Hankel multipliers and transplantation operators

by

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Dedicated to Professor Satoru Igari on the occasion of his 60th birthday

Abstract. Connections between Hankel transforms of different order for $L^p$-functions are examined. Well known are the results of Guy [Guy] and Schindler [Sch]. Further relations result from projection formulae for Bessel functions of different order. Consequences for Hankel multipliers are exhibited and implications for radial Fourier multipliers on Euclidean spaces of different dimensions indicated.

1. Introduction. It is well known that harmonic analysis of radial functions on the Euclidean space $\mathbb{R}^n$, $n \geq 1$, reduces to studying appropriate function spaces on the half-line equipped with the measure $x^{n-1}dx$. The Fourier transform is then replaced by the modified Hankel transform of order $(n - 2)/2$. The aim of this paper is to show, among other things, that also studying the non-modified Hankel transform of an arbitrary order $\nu \geq -1/2$ within an appropriate weighted setting leads to corresponding results for the Fourier transform of radial functions. This is seen, for instance, in Section 2 where we discuss multiplier results for the modified Hankel transform. It occurs that they are closely related to two transference theorems of Rubio de Francia for the Fourier transform on Euclidean spaces.

Given $\nu \geq -1/2$ and $f$, an integrable function on $\mathbb{R}_+ = (0, \infty)$, its (non-modified) Hankel transform is defined by

\[ \mathcal{H}_\nu f(x) = \int_0^\infty (xy)^{1/2} J_\nu(xy) f(y) \, dy, \quad x > 0. \]

Here $J_\nu(x)$ denotes the Bessel function of the first kind of order $\nu$ [Sz, (1.7.1)]. For $\nu = -1/2$ or $\nu = 1/2$ one recovers the cosine and sine transfor...
forms on the half-line. The modified Hankel transform is given by
\begin{equation}
H_{\nu}f(x) = \int_0^\infty J_\nu(xy)f(y)y^{2\nu+1}dy, \quad x > 0.
\end{equation}

Due to the estimates on the Bessel function
\begin{equation}
J_\nu(x) = O(x^\nu), \quad J_\nu(x) = O(x^{-1/2}),
\end{equation}
valid for $z \to 0$ and $z \to \infty$ respectively, $H_{\nu}f$ is well defined for every function $f$ in $L^p(\mathbb{R}_+, x^{2\nu+1}dx)$, $1 \leq p < 4(\nu + 1)/(2\nu + 3)$. Clearly both transforms are related to each other by
\begin{equation}
H_{\nu}f(x) = x^{\nu+1/2}H_\nu((\nu + 1/2)f')'(x)
\end{equation}
whenever $f$ is an integrable function on $\mathbb{R}_+$, and $\int_0^\infty |f(x)|x^{\nu+1/2}dx < \infty$, for instance. Moreover, for $\nu \geq -1/2$ the inversion formulae
\begin{equation}
f(x) = \int_0^\infty (xy)^{1/2}J_\nu(xy)H_{\nu}f(x)dx
\end{equation}
and
\begin{equation}
f(x) = \int_0^\infty J_\nu(xy)H_{\nu}f(x)x^{2\nu+1}dx
\end{equation}
hold: (1.5) holds, for instance, for every $C^1$ function $f \in L^1(\mathbb{R}_+, dx)$ with $H_{\nu}f \in L^1(\mathbb{R}_+, dx)$; (1.6) holds for every $C^1$ function $f \in L^1(\mathbb{R}_+, x^{2\nu+1}dx)$ with $H_{\nu}f \in L^1(\mathbb{R}_+, x^{2\nu+1}dx)$ (cf. [W, p. 456]). More can be said: $H_{\nu}$ is a bijection on $L^2(\mathbb{R}_+)$, the space of infinitely differentiable even functions on $\mathbb{R}$ with rapidly decreasing derivatives, while $H_{\nu}$ is a bijection on the Zemanian space $Z_\nu$ of all $C^\infty$ functions on $\mathbb{R}_+$ for which the quantity
\[\sup_{x > 0} \left| x^n \frac{1}{x} \frac{d}{dx} \frac{1}{x} \frac{d}{dx} f(x) \right| \]
is finite for every $n$, $k \in \mathbb{N} = \{0, 1, 2, \ldots\}$ (see [Z1, Z2]). Note at this point that $C_0^\infty = C_0^\infty(\mathbb{R}_+)$, the space of compactly supported $C^\infty$ functions on $\mathbb{R}_+$, is contained in every $Z_\nu$. The kernels $\varphi_\nu(y) = (xy)^{1/2}J_\nu(xy)$, $y > 0$, appearing in (1.1) satisfy
\begin{equation}
\left( \frac{d^2}{dy^2} + \frac{1 - 4\nu^2}{y^2} \right) \varphi_\nu(y) = -x^2 \varphi_\nu(y), \quad x > 0,
\end{equation}
while the kernels $\varphi_\nu(y) = (xy)^{-\nu}J_\nu(xy)$, $y > 0$, appearing in (1.2) fulfill
\begin{equation}
\left( \frac{d^2}{dy^2} + \frac{2\nu + 1}{y} \frac{d}{dy} \right) \varphi_\nu(y) = -x^2 \varphi_\nu(y), \quad x > 0.
\end{equation}
The differential operators on the left sides of (1.7) and (1.8) are symmetric in $L^2(\mathbb{R}_+, dx)$ and $L^2(\mathbb{R}_+, x^{2\nu+1}dx)$ respectively.

As usual we use $C$ or $c$ with or without subscripts for a constant which is not necessarily the same at each occurrence.

