Contents of Volume 126, Number 1

STUDIA MATHEMATICA

Executive Editors: **Z. Ciesielski, A. Pełczyński, W. Żelazko**

The journal publishes original papers in English, French, German and Russian, mainly in functional analysis, abstract methods of mathematical analysis and probability theory. Usually 3 issues constitute a volume.

Detailed information for authors is given on the inside back cover. Manuscripts and correspondence concerning editorial work should be addressed to

STUDIA MATHEMATICA

Śniadeckich 8, P.O. Box 137, 00-950 Warszawa, Poland, fax 48-22-6293997
E-mail: studia@impan.gov.pl

Subscription information (1997): Vols. 122–126 (15 issues); $30 per issue.

Correspondence concerning subscription, exchange and back numbers should be addressed to

Institute of Mathematics, Polish Academy of Sciences
Publications Department

Śniadeckich 8, P.O. Box 137, 00-950 Warszawa, Poland, fax 48-22-6293997
E-mail: publ@impan.gov.pl

Published by the Institute of Mathematics, Polish Academy of Sciences
Typeset in TEX at the Institute
Printed and bound by

02-240 Warszawa, ul. Jakobinów 23, tel: 846-79-66, tel/fax: 49-89-95

PRINTED IN POLAND

ISSN 0039-3223

STUDIA MATHEMATICA 126 (1) (1997)

A restriction theorem for the Heisenberg motion group

by

P. K. RATNAKUMAR, RAMA RAWAT and S. THANGAVELU (Bangalore)

Abstract. We prove a restriction theorem for the class-1 representations of the Heisenberg motion group. This is done using an improvement of the restriction theorem for the special Hermite projection operators proved in [13]. We also prove a restriction theorem for the Heisenberg group.

1. Introduction. The inversion formula for the Fourier transform on $\mathbb{R}^n$ can be written in the form

$$f(x) = C_n \int_0^\infty f * \varphi_\lambda(x) \lambda^{n-1}\, d\lambda$$

where φ_λ is the Bessel function given by

$$\varphi_\lambda(x) = (\lambda|x|)^{-n/2+1} J_{n/2-1}(\lambda|x|).$$

Then for $f \in L^p(\mathbb{R}^n), 1 \le p \le 2(n+1)/(n+3)$, there follows the inequality

$$\|f * \varphi_\lambda\|_p \le C_\lambda \|f\|_p.$$

From this one gets the Stein–Tomas restriction theorem for the Fourier transform [11]:

$$\int_{|\xi|=1} |\widehat{f}(\xi)|^2\, d\sigma \le C\|f\|_p^2,$$

for $f \in L^p(\mathbb{R}^n), 1 \le p \le 2(n+1)/(n+3)$. The restriction theorem finds applications in the study of Bochner–Riesz means for the Laplacian.

Analogues of the above restriction theorem have been studied in various set ups. As $f * \varphi_\lambda$ are eigenfunctions of the Laplacian Δ on $\mathbb{R}^n$, it is natural to study the L^p-L^2 mapping properties of projection operators associated

1991 *Mathematics Subject Classification*: Primary 42C15, 43A80; Secondary 43A99.
Key words and phrases: Hermite function, special Hermite function, Laguerre function, class-1 representation, Heisenberg motion group.

with eigenfunction expansions. In the case of spherical harmonics and eigenfunction expansions on compact Riemannian manifolds such theorems have been proved by Sogge in [9] and [10]. In the non-compact set up, restriction theorems for Hermite and special Hermite projection operators have been studied by Thangavelu in [13].

Restriction theorems have also been studied in the case of the Heisenberg group H^n. Let

$$f = \int_0^\infty P_\lambda f \, d\lambda$$

stand for the Strichartz decomposition [12] of f in terms of eigenfunctions of the sublaplacian $\mathcal{L}$ on H^n. In [5] Müller has studied mapping properties of P_λ. Some extensions have been treated in [14] and [15] and the restriction theorem has been found useful in the study of Bochner–Riesz means for the sublaplacian [6].

Our aim in this note is to prove a restriction theorem for class-1 representations of the Heisenberg motion group. The main theorem should be compared with the corresponding theorem for the spherical harmonic projections stated and proved in Sogge [9] in the language of representation theory. To prove the main theorem we need a restriction theorem for special Hermite projection operators proved in [13]. We take this opportunity to present a simpler proof of a crucial estimate used in [13] and also to show that the restriction theorem is valid in a slightly bigger range of p than established in [13]. In the last section we also prove a restriction theorem for the Heisenberg group by considering individual projections.

For many facts we use regarding the Heisenberg group and special Hermite expansions we refer to the monographs [1] and [16] and also to the paper of Strichartz [12].

