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Tauberian operators on $L_1(\mu)$ spaces

by

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Abstract. We characterize tauberian operators $T : L_1(\mu) \to Y$ in terms of the images of disjoint sequences and in terms of the image of the dyadic tree in $L_1[0,1]$. As applications, we show that the class of tauberian operators is stable under small norm perturbations and that its perturbation class coincides with the class of all weakly precompact operators. Moreover, we prove that the second conjugate of a tauberian operator $T : L_1(\mu) \to Y$ is also tauberian, and the induced operator $\widetilde{T} : L_1(\mu)^{**}/L_1(\mu) \to Y^{**}/Y$ is an isomorphism into. Also, we show that $L_1(\mu)$ embeds isomorphically into the quotient of $L_1(\mu)$ by any of its reflexive subspaces.

1. Introduction. Tauberian operators were introduced by Kalton and Wilansky [14] to solve a summability problem. Since then, they have found many applications in Banach space theory: preservation of isomorphic properties [20], equivalence between the Radon–Nikodym property and the Krein–Milman property [24], and factorization of operators [3], for example. Given Banach spaces X and Y, an operator $T : X \to Y$ is said to be *tauberian* if $T^{**-1}(Y) \subset X$, where $T^{**} : X^{**} \to Y^{**}$ is the second conjugate of T. The class of tauberian operators presents some deficiencies: it is not open in the class of all operators, and has bad behaviour under duality. These deficiencies led Tacon [25] to introduce a smaller class, the supertauberian operators.

In this paper we give some characterizations for tauberian operators from $L_1(\mu)$ into a Banach space, and derive some consequences.

We assume that μ is a non-purely atomic, finite measure in order to simplify the exposition. For ν a purely atomic measure, our results are valid but trivial, because in this case $L_1(\nu)$ has the Schur property: weakly convergent sequences are convergent; hence it contains no infinite-dimensional reflexive subspaces. Therefore, by [9, Theorem 4.2] any tauberian operator

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$T : L_1(\nu) \to Y$ is upper semi-Fredholm; i.e., it has finite-dimensional kernel and closed range.

In Section 2, we characterize the tauberian operators $T : L_1(\mu) \to Y$ as those operators T such that $\liminf_n \|Tf_n\| > 0$ for every disjoint normalized sequence (f_n) in $L_1(\mu)$; or equivalently, the kernel $N(T^{**})$ of the second conjugate of T coincides with $N(T)$. As a consequence, we prove that $L_1(\mu)$ is contained isomorphically in every quotient of $L_1(\mu)$ by any of its reflexive subspaces, and that the class of all tauberian operators from $L_1(\mu)$ into Y is open. We give several examples of operators in this class, and show that every tauberian operator $T : L_1(\mu) \to Y$ is supertauberian (i.e., any ultrapower $T_{\mathcal{U}}$ of T is tauberian) and its second conjugate T^{**} is also tauberian. Moreover, we prove that the operator $\widetilde{T} : L_1(\mu)^{**}/L_1(\mu) \to Y^{**}/Y$ given by $\widetilde{T}(x^{**} + L_1(\mu)) := T^{**}(x^{**}) + Y$ is an isomorphism into. Another proof of this fact may be obtained in a more general way which requires the use of supertauberian operators and a class of operators recently introduced by H. Rosenthal [23], the strongly tauberian operators.

In the case where the measure space (Ω, Σ, μ) has no atoms, we prove that for every tauberian operator $T : L_1(\mu) \to Y$ we can find a finite partition $\{\Omega_1, \ldots, \Omega_n\}$ of Ω so that the restrictions of T to the subspaces $L_1(\Omega_i)$ of functions supported in Ω_i are isomorphisms (into). Finally in this section, we show that $T : L_1[0,1] \to Y$ is tauberian if and only if for every sequence (f_n) contained in the dyadic tree of $L_1[0,1]$ and equivalent to the unit vector basis of ℓ_1, $(Tf_n)_{n\geq k}$ is also equivalent to the unit vector basis of ℓ_1 for some k.

In Section 3 we identify the perturbation class for the tauberian operators acting on $L_1(\mu)$. We show that an operator $K : L_1(\mu) \to Y$ is weakly precompact if and only if $T + K$ is tauberian for every tauberian operator $T : L_1(\mu) \to Y$.