### 2. Hankel multipliers

In this section we fix $\nu \geq -1/2$ and consider weighted Lebesgue spaces on $\mathbb{R}_+$ with respect to the Lebesgue measure $dx$ on one occasion and the measure
\[dm_u(x) = x^{2\nu+1}dx\]
on another one. Hence, in what follows we use the notation
\[\|f\|_{p,\alpha} = \left( \int_0^\infty |f(x)|^p x^{\alpha \nu} dx \right)^{1/p}\]
and
\[\|f\|_{L^p(\mathbb{R}_+, dm_u)} = \left( \int_0^\infty |f(x)|^p x^{\alpha \nu} dm_u(x) \right)^{1/p}\]
for $1 \leq p < \infty$ with the usual modification when $p = \infty$. By $L^{p,\alpha}(dx)$ and $L^{p,\alpha}(dm_u)$ we denote the weighted Lebesgue spaces of functions for which the above quantities are finite. If $\alpha = 0$ we write $L^p$ instead of $L^{p,0}$. By $M_{p,\alpha}^{\nu}$ and $M_{p,\alpha}^{\nu}$ we denote the spaces of weighted $p$-multipliers for the Hankel and modified Hankel transform. Thus, a bounded measurable function $m(x)$ on $\mathbb{R}_+$ is in $M_{p,\alpha}^{\nu}$ provided
\[\|H_{\nu}(m \cdot H_{\nu}f)\|_{p,\alpha} \leq C \|f\|_{p,\alpha},\]
where $C$ is a constant independent of $f$ in $H_{\nu}(C_0^\infty)$, the image of $C_0^\infty$ under the action of $H_{\nu}$. The least constant $C$ for which the above inequality holds is called the multiplier norm of $m$. The multiplier space $M_{p,\alpha}^{\nu}$ has a similar definition, now with the norm $\| \cdot \|_{L^{p,\alpha}(\mathbb{R}_+, dm_u)}$ in use, and with $H_{\nu}(C_0^\infty)$ as the testing function space.

We postpone to Section 4 the proof of the fact that $H_{\nu}(C_0^\infty)$ is dense in $L^{p,\alpha}(dx)$ if $1 < p < \infty$ and $\alpha > 1$ while $H_{\nu}(C_0^\infty)$ is dense in $L^{p,\alpha}(dx)$ if $1 < p < \infty$ and $\alpha > -1 - p/(\nu + 1/2)$ (the case $p = 1$ requires additional assumptions). This is the content of Theorem 4.7 and Corollary 4.8.

The following is Guy’s transplantation theorem for the Hankel transform (cf. also [Sch] for an alternative proof).

**Theorem** ([Guy, Lemma 8C]). Let $\mu, \nu \geq -1/2$, $1 < p < \infty$ and $-1 < \alpha < p - 1$. Then
\[C^{-1}\|H_{\nu}f\|_{p,\alpha} \leq \|H_{\nu}f\|_{p,\alpha} \leq C\|H_{\nu}f\|_{p,\alpha}\]
with $C = C(\mu, \nu, p, \alpha)$ independent of $f \in L^1(\mathbb{R}_+, dx)$.

As an immediate consequence one obtains
COROLLARY 2.1. Let $\mu, \nu \geq -1/2$, $1 < p < \infty$ and $-1 < \alpha < p - 1$. Then
\begin{equation}
M_p^{\mu, \alpha} = M_p^{\nu, \alpha}.
\end{equation}

Proof. Assuming $m$ is in $M_p^{\mu, \alpha}$ and $f$ is in $H_p(C_0^\infty)$ and using the fact that $H_p \in L^1(dx)$, hence $H_p H_p f \in H_p(C_0^\infty)$, we write
\[
\|H_p(m \cdot H_p f)\|_{p, \alpha} \leq C\|H_p(m \cdot H_p (H_p H_p f))\|_{p, \alpha}
\leq C C_{\mu, \alpha} \|H_p H_p f\|_{p, \alpha} \leq C^2 C_{\mu, \alpha} \|f\|_{p, \alpha},
\]
where $C_{\mu, \alpha}$ denotes the operator norm of the multiplier $m$ in $M_p^{\mu, \alpha}$. Thus $M_p^{\mu, \alpha} \subseteq M_p^{\nu, \alpha}$. Analogously the opposite inclusion follows.

COROLLARY 2.2. Let $\mu, \nu \geq -1/2$ and $1 < p < \infty$. Assume further that
$-1 < \beta + (\nu + 1/2)(2 - p) < p - 1$ and define $\beta^* = \beta + (\nu - \mu)(2 - p)$. Then
\begin{equation}
M_p^{\mu, \beta} = M_p^{\nu, \beta^*}.
\end{equation}

Proof. The identity (1.4), the fact that $C_0^{\infty} = C_0^\infty$ and the definition of multiplier spaces immediately give
\[
M_p^{\mu, \beta} = M_p^{\nu, \beta^* + p(2\nu + 1)(1/\nu - 1/2)}
\]
and then (2.1) produces (2.2).

In particular, (2.2) for $p = 2$ gives

COROLLARY 2.3. Let $\mu, \nu \geq -1/2$ and $-1 < \beta < 1$. Then
\begin{equation}
M_2^{\mu, \beta} = M_2^{\nu, \beta^*}.
\end{equation}

In some sense (2.3) may be viewed as a “radial” generalization of Rubio de Francia’s transference result [RdF, Theorem 2.1], which states that given $-1 < \beta < 1$ and $m \in L^\infty(\mathbb{R}_+^n)$, if $m(\cdot|x|)$ is a Fourier multiplier on $L^2(\mathbb{R}^n, |x|^n dx)$, then $m(\cdot|x|)$ is a Fourier multiplier on $L^2(\mathbb{R}^n, |x|^n dx)$, $n \geq 2$ ($\cdot|\cdot|$ denotes here the Euclidean norm in the appropriate Euclidean space, and $dx$ the Lebesgue measure). When restricted to the space of radial functions on which the multiplier acts, (2.3) states that the opposite implication also holds. In particular, this implies that one can jump between Euclidean spaces of arbitrary dimension, in contrast to the preceding situation where a jump was allowed only between $\mathbb{R}$ and $\mathbb{R}^n$.