2. A restriction theorem for the Heisenberg motion group. Consider the Heisenberg group $H^n = \mathbb{C}^n \times \mathbb{R}$ equipped with the group law

$$(z,t)(w,s) = \big(z+w, t+s+\tfrac{1}{2}\operatorname{Im}(z.\overline{w})\big).$$

The group $U(n)$ of $n \times n$ complex unitary matrices acts on H^n by the automorphisms

$$\sigma(z,t) = (\sigma z, t), \qquad \sigma \in U(n).$$

The Heisenberg motion group is then the semi-direct product $G = H^n \ltimes U(n)$ which acts on H^n in the following way:

$$(\sigma, z, t)(w,s) = \big(z + \sigma w, t+s+\tfrac{1}{2}\operatorname{Im}(\sigma w.\overline{z})\big).$$

Functions on H^n can be viewed as right $U(n)$-invariant functions on the Heisenberg motion group G. To formulate our restriction theorem for G we need to recall a family of class-1 representations of G which have been studied in [8] and [12].

For each $\lambda \in \mathbb{R}, \lambda \neq 0$ we have an irreducible unitary representation π_λ of H^n which is realised on $L^2(\mathbb{R}^n)$ and acts by

$$\pi_\lambda(z,t)\varphi(\xi) = e^{i\lambda t} e^{i\lambda(x\cdot\xi + \frac{1}{2}x\cdot y)}\varphi(\xi+y),$$

for $\varphi \in L^2(\mathbb{R}^n)$. Up to unitary equivalence these π_λ give all the infinite-dimensional irreducible representations of H^n. Let Φ_α, $\alpha \in \mathbb{N}^n$, be the normalised Hermite functions on $\mathbb{R}^n$. (For the explicit definition of Φ_α we refer to [16].) For $\lambda \neq 0$ define $\Phi_\alpha^\lambda(x) = |\lambda|^{n/4}\Phi_\alpha(|\lambda|^{1/2}x)$ and let

$$E^\lambda_{\alpha,\beta}(z,t) = (\pi_\lambda(z,t)\Phi^\lambda_\alpha, \Phi^\lambda_\beta)$$

be the entry functions of the representation π_λ. The functions $\Phi_{\alpha,\beta}(z) = (2\pi)^{-n/2}E^1_{\alpha,\beta}(z,0)$ are called the *special Hermite functions* and it is well known that $\{\Phi_{\alpha\beta} : \alpha, \beta \in \mathbb{N}^n\}$ forms an orthonormal basis for $L^2(\mathbb{C}^n)$.

We recall some general facts about the class-1 representations. Let N be a locally compact topological group and K_0 be a compact subgroup of N. Let π be an irreducible unitary representation of N on a Hilbert space H. We say that π is a *class-1 representation* for the pair (N, K_0) if the space H_0 of K_0-fixed vectors in H, i.e. $H_0 = \{v \in H : \pi(k)v = v \ \forall k \in K_0\}$, is not $\{0\}$.

In case (N, K_0) is a Gelfand pair, i.e. if the algebra $\{f \in L^1(N) : f(k_1 x k_2) = f(x) \ \forall k_1, k_2 \in K_0, \ x \in N\}$ is commutative with respect to the usual convolution on N, it is known (see [2]) that for π, H, H_0 as above, $\dim H_0 = 1$.

We now list a family of class-1 representations for the pair $(G, U(n))$. For each $\lambda \neq 0$ and $k \in \mathbb{N}$, let $\mathcal{H}^\lambda_k$ be the Hilbert space for which an orthonormal basis is given by

$$\{E^\lambda_{\alpha,\beta}(z,t) : \alpha, \beta \in \mathbb{N}^n, \ |\beta| = k\}$$

and the inner product being

$$(f,g) = (2\pi)^{-n}|\lambda|^n \int_{\mathbb{C}^n} f(z,0)\overline{g}(z,0)\,dz.$$

The space $\mathcal{H}^\lambda_k$ can be characterised as a certain eigenspace of the sublaplacian (see Strichartz [12]). On $\mathcal{H}^\lambda_k$ define a representation ϱ^λ_k of G by

$$\varrho^\lambda_k(\sigma, z, t)\varphi(w,s) = \varphi((\sigma, z, t)^{-1}(w,s))$$

for $\varphi \in \mathcal{H}^\lambda_k$ and $(w,s) \in H^n$. Then ϱ^λ_k is an irreducible unitary representation of G. As noticed in [8], the vector

$$e^\lambda_k(z,t) = (2\pi)^{-n/2}\sum_{|\mu|=k}(\pi_\lambda(z,t)\Phi^\lambda_\mu, \Phi^\lambda_\mu)$$

is a $U(n)$-fixed vector. As $(G, U(n))$ is a Gelfand pair (see [3]), we conclude that e_k^λ is the unique (up to a scalar multiple) $U(n)$-fixed vector in $\mathcal{H}_k^\lambda$.

