \infty$, for instance. Moreover, for $\nu \ge -1/2$ the inversion formulae

$$f(y) = \int_0^\infty (xy)^{1/2} J_\nu(xy) \mathcal{H}_\nu f(x)\, dx \tag{1.5}$$

and

$$f(y) = \int_0^\infty \frac{J_\nu(xy)}{(xy)^\nu} H_\nu f(x) x^{2\nu+1}\, dx \tag{1.6}$$

hold: (1.5) holds, for instance, for every C^1 function $f \in L^1(\mathbb{R}_+, dx)$ with $\mathcal{H}_\nu f \in L^1(\mathbb{R}_+, dx)$; (1.6) holds for every C^1 function $f \in L^1(\mathbb{R}_+, x^{2\nu+1}dx)$ with $H_\nu f \in L^1(\mathbb{R}_+, x^{2\nu+1}dx)$ (cf. [W, p. 456]). More can be said: H_ν is a bijection on $\mathcal{S}(\mathbb{R}_+)$, the space of infinitely differentiable even functions on $\mathbb{R}$ with rapidly decreasing derivatives, while $\mathcal{H}_\nu$ is a bijection on the Zemanian space $\mathcal{Z}_\nu$ of all C^∞ functions f on $\mathbb{R}_+$ for which the quantity

$$\sup_{x>0} \left| x^n \left(\frac{1}{x}\frac{d}{dx}\right)^k (x^{-\nu-1/2} f(x)) \right|$$

is finite for every $n, k \in \mathbb{N} = \{0, 1, 2, \ldots\}$ (see [Z1, Z2]). Note at this point that $C_0^\infty = C_0^\infty(\mathbb{R}_+)$, the space of compactly supported C^∞ functions on $\mathbb{R}_+$, is contained in every $\mathcal{Z}_\nu$. The kernels $\varphi_x^\nu(y) = (xy)^{1/2} J_\nu(xy)$, $y > 0$, appearing in (1.1) satisfy

$$\left(\frac{d^2}{dy^2} + \frac{1/4 - \nu^2}{y^2}\right) \varphi_x^\nu(y) = -x^2 \varphi_x^\nu(y), \qquad x > 0, \tag{1.7}$$

while the kernels $\phi_x^\nu(y) = (xy)^{-\nu} J_\nu(xy)$, $y > 0$, appearing in (1.2) fulfil

$$\left(\frac{d^2}{dy^2} + \frac{2\nu+1}{y}\frac{d}{dy}\right) \phi_x^\nu(y) = -x^2 \phi_x^\nu(y), \qquad x > 0. \tag{1.8}$$

The differential operators on the left sides of (1.7) and (1.8) are symmetric in $L^2(\mathbb{R}_+, dx)$ and $L^2(\mathbb{R}_+, x^{2\nu+1}dx)$ respectively.

As usual we use C or c with or without subscripts for a constant which is not necessarily the same at each occurrence.

2. Hankel multipliers. In this section we fix $\nu \ge -1/2$ and consider weighted Lebesgue spaces on $\mathbb{R}_+$ with respect to the Lebesgue measure dx on one occasion and the measure

$$dm_\nu(x) = x^{2\nu+1} dx$$

on another one. Hence, in what follows we use the notation

$$\|f\|_{p,\alpha} = \left(\int_0^\infty |f(x)|^p x^\alpha\, dx\right)^{1/p}$$

and

$$\|f\|_{L^p(x^\alpha dm_\nu)} = \left(\int_0^\infty |f(x)|^p x^\alpha\, dm_\nu(x)\right)^{1/p}$$

for $1 \le p < \infty$ with the usual modification when $p = \infty$. By $L^{p,\alpha}(dx)$ and $L^{p,\alpha}(dm_\nu)$ we denote the weighted Lebesgue spaces of functions for which the above quantities are finite. If $\alpha = 0$ we write L^p instead of $L^{p,0}$. By $\mathcal{M}_p^{\nu,\alpha}$ and $M_p^{\nu,\alpha}$ we denote the spaces of weighted p-multipliers for the Hankel and modified Hankel transform. Thus, a bounded measurable function $m(x)$ on $\mathbb{R}_+$ is in $\mathcal{M}_p^{\nu,\alpha}$ provided

$$\|\mathcal{H}_\nu(m \cdot \mathcal{H}_\nu f)\|_{p,\alpha} \le C\|f\|_{p,\alpha},$$

where C is a constant independent of f in $\mathcal{H}_\nu(C_0^\infty)$, the image of C_0^∞ under the action of $\mathcal{H}_\nu$. The least constant C for which the above inequality holds is called the *multiplier norm* of m. The multiplier space $M_p^{\nu,\alpha}$ has a similar definition, now with the norm $\|\cdot\|_{L^p(x^\alpha dm_\nu)}$ in use, and with $H_\nu(C_0^\infty)$ as the testing function space.

We postpone to Section 4 the proof of the fact that $H_\nu(C_0^\infty)$ is dense in $L^{p,\alpha}(dx)$ if $1 < p < \infty$ and $\alpha > -1$ while $\mathcal{H}_\nu(C_0^\infty)$ is dense in $L^{p,\alpha}(dx)$ if $1 < p < \infty$ and $\alpha > -1 - p(\nu + 1/2)$ (the case $p = 1$ requires additional assumptions). This is the content of Theorem 4.7 and Corollary 4.8.

The following is Guy's *transplantation theorem* for the Hankel transform (cf. also [Sch] for an alternative proof).

THEOREM ([Guy, Lemma 8C]). *Let $\mu, \nu \ge -1/2$, $1 < p < \infty$ and $-1 < \alpha < p - 1$. Then*

$$C^{-1}\|\mathcal{H}_\nu f\|_{p,\alpha} \le \|\mathcal{H}_\mu f\|_{p,\alpha} \le C\|\mathcal{H}_\nu f\|_{p,\alpha}$$

with $C = C(\mu, \nu, p, \alpha)$ independent of $f \in L^1(\mathbb{R}_+, dx)$.