Let $T_R$, $R > 0$, denote the multiplier operator corresponding to the characteristic function of the interval $(0, R)$. By a homogeneity argument, for every $\mu, p$ and $\alpha$, the operator norms of $T_R$ as members of $M_2^{\mu, \alpha}$ or $M_p^{\mu, \alpha}$ are independent of $R > 0$. Hence, in what follows we consider $T = T_1$ only. Hirschman’s “one-dimensional” weighted multiplier result [Hi1] says that $T \in M_p^{1/2, \alpha} = M_p^{1/2, \alpha}$ for every $1 < p < \infty$ and $-1 < \alpha < p - 1$. Thus (2.1) further gives that $T \in M_p^{\nu, \alpha}$ for every $\nu > -1/2, 1 < p < \infty$ and $-1 < \alpha < p - 1$, which, for $\alpha = 0$, was proved by Wing [Wi]. Similarly, Herz’ result [He] which says that $T \in M_p^{\nu, \alpha}$ for $\nu > -1/2$ provided
\[
\frac{4(\nu + 1)}{2\nu + 3} < p < \frac{4(\nu + 1)}{2\nu + 1}
\]
may be recovered from (2.2) and Hirschman’s result just mentioned.

The next corollary may be considered as a “radial” extension of another result due to Hirschman [Hi2].

COROLLARY 2.4. Let $\mu \geq -1/2$ and $-1 < \beta < 1$. Then $T \in M_2^{\mu, \beta}$.

Proof. Combine (2.3) and the fact that $T \in M_2^{1/2, \beta}$.

As already mentioned, Schindler gave an alternative proof of Guy’s result. Besides, in the special case $\mu = \nu + 2k$, $k = 1, 2, \ldots$, her approach allowed taking into account both endpoints $p = 1$ and $p = \infty$ and, in addition, obtaining a range of $\alpha$’s different from the $A_p$ range from Guy’s theorem.

THEOREM ([Sch, Theorems 3 and 4]). Let $\nu \geq -1/2$, $1 \leq p < \infty$, $k = 1, 2, \ldots$ and $-p(\nu + 1/2) < \alpha < p(\nu + 1/2)$. Then
\[
C^{-1} \|H_p f\|_{p, \alpha} \leq \|H_{p+2k} f\|_{p, \alpha} \leq C \|H_p f\|_{p, \alpha}
\]
with $C = C(\nu, p, \alpha)$ independent of $f \in L^1(\mathbb{R}^+, dx)$.

In consequence, the analogue to (2.1) is now the identity
\begin{equation}
M_p^{\mu, \beta} = M_p^{\nu+2k, \beta},
\end{equation}
where $\nu \geq -1/2$, $1 \leq p < \infty$, $k = 1, 2, \ldots$ and $-p(\nu + 1/2) < \alpha < p(\nu + 1/2)$. Moreover, in the case $p > 1$, considering the above hypotheses with those from Corollary 2.1 allows us to enlarge the range of $\alpha$’s for which (2.4) holds to
\[
-\max\{p(\nu + 1/2), 1\} \leq \alpha \leq \max\{p(\nu + 1/2), p - 1\}.
\]

Similarly, the analogue to (2.2) is
\begin{equation}
M_p^{\mu, \beta} = M_p^{\nu+2k, \beta},
\end{equation}
where $\nu \geq -1/2, 1 < p < \infty$, $k = 1, 2, \ldots$, $\beta^* = \beta - 2k(2 - p)$ and
\[
-\max\{p(\nu + 1/2), 1\} \leq \beta + p(2\nu + 1) \left(\frac{1}{p} - \frac{1}{2}\right) \leq \max\{p(\nu + 1/2), p - 1\}.
\]

In particular, (2.5) for $p = 2$ gives

COROLLARY 2.5. Let $\nu \geq -1/2, k = 1, 2, \ldots$ and $-\max\{2\nu + 1, 1\} < \beta < \max\{2\nu + 1, 1\}$. Then
\[
M_2^{\mu, \beta} = M_2^{\nu+2k, \beta}.
\]
The above stands in a relationship with another Rubio de Francia transference result, [RdF, Theorem 2.2], in the same way as (2.3) "generalizes" [RdF, Theorem 2.1]. This result says that given $m \in L^\infty(R_+)$ and $w(s)$, a nonnegative measurable function on $R_+$, if $m(\|w\|)$ is a Fourier multiplier on $L^p(R^n, w(\|x\|)dx)$ for some $n \geq 2$ then $m(\|x\|)$ is a Fourier multiplier on $L^p(R^{n-k}, w(\|x\|)dx)$ for any $k = 1, 2, \ldots$. Corollary 2.5 says that when restricted to the power weights indicated and spaces of radial functions on which the multipliers act, the converse also holds provided the difference of the Euclidean dimensions is a multiple of 4 (raising $v$ by 2 changes the Euclidean dimension by 4).Speaking less precisely, this means that under appropriately modified assumptions we can exchange radial Fourier multipliers, in both directions, between Euclidean spaces whose difference in dimensions is a multiple of 4 (the one-dimensional situation is now included!).

3. Weighted estimates for the transference operators. Throughout this section all the functions we are dealing with are assumed to be in $C_0^\infty$. Let

$$L_{\nu} = -\left( \frac{d^2}{dy^2} + \frac{2\nu + 1}{y} \frac{d}{dy} \right), \quad \nu \geq -1/2,$$

be the differential operator appearing in (1.8). Clearly

$$H_{\nu}(L_{\nu}f)(y) = y^{2\nu}H_{\nu}f(y).$$

Hence, in terms of the modified Hankel transform,

$$H_{\nu}(L_{\nu}^\delta f)(y) = y^{2\nu}H_{\nu}f(y)$$

is a well motivated definition of $L_{\nu}^\delta$, the $\delta$-fractional power of $L_{\nu}$. Rewriting in terms of the modified Hankel transform the inequality

$$\|H_{\mu}H_{\nu}f\|_{p,\alpha} \leq C\|f\|_{p,\alpha},$$

which follows from Guy's transplantation theorem, gives

$$\left( \int_0^\infty |H_{\mu}H_{\nu}h(x)|p_{\alpha} x^{\gamma} dx \right)^{1/p} \leq C \left( \int_0^\infty |L_{\nu}(\mu-\nu/2)h(x)|p_{\alpha} x^{\delta} dx \right)^{1/p},$$

where $\gamma = p(\mu + 1/2) + \alpha$, $\delta = p(\nu + 1/2) + \alpha$ for $1 < p < \infty$, $-1 < \alpha < p-1$ and $\mu, \nu \geq -1/2$.