Let $f \in L^1(H^n)$; viewing f as a right $U(n)$-invariant function on G we can define the operator

$$\varrho_k^\lambda(f) = \int_G f(z,t)\varrho_k^\lambda(\sigma, z, t)\, d\sigma\, dz\, dt$$

which acts on the Hilbert space $\mathcal{H}_k^\lambda$. It is easy to calculate the action of $\varrho_k^\lambda(f)$ on a function $\varphi \in \mathcal{H}_k^\lambda$. In fact, letting

$$\varphi^\#(z) = \int_{U(n)} \varphi(\sigma z, 0)\, d\sigma$$

be the *radialisation* of φ,

$$f^\lambda(z) = \int e^{i\lambda t} f(z,t)\, dt$$

be the inverse Fourier transform of f in the t-variable and

$$g *_\lambda h(z) = \int_{\mathbb{C}^n} g(z-w)h(w)e^{i(\lambda/2)\operatorname{Im}(z.\bar{w})}\, dw$$

be the λ-*twisted convolution* of g and h we can show that

$$\varrho_k^\lambda(f)\varphi(z,t) = e^{i\lambda t} f^{-\lambda} *_\lambda \varphi^\#(z).$$

It is easy to see that $\varrho_k^\lambda(f)$ is a bounded operator on $\mathcal{H}_k^\lambda$. In fact, since ϱ_k^λ is a unitary operator we have the norm estimate

$$(*) \qquad \|\varrho_k^\lambda(f)\|_\infty \le \int_{H^n} |f(z,t)|\, dz\, dt$$

where we have used $\|\cdot\|_\infty$ to denote the operator norm.

When $f \in L^1 \cap L^2(H^n)$ we can say more about the operator $\varrho_k^\lambda(f)$. Let L_k^{n-1} be the kth Laguerre polynomial of type $n-1$ and let

$$\varphi_k(z) = L_k^{n-1}(|z|^2/2)e^{-|z|^2/4}$$

be the Laguerre function. Let $\varphi_k^\lambda(z) = \varphi_k(|\lambda|^{1/2}z)$.

PROPOSITION 2.1. *For $f \in L^1 \cap L^2(H^n)$, $\varrho_k^\lambda(f)$ is a Hilbert–Schmidt operator on $\mathcal{H}_k^\lambda$ and*

$$\|\varrho_k^\lambda(f)\|_2^2 = (2\pi)^{-n}|\lambda|^n \frac{k!(n-1)!}{(k+n-1)!} \int_{\mathbb{C}^n} |f^{-\lambda} *_\lambda \varphi_k^\lambda(z)|^2\, dz.$$

Here $\|\cdot\|_2$ denotes the Hilbert–Schmidt norm.

Proof. We calculate the norm of $\varrho_k^\lambda(f)\varphi$ when $\varphi = E_{\alpha,\beta}^\lambda$, $|\beta| = k$. Since

$$e_k^\lambda(z,t) = (2\pi)^{-n/2} \sum_{|\mu|=k} (\pi_\lambda(z,t)\Phi_\mu^\lambda, \Phi_\mu^\lambda)$$

is the essentially unique $U(n)$-invariant function in $\mathcal{H}_k^\lambda$, the radialisation

$$e^{i\lambda t}\varphi^\#(z) = \int_{U(n)} E_{\alpha,\beta}^\lambda(\sigma z, t)\, d\sigma$$

should be a constant multiple of $e_k^\lambda(z,t)$. From the definition of $E_{\alpha,\beta}^\lambda$ we infer that $E_{\alpha,\beta}^\lambda(0,t) = 0$ for $\alpha \ne \beta$ and consequently

$$\varrho_k^\lambda(f)E_{\alpha,\beta}^\lambda = 0, \qquad \alpha \ne \beta.$$

When $\alpha = \beta$ we have

$$\int_{U(n)} E_{\alpha,\alpha}^\lambda(\sigma z, 0)\, d\sigma = Ae_k^\lambda(z,0)$$

where A is a constant. We evaluate the constant by taking $z = 0$:

$$A(2\pi)^{-n/2}\Big(\sum_{|\mu|=k} 1\Big) = \int_{U(n)} d\sigma = 1.$$

This gives

$$\int_{U(n)} E_{\alpha,\alpha}^\lambda(\sigma z, 0)\, d\sigma = (2\pi)^{n/2}\frac{k!(n-1)!}{(k+n-1)!}e_k^\lambda(z,0).$$