We use standard notations: X and Y are Banach spaces, B_X the closed unit ball of X, S_X the unit sphere of X, $\mathcal{B}(X,Y)$ the class of bounded linear operators from X into Y, X^* the dual of X, $T^* : Y^* \to X^*$ the conjugate operator of $T \in \mathcal{B}(X,Y)$, and $R(T)$ and $N(T)$ the range and kernel of T. We identify X with a subspace of X^{**}. If $A \subset X$, then $\overline{A}^{w^*}$ is the weak-star closure of A in X^{**}. We denote the set of all positive integers by $\mathbb{N}$, and the set of all real numbers by $\mathbb{R}$.

Let (Ω, Σ, μ) be a finite measure space. A set $A \in \Sigma$ is said to be an *atom* if $\mu(A) \neq 0$ and for every $M \in \Sigma$, either $\mu(A \cap M) = 0$ or $\mu(A \cap M) = \mu(A)$. We say that $A \in \Sigma$ is *non-purely atomic* if it is not a union of atoms. Henceforth we assume that μ is non-purely atomic; i.e., Ω is non-purely atomic. We denote by χ_A the characteristic function of $A \in \Sigma$. For a function $f : \Omega \to \mathbb{R}$, we write $D(f) := \{x \in \Omega : f(x) \neq 0\}$. A sequence $(f_n) \subset L_1(\mu)$ is said to be *disjoint* if $f_k(x) \cdot f_m(x) = 0$ a.e. for $k \neq m$. Observe that given a disjoint sequence $(f_n) \subset L_1(\mu)$, we have $\lim_n \mu(D(f_n)) = 0$.

2. Characterizing tauberian operators on $L_1(\mu)$. The following technical lemma was obtained by Kadec and Pełczyński [13] for $L_1[0,1]$. Their proof is essentially valid for all $L_1(\mu)$ spaces, with μ a finite measure.

LEMMA 1. *Let (f_n) be a bounded sequence in $L_1(\mu)$. Then there exists a subsequence $(f_{n_k}) \subset (f_n)$ and sequences (y_k), (z_k) in $L_1(\mu)$ such that $f_{n_k} = y_k + z_k$, (z_k) is weakly convergent, and (y_k) is disjoint.*

Recall that an operator $T \in \mathcal{B}(X,Y)$ is *tauberian* [14] if $T^{**-1}(Y) \subset X$, or equivalently [14, Theorem 3.2], if any bounded sequence $(x_n) \subset X$ admits a weakly convergent subsequence (x_{n_k}) whenever (Tx_n) is weakly convergent. We denote the class of all tauberian operators from X into Y by $\mathcal{T}(X,Y)$.

Next we characterize tauberian operators on $L_1(\mu)$ in terms of their action over disjoint sequences.

THEOREM 2. *For $T \in \mathcal{B}(L_1(\mu), Y)$, the following statements are equivalent*:

(1) *T is tauberian*;

(2) $N(T) = N(T^{**})$;

(3) $\liminf_n \|Tf_n\| > 0$ *for every normalized disjoint sequence (f_n) in $L_1(\mu)$*;

(4) *there exists $r > 0$ such that* $\liminf_n \|Tf_n\| > r$ *for every normalized disjoint sequence (f_n) in $L_1(\mu)$.*

Proof. (1)⇒(2). This follows directly from the definition of tauberian operator.

(2)⇒(3). Let (f_n) be a normalized disjoint sequence in $L_1(\mu)$ and assume that $\lim_n \|Tf_n\| = 0$. Since (f_n) is equivalent to the unit vector basis of ℓ_1, there exists a vector $z \in \overline{\{f_n\}}^{w^*} \setminus L_1(\mu)$ such that $T^{**}z = 0$, which proves that $N(T) \neq N(T^{**})$.

(3)⇒(4). We may assume that $\|T\| = 1$. Suppose that (4) is false. Then for every positive integer k there exists a normalized disjoint sequence $(f_n^k)_n \subset L_1(\mu)$ such that $\|Tf_n^k\| < 1/k$. We will find a disjoint sequence $(f_n) \subset L_1(\mu)$ such that $1/2 < \|f_n\| \leq 1$ and $\|Tf_n\| \leq 2/n$ for all n, and the proof will be done.

First we take $g_1 := f_1^1$. Since $\lim_k \int_{D(f_k^2)} |g_1|\,d\mu = 0$, we can select k_2 satisfying $\int_{D(f_{k_2}^2)} |g_1|\,d\mu < 1/2^2$, and we take $g_2 := f_{k_2}^2$. At the nth stage we apply the argument to the functions $|g_1|, \ldots, |g_{n-1}|$. In this way we obtain a normalized sequence $(g_n) \subset L_1(\mu)$ such that $\|Tg_n\| < 1/n$ for all n, and $\int_{D(g_k)} |g_n|\, d\mu < 1/2^{2k}$ for $k > n$.