In this section we prove two weighted $L^p-L^q$ inequalities for the transplantation operator

$$T_{\nu}^\mu = H_{\mu} \circ H_{\nu}.$$

This is achieved first by using appropriately chosen integral formulae for Bessel functions that generate nice representations of $T_{\nu}^\mu$ (with necessary restrictions on $\nu$ and $\mu$) and then applying some weighted norm inequalities for the Riemann–Liouville and Weyl fractional integral operators.

**Theorem 3.1.** Let $-1/2 \leq \nu < \mu$, $1 < p < q < \infty$, $p(\mu + 1) \geq 1$ and

$$\frac{\mu + 1}{q} = \frac{\nu + 1}{p}.$$ 

Then

$$\left( \int_0^\infty |T_{\nu}^\mu g(x)|^{q_{\alpha}} x^{\nu+1} dx \right)^{1/q} \leq C \left( \int_0^\infty |L_{\nu}(\mu-\nu)g(x)|^{p_{\alpha}} x^{\nu+1} dx \right)^{1/p}.$$

**Proof.** By using a homogeneity argument it is easy to see that (3.2) is necessary for (3.3) to hold. If $\vartheta_f g(x) = (1/r)g(x/r)$, $r > 0$, then $H_{\nu}(\vartheta_f g)(x) = r^{\nu+1}H_{\nu}g(rx)$ and

$$L_{\nu}^{\nu-\nu}(\vartheta_f g)(x) = r^{(\nu-\nu)}(\vartheta_f g)(x).$$

Hence, considering (3.3) with $\vartheta_f g$ in place of $g$ and allowing $r$ to be small and large gives (3.2).

Evaluating the formula [EMOT, 8.5 (33)] at $y = 1$ and writing $\mu$ in place of $\mu + \nu + 1$ produces

$$\frac{J_{\mu}(a)}{a^\nu} = c_{\nu,\mu} \int_0^a \left( a^2 - t^2 \right)^{\mu-\nu-1} J_{\nu}(t) t^{\nu+1} dt$$

for $-1 < \nu < \mu$ and $a > 0$. Hence, a change of variable and Fubini's theorem give

$$T_{\nu}^\mu g(x) = \int_0^\infty H_{\nu}g(y) \frac{J_{\mu}(xy)}{(xy)^\nu} x^{\nu+1} dy$$

and

$$= c_{\nu,\mu} \int_0^\infty H_{\nu}g(y) \frac{1}{(xy)^\nu}$$

$$\times \left( (xy)^2 - u^2 \right)^{\nu-\nu-1} J_{\nu}(t) t^{\nu+1} dt y^{\nu+1} dy$$

$$= c_{\nu,\mu} \int_0^\infty \frac{1}{u^{\nu+1}}$$

$$\times \left( u^2 - y^2 \right)^{\nu-\nu-1} J_{\nu}(t) t^{\nu+1} dt u^{\nu+1} du.$$
Taking into account (3.1) and the inversion formula for the modified Hankel transform (recall that $\nu \geq -1/2$) we arrive at

$$T_\nu^* g(x) = c_{\nu, \mu} \int_0^\infty \left( \int_0^x (x^2 - y^2)^{-\nu - 1} L_0^{\mu - \nu} g(y) y^{2\nu + 1} dy \right)^{1/q} \frac{1}{x^{2\nu + 1}} x dx.$$ 

What we now need is the inequality

$$\left( \int_0^\infty \left( \int_0^x (x^2 - y^2)^{-\nu - 1} G(y) y^{2\nu + 1} dy \right)^{1/q} x^{2\nu + 1} dx \right)^{1/q} \leq C \left( \int_0^\infty G(x) x^{2\nu + 1} dx \right)^{1/p},$$

say, for all nonnegative functions $G$. Elementary variable changes show that the above inequality is equivalent to

$$\left( \int_0^\infty \left( \int_0^t (t - s)^{-\nu - 1} h(s) ds \right)^{q \nu/(1-q)} dt \right)^{1/q} \leq C \left( \int_0^\infty h(t) t^{(1-p)/(1-q)} dt \right)^{1/p},$$

for $h$ nonnegative. We use the following criterion for $L^p$-$L^q$ weighted estimates for the Riemann–Liouville operator

$I_\alpha^\nu h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} h(s) ds$

and Weyl

$I_\alpha^{-\nu} h(t) = \frac{1}{\Gamma(\alpha)} \int_t^\infty (s - t)^{\alpha - 1} h(s) ds$

fractional integral operators.

**Lemma (SKM, Theorem 5.4).** Let $\alpha > 0$, $p \geq 1$ and $p \leq q \leq p/(1 - \alpha)$ when $1 \leq p < 1/\alpha$ (if $p = 1$ then the right endpoint $p/(1 - \alpha)$ is excluded), or $p \leq q < \infty$ when $p \geq 1/\alpha$. Suppose also that $-\infty < N < \infty$ and $M < p - 1$ when considering $I_\alpha^\nu$, or $M > \alpha p - 1$ when considering $I_\alpha^{-\nu}$, and

$$\frac{N + 1}{q} = \frac{M + 1}{p} - \alpha.$$ 

Then

$$\left( \int_0^\infty |I_\alpha^\nu h(t)|^{q t^N} dt \right)^{1/q} \leq C \left( \int_0^\infty |h(t)|^{q t^M} dt \right)^{1/p}. $$

To see that the lemma gives (3.4) and thus (3.3) assume that the hypotheses of Theorem 3.1 are satisfied and take $\alpha = \mu - \nu$, $N = \mu(1 - q)$ and $M = \nu/(1 - p)$. Clearly (3.2) gives (3.5) and $M < p - 1$ holds provided $\nu > -1$. Moreover, if $1 < p < 1/\alpha$ then the condition $p(\mu + 1) \geq 1$ implies $q \leq p/(1 - \alpha)$. Thus, the conclusion of the above lemma, (3.6), holds for the operator $I_\alpha^\nu$ and, in consequence, (3.4) is valid. This concludes the proof of Theorem 3.1.