It is well known (see [8]) that

$$e_k^\lambda(z,t) = (2\pi)^{-n/2}e^{i\lambda t}\varphi_k^\lambda(z)$$

and consequently

$$\varrho_k^\lambda(f)E_{\alpha,\alpha}^\lambda(z,t) = \frac{k!(n-1)!}{(k+n-1)!}e^{i\lambda t}f^{-\lambda} *_\lambda \varphi_k^\lambda(z)$$

for $|\alpha| = k$. Finally,

$$\|\varrho_k^\lambda(f)\|_2^2 = \sum_{|\alpha|=k} \|\varrho_k^\lambda(f)E_{\alpha,\alpha}^\lambda\|_{\mathcal{H}_k^\lambda}^2$$

and after simplification we obtain

$$\|\varrho_k^\lambda(f)\|_2^2 = \frac{k!(n-1)!}{(k+n-1)!}|\lambda|^n(2\pi)^{-n}\int_{\mathbb{C}^n} |f^{-\lambda} *_\lambda \varphi_k^\lambda(z)|^2\, dz.$$

This proves the proposition.

For $0 < q \le \infty$, let S_q stand for the Schatten–von Neumann class of operators on $\mathcal{H}_k^\lambda$ whose singular numbers belong to ℓ^q. In particular, S_2 will denote the class of Hilbert–Schmidt operators. Let $\|\cdot\|_q$ stand for the norm in S_q. We are now ready to state the following restriction theorem for ϱ_k^λ.

Let $L^{(p,1)}(H^n)$ stand for the space of all functions on H^n for which

$$\|f\|_{(p,1)} = \Big(\int_{\mathbb{C}^n} \Big(\int_{-\infty}^{\infty} |f(z,t)|\,dt\Big)^p dz\Big)^{1/p} < \infty.$$

THEOREM 2.1. *Let $f \in L^{(p,1)}(H^n)$, $1 \le p < 2(3n+1)/(3n+4)$, and let $q < (3n-2)p'/(n+1)$. Then $\varrho_k^\lambda(f) \in S_q$ and*

$$\|\varrho_k^\lambda(f)\|_q \le C|\lambda|^{\frac{3n+4}{3n+1}\cdot\frac{n}{q}} k^{-\frac{3n-2}{3n+1}\cdot\frac{n}{q}} \|f\|_{(p,1)}.$$

To prove the theorem we need the following restriction theorem for the special Hermite projections. For functions f on $\mathbb{C}^n$ let

$$f \times \varphi_k(z) = \int_{\mathbb{C}^n} f(z-w) e^{(i/2)\operatorname{Im}(z\cdot\bar{w})} \varphi_k(w)\,dw,$$

which is called the *twisted convolution* of f with φ_k. We need

PROPOSITION 2.2. *Let $f \in L^p(\mathbb{C}^n)$ and $1 \le p < 2(3n+1)/(3n+4)$. Then*

$$\|f \times \varphi_k\|_2 \le C k^{n(1/p-1/2)-1/2} \|f\|_p.$$

We postpone the proof of this proposition to the next section. Assuming it for a moment we will prove the theorem. From equation $(*)$ we have

$$\|\varrho_k^\lambda(f)\|_\infty \le \|f\|_{(1,1)}.$$

Assuming $\lambda > 0$ for definiteness we see that

$$f^{-\lambda} *_\lambda \varphi_k^\lambda(z) = \lambda^{-n} f_\lambda^{-\lambda} \times \varphi_k(\lambda^{1/2} z)$$

where

$$f_\lambda^{-\lambda}(z) = f^{-\lambda}(\lambda^{-1/2} z).$$

Applying the proposition we get

$$\begin{aligned} \|f^{-\lambda} *_\lambda \varphi_k^\lambda\|_2 &\le C|\lambda|^{n(1/p-1/2)} k^{n(1/p-1/2)-1/2} \|f^\lambda\|_p \\ &\le C|\lambda|^{n(1/p-1/2)} k^{n(1/p-1/2)-1/2} \|f\|_{(p,1)}. \end{aligned}$$

Using this estimate in Proposition 2.1 we get

$$\|\varrho_k^\lambda(f)\|_2 \le C|\lambda|^{n/p} k^{n/p-n} \|f\|_{(p,1)}.$$

We pretend as if Proposition 2.2 is true at the end point $p_0 = 2(3n+1)/(3n+4)$. (A slight modification required is left to the reader.) This gives

$$\|\varrho_k^\lambda(f)\|_2 \le C|\lambda|^{\frac{n(3n+4)}{2(3n+1)}} k^{-\frac{n(3n-2)}{2(3n+1)}} \|f\|_{(p_0,1)}.$$

Appealing to the non-commutative interpolation theorem of Peetre–Sparr [7] we obtain for $1 \le p \le 2(3n+1)/(3n+4)$,

$$\|\varrho_k^\lambda(f)\|_q \le C|\lambda|^{\frac{n(3n+4)}{2(3n+1)}\theta} k^{-\frac{n(3n-2)}{2(3n+1)}\theta} \|f\|_{(p,1)}$$

where p, q and θ are related by

$$\frac{1}{q} = \frac{\theta}{2}, \qquad \theta = \frac{2(3n+1)}{3n-2}\cdot\frac{1}{p'}.$$

Simplifying we see that $q = (3n-2)p'/(3n+1)$, thus completing the proof of the theorem.