As an immediate consequence one obtains

COROLLARY 2.1. *Let* $\mu, \nu \geq -1/2$, $1 < p < \infty$ *and* $-1 < \alpha < p-1$. *Then*

$$\mathcal{M}_p^{\nu,\alpha} = \mathcal{M}_p^{\mu,\alpha}. \tag{2.1}$$

Proof. Assuming m is in $\mathcal{M}_p^{\nu,\alpha}$ and f is in $\mathcal{H}_\mu(C_0^\infty)$ and using the fact that $\mathcal{H}_\mu f \in C_0^\infty \subset L^1(dx)$, hence $\mathcal{H}_\nu \mathcal{H}_\mu f \in \mathcal{H}_\nu(C_0^\infty)$, we write

$$\begin{aligned}\|\mathcal{H}_\mu(m \cdot \mathcal{H}_\mu f)\|_{p,\alpha} &\leq C\|\mathcal{H}_\nu(m \cdot \mathcal{H}_\nu(\mathcal{H}_\nu\mathcal{H}_\mu f))\|_{p,\alpha} \\ &\leq CC_{\nu,m}\|\mathcal{H}_\nu\mathcal{H}_\mu f\|_{p,\alpha} \leq C^2 C_{\nu,m}\|f\|_{p,\alpha},\end{aligned}$$

where $C_{\nu,m}$ denotes the operator norm of the multiplier $m \in \mathcal{M}_p^{\nu,\alpha}$. Thus $\mathcal{M}_p^{\nu,\alpha} \subset \mathcal{M}_p^{\mu,\alpha}$. Analogously the opposite inclusion follows.

COROLLARY 2.2. *Let* $\mu, \nu \geq -1/2$ *and* $1 < p < \infty$. *Assume further that* $-1 < \beta + (\nu + 1/2)(2-p) < p-1$ *and define* $\beta^* = \beta + (\nu - \mu)(2-p)$. *Then*

$$M_p^{\nu,\beta} = M_p^{\mu,\beta^*}. \tag{2.2}$$

Proof. The identity (1.4), the fact that $x^{\nu+1/2}C_0^\infty = C_0^\infty$ and the definition of multiplier spaces immediately give

$$M_p^{\nu,\beta} = \mathcal{M}_p^{\nu,\beta+p(2\nu+1)(1/p-1/2)}$$

and then (2.1) produces (2.2).

In particular, (2.2) for $p = 2$ gives

COROLLARY 2.3. *Let* $\mu, \nu \geq -1/2$ *and* $-1 < \beta < 1$. *Then*

$$M_2^{\nu,\beta} = M_2^{\mu,\beta}. \tag{2.3}$$

In some sense (2.3) may be viewed as a "radial" generalization of Rubio de Francia's transference result [RdF, Theorem 2.1], which states that given $-1 < \beta < 1$ and $m \in L^\infty(\mathbb{R}_+)$, if $m(|x|)$ is a Fourier multiplier on $L^2(\mathbb{R}, |x|^\beta\,dx)$, then $m(\|x\|)$ is a Fourier multiplier on $L^2(\mathbb{R}^n, \|x\|^\beta\,dx)$, $n \geq 2$ ($\|\cdot\|$ denotes here the Euclidean norm in the appropriate Euclidean space, and dx the Lebesgue measure). When restricted to the space of radial functions on which the multiplier acts, (2.3) states that the opposite implication also holds. In particular, this implies that one can jump between Euclidean spaces of arbitrary dimension, in contrast to the preceding situation where a jump was allowed only between $\mathbb{R}$ and $\mathbb{R}^n$.

Let T_R, $R > 0$, denote the multiplier operator corresponding to the characteristic function of the interval $(0, R)$. By a homogeneity argument, for every μ, p and α, the operator norms of T_R as members of $\mathcal{M}_p^{\mu,\alpha}$ or $M_p^{\mu,\alpha}$ are independent of $R > 0$. Hence, in what follows we consider $T = T_1$ only. Hirschman's "one-dimensional" weighted multiplier result [Hi1] says that $T \in \mathcal{M}_p^{-1/2,\alpha} = M_p^{-1/2,\alpha}$ for every $1 < p < \infty$ and $-1 < \alpha < p-1$. Thus (2.1) further gives that $T \in \mathcal{M}_p^{\nu,\alpha}$ for every $\nu > -1/2$, $1 < p < \infty$ and $-1 < \alpha < p-1$, which, for $\alpha = 0$, was proved by Wing [Wi]. Similarly, Herz' result [He] which says that $T \in M_p^{\nu,0}$ for $\nu > -1/2$ provided

$$\frac{4(\nu+1)}{2\nu+3} < p < \frac{4(\nu+1)}{2\nu+1}$$

may be recovered from (2.2) and Hirschman's result just mentioned.

The next corollary may be considered as a "radial" extension of another result due to Hirschman [Hi2].

COROLLARY 2.4. *Let* $\mu \geq -1/2$ *and* $-1 < \beta < 1$. *Then* $T \in M_2^{\mu,\beta}$.

Proof. Combine (2.3) and the fact that $T \in M_2^{-1/2,\beta}$.

As already mentioned, Schindler gave an alternative proof of Guy's result. Besides, in the special case $\mu = \nu + 2k$, $k = 1, 2, \ldots$, her approach allowed taking into account both endpoints $p = 1$ and $p = \infty$ and, in addition, obtaining a range of α's different from the A_p range from Guy's theorem.

THEOREM ([Sch, Theorems 3 and 4]). *Let* $\nu \geq -1/2$, $1 \leq p < \infty$, $k = 1, 2, \ldots$ *and* $-p(\nu + 1/2) < \alpha < p(\nu + 1/2)$. *Then*

$$C^{-1}\|\mathcal{H}_\nu f\|_{p,\alpha} \leq \|\mathcal{H}_{\nu+2k} f\|_{p,\alpha} \leq C\|\mathcal{H}_\nu f\|_{p,\alpha}$$

with $C = C(\nu, k, p, \alpha)$ *independent of* $f \in L^1(\mathbb{R}_+, dx)$.