Now, defining $A_n := D(g_n) \setminus \bigcup_{k=n+1}^{\infty} D(g_k)$, it is enough to take $f_n := g_n \cdot \chi_{A_n}$. In fact, clearly (f_n) is disjoint. Moreover,

$$\|f_n\| \ge 1 - \sum_{k=n+1}^{\infty} \int_{D(g_k)} |g_n|\, d\mu \ge 1 - \sum_{k=n+1}^{\infty} 2^{-2k} \ge 1/2,$$

and

$$\|Tf_n\| \le \|Tg_n\| + \|T\| \sum_{k=n+1}^{\infty} \int_{D(g_k)} |g_n|\, d\mu \le \frac{1}{n} + 2^{-n} \le \frac{2}{n}.$$

(4)⇒(1). Suppose T is not tauberian. Then we can find a sequence (g_n) in $B_{L_1(\mu)}$ without weakly convergent subsequences and such that (Tg_n) is weakly convergent. By Lemma 1, there is a subsequence $(g_{n_k}) \subset (g_n)$ and sequences (u_k), (v_k) in $L_1(\mu)$ such that (v_k) is weakly convergent, (u_k) is disjoint and $g_{n_k} = u_k + v_k$. Note that $\liminf_n \|u_n\| > 0$. Since $v_{2k} - v_{2k-1}$ and $T(g_{n_{2k}} - g_{n_{2k-1}})$ are weakly null, there is an increasing sequence of positive integers $k_1 < k_2 < \ldots$ and a sequence of real numbers $\alpha_n \ge 0$ with $\sum_{i=k_n+1}^{k_{n+1}} \alpha_i = 1$ such that taking $x_n := \sum_{i=k_n+1}^{k_{n+1}} \alpha_i (g_{n_{2i}} - g_{n_{2i-1}})$ and $z_n := \sum_{i=k_n+1}^{k_{n+1}} \alpha_i (v_{2i} - v_{2i-1})$ we obtain $\lim_n \|Tx_n\| = 0$ and $\lim_n \|z_n\| = 0$. Obviously, the sequence $y_n := \sum_{i=k_n+1}^{k_{n+1}} \alpha_i (u_{2i} - u_{2i-1})$ is disjoint, $\liminf_n \|y_n\| > 0$ and $\lim_n \|Ty_n\| = 0$. In this way we obtain a normalized disjoint sequence $f_n := \|y_n\|^{-1} y_n$ satisfying $\lim_n \|Tf_n\| = 0$, in contradiction with (4). ■

Remark. Condition (2) is not enough in general for an operator $T \in \mathcal{B}(X, Y)$ to be tauberian, as is shown by $L : c_0 \ni (x_n) \to (x_n/n) \in \ell_2$.

Let us see some consequences of Theorem 2. Given $T : L_1(\mu) \to Y$ we set

$$\beta_T := \inf \{\liminf_n \|Tf_n\| : (f_n) \subset L_1(\mu) \text{ normalized and disjoint}\}.$$

COROLLARY 3. *The class $\mathcal{T}(L_1(\mu), Y)$ of tauberian operators is open in $\mathcal{B}(L_1(\mu), Y)$.*

Proof. If $T : L_1(\mu) \to Y$ is tauberian, then by Theorem 2(4) we have $\beta_T > 0$. Let $S : L_1(\mu) \to Y$ be an operator such that $\|T - S\| = \alpha < \beta_T$. Then, for each normalized and disjoint sequence $(f_n) \subset L_1(\mu)$, we have $\liminf_n \|Sf_n\| \ge \beta_T - \alpha > 0$, which implies S is tauberian, according to Theorem 2. ■

Note that the class $\mathcal{T}(X, Y)$ is not open in general [1], [25].

Given a measurable subset $C \subset \Omega$, we denote by $L_1(C)$ the subspace of $L_1(\mu)$ which consists of all functions f with $D(f) \subset C$.