**Theorem 3.2.** Let $-1/2 \leq \mu < \nu$, $1 < p \leq q < \infty$ and $p \mu + 1 \geq 0$. Suppose also that

$$\nu + \frac{\mu + 1}{q} = \mu + \frac{\nu + 1}{p},$$

Then

$$\left( \int_0^\infty |T_\nu^* g(x)|^p x^{2\nu + 1} dx \right)^{1/q} \leq C \left( \int_0^\infty |g(x)|^p x^{2\nu + 1} dx \right)^{1/p}.$$  

**Proof.** A homogeneity argument similar to that from the beginning of the proof of Theorem 3.1 also shows that (3.7) is necessary for (3.8) to hold.

Evaluating the formula [EMOT, 8.5 (32)] at $y = 1$ and writing $\nu - \mu - 1$ in place of $\mu$ produces

$$J_\nu(a) = c_{\nu, \mu} \int_0^\infty t^{-\nu} (t^2 - a^2)^{\nu - 1} J_\nu(a) t dt$$

for arbitrary $\nu, \mu$ satisfying $\Re \mu < \Re \nu < 2\Re \mu + 3/2$ and $\alpha > 0$. Note at this point that only with the stronger assumption $\Re \mu < \Re \nu < 2\Re \mu + 1/2$, $\Re \mu > -1/2$, is the above integral Lebesgue integrable; otherwise it converges in the Riemann sense. Hence, considering first the case $\mu > -1/2$, for real $\nu, \mu$ that satisfy $-1/2 < \mu < \nu < 2\mu + 1/2$, a change of variable and Fubini's theorem give

$$T_\nu^* g(x) = \int_0^\infty H_\nu g(y) J_\nu(xy) y^{2\nu + 1} dy$$

$$= c_{\nu, \mu} \int_0^\infty H_\nu g(y) \int_0^\infty t^{1-\nu} (y^2 - (xy)^2)^{\nu - 1} J_\nu(t) dt y^{2\nu + 1} dy$$

$$= c_{\nu, \mu} \int_t^\infty u^{1-\nu} (u^2 - (xy)^2)^{\nu - 1} H_\nu g(y) J_\nu(uy) y^{2\nu + 1} dy du$$

$$= c_{\nu, \mu} \int_t^\infty u(u^2 - x^2)^{\nu - 1} H_\nu g(y) J_\nu(uy) y^{2\nu + 1} dy du.$$  

An application of Fubini's theorem is allowed at this point since $|J_\nu(s)| \leq Cs^{-1/2}$ on $(0, \infty)$, the function $y^{\nu + 1/2} H_\nu g(y)$ is integrable on $(0, \infty)$ and
provided that $H_n m_0 \in C_0^\infty (0, \infty)$. Denote now by $L^p_{rad}(\mathbb{R}^n)$ the set of radial $L^p$-functions on $\mathbb{R}^n$, $f(x) = f_0(\|x\|)$, with standard $L^p(\mathbb{R}^n, dx)$ norm and by $[L^p_{rad}(\mathbb{R}^n)]^\wedge$ the set of its Fourier transforms, i.e. in the case $1 \leq p < 2n/(n+1)$,

$$\hat{f}(\xi) = c_n H_{n-2} f_0(\|\xi\|).$$

By the convolution inequality it follows for $m \in [L^p_{rad}(\mathbb{R}^n)]^\wedge$ that

$$T_m : L^p \to L^p, \quad (T_m \varphi) \hat{\xi} = m(\|\xi\|) \hat{\varphi}(\xi), \quad \varphi \in S(\mathbb{R}^n),$$

is bounded and it is well known that if a bounded convolution operator from $L^2(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, $1 < p < \infty$, is generated by some radial $m$ then $m \in [L^p_{rad}(\mathbb{R}^n)]^\wedge$. We then say that $m(\|\xi\|) \in L^1_{rad}(\mathbb{R}^n)$ and define $\|m(\|\xi\|)\|_{L^1_{rad}(\mathbb{R}^n)}$ to be the operator norm of $T_m$ which is equal to the $L^p(\mathbb{R}^n, dx)$-norm of $H_{n-2} m(\|x\|)$. Setting $\xi = (\xi, \xi_{n+1})$ and $\check{\xi} = (\check{\xi}, \xi_{n+2})$ we have

**COROLLARY 3.3.** Let $1 < p < q < \infty$ and $n \geq 2$ be an integer.

- (a) $\|m(\|\xi\|)\|_{L^1_{rad}(\mathbb{R}^n)} \leq C(\|\xi\|^2 m(\|\xi\|) \|_{M^f_2(\mathbb{R}^n)}, \frac{1}{q} = 1 - \frac{2}{n+2} + \frac{1}{(n+2)p}$.

- (b) $\|m(\|\xi\|)\|_{L^1_{rad}(\mathbb{R}^n)} \leq C(\|\xi\|^2 m(\|\xi\|) \|_{M^f_2(\mathbb{R}^n)}, \frac{1}{q} = 1 - \frac{1}{n(p')}$.

**Proof.** For (a) choose $\nu = (n-2)/2$ and $\mu = n/2$ for an integer $n \geq 2$ in Theorem 3.1. The assertion there that $m$ is smooth may be dropped since any $H_n m(\|x\|) \in L^p(\mathbb{R}^n)$ can be approximated in $L^p(\mathbb{R}^n)$ by smooth rapidly decreasing $H_n m_0(\|x\|)$ with $m_0 \to m$ in $S(\mathbb{R}^n)$, thus (3.11) gives the assertion (a) for an arbitrary radial $m \in M^f_{2,p}(\mathbb{R}^n)$. Part (b) follows similarly from Theorem 3.2 upon choosing $\mu = (n-2)/2$ and $\nu = (n-2)/4$.

**Remarks.** 1) The results of Corollary 3.3 are best possible for $1 < p < q < 2n/(n+1)$ in the following sense. For (a) consider the example

$$m(t) = t^{-(n+2)/p'} (1 + \log^2 t)^{-1} = t^{-(n+2)/p} + 1 + \log^2 t. t^{-1}.$$

By a criterion in [T] we have $m(\|\xi\|) \in L^p(\mathbb{R}^{n+2}) \wedge$ but $m(\|\xi\|)$ does not belong to any other space $[L^p(\mathbb{R}^{n+2})]^\wedge, r \neq q$, which follows directly by (3.12) on account of Hölder’s inequality. The same example, this time written in the form

$$m(t) = t^{-(n+1)/p'} + 1 + \log^2 t = t^{-n/p'} (1 + \log^2 t)^{-1},$$

again shows that in the same sense (b) is also best possible.