In consequence, the analogue to (2.1) is now the identity

$$\mathcal{M}_p^{\nu,\alpha} = \mathcal{M}_p^{\nu+2k,\alpha}, \tag{2.4}$$

where $\nu \geq -1/2$, $1 \leq p < \infty$, $k = 1, 2, \ldots$ and $-p(\nu+1/2) < \alpha < p(\nu+1/2)$. Moreover, in the case $p > 1$, comparing the above hypotheses with those from Corollary 2.1 allows us to enlarge the range of α's for which (2.4) holds to

$$-\max\{p(\nu + 1/2), 1\} < \alpha < \max\{p(\nu + 1/2), p - 1\}.$$

Similarly, the analogue to (2.2) is

$$M_p^{\nu,\beta} = M_p^{\nu+2k,\beta^*}, \tag{2.5}$$

where $\nu \geq -1/2$, $1 < p < \infty$, $k = 1, 2, \ldots$, $\beta^* = \beta - 2k(2-p)$ and

$$-\max\{p(\nu + 1/2), 1\} < \beta + p(2\nu + 1)\left(\frac{1}{p} - \frac{1}{2}\right) < \max\{p(\nu + 1/2), p - 1\}.$$

In particular, (2.5) for $p = 2$ gives

COROLLARY 2.5. *Let* $\nu \geq -1/2$, $k = 1, 2, \ldots$ *and* $-\max\{2\nu + 1, 1\} < \beta < \max\{2\nu + 1, 1\}$. *Then*

$$M_2^{\nu,\beta} = M_2^{\nu+2k,\beta}.$$

The above stands in a relationship with another Rubio de Francia transference result, [RdF, Theorem 2.2], in the same way as (2.3) "generalizes" [RdF, Theorem 2.1]. This result says that given $m \in L^\infty(\mathbb{R}_+)$ and $w(s)$, a nonnegative measurable function on $\mathbb{R}_+$, if $m(\|x\|)$ is a Fourier multiplier on $L^2(\mathbb{R}^n, w(\|x\|)dx)$ for some $n \ge 2$ then $m(\|x\|)$ is a Fourier multiplier on $L^2(\mathbb{R}^{n+2k}, w(\|x\|)\,dx)$ for any $k = 1, 2, \ldots$ Corollary 2.5 says that when restricted to the power weights indicated and spaces of radial functions on which the multipliers act, the converse also holds provided the difference of the Euclidean dimensions is a multiple of 4 (raising ν by 2 changes the Euclidean dimension by 4). Speaking less precisely, this means that under appropriately modified assumptions we can exchange radial Fourier multipliers, in both directions, between Euclidean spaces whose difference in dimensions is a multiple of 4 (the one-dimensional situation is now included!).

3. Weighted estimates for the transference operators. Throughout this section all the functions we are dealing with are assumed to be in C_0^∞. Let

$$L_\nu = -\left(\frac{d^2}{dy^2} + \frac{2\nu+1}{y}\frac{d}{dy}\right), \quad \nu \ge -1/2,$$

be the differential operator appearing in (1.8). Clearly

$$H_\nu(L_\nu f)(y) = y^2 H_\nu f(y).$$

Hence, in terms of the modified Hankel transform,

$$H_\nu(L_\nu^\delta f)(y) = y^{2\delta} H_\nu f(y) \tag{3.1}$$

is a well motivated definition of L_ν^δ, the δ-fractional power of L_ν. Rewriting in terms of the modified Hankel transform the inequality

$$\|\mathcal{H}_\mu \mathcal{H}_\nu f\|_{p,\alpha} \le C\|f\|_{p,\alpha},$$

which follows from Guy's transplantation theorem, gives

$$\Big(\int_0^\infty |H_\mu H_\nu h(x)|^p x^\gamma\,dx\Big)^{1/p} \le C\Big(\int_0^\infty |L_\nu^{(\mu-\nu)/2} h(x)|^p x^\delta\,dx\Big)^{1/p},$$

where $\gamma = p(\mu+1/2)+\alpha$, $\delta = p(\nu+1/2)+\alpha$ for $1<p<\infty$, $-1<\alpha<p-1$ and $\mu, \nu \ge -1/2$.

In this section we prove two weighted L^p-L^q inequalities for the *transplantation operator*

$$T_\nu^\mu = H_\mu \circ H_\nu.$$

This is achieved first by using appropriately chosen integral formulae for Bessel functions that generate nice representations of T_ν^μ (with necessary restrictions on ν and μ) and then applying some weighted norm inequalities for the Riemann–Liouville and Weyl fractional integral operators.

THEOREM 3.1. *Let* $-1/2 \le \nu < \mu$, $1 < p < q < \infty$, $p(\mu+1) \ge 1$ *and*

$$\frac{\mu+1}{q} = \frac{\nu+1}{p}. \tag{3.2}$$

Then

$$\Big(\int_0^\infty |T_\nu^\mu g(x)|^q x^{2\mu+1}\,dx\Big)^{1/q} \le C\Big(\int_0^\infty |L_\nu^{\mu-\nu} g(x)|^p x^{2\nu+1}\,dx\Big)^{1/p}. \tag{3.3}$$

Proof. By using a homogeneity argument it is easy to see that (3.2) is necessary for (3.3) to hold. If $\varrho_r g(x) = (1/r)g(x/r)$, $r>0$, then $H_\nu(\varrho_r g)(x) = r^{2\nu+1} H_\nu g(rx)$ and

$$L_\nu^{\mu-\nu}(\varrho_r g)(x) = r^{2(\nu-\mu)}\varrho_r(L_\nu^{\mu-\nu} g)(x).$$

Hence, considering (3.3) with $\varrho_r g$ in place of g and allowing r to be small and large gives (3.2).

Evaluating the formula [EMOT, 8.5 (33)] at $y=1$ and writing μ in place of $\mu+\nu+1$ produces

$$\frac{J_\mu(a)}{a^\mu} = c_{\nu,\mu}\frac{1}{a^{2\mu}}\int_0^a (a^2-t^2)^{\mu-\nu-1}\frac{J_\nu(t)}{t^\nu}t^{2\nu+1}\,dt$$

for $-1<\nu<\mu$ and $a>0$. Hence, a change of variable and Fubini's theorem give

$$\begin{aligned} T_\nu^\mu g(x) &= \int_0^\infty H_\nu g(y)\frac{J_\mu(xy)}{(xy)^\mu}y^{2\mu+1}\,dy \\ &= c_{\nu,\mu}\int_0^\infty H_\nu g(y)\frac{1}{(xy)^{2\mu}} \\ &\quad\times \int_0^{xy} ((xy)^2-t^2)^{\mu-\nu-1}\frac{J_\nu(t)}{t^\nu}t^{2\nu+1}\,dt\,y^{2\mu+1}\,dy \\ &= c_{\nu,\mu}\frac{1}{x^{2\mu}}\int_0^x (x^2-u^2)^{\mu-\nu-1} \\ &\quad\times \int_0^\infty y^{2(\mu-\nu)}H_\nu g(y)\frac{J_\nu(uy)}{(uy)^\nu}y^{2\nu+1}\,dy\,u^{2\nu+1}\,du. \end{aligned}$$

Note that an application of Fubini's theorem is possible due to the boundedness of $J_\nu(s)/s^\nu$ on $(0,\infty)$, integrability of $y^{2\mu}H_\nu g(y)$ on $(0,\infty)$ (with respect to the Lebesgue measure) and the assumption $\mu-\nu>0$.