COROLLARY 4. *Let $T \in \mathcal{B}(L_1(\mu), Y)$ be a tauberian operator. For every non-purely atomic measurable set $A \subset \Omega$ with $\mu(A) > 0$ there is a non-purely atomic subset $C \subset A$ with $\mu(C) > 0$ so that the restriction $T|_{L_1(C)}$ is an isomorphism.*

Proof. Assume the result is false, and let $(C_n)_n$ be a sequence of disjoint non-purely atomic measurable subsets of A with $\mu(C_n) > 0$ for every $n \in \mathbb{N}$. Since T restricted to $L_1(C_n)$ is not an isomorphism, there is a function $f_n \in L_1(C_n)$ such that $\|f_n\| = 1$ and $\|Tf_n\| < 1/n$, in contradiction with Theorem 2(3). ■

If a measurable subset C with $\mu(C) > 0$ is non-purely atomic then $L_1(C)$ is isomorphic to $L_1(\mu)$. This allows us to give the next result.

COROLLARY 5. *The class $\mathcal{T}(L_1(\mu), Y)$ is non-empty if and only if Y contains a subspace isomorphic to $L_1(\mu)$. In particular, if M is a reflexive subspace of $L_1(\mu)$ then $L_1(\mu)/M$ contains a subspace isomorphic to $L_1(\mu)$.*

Now we present some examples of tauberian operators on $L_1(\mu)$.

EXAMPLES. (a) Operators with reflexive kernel and closed range are tauberian [14]. Hence, for every reflexive subspace R of $L_1(\mu)$, the quotient map $Q : L_1(\mu) \to L_1(\mu)/R$ is tauberian. In fact, we have $\beta_Q = 1$.

Indeed, take $0 < \varepsilon < 1$, and let (f_n) be a normalized disjoint sequence in $L_1(\mu)$. Since R is reflexive, the set $3B_R$ is equiintegrable [2, Proposition V.2.2]. So there exists $\delta > 0$ such that $\int_A |g|\, d\mu < \varepsilon$ for every $g \in 3B_R$ and every measurable set A with $\mu(A) < \delta$. Take $n_0 \in \mathbb{N}$ such that $\mu(D(f_n)) < \delta$ for all $n \ge n_0$. Then for every $g \in 3B_R$ we have

$$\|f_n - g\| \ge \int_{D(f_n)} |f_n|\, d\mu - \int_{D(f_n)} |g|\, d\mu \ge 1 - \varepsilon$$

for all $n \ge n_0$. Note that $\|Qf_n\| = \inf\{\|f_n - g\| : g \in 3B_R\} \le 1$. So we have $\lim_n \|Qf_n\| = 1$, hence $\beta_Q = 1$.

The space $L_1[0,1]$ contains a large list of reflexive subspaces. For instance, the closed space generated by the Rademacher functions on $[0,1]$, which are given by $r_n(t) = \operatorname{sgn} \sin 2^n \pi t$ for $n \in \mathbb{N}$, is isomorphic to ℓ_2 (see [16, Theorem 2.b.3]). Also, it is known [17, Theorem 2.f.5] that for $1 < r < 2$ there exists a closed subspace of $L_1[0,1]$ isomorphic to $L_r[0,1]$.

(b) Let $T \in \mathcal{B}(X, Y)$ be a tauberian operator. If $K \in \mathcal{B}(X, Y)$ is weakly compact then it is easy to check that $T + K$ is also tauberian. Moreover, in Section 3 we will see that, in the case $X = L_1(\mu)$, the class of tauberian operators is stable under weakly precompact perturbations.

(c) If $T \in \mathcal{B}(L_1(\mu), Y)$ is tauberian and $S \in \mathcal{B}(L_1(\mu), Y)$ satisfies $\|S\| < \beta_T$, then the proof of Corollary 3 shows us that $T+S$ is tauberian.

Corollary 4 can be greatly improved using the following characterization of tauberian operators on $L_1(\mu)$.

THEOREM 6. *For $T \in \mathcal{B}(L_1(\mu), Y)$, the following statements are equivalent:*

(1) *T is tauberian;*

(2) *for every normalized sequence (f_n) in $L_1(\mu)$ such that $\lim_n \mu(D(f_n)) = 0$ we have $\liminf_n \|Tf_n\| > 0$;*

(3) *there exists $r > 0$ such that for every normalized vector $f \in L_1(\mu)$ with $\mu(D(f)) < r$ we have $\|Tf\| > r$.*

Proof. (1)⇒(2). A normalized sequence (f_n) in $L_1(\mu)$ with $\lim_n \mu(D(f_n)) = 0$ has no equiintegrable subsequences; hence it has no weakly convergent subsequences.