3. Special Hermite projection operators. In this section we prove Proposition 2.2. By the *special Hermite expansion* we mean the series

$$f(z) = (2\pi)^{-n} \sum_{k=0}^{\infty} f \times \varphi_k(z)$$

which converges in L^2 norm for $f \in L^2(\mathbb{C}^n)$. The above is the compact form of the expansion in terms of the special Hermite functions, namely

$$f(z) = \sum_\alpha \sum_\beta (f, \Phi_{\alpha\beta}) \Phi_{\alpha\beta}.$$

Summability and multipliers for the above expansions have been studied in [13]. A crucial ingredient for proving summability results is the L^p-L^2 restriction theorem stated in Proposition 2.2.

The proposition was proved in [13] for the slightly smaller range $1 \le p \le 2n/(n+1)$. The main idea of the proof is to embed the operator $f \to f \times \varphi_k$ into an analytic family of operators, get estimates at the end points and then appeal to Stein's analytic interpolation theorem. The analytic family used for this purpose is the one given by twisted convolution with the Laguerre function

$$\psi_k^\alpha(z) = \frac{\Gamma(k+1)\Gamma(\alpha+1)}{\Gamma(k+\alpha+1)} L_k^\alpha(|z|^2/2) e^{-|z|^2/4}.$$

Here these functions can be defined even for complex α with $\operatorname{Re}\alpha > -1$. In [13] it was shown that $\psi_k^\alpha(z)$ are bounded uniformly in k and z provided $\operatorname{Re}\alpha \ge 0$. In the following proposition we show that the same is true as long as $\operatorname{Re}\alpha > -1/3$.

PROPOSITION 3.1. *Let $\alpha = \sigma + i\tau$ with $-1 < \sigma \le n$. Then with a constant C independent of k we have*

$$\sup_z |\psi_k^\alpha(z)| \le C(1+|\tau|)^{2/3}$$

provided $\sigma > -1/3$.

Proof. This proposition was proved in [13] for $\sigma \ge 0$ by expressing L_k^α in terms of Hermite functions and then using estimates for the latter. Here we use the following formula which connects Laguerre polynomials of different types:

$$L_k^{\mu+\nu}(t) = \frac{\Gamma(k+\mu+\nu+1)}{\Gamma(\nu)\Gamma(k+\mu+1)} \int_0^1 s^{\mu}(1-s)^{\nu-1} L_k^{\mu}(ts)\, ds.$$

From the above formula we have

$$\psi_k^{\alpha}(z) = \frac{\Gamma(\alpha+1)}{\Gamma(-1/3)\Gamma(\alpha+1/3)} \int_0^1 s^{-1/3}(1-s)^{\alpha-2/3}\psi_k^{-1/3}(\sqrt{s}\,z)e^{-(1-s)|z|^2/4}\, ds$$

for $\alpha > -1/3$ and it is clear that the above can be defined even for complex α provided $\operatorname{Re}\alpha > -1/3$.

Using Stirling's formula for the Gamma function we can show that

$$\left|\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1/3)}\right| \le C(1+|\tau|)^{2/3}.$$

For the Laguerre functions $\psi_k^{\alpha}(z)$ various L^p estimates are known (see Markett [4]). From the Lemma of [4] we can infer that

$$\sup_z |\psi_k^{-1/3}(z)| \le C$$

where C is independent of k. This completes the proof of the proposition.

Once we have the proposition we look at the analytic family of operators

$$G_k^{\alpha} f = f \times \psi_k^{-1/3+(n+1/3)\alpha}.$$

Then by interpolating between the cases $\operatorname{Re}\alpha > 0$ and $\operatorname{Re}\alpha = 1$ we get the desired result. For details we refer to [13].

As we have already mentioned the restriction theorems are useful in the study of Bochner–Riesz means. Recall that

$$S_R^{\delta} f = (2\pi)^{-n} \sum \left(1 - \frac{2k+n}{R}\right)_+^{\delta} f \times \varphi_k$$

are called the *Bochner–Riesz means* of order $\delta \ge 0$ associated with the special Hermite expansions. Using Proposition 2.2 we can prove

THEOREM 3.1. *Let $1 \le p < 2(3n+1)/(3n+4)$ and $\delta > \delta(p) = 2n(1/p - 1/2) - 1/2$. Then $S_R^{\delta} f$ are uniformly bounded on $L^p(\mathbb{C}^n)$.*

The theorem was proved in [13] for the smaller range $1 \le p \le 2n/(n+1)$. The same proof yields the above theorem in view of Proposition 2.2.