Taking into account (3.1) and the inversion formula for the modified Hankel transform (recall that $\nu \ge -1/2$) we arrive at

$$T_\nu^\mu g(x) = c_{\nu,\mu} \cdot \frac{1}{x^{2\mu}} \int_0^x (x^2-y^2)^{\mu-\nu-1} L_\nu^{\mu-\nu} g(y) y^{2\nu+1}\,dy.$$

What we now need is the inequality

$$\Big(\int_0^\infty \Big(\frac{1}{x^{2\mu}}\int_0^x (x^2-y^2)^{\mu-\nu-1} G(y) y^{2\nu+1}\,dy\Big)^q x^{2\mu+1}\,dx\Big)^{1/q} \le C\Big(\int_0^\infty G(x)^p x^{2\nu+1}\,dx\Big)^{1/p},$$

say, for all nonnegative functions G. Elementary variable changes show that the above inequality is equivalent to

$$(3.4)\qquad \Big(\int_0^\infty \Big(\int_0^t (t-s)^{\mu-\nu-1} h(s)\,ds\Big)^q t^{\mu(1-q)}\,dt\Big)^{1/q} \le C\Big(\int_0^\infty h(t)^p t^{\nu(1-p)}\,dt\Big)^{1/p}$$

for h nonnegative. We use the following criterion for L^p-L^q weighted estimates for the *Riemann–Liouville*

$$I_+^\alpha h(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} h(s)\,ds$$

and *Weyl*

$$I_-^\alpha h(t) = \frac{1}{\Gamma(\alpha)}\int_t^\infty (s-t)^{\alpha-1} h(s)\,ds$$

fractional integral operators.

LEMMA ([SKM, Theorem 5.4]). *Let $\alpha > 0$, $p \ge 1$ and $p \le q \le p/(1-p\alpha)$ when $1 \le p < 1/\alpha$ (if $p = 1$ then the right endpoint $p/(1-p\alpha)$ is excluded), or $p \le q < \infty$ when $p \ge 1/\alpha$. Suppose also that $-\infty < N < \infty$ and $M < p-1$ when we consider I_+^α, or $M > \alpha p - 1$ when considering I_-^α, and*

$$(3.5)\qquad \frac{N+1}{q} = \frac{M+1}{p} - \alpha.$$

Then

$$(3.6)\qquad \Big(\int_0^\infty |I_\pm^\alpha h(t)|^q t^N\,dt\Big)^{1/q} \le C\Big(\int_0^\infty |h(t)|^p t^M\,dt\Big)^{1/p}.$$

To see that the lemma gives (3.4) and thus (3.3) assume that the hypotheses of Theorem 3.1 are satisfied and take $\alpha = \mu - \nu$, $N = \mu(1-q)$ and $M = \nu(1-p)$. Clearly (3.2) gives (3.5) and $M < p-1$ holds provided $\nu > -1$. Moreover, if $1 < p < 1/\alpha$ then the condition $p(\mu+1) \ge 1$ implies $q \le p/(1-p\alpha)$. Thus, the conclusion of the above lemma, (3.6), holds for the operator I_+^α and, in consequence, (3.4) is valid. This concludes the proof of Theorem 3.1.

THEOREM 3.2. *Let $-1/2 \le \mu < \nu$, $1 \le p \le q < \infty$ and $p\mu + 1 \ge 0$. Suppose also that*

$$(3.7)\qquad \nu + \frac{\mu+1}{q} = \mu + \frac{\nu+1}{p}.$$

Then

$$(3.8)\qquad \Big(\int_0^\infty |T_\nu^\mu g(x)|^q x^{2\mu+1}\,dx\Big)^{1/q} \le C\Big(\int_0^\infty |g(x)|^p x^{2\nu+1}\,dx\Big)^{1/p}.$$

Proof. A homogeneity argument similar to that from the beginning of the proof of Theorem 3.1 also shows that (3.7) is necessary for (3.8) to hold.

Evaluating the formula [EMOT, 8.5 (32)] at $y = 1$ and writing $\nu - \mu - 1$ in place of μ produces

$$\frac{J_\mu(a)}{a^\mu} = c_{\nu,\mu}\int_a^\infty t^{1-\nu}(t^2-a^2)^{\nu-\mu-1} J_\nu(t)\,dt$$

for arbitrary ν, μ satisfying $\Re\mu < \Re\nu < 2\Re\mu + 3/2$ and $a > 0$. Note at this point that only with the stronger assumption $\Re\mu < \Re\nu < 2\Re\mu + 1/2$, $\Re\mu > -1/2$, is the above integral Lebesgue integrable; otherwise it converges in the Riemann sense. Hence, considering first the case $\mu > -1/2$, for real ν, μ that satisfy $-1/2 < \mu < \nu < 2\mu + 1/2$, a change of variable and Fubini's theorem give

$$\begin{aligned} T_\nu^\mu g(x) &= \int_0^\infty H_\nu g(y) \frac{J_\mu(xy)}{(xy)^\mu} y^{2\mu+1}\,dy \\ &= c_{\nu,\mu}\int_0^\infty H_\nu g(y) \int_{xy}^\infty t^{1-\nu}(t^2-(xy)^2)^{\nu-\mu-1} J_\nu(t)\,dt\, y^{2\mu+1}\,dy \\ &= c_{\nu,\mu}\int_x^\infty u^{1-\nu}(u^2-x^2)^{\nu-\mu-1}\int_0^\infty H_\nu g(y) J_\nu(uy) y^{2\nu+1}\,dy\,du \\ &= c_{\nu,\mu}\int_x^\infty u(u^2-x^2)^{\nu-\mu-1}\int_0^\infty H_\nu g(y)\frac{J_\nu(uy)}{(uy)^\nu} y^{2\nu+1}\,dy\,du. \end{aligned}$$

An application of Fubini's theorem is allowed at this point since $|J_\nu(s)| \le Cs^{-1/2}$ on $(0,\infty)$, the function $y^{\nu+1/2} H_\nu g(y)$ is integrable on $(0,\infty)$ and

$u^{1/2-\nu}(u^2-x^2)^{\nu-\mu-1}$ is integrable on (x,∞) (both with respect to the Lebesgue measure).