2) Note that (a) is in the spirit of the following result due to Coifman and Weiss [CW, pp. 33-45]:

$$\|m(\|\xi\|)\|_{M^p_{2,p}(\mathbb{R}^n)} \leq C \|m(\|\xi\|)\|_{M^p_{2,p}(\mathbb{R}^n)},$$

where $m(\|\xi\|)$ is radial.
which is only good for \( p \) near 1 or infinity. That the right side of (3.13) contains an expression of type \( tm'(t) \) is only natural in view of the necessary conditions for radial Fourier multipliers in [GT, p. 412] (for \( p \) near 1). These conditions also indicate that part (a) of Corollary 3.3 is a natural result (\( p \leq q < 2n/(n+1) \)) for the necessary conditions arising from the right side guarantee quite precisely the necessary conditions arising from the left side. Part (b) of the corollary is in the spirit of the well known deLeeuw restriction result for Fourier multipliers (see e.g. [To], p. 265), which by duality and the Riesz interpolation theorem implies

\[
\|m(\langle |\xi| \rangle^\delta)\|_{M^{p,q}(\mathbb{R}^n)} \leq C\|m(\langle |\xi| \rangle^\delta)\|_{M^{p,q}(\mathbb{R}^{n+1})},
\]

\[1 \leq \min\{p,p', q \leq \max\{p,p'\} \leq \infty.\]

4. Density theorems. In this section we prove density theorems which were announced and used in §2. Because they are of some independent interest and, perhaps, could be used for other purposes, we prove these theorems in a more general form. The results we obtain are generalizations to the Hankel transform setting of density theorems proved by Muckenhoupt, Wheeden and Young [MWY]. In other words, we extend the results of Section 2 of [MWY] to the cosine transform setting that corresponds to the case \( \nu = -1/2 \) to general \( \nu \geq -1/2 \). Hence in what follows we restrict the attention to \( \nu > -1/2 \) only. Needless to say, we follow the ideas of [MWY] closely.

If not otherwise stated the letter \( k \) will always denote an integer. Recall that \( S(\mathbb{R}_+^n) \) denotes the space of restrictions to \((0,\infty)\) of even Schwartz functions on \( \mathbb{R}_+^n \) and \( C_0^\infty \) denotes the space of \( C^\infty \) functions with compact support in \((0,\infty)\). Recall also that \( H_\nu \) is a bijection on \( S(\mathbb{R}_+^n) \). Observing that for even \( f \in S(\mathbb{R}_+^n) \) we have \( f(0) = 0 \) it is readily checked that the differential operator \( L_\nu \) can be extended to even Schwartz functions by setting \( L_\nu f(0) = 2(\nu + 1)f'(0) \). Thus, if the powers of the operator \( L_\nu \) are now defined in the usual way: \( L_\nu^k = L_\nu \) and \( L_\nu^k = L_\nu(L_\nu^{k-1}) \), \( k > 1 \), iterating the process we can regard \( L_\nu^k f, k = 0, 1, \ldots, \) to be a function in \( S(\mathbb{R}_+^n) \).

**Lemma 4.1.** If \( f \in S(\mathbb{R}_+^n) \) satisfies

\[
(4.1) \quad \int_0^\infty x^{2j} f(x)x^{2\nu+1} dx = 0, \quad j = 0, 1, \ldots, k,
\]

then \( (H_\nu f)^{(j)}(0) \), the derivatives of \( H_\nu f \) at zero, vanish for \( j = 0, 1, \ldots, 2k \).

**Proof.** It follows from (1.8) that

\[
(4.2) \quad L_\nu^j H_\nu f(x) = (-1)^j H_\nu((-2j)f)(x), \quad x > 0.
\]

Hence \( L_\nu^j H_\nu f(0) = 0 \) for \( j = 0, 1, \ldots, k \), which implies \( (H_\nu f)^{(j)}(0) = 0 \) for \( j = 0, 2, \ldots, 2k \). It is obvious that the same holds for odd \( j \).

By \( Q_{k}(\nu), k \geq 0, \) we will denote the set of functions \( f \in L^2(\mathbb{R}^n) \cap L^{1,\infty}(\mathbb{R}^n) \) that satisfy (4.1) and, if \( k < 0 \), we set \( Q_k = L^2(\mathbb{R}^n) \); then we define \( C_0^\infty(\mathbb{R}^n) = C_0^\infty \cap Q_k(\nu) \).

**Lemma 4.2.** If \( 1 \leq p < \infty, \gamma > -1 \) and \( 2k > -(\gamma + 1)/p \) then every function \( f \in C_0^\infty(\mathbb{R}^n) \) can be approximated by functions from \( H_\nu(\mathbb{R}^n) \) in both \( L^{p,\gamma}(dx) \) and \( L^2(dx) \) norms.

**Proof.** Let \( \phi_n(x) \) be the sequence of functions on \((0,\infty)\) defined as in Lemma 6.2 in [MWY]: \( \phi_n(x) = \phi(nx) \) if \( 0 < x \leq 1/n, \phi_n(x) = \phi(x/n) \) if \( x \geq n \) and \( \phi_n(x) = 0 \) if \( n^{-1} \leq x \leq n \), where \( \phi \) is a fixed \( C^\infty \) function on \((0,\infty)\) with \( \phi(x) = 0 \) for \( 1/2 < x < 2, \phi(x) = 1 \) for \( 0 < x \leq 1/4 \) and \( \phi \geq 0 \). Given \( f \in C_0^\infty(\mathbb{R}^n) \) define \( f_n = H_\nu(f(x)\cdot(1 - \phi_n(x))) \). Since \( 1 - \phi_n \in C_0^\infty \) we get \( f_n \in H_\nu(C_0^\infty) \). The convergence of \( f_n \) to \( f \) in \( L^2(dx) \) is immediate. To prove that \( f_n \) approaches \( f \) in \( L^{p,\gamma}(dx) \) norm we write