In the case of radial functions the estimate of the proposition remains true in the bigger range $1 \le p < 4n/(2n+1)$. This has been observed in [13]. Based on that it was conjectured that the same is true for all functions. In what follows we show that the proposition is not true above a certain value of p. More precisely, we have the following theorem:

THEOREM 3.2. *The estimates of Proposition 2.2 are not valid for $p > 2(n+1)/(n+2)$.*

Proof. The proof is by contradiction. Assume that the estimate

$$(**) \qquad \|f \times \varphi_k\|_2 \le C k^{n(1/p-1/2)-1/2} \|f\|_p$$

is valid in the range $4/3 < p < 4n/(2n+1)$. Recall that if f is polyradial, i.e., $f(z) = f(|z_1|, |z_2|, \ldots, |z_n|)$, then

$$f \times \varphi_k(z) = \sum_{|\alpha|=k} \Big(\int f(w)\Phi_{\alpha\alpha}(w)\, dw\Big)\Phi_{\alpha\alpha}(z)$$

and the $\Phi_{\alpha\alpha}(z)$ are expressible in terms of Laguerre functions of type 0:

$$\Phi_{\alpha\alpha}(z) = \prod_{j=1}^{n} \psi_{\alpha_j}^0(z_j).$$

By taking

$$f(z) = g(|z_1|)e^{-|z'|^2/4}, \quad z = (z_1, z'),$$

and using the orthogonality properties of the Laguerre functions we get

$$f \times \varphi_k(z) = C\Big(\int_0^{\infty} g(r)L_k^0(r^2/2)e^{-r^2/4} r\, dr\Big)\psi_k^0(z_1)^{-|z'|^2/4}.$$

The estimate $(**)$ now gives us

$$\Big|\int_0^{\infty} g(r)L_k^0(r^2/2)e^{-r^2/4} r\, dr\Big| \le C k^{n(1/p-1/2)-1/2}\Big(\int_0^{\infty} |g(r)|^p r\, dr\Big)^{1/p}.$$

Taking supremum over all $g \in L^p(\mathbb{C})$ with unit norm we get

$$\Big(\int_0^{\infty} |L_k^0(r)e^{-r/2}|^{p'}\, dr\Big)^{1/p'} \le C k^{n(1/p-1/2)-1/2}.$$

From the Lemma of [4] already mentioned we infer that the left-hand side of the above equation behaves like $k^{1/2-1/p}$ and hence the estimate cannot be valid unless $p \le 2(n+1)/(n+2)$.

4. Heisenberg group revisited. In this last section we briefly recall Müller's restriction theorem for the Heisenberg group and then prove a slightly different restriction theorem for individual projections. The spectral decomposition of a function on H^n is given by

$$f(z,t) = (2\pi)^{-n-1} \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} f * e_k^{\lambda}(z,t)|\lambda|^n\, d\lambda.$$

Defining

$$\tilde{e}_k^{\lambda}(z,t) = e_k^{\lambda/(2k+n)}(z,t)$$

we can write the above as

$$f(z,t)=(2\pi)^{-n-1}\sum_{k=0}^{\infty}(2k+n)^{-n-1}\int_{-\infty}^{\infty} f*\tilde{e}_k^\lambda(z,t)|\lambda|^n\,d\lambda$$

or in a more compact form

$$f(z,t)=\int_{-\infty}^{\infty} P_\lambda f(z,t)|\lambda|^n\,d\lambda$$

where

$$P_\lambda f(z,t)=(2\pi)^{-n-1}\sum_{k=0}^{\infty}(2k+n)^{-n-1} f*\tilde{e}_k^\lambda(z,t).$$

The restriction theorem of Müller states that

$$\|P_\lambda f\|_{(p',\infty)}\le c_\lambda\|f\|_{(p,1)},\quad 1\le p<2.$$

Instead of considering all $f*e_k^\lambda$ together we will consider them separately. So we define

$$P_{k,a}f(z,t)=\int_{-a}^{a} f*e_k^\lambda(z,t)|\lambda|^n\,d\lambda$$

and will see what L^p-L^2 mapping properties these projections possess. Our result is:

THEOREM 4.1. *Let $f\in L^p(H^n)$ and $1\le p<2(3n+1)/(3n+4)$. Then*

$$\|P_{k,a}f\|_2\le Ck^{n(1/p-1/2)-1/2}a^{(n+1)(1/p-1/2)}\|f\|_p.$$

For the proof we need the following simple lemma:

LEMMA 4.1. *For $f\in L^p(\mathbb{R})$ with $1\le p<2$ one has*

$$\Big(\int_{-a}^{a}|\widehat{f}(\lambda)|^2\,d\lambda\Big)^{1/2}\le Ca^{1/p-1/2}\|f\|_p.$$

Proof. Let χ_a be the characteristic function of the interval $-a\le t\le a$ so that $\widehat{\chi}_a(\lambda)=\lambda^{-1}\sin a\lambda$. By Plancherel and Young,

$$\Big(\int_{-a}^{a}|\widehat{f}(\lambda)|^2\,d\lambda\Big)^{1/2}=C\Big(\int|f*\widehat{\chi}_a(t)|^2\,dt\Big)^{1/2}\le C\|f\|_p\|\widehat{\chi}_a\|_q$$

where $1/p+1/q-1=1/2$. But

$$\Big(\int|\widehat{\chi}_a(t)|^q\,dt\Big)^{1/q}=a\Big(\int\Big|\frac{\sin at}{at}\Big|^q\,dt\Big)^{1/q},$$

which equals a constant times $a^{1-1/q}$. Since $1/q-1=1/2-1/p$ the lemma follows.

Coming to the proof of the theorem we have, by a simple calculation,

$$\int_{-a}^{a} f*e_k^\lambda(z,t)|\lambda|^n\,d\lambda=\int_{-a}^{a} e^{i\lambda t}f^\lambda*_\lambda\varphi_k^\lambda(z)|\lambda|^n\,d\lambda.$$

Therefore,

$$\Big\|\int_{-a}^{a} f*e_k^\lambda|\lambda|^n\,d\lambda\Big\|_2^2=C\int_{\mathbb{C}^n}\int_{-a}^{a}|f^\lambda*_\lambda\varphi_k^\lambda(z)|^2|\lambda|^{2n}\,d\lambda\,dz.$$

In view of Proposition 2.2 a simple calculation shows that

$$\int_{\mathbb{C}^n}|f^\lambda*_\lambda\varphi_k^\lambda(z)|^2\,dz\le Ck^{2n(1/p-1/2)-1}\lambda^{-3n+2n/p}\Big(\int|f^\lambda(z)|^p\,dz\Big)^{2/p}.$$

Thus

$$\Big\|\int_{-a}^{a} f*e_k^\lambda|\lambda|^n\,d\lambda\Big\|_2^2\le Ck^{2n(1/p-1/2)-1}\int_{-a}^{a}|\lambda|^{-n+2n/p}\Big(\int|f^\lambda(z)|^p\,dz\Big)^{2/p}\,d\lambda.$$

Now applying Minkowski's integral inequality we get

$$\begin{aligned}\Big(\int_{-a}^{a}|\lambda|^{-n+2n/p}\Big(\int|f^\lambda(z)|^p\,dz\Big)^{2/p}\,d\lambda\Big)^{p/2}&\le\int dz\Big(\int_{-a}^{a}|\lambda|^{-n+2n/p}|f^\lambda(z)|^2\,d\lambda\Big)^{p/2}\\&\le a^{n-np/2}\int dz\Big(\int_{-a}^{a}|f^\lambda(z)|^2\,d\lambda\Big)^{p/2}\\&\le Ca^{n-np/2}a^{1-p/2}\iint|f(z,t)|^p\,dz\,dt\end{aligned}$$

where we have used the lemma. Finally,

$$\Big\|\int_{-a}^{a} f*e_k^\lambda|\lambda|^n\,d\lambda\Big\|_2\le Ck^{n(1/p-1/2)-1/2}a^{(n+1)(1/p-1/2)}\|f\|_p$$

follows.

COROLLARY. *Let $Q=2n+2$ be the homogeneous dimension of H^n. Then for $f\in L^p(H^n)$ and $1\le p<2(3n+1)/(3n+4)$ we have*

$$\Big\|\int_{-k^{2/Q}}^{k^{2/Q}} f*e_k^\lambda|\lambda|^n\,d\lambda\Big\|_2\le k^{(Q/2)(1/p-1/2)-1/2}\|f\|_p.$$