Since $\nu>-1/2$ the inversion formula for the modified Hankel transform gives

$$T_\nu^\mu g(x)=c_{\nu,\mu}\int_x^\infty (y^2-x^2)^{\nu-\mu-1}yg(y)\,dy. \tag{3.9}$$

It is now easy to see that the above argument, in particular the inversion formula, remains valid for complex ν,μ satisfying $-1/2<\Re\mu<\Re\nu<2\Re\mu+1/2$. Moreover, for any fixed $g\in\mathcal{S}(\mathbb{R}_+)$, $x\in\mathbb{R}_+$ and μ with $\Re\mu>-1/2$, both sides of (3.9) are analytic functions of the complex variable ν, $\Re\nu>\Re\mu$ (analyticity of the coefficient $c_{\nu,\mu}$ follows from its explicit form, cf. [EMOT, 8.5(32)]). Hence, the validity of (3.9) follows for every real ν,μ with $-1/2<\mu<\nu$, by an analytic continuation method. For the remaining case $\mu=-1/2$ we need some parameter interval to do analytic continuation. Fortunately, the interchange of integration we did above is also allowed in the range $-1/2<\nu<1/2$ when $\mu=-1/2$: for this more subtle argument see the remarks in [GT] following the proof of [GT, (1.5)].

To prove (3.8) we now need the inequality

$$\Big(\int_0^\infty\Big(\int_x^\infty (y^2-x^2)^{\nu-\mu-1}yG(y)\,dy\Big)^q x^{2\mu+1}\,dx\Big)^{1/q}\le C\Big(\int_0^\infty G(x)^p x^{2\nu+1}\,dx\Big)^{1/p},$$

which, after an elementary change of variable, turns out to be equivalent to

$$\Big(\int_0^\infty\Big(\int_t^\infty (s-t)^{\nu-\mu-1}h(s)\,ds\Big)^q t^\mu\,dt\Big)^{1/q}\le C\Big(\int_0^\infty h(s)^p s^\nu\,ds\Big)^{1/p}. \tag{3.10}$$

Both G and h are assumed to be nonnegative functions. (3.10) is now a consequence of (3.6) for the operator I_-^α if we take $\alpha=\nu-\mu$, $N=\mu$ and $M=\nu$. Indeed, assuming the hypotheses of Theorem 3.2 to be satisfied it is easy to see that (3.7) forces the condition $p<(\nu+1)/(\nu-\mu)$, which is nothing but $M>\alpha p-1$. Further, in the case $p<1/(\nu-\mu)$ the assumption $p\mu+1\ge 0$ guarantees $q\le p/(1-p\alpha)$ to hold. This finishes the proof of Theorem 3.2.

Theorems 3.1 and 3.2 also have applications to radial Fourier multipliers. Upon setting $H_\nu g(y)=m(y)$ Theorem 3.1 states (under the relevant conditions on the parameters) that

$$\|H_\mu m\|_{L^q(dm_\mu)}\le C\|H_\nu(y^{2(\mu-\nu)}m)\|_{L^p(dm_\nu)} \tag{3.11}$$

provided that $H_\nu m\in C_0^\infty(0,\infty)$. Denote now by $L^p_{\mathrm{rad}}(\mathbb{R}^n)$ the set of radial L^p-functions on $\mathbb{R}^n$, $f(x)=f_0(\|x\|)$, with standard $L^p(\mathbb{R}^n,dx)$ norm and by $[L^p_{\mathrm{rad}}(\mathbb{R}^n)]^\wedge$ the set of its Fourier transforms, i.e. in the case $1\le p<2n/(n+1)$,

$$\widehat{f}(\xi)=c_nH_{(n-2)/2}f_0(\|\xi\|). \tag{3.12}$$

By the convolution inequality it follows for $m\in[L^p_{\mathrm{rad}}(\mathbb{R}^n)]^\wedge$ that

$$T_m:L^1\to L^p,\qquad [T_m\varphi]^\wedge(\xi):=m(\|\xi\|)\,\widehat{\varphi}(\xi),\qquad \varphi\in\mathcal{S}(\mathbb{R}^n),$$

is bounded and it is well known that if a bounded convolution operator from $L^1(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, $1<p<\infty$, is generated by some radial m then $m\in[L^p_{\mathrm{rad}}(\mathbb{R}^n)]^\wedge$. We then say that $m(\|\xi\|)\in M^{1,p}(\mathbb{R}^n)$ and define $\|m(\|\xi\|)\|_{M^{1,p}(\mathbb{R}^n)}$ to be the operator norm of T_m which is equal to the $L^p(\mathbb{R}^n,dx)$-norm of $H_{(n-2)/2}m(\|x\|)$. Setting $\overline{\xi}=(\xi,\xi_{n+1})$ and $\overline{\overline{\xi}}=(\overline{\xi},\xi_{n+2})$ we have

Corollary 3.3. *Let $1<p<q<\infty$ and $n\ge 2$ be an integer.*

(a) $$\|m(\|\overline{\overline{\xi}}\|)\|_{M^{1,q}(\mathbb{R}^{n+2})}\le C\|\|\xi\|^2m(\|\xi\|)\|_{M^{1,p}(\mathbb{R}^n)},\qquad \frac1q=\frac1p-\frac{2}{(n+2)p};$$

(b) $$\|m(\|\xi\|)\|_{M^{1,q}(\mathbb{R}^n)}\le C\|m(\|\overline{\xi}\|)\|_{M^{1,p}(\mathbb{R}^{n+1})},\qquad \frac1q=\frac1p-\frac{1}{np'}.$$

Proof. For (a) choose $\nu=(n-2)/2$ and $\mu=n/2$ for an integer $n\ge2$ in Theorem 3.1. The assumption there that m is smooth may be dropped since any $H_\nu m(\|x\|)\in L^p(\mathbb{R}^n)$ can be approximated in $L^p(\mathbb{R}^n)$ by smooth rapidly decreasing $H_\nu m_k(\|x\|)$ with $m_k\to m$ in $S'(\mathbb{R}^n)$, thus (3.11) gives the assertion (a) for an arbitrary radial $m\in M^{1,p}(\mathbb{R}^n)$. Part (b) follows similarly from Theorem 3.2 upon choosing $\mu=(n-2)/2$ and $\nu=(n-1)/2$.