\[
\|f - f_n\|_{L^{p,\gamma}} \leq \|f - f_n(x)(1 + x)^{2(k+1)}\|_{L^{p,\gamma}} ||(1 + x)^{-2(k+1)}||_{L^{p,\gamma}}
\]

and note that the last norm on the right is finite due to assumptions on \( p, \gamma \) and \( k \). Moreover, by (4.2),

\[
\|(f - f_n)(1 + x)^{2(k+1)}||_\infty \leq C||f - f_n||_\infty + C\|x^{2(k+1)}(f - f_n)||_\infty
\]

\[
\leq C||H_\nu(f - f_n)||_1 + C\|L^{k+1}_\nu(H_\nu f - H_\nu f_n)||_1,
\]

where \( \| \cdot \|_1 \) denotes the norm in \( L^1(\mathbb{R}^n) \). Since \( H_\nu f - H_\nu f_n = H_\nu f \cdot \phi_n \) we have \( ||H_\nu(f - f_n)||_1 \to 0 \) as \( n \to \infty \). To estimate the remaining term we use the following Leibniz’ rule for the \((k + 1)\)th power of the operator \( L_\nu \):

\[
L^{k+1}_\nu(H_\nu f \cdot \phi_n) = \sum_{1 \leq i + j \leq 2(k+1)} c_{i,j} x^{-2(k+1)+i+2j} (H_\nu f)^{(i)}(\phi_n^{(j)}).
\]

This may be proved by induction. We now consider the \( L^1(\mathbb{R}^n) \) norm of each summand in the above sum separately. Fixing \( i, j \) with \( 1 \leq i + j \leq 2(k+1) \) we have to show that the quantities

\[
n^{-j} \int_0^\infty \|(H_\nu f)^{(i)}(x)\phi_n^{(j)}(x/n)||x^{-2(k+1)+i+j+2(j+1)} dx
\]

and

\[
n^{-j} \int_0^\infty \|(H_\nu f)^{(i)}(x)\phi_n^{(j)}(x/n)||x^{-2(k+1)+i+j+2(j+1)} dx
\]

tend to 0 as \( n \to \infty \). This is easily seen for (4.3) since \( \phi_n^{(j)} \) is bounded and \( (H_\nu f)^{(i)} \) is of rapid decrease at \( \infty \). For (4.4), consider first the case \( i = 2(k+1) \). Then \( f = 0 \) and (4.4) is bounded by \( Cn^{-2(k+1)} \). If \( 0 \leq i \leq 2k+1 \), then by Taylor’s formula and Lemma 4.1 the estimate \( ||(H_\nu f)^{(i)}(x)|x^{-2(k+1)+i+j+2(j+1)} dx|| \leq C_{2k+1-i} \)

follows. This shows that (4.4) is bounded by $Cn^{-(2\nu+1)}$ and finishes the proof of Lemma 4.2.

**Lemma 4.3.** If $1 \leq p < \infty$ and $\gamma > -1$, then every function $f$ in $Q_k(\nu) \cap L^p(\nu)\,(dx)$ can be approximated by functions from $C_0^\infty(k, \nu)$ in both $L^p(\nu)\,(dx)$ and $L^2(\nu)\,(dx)$ norms.

The proof of Lemma 4.3, with minor changes, is the same as the proof of Lemma 6.6 of [MWY]. Let us mention at this point that for our future purposes we will use, for given $k$, a sequence $\{\alpha_j(x)\}_{j=0}^n$ of $C^\infty$ functions, the same as in Lemma 6.5 of [MWY] except for the fact that now their supports are separated from zero, say, they are contained in $1/4 \leq x \leq 3/4$. It can be checked that this requirement is not essential. Recall that an important feature of $\alpha_j(x)$'s is the fact that

$$\int_0^\infty x^i \alpha_j(x)\,dx = \delta_{ij},$$

$0 \leq i, j \leq k$, where $\delta_{ij}$ is the Kronecker delta.

**Lemma 4.4.** If $1 \leq p < \infty$, $\gamma > -1$ and $2k < -1+(\gamma+1)/p-(2\nu+1)$ then every function $f$ in $C_0^\infty$ can be approximated by functions from $C_0^\infty(k, \nu)$ in $L^p(\nu)(dx)$ norm.

**Proof.** If $k$ is negative the statement is obvious. Let $k \geq 0$ and take $\alpha_0, \alpha_1, \ldots, \alpha_{2k}$ satisfying (4.5) for $0 \leq i, j \leq 2k$, supported in $1/4 \leq x \leq 3/4$ and, given $f \in C_0^\infty$, define

$$f_n(x) = f(x) - \sum_{i=0}^{k} n^{2i+1} \alpha_{2i}(nx)x^{-(2\nu+1)} \int_0^\infty f(t) t^{2i+2\nu+1} \,dt.$$  

Then $f_n \in C_0^\infty(k, \nu)$ and the required convergence $f_n \to f$ as $n \to \infty$ holds in $L^p(\nu)(dx)$.

**Lemma 4.5.** If $1 \leq p < \infty$, $\gamma > -1$ and $2k < -3+(\gamma+1)/p-(2\nu+1)$ then every function $f$ in $Q_k(\nu) \cap L^p(\nu)(dx)$ can be approximated by functions from $C_0^\infty(k+1, \nu)$ in both $L^p(\nu)(dx)$ and $L^2(\nu)(dx)$ norms.

**Proof.** By Lemma 4.3 we can assume that $f$ is in $C_0^\infty(k, \nu)$. Again the statement is obvious if $k < -2$. Hence, assume $k \geq -1$, take $\alpha_0, \alpha_1, \ldots, \alpha_{2(k+1)}$ satisfying (4.5) for $0 \leq i, j \leq 2(k+1)$ and define

$$f_n(x) = f(x) - n^{-2(k+1)+1} \alpha_{2(k+1)}(x/n)x^{-(2\nu+1)} \int_0^\infty f(t) t^{2(k+1)+2\nu+1} \,dt.$$  

Then $f_n \in C_0^\infty(k+1, \nu)$ and $f_n$ converges to $f$ in $L^p(\nu)(dx)$ and $L^2(\nu)(dx)$.