(2)⇒(3). If (3) fails, then we can select a normalized sequence (f_n) so that $\mu(D(f_n)) < n^{-1}$ and $\|Tf_n\| < n^{-1}$; hence (2) fails.

(3)⇒(1). Assume T is not tauberian. By Theorem 2 we can select a normalized disjoint sequence (f_n) so that $\lim_n \|Tf_n\| = 0$. Since $\lim_n \mu(D(f_n)) = 0$, (3) fails. ■

COROLLARY 7. *Assume that the measure space (Ω, Σ, μ) has no atoms. Then for every $T \in \mathcal{T}(L_1(\mu), Y)$ we can find a finite partition $\{\Omega_1, \ldots, \Omega_n\}$ of Ω so that the restrictions $T|_{L_1(\Omega_i)}$ are isomorphisms (into).*

Proof. This follows from statement (3) in the previous theorem: since Ω has finite measure and contains no atoms, we can find a finite partition of Ω into measurable subsets of measure smaller than r. ■

Recall that an operator $T \in \mathcal{B}(X, Y)$ is said to be *supertauberian* if for every $0 < \varepsilon < 1$ there exists a positive integer $n \in \mathbb{N}$ for which there are no families $\{x_1, \ldots, x_n\} \subset S_X$, $\{f_1, \ldots, f_n\} \subset S_{X^*}$ satisfying $f_k(x_m) > \varepsilon$ for $1 \le k \le m \le n$, $f_k(x_m) = 0$ for $1 \le m < k \le n$ and $\|Tx_k\| < 1/k$ for $k = 1, \ldots, n$. These operators, introduced by Tacon [25], can be characterized in terms of ultrapowers: T is supertauberian if and only if every ultrapower $T_{\mathcal{U}}$ of T is tauberian [8, Theorem 9]. Clearly, supertauberian operators are tauberian. We refer to [18] for a detailed study of supertauberian operators.

PROPOSITION 8. *An operator $T \in \mathcal{B}(L_1(\mu), Y)$ is tauberian if and only if T is supertauberian.*

Proof. This is a consequence of two non-trivial facts: (1) every reflexive subspace of $L_1(\mu)$ is superreflexive [21], and (2) for a Banach space X whose reflexive subspaces are superreflexive, we know that $T \in \mathcal{B}(X, Y)$ is tauberian if and only if it is supertauberian [8, Theorem 17]. ■

The class of tauberian operators $\mathcal{T}(X, Y)$ has bad behaviour under duality: there exists a tauberian operator T whose second conjugate T^{**} is not tauberian [1, Proposition 5]. However, supertauberian operators behave better. We denote the nth conjugate of T by T^{n*} for $n \ge 2$.

COROLLARY 9. *If $T \in \mathcal{B}(L_1(\mu), Y)$ is tauberian then T^{2n*} is tauberian for all $n \in \mathbb{N}$.*

Proof. If T is supertauberian then T^{2n*} is supertauberian [26, Theorem 1]. ■

Corollary 3 can also be derived from Proposition 8, because the class of supertauberian operators in $\mathcal{B}(X, Y)$ is open [8, Proposition 13].

Given an operator $T \in \mathcal{B}(X, Y)$, we denote by

$$\widetilde{T} : X^{**}/X \to Y^{**}/Y$$

the induced operator defined by $\widetilde{T}(x^{**} + X) := T^{**}(x^{**}) + Y$ for every $x^{**} \in X^{**}$. Note that T is tauberian if and only if $\widetilde{T}$ is injective. H. Rosenthal [23] has recently studied the operators $T \in \mathcal{B}(X, Y)$ for which $\widetilde{T}$ is an isomorphism into, calling them *strongly tauberian operators*. Obviously, strongly tauberian operators are tauberian. Moreover, Rosenthal [23] proves that supertauberian operators are strongly tauberian. Thus, if every reflexive subspace of X is superreflexive then, reasoning as in Proposition 8, we see that every tauberian operator $T \in \mathcal{B}(X, Y)$ is supertauberian, therefore, it is strongly tauberian. The above argument can be applied to tauberian operators on $L_1(\mu)$, but this particular case admits a direct proof without using general principles, which is given in the next proposition. The main ingredients of the proof are the result of Kadec and Pełczyński given in Lemma 1 and a refinement of an argument of James [11, 12] used in his sequential characterization of non-reflexive spaces (see also [19]). First we need a technical lemma.