Remarks. 1) The results of Corollary 3.3 are best possible for $1<p<q<2n/(n+1)$ in the following sense. For (a) consider the example

$$m(t)=t^{-(n+2)/q'}(1+\log^2t)^{-1}=t^{-2}t^{-n/p'}(1+\log^2t)^{-1}.$$

By a criterion in [T] we have $m(\|\overline{\overline{\xi}}\|)\in[L^q(\mathbb{R}^{n+2})]^\wedge$ but $m(\|\overline{\overline{\xi}}\|)$ does not belong to any other space $[L^r(\mathbb{R}^{n+2})]^\wedge$, $r\ne q$, which follows directly by (3.12) on account of Hölder's inequality. The same example, this time written in the form

$$m(t)=t^{-(n+1)/p'}(1+\log^2t)^{-1}=t^{-n/q'}(1+\log^2t)^{-1},$$

again shows that in the same sense (b) is also best possible.

2) Note that (a) is in the spirit of the following result due to Coifman and Weiss [CW, pp. 33–45]:

$$\|m(\|\overline{\overline{\xi}}\|)\|_{M^{p,p}(\mathbb{R}^{n+2})}\le C\|nm(\|\xi\|)+\|\xi\|m'(\|\xi\|)\|_{M^{p,p}(\mathbb{R}^n)}, \tag{3.13}$$

which is only good for p near 1 or infinity. That the right side of (3.13) contains an expression of type $tm'(t)$ is only natural in view of the necessary conditions for radial Fourier multipliers in [GT, p. 412] (for p near 1). These conditions also indicate that part (a) of Corollary 3.3 is a natural result ($p \le q < 2n/(n+1)$) for the necessary conditions arising from the right side guarantee quite precisely the necessary conditions arising from the left side. Part (b) of the corollary is in the spirit of the well known deLeeuw restriction result for Fourier multipliers (see e.g. [To], p. 265), which by duality and the Riesz interpolation theorem implies

$$\|m(\|\xi\|)\|_{M^{q,q}(\mathbb{R}^n)} \le C\|m(\|\bar{\xi}\|)\|_{M^{p,p}(\mathbb{R}^{n+1})},$$

$1 \le \min\{p,p'\} \le q \le \max\{p,p'\} \le \infty$.

4. Density theorems. In this section we prove density theorems which were announced and used in §2. Because they are of some independent interest and, perhaps, could be used for other purposes, we prove these theorems in a more general form. The results we obtain are generalizations to the Hankel transform setting of density theorems proved by Muckenhoupt, Wheeden and Young [MWY]. In other words, we extend the results of Section 2 of [MWY] from the cosine transform setting that corresponds to the case $\nu = -1/2$ to general $\nu \ge -1/2$. Hence in what follows we restrict the attention to $\nu > -1/2$ only. Needless to say, we follow the ideas of [MWY] fairly closely.

If not otherwise stated the letter k will always denote an integer. Recall that $\mathcal{S}(\mathbb{R}_+)$ denotes the space of restrictions to $(0,\infty)$ of even Schwartz functions on $\mathbb{R}_+$ and C_0^∞ denotes the space of C^∞ functions with compact support in $(0,\infty)$. Recall also that H_ν is a bijection on $\mathcal{S}(\mathbb{R}_+)$. Observing that for even f in $\mathcal{S}(\mathbb{R}_+)$ we have $f'(0) = 0$ it is readily checked that the differential operator L_ν can be extended to even Schwartz functions by setting $L_\nu f(0) = 2(\nu+1)f''(0)$. Thus, if the powers of the operator L_ν are now defined in the usual way: $L_\nu^1 = L_\nu$ and $L_\nu^k = L_\nu(L_\nu^{k-1})$, $k > 1$, iterating the process we can regard $L_\nu^k f$, $k = 0,1,\ldots$, to be a function in $\mathcal{S}(\mathbb{R}_+)$.

LEMMA 4.1. *If $f \in \mathcal{S}(\mathbb{R}_+)$ satisfies*

$$\int_0^\infty x^{2j} f(x) x^{2\nu+1}\,dx = 0, \qquad j = 0,1,\ldots,k, \tag{4.1}$$

then $(H_\nu f)^{(j)}(0)$, the derivatives of $H_\nu f$ at zero, vanish for $j = 0,1,\ldots,2k$.

Proof. It follows from (1.8) that

$$L_\nu^j H_\nu f(x) = (-1)^j H_\nu((\cdot)^{2j} f)(x), \qquad x > 0. \tag{4.2}$$

Hence $L_\nu^j H_\nu f(0) = 0$ for $j = 0,1,\ldots,k$, which implies $(H_\nu f)^{(j)}(0) = 0$ for $j = 0,2,\ldots,2k$. It is obvious that the same holds for odd j's.

By $Q_k(\nu)$, $k \ge 0$, we will denote the set of functions f in $L^2(dm_\nu) \cap L^{1,2k}(dm_\nu)$ that satisfy (4.1) and, if $k < 0$, we set $Q_k = L^2(dm_\nu)$; then we define $C_0^\infty(k,\nu) = C_0^\infty \cap Q_k(\nu)$.