**Lemma 4.6.** If $1 < p < \infty$, $\gamma > -1$ and $2k < -3+(\gamma+1)/p-(2\nu+1)$ then every function $f$ in $Q_k(\nu) \cap L^p(\nu)(dx)$ can be approximated by functions from $Q_{k+1}(\nu) \cap L^p(\nu)(dx)$ in both $L^p(\nu)(dx)$ and $L^2(\nu)(dx)$ norms.

**Proof.** We can consider the case $k \geq 1$ only. Take $\alpha_0, \alpha_1, \ldots, \alpha_{2(k+1)}$ satisfying (4.5) for $0 \leq i, j \leq 2(k+1)$ and define

$$g_n(x) = \frac{X_n(z)}{x^2(k+1)+1} \frac{x^{-(2\nu+1)}}{\log x \log \log x} \frac{\sum_{i=0}^{k} \alpha_{2i}(x)x^{2i+2\nu+1}}{\log \log x} \int_1^\infty \frac{t^{2i-2(k+1)-1}}{\log t} \,dt.$$  

Then $g_n(x)x^{2i+2\nu+1} \,dx$ equals $0$ for $i = 0, 1, \ldots, k$ and is in $1$ for $i = k+1$. An argument shows that $g_n \to 0$ in $L^2(\nu)(dx)$ and convergence of $g_n$ to $0$ in $L^p(\nu)(dx)$ follows from the fact that $\int_1^\infty x^{\alpha} \nu(\alpha) \,dx$ is finite for $0 < p > 1$. By Lemma 4.3 we can assume $f$ to be in $C_0^\infty(k+1, \nu)$. Define

$$f_n(x) = f(x) - \int_0^\infty f(t) t^{2(k+1)+2\nu+1} \,dt.$$  

Then $f_n \in C_0^\infty(k+1, \nu)$ and the properties of $g_n$ imply the desired convergence for $f_n$.

**Theorem 4.7.** If $1 < p < \infty$ and $\gamma > -1$ then $H_\nu(C_0^\infty)$ is dense in $L^p(\nu)(dx)$. If in addition $\gamma$ is not of the form $2k+2\nu+1$ then $H_\nu(C_0^\infty)$ is dense in $L^{1,\gamma}(\nu)(dx)$.

**Proof.** Fix $p$ and $\gamma$ and choose an integer $k$ satisfying $-3+(\gamma+1)/p-(2\nu+1) < 2k < -1+(\gamma+1)/p-(2\nu+1)$ if $p > 1$ and $-2+\gamma-(2\nu+1) < 2k < -\gamma-(2\nu+1)$ if $p = 1$. Since $C_0^\infty$ is dense in $L^p(\nu)(dx)$ it is sufficient to approximate $C_0^\infty$ functions only. Lemma 4.4 allows us further to restrict attention to $C_0^\infty(k, \nu)$ functions. By applying Lemma 4.5 or Lemma 4.6 several times and then applying Lemma 4.3 if necessary, we conclude that every $C_0^\infty(k, \nu)$ function can be approximated by $C_0^\infty(k+\nu, \nu)$ functions, where $\nu$ is a positive integer such that $2(k+\nu) > -2+(\gamma+1)/p$. Using Lemma 4.2 finishes the proof of the theorem.

**Corollary 4.8.** If $1 < p < \infty$ and $\beta > -1-p(\nu+1/2)$ then $H_\nu(C_0^\infty)$ is dense in $L^p(\nu)(dx)$. If in addition $\beta$ is not of the form $2k+\nu+1/2$ then $H_\nu(C_0^\infty)$ is dense in $L^{1,\beta}(\nu)(dx)$.

**Proof.** The corollary follows from Theorem 4.7 by using (1.4), the fact that $\alpha$ has $x^{\nu+1/2}C_0^\infty$ and the remark that multiplication by $x^{\nu+1/2}$ is an isometric bijection between $L^p(\nu)(dx)$ and $L^{p,\nu-\nu/2}(\nu)(dx)$.
References


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Two-weight norm inequalities for maximal functions on homogeneous spaces and boundary estimates

by

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Abstract. Let D be an open subset of a homogeneous space (X, d, µ). Consider the maximal function
\[ M_\varphi f(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f| \, d\nu, \quad x \in D, \]
where the supremum is taken over all balls of the form B = B(a(x), r) with r > t(x) = \( d(x, \partial D) \), \( a(x) \in \partial D \) is such that \( d(a(x), x) < \frac{3}{2} t(x) \) and \( \varphi \) is a nonnegative set function defined for all Borel sets of X satisfying the quasi-monotonicity and doubling properties. We give a necessary and sufficient condition on the weights \( w \) and \( \nu \) for the weighted norm inequality
\[ \left( \int_D |M_\varphi f|^q \, d\mu \right)^{1/q} \leq c \left( \int_D |f|^p \, d\nu \right)^{1/p} \]
to hold when 1 < p < q < \infty, \( \sigma \, d\nu = \nu^{1-p} \, d\nu \) is a doubling weight, and \( d\mu \) is a doubling measure, and give a sufficient condition for (0.1) when 1 < p < q < \infty without assuming that \( \sigma \) is a doubling weight but with an extra assumption on \( \nu \). Another characterization for (0.1) is also provided for 1 < p < q < \infty and D of the form Y x (0, \infty), where Y is a homogeneous space with group structure. These results generalize some known theorems in the case when \( M_\varphi \) is the fractional maximal function in \( R_n^{n+1} \), that is, when
\[ M_\varphi f(x, t) = M_\varphi f(x, t) = \sup_{r > t} \frac{1}{\nu(B(z, r))^{1-\gamma}} \int_{B(z, r)} |f| \, d\nu, \]
where \((x, t) \in R_n^{n+1}, 0 < \gamma < 1, \) and \( \nu \) is a doubling measure in \( R^n \).

1. Introduction. We consider a homogeneous space \((X, d, \mu)\) where \( d : X \times X \to [0, \infty) \) satisfies:

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