References

[1] G. B. Folland, *Harmonic Analysis in Phase Space*, Ann. of Math. Stud. 122, Princeton Univ. Press, Princeton, N.J., 1989.
[2] R. Gangolli, *Spherical functions on semisimple Lie groups*, in: Symmetric Spaces, W. Boothby and G. Weiss (eds.), Dekker, New York, 1972, 41–92.
[3] A. Hulanicki and F. Ricci, *A Tauberian theorem and tangential convergence for bounded harmonic functions on balls in* $\mathbb{C}^n$, Invent. Math. 62 (1980), 325–331.
[4] C. Markett, *Mean Cesàro summability of Laguerre expansions and norm estimates with shifted parameter*, Anal. Math. 8 (1982), 19–37.
[5] D. Müller, *A restriction theorem for the Heisenberg group*, Ann. of Math. 131 (1990), 567–587.
[6] —, *On Riesz means of eigenfunction expansions for the Kohn-Laplacian*, J. Reine Angew. Math. 401 (1989), 113–121.
[7] J. Peetre and G. Sparr, *Interpolation and non-commutative integration*, Ann. Mat. Pura Appl. 104 (1975), 187–207.
[8] R. Rawat, *A theorem of the Wiener–Tauberian type for* $L^1(H^n)$, Proc. Indian Acad. Sci. Math. Sci. 106 (1996), 369–377.
[9] C. Sogge, *Oscillatory integrals and spherical harmonics*, Duke Math. J. 53 (1986), 43–65.
[10] —, *Concerning the* L^p *norm of spectral clusters for second-order elliptic operators on compact manifolds*, J. Funct. Anal. 77 (1988), 123–134.
[11] E. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Univ. Press, Princeton, N.J., 1993.
[12] R. Strichartz, L^p *harmonic analysis and Radon transforms on the Heisenberg group*, J. Funct. Anal. 96 (1991), 350–406.
[13] S. Thangavelu, *Weyl multipliers, Bochner–Riesz means and special Hermite expansions*, Ark. Mat. 29 (1991), 307–321.
[14] —, *Restriction theorems for the Heisenberg group*, J. Reine Angew. Math. 414 (1991), 51–65.
[15] —, *Some restriction theorems for the Heisenberg group*, Studia Math. 99 (1991), 11–21.
[16] —, *Lectures on Hermite and Laguerre Expansions*, Math. Notes 42, Princeton Univ. Press, Princeton, N.J., 1993.

Statistics and Mathematics unit
Indian Statistical Institute
8th mile, Mysore Road
Bangalore 560 059, India
E-mail: ratna@isibang.ernet.in
rawat@isibang.ernet.in
veluma@isibang.ernet.in

Received October 5, 1995
Revised version June 10, 1997 (3538)

STUDIA MATHEMATICA 126 (1) (1997)

The Minlos lemma for positive-definite functions on additive subgroups of $\mathbb{R}^n$

by

W. BANASZCZYK (Łódź)

Abstract. Let H be a real Hilbert space. It is well known that a positive-definite function φ on H is the Fourier transform of a Radon measure on the dual space if (and only if) φ is continuous in the Sazonov topology (resp. the Gross topology) on H. Let G be an additive subgroup of H and let $G^\wedge_{\mathrm{pc}}$ (resp. $G^\wedge_{\mathrm{b}}$) be the character group endowed with the topology of uniform convergence on precompact (resp. bounded) subsets of G. It is proved that if a positive-definite function φ on G is continuous in the Gross topology, then φ is the Fourier transform of a Radon measure μ on $G^\wedge_{\mathrm{pc}}$; if φ is continuous in the Sazonov topology, μ can be extended to a Radon measure on $G^\wedge_{\mathrm{b}}$.

1. Introduction. Every continuous positive-definite function on an LCA group G is the Fourier transform of a (unique) Radon measure on the character group $G^\wedge$. This fact, known as the Bochner theorem, has been generalized to certain abelian topological groups which are not locally compact; a brief survey can be found in [1, Sec. 11], see also Remark 1.5. In particular, R. A. Minlos [7] proved that the Bochner theorem remains valid if G is a nuclear locally convex space. In what follows, D is an n-dimensional ellipsoid in $\mathbb{R}^n$ with centre at 0 and principal semiaxes of lengths $\lambda_1, \ldots, \lambda_n$. By $x \cdot y$ we denote the euclidean inner product of vectors $x, y \in \mathbb{R}^n$. The proof of the Minlos theorem is based on the following fact (see Lemma 4.1 in [11, Ch. VI]):

LEMMA 1.1 (R. A. Minlos). *Let μ be a probability measure on $\mathbb{R}^n$ and $\widehat{\mu}$ the characteristic functional of μ:*

$$\widehat{\mu}(x) = \int_{\mathbb{R}^n} e^{ix\cdot y}\, d\mu(y), \qquad x \in \mathbb{R}^n.$$

1991 *Mathematics Subject Classification*: 43A35, 46C05, 52A40, 60B11, 60B15.
Supported by KBN Grant 2 P301 019 04; the author wishes to thank the Department of Applied Mathematics, University of Vigo, where the final part of this paper was written, for the invitation and hospitality.