LEMMA 4.2. *If $1 \le p < \infty$, $\gamma > -1$ and $2k > -2 + (\gamma+1)/p$ then every function f in $C_0^\infty(k,\nu)$ can be approximated by functions from $H_\nu(C_0^\infty)$ in both $L^{p,\gamma}(dx)$ and $L^2(dx)$ norms.*

Proof. Let $\phi_n(x)$ be the sequence of functions on $(0,\infty)$ defined as in the proof of Lemma 6.2 in [MWY]: $\phi_n(x) = \phi(nx)$ if $0 < x \le 1/n$, $\phi_n(x) = \phi(x/n)$ if $x \ge n$ and $\phi_n(x) = 0$ if $n^{-1} \le x \le n$, where ϕ is a fixed C^∞ function on $(0,\infty)$ with $\phi(x) = 0$ for $1/2 \le x \le 2$, $\phi(x) = 1$ for $0 < x \le 1/4$ and $x \ge 4$ and $0 \le \phi(x) \le 1$ elsewhere. Given f in $C_0^\infty(k,\nu)$ define $f_n = H_\nu(H_\nu f \cdot (1-\phi_n))$. Since $1-\phi_n \in C_0^\infty$ we get $f_n \in H_\nu(C_0^\infty)$. The convergence of f_n to f in $L^2(dx)$ is immediate. To prove that f_n approaches f in $L^{p,\gamma}(dx)$ norm we write

$$\|f - f_n\|_{p,\gamma} \le \|(f(x) - f_n(x))(1+x)^{2(k+1)}\|_\infty \|(1+x)^{-2(k+1)}\|_{p,\gamma}$$

and note that the last norm on the right is finite due to assumptions on p,γ and k. Moreover, by (4.2),

$$\begin{aligned}\|(f-f_n)(1+x)^{2(k+1)}\|_\infty &\le C\|f-f_n\|_\infty + C\|x^{2(k+1)}(f-f_n)\|_\infty\\ &\le C\|H_\nu(f-f_n)\|_1 + C\|L_\nu^{k+1}(H_\nu f - H_\nu f_n)\|_1,\end{aligned}$$

where $\|\cdot\|_1$ denotes the norm in $L^1(dm_\nu)$. Since $H_\nu f - H_\nu f_n = H_\nu f \cdot \phi_n$ we have $\|H_\nu(f-f_n)\|_1 \to 0$ as $n \to \infty$. To estimate the remaining term we use the following Leibniz' rule for the $(k+1)$th power of the operator L_ν:

$$L_\nu^{k+1}(H_\nu f \cdot \phi_n) = \sum_{1 \le i+j \le 2(k+1)} c_{ij} x^{-2(k+1)+i+j} (H_\nu f)^{(i)} \phi_n^{(j)}.$$

This may be proved by induction. We now consider the $L^1(dm_\nu)$ norm of each summand in the above sum separately. Fixing i,j with $1 \le i+j \le 2(k+1)$ we have to show that the quantities

$$n^{-j}\int_n^\infty |(H_\nu f)^{(i)}(x)\phi^{(j)}(x/n)| x^{-2(k+1)+i+j+2\nu+1}\,dx \tag{4.3}$$

and

$$n^j \int_0^{1/n} |(H_\nu f)^{(i)}(x)\phi^{(j)}(xn)| x^{-2(k+1)+i+j+2\nu+1}\,dx \tag{4.4}$$

tend to 0 as $n \to \infty$. This is easily seen for (4.3) since $\phi^{(j)}$ is bounded and $(H_\nu f)^{(i)}$ is of rapid decrease at ∞. For (4.4), consider first the case $i = 2(k+1)$. Then $j = 0$ and (4.4) is bounded by $Cn^{-(2\nu+2)}$. If $0 \le i \le 2k+1$ then by Taylor's formula and Lemma 4.1 the estimate $|(H_\nu f)^{(i)}(x)| \le Cx^{2k+1-i}$

follows. This shows that (4.4) is bounded by $Cn^{-(2\nu+1)}$ and finishes the proof of Lemma 4.2.

LEMMA 4.3. *If $1 \le p < \infty$ and $\gamma > -1$, then every function f in $Q_k(\nu) \cap L^{p,\gamma}(dx)$ can be approximated by functions from $C_0^\infty(k,\nu)$ in both $L^{p,\gamma}(dx)$ and $L^2(dx)$ norms.*

The proof of Lemma 4.3, with minor changes, is the same as the proof of Lemma 6.6 of [MWY]. Let us mention at this point that for our future purposes we will use, for given k, a sequence $\{\alpha_j(x)\}_{j=0}^k$ of C^∞ functions, the same as in Lemma 6.5 of [MWY] except for the fact that now their supports are separated from zero, say, they are contained in $1/4 \le x \le 3/4$. It can be checked that this requirement is not essential. Recall that an important feature of α_j's is the fact that

$$\int_0^\infty x^i \alpha_j(x)\,dx = \delta_{i,j}, \tag{4.5}$$

$0 \le i, j \le k$, where $\delta_{i,j}$ is the Kronecker delta.

LEMMA 4.4. *If $1 \le p < \infty$, $\gamma > -1$ and $2k < -1+(\gamma+1)/p-(2\nu+1)$ then every function f in C_0^∞ can be approximated by functions from $C_0^\infty(k,\nu)$ in $L^{p,\gamma}(dx)$ norm.*

Proof. If k is negative the statement is obvious. Let $k \ge 0$ and take $\alpha_0, \alpha_1, \ldots, \alpha_{2k}$ satisfying (4.5) for $0 \le i, j \le 2k$, supported in $1/4 \le x \le 3/4$ and, given $f \in C_0^\infty$, define

$$f_n(x) = f(x) - \sum_{i=0}^k n^{2i+1}\alpha_{2i}(nx)x^{-(2\nu+1)} \int_0^\infty f(t)t^{2i}t^{2\nu+1}\,dt.$$

Then $f_n \in C_0^\infty(k,\nu)$ and the required convergence $f_n \to f$ as $n \to \infty$ holds in $L^{p,\gamma}(dx)$.

LEMMA 4.5. *If $1 \le p < \infty$, $\gamma > -1$ and $2k > -3 + (\gamma+1)/p - (2\nu+1)$ then every function f in $Q_k(\nu) \cap L^{p,\gamma}(dx)$ can be approximated by functions from $C_0^\infty(k+1,\nu)$ in both $L^{p,\gamma}(dx)$ and $L^2(dx)$ norms.*

Proof. By Lemma 4.3 we can assume that f is in $C_0^\infty(k,\nu)$. Again the statement is obvious if $k < -2$. Hence, assume $k \ge -1$, take $\alpha_0, \alpha_1, \ldots$ $\ldots, \alpha_{2(k+1)}$ satisfying (4.5) for $0 \le i, j \le 2(k+1)$ and define

$$f_n(x) = f(x) - n^{-2(k+1)-1}\alpha_{2(k+1)}(x/n)x^{-(2\nu+1)} \int_0^\infty f(t)t^{2(k+1)}t^{2\nu+1}\,dt.$$

Then f_n is in $C_0^\infty(k+1,\nu)$ and f_n converges to f in $L^{p,\gamma}(dx)$ and $L^2(dx)$.

LEMMA 4.6. *If $1 < p < \infty$, $\gamma > -1$ and $2k = -3 + (\gamma+1)/p - (2\nu+1)$ then every function f in $Q_k(\nu) \cap L^{p,\gamma}(dx)$ can be approximated by functions from $Q_{k+1}(\nu) \cap L^{p,\gamma}(dx)$ in both $L^{p,\gamma}(dx)$ and $L^2(dx)$ norms.*

Proof. We can consider the case $k \ge -1$ only. Take $\alpha_0, \alpha_1, \ldots, \alpha_{2(k+1)}$ satisfying (4.5) for $0 \le i, j \le 2(k+1)$ and define

$$g_n(x) = \frac{\chi_{[e,n]}(x)x^{-(2\nu+1)}}{x^{2(k+1)+1}\log x \log\log n} - \sum_{i=0}^k \frac{\alpha_{2i}(x)x^{-(2\nu+1)}}{\log\log n} \int_e^n \frac{t^{2i-2(k+1)-1}}{\log t}\,dt.$$

Then $\int_0^\infty g_n(x)x^{2i}x^{2\nu+1}\,dx$ equals 0 for $i = 0, 1, \ldots, k$ and is 1 for $i = k+1$. An argument shows that $g_n \to 0$ in $L^2(dx)$ and convergence of g_n to 0 in $L^{p,\gamma}(dx)$ follows from the fact that $\int_e^\infty (x(\log x)^p)^{-1}\,dx < \infty$ for $p > 1$. By Lemma 4.3 we can assume f to be in $C_0^\infty(k+1,\nu)$. Define

$$f_n(x) = f(x) - g_n(x)\int_0^\infty f(t)t^{2(k+1)}t^{2\nu+1}\,dt.$$

Then $f_n \in C_0^\infty(k+1,\nu)$ and the properties of g_n imply the desired convergence for f_n.

THEOREM 4.7. *If $1 < p < \infty$ and $\gamma > -1$ then $H_\nu(C_0^\infty)$ is dense in $L^{p,\gamma}(dx)$. If in addition γ is not of the form $2k + 2\nu + 1$ then $H_\nu(C_0^\infty)$ is dense in $L^{1,\gamma}(dx)$.*

Proof. Fix p and γ and choose an integer k satisfying $-3 + (\gamma+1)/p$ $-(2\nu+1) \le 2k < -1+(\gamma+1)/p-(2\nu+1)$ if $p > 1$ and $-2+\gamma-(2\nu+1) < 2k$ $< \gamma - (2\nu+1)$ if $p = 1$. Since C_0^∞ is dense in $L^{p,\gamma}(dx)$ it is sufficient to approximate C_0^∞ functions only. Lemma 4.4 allows us further to restrict attention to $C_0^\infty(k,\nu)$ functions. By applying Lemma 4.5 or Lemma 4.6 several times and then applying Lemma 4.3 if necessary, we conclude that every $C_0^\infty(k,\nu)$ function can be approximated by $C_0^\infty(k+r,\nu)$ functions, where r is a positive integer such that $2(k+r) > -2 + (\gamma+1)/p$. Using Lemma 4.2 finishes the proof of the theorem.

COROLLARY 4.8. *If $1 < p < \infty$ and $\beta > -1 - p(\nu+1/2)$ then $\mathcal{H}_\nu(C_0^\infty)$ is dense in $L^{p,\beta}(dx)$. If in addition β is not of the form $2k + \nu + 1/2$ then $\mathcal{H}_\nu(C_0^\infty)$ is dense in $L^{1,\beta}(dx)$.*

Proof. The corollary follows from Theorem 4.7 by using (1.4), the fact that one has $x^{\nu+1/2}C_0^\infty = C_0^\infty$ and the remark that multiplication by $x^{\nu+1/2}$ is an isometric bijection between $L^{p,\gamma}(dx)$ and $L^{p,\gamma-p(\nu+1/2)}(dx)$.

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Two-weight norm inequalities for maximal functions on homogeneous spaces and boundary estimates

by

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Abstract. Let D be an open subset of a homogeneous space (X,d,μ). Consider the maximal function

$$M_\varphi f(x)=\sup\frac{1}{\varphi(B)}\int\limits_{B\cap\partial D}|f|\,d\nu,\quad x\in D,$$

where the supremum is taken over all balls of the form $B=B(a(x),r)$ with $r>t(x)=d(x,\partial D)$, $a(x)\in\partial D$ is such that $d(a(x),x)<\frac{3}{2}t(x)$ and φ is a nonnegative set function defined for all Borel sets of X satisfying the quasi-monotonicity and doubling properties. We give a necessary and sufficient condition on the weights w and v for the weighted norm inequality

$$\Big(\int\limits_D[M_\varphi(f)]^q w\,d\mu\Big)^{1/q}\le c\Big(\int\limits_{\partial D}|f|^p v\,d\nu\Big)^{1/p} \tag{0.1}$$

to hold when $1<p<q<\infty$, $\sigma d\nu=v^{1-p'}d\nu$ is a doubling weight, and $d\nu$ is a doubling measure, and give a sufficient condition for (0.1) when $1<p\le q<\infty$ without assuming that σ is a doubling weight but with an extra assumption on φ. Another characterization for (0.1) is also provided for $1<p\le q<\infty$ and D of the form $Y\times(0,\infty)$, where Y is a homogeneous space with group structure. These results generalize some known theorems in the case when M_φ is the fractional maximal function in $\mathbb{R}^{n+1}_+$, that is, when

$$M_\varphi f(x,t)=M_\gamma f(x,t)=\sup_{r>t}\frac{1}{\nu(B(x,r))^{1-\gamma}}\int\limits_{B(x,r)}|f|\,d\nu,$$

where $(x,t)\in\mathbb{R}^{n+1}_+$, $0<\gamma<1$, and ν is a doubling measure in $\mathbb{R}^n$.

1. Introduction. We consider a homogeneous space (X,d,μ) where $d:X\times X\to[0,\infty)$ satisfies:

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