LEMMA 10 [22]. *If $T \in \mathcal{B}(X, Y)$, $z \in \operatorname{int} B_{X^{**}}$ and $y \in Y$ satisfy $\|T^{**}z - y\| < \varepsilon$, then $z \in \overline{L}^{w^*}$, where $L := \{x \in B_X : \|Tx - y\| < \varepsilon\}$.*

Proof. Assume that the result fails and $L \neq \emptyset$. The Hahn–Banach Theorem gives $f \in X^*$ and $a < b =: |z(f)|$ such that $|f(x)| \le a$ for all $x \in L$. Therefore, if we define $W := \{x \in X : \|x\| < 1 \text{ and } 2^{-1}(a+b) < |f(x)|\}$, then $z \in \overline{W}^{w^*}$ and $W \cap L = \emptyset$. Thus, $T^{**}z \in \overline{T(W)}^{w^*}$ and $\|Tw - y\| \ge \varepsilon$ for all $w \in W$. By the Hahn–Banach Theorem, there is $g \in S_{Y^*}$ such that $\varepsilon \le |g(Tw - y)|$ for all $w \in W$. But $T^{**}z - y \in \overline{(T(W) - y)}^{w^*}$, so $\|T^{**}z - y\| \ge |g(T^{**}z - y)| \ge \varepsilon$, a contradiction.

For the case $L = \emptyset$, taking $W := \{x \in X : \|x\| < 1\}$, and applying a similar argument to the above, we get contradiction. ∎

PROPOSITION 11. *An operator $T \in \mathcal{B}(L_1(\mu), Y)$ is tauberian if and only if the induced operator $\widetilde{T} : L_1(\mu)^{**}/L_1(\mu) \to Y^{**}/Y$ is an isomorphism into.*

Proof. The "if" part is trivial. For the converse, assume that $T \in \mathcal{B}(L_1(\mu), Y)$ is tauberian. It follows from Theorem 2 that there is a real number $r > 0$ such that for every normalized disjoint sequence $(f_n) \subset L_1(\mu)$, we have

$$\liminf_n \|Tf_n\| > r. \tag{1}$$

Assume that $\widetilde{T}$ is not an isomorphism. Then we can find $x^{**} \in L_1(\mu)^{**}$ such that $\|x^{**}+L_1(\mu)\| = 1$ and $\|\widetilde{T}(x^{**}+L_1(\mu))\| < r/4$. We select a vector $y \in Y$ so that $\|T^{**}(x^{**})+y\| < r/4$ and a number ε such that $6/7 < \varepsilon < 1$.

We are going to obtain normalized sequences $(g_n)_n \subset L_1(\mu)$, $(f_n)_n \subset L_1(\mu)^*$ such that $f_i(g_j) > \varepsilon$ for $i \le j$, $f_i(g_j) = 0$ for $j < i$ and $\|T(g_n)+y\| < r/4$ for all n. First, take $f_1 \in S_{L_1(\mu)^*}$ so that $x^{**}(f_1) > \varepsilon$. By Lemma 10 there is $g_1 \in S_{L_1(\mu)}$ such that $f_1(g_1) > \varepsilon$ and $\|T(g_1)+y\| < r/4$. Suppose we already have families $(g_k)_{k=1}^{n-1} \subset S_{L_1(\mu)}$, $(f_k)_{k=1}^{n-1} \subset S_{L_1(\mu)^*}$ satisfying the conditions

$$\begin{aligned} &f_i(g_j) > \varepsilon \quad \text{if } 1 \le i \le j \le n-1,\\ &f_i(g_j) = 0 \quad \text{if } 1 \le j < i \le n-1,\\ &x^{**}(f_k) > \varepsilon \quad \text{for } k = 1,\dots,n-1,\\ &\|T(g_k)+y\| < r/4 \quad \text{for } k = 1,\dots,n-1. \end{aligned}$$

Since the quotient $L_1(\mu)^{**}/\langle g_1,\dots,g_{n-1}\rangle$ is isometric to $(\langle g_1,\dots,g_{n-1}\rangle^{\perp})^*$, we can take $f_n \in S_{L_1(\mu)^*}$ so that $f_n(g_k) = 0$ for $k = 1,\dots,n-1$ and $x^{**}(f_n) > \varepsilon$. By Lemma 10, there is $g_n \in S_{L_1(\mu)}$ such that $f_k(g_n) > \varepsilon$ for $k = 1,\dots,n$ and $\|T(g_n)+y\| < r/4$. We have just proved the existence of the required normalized sequences $(g_n) \subset L_1(\mu)$ and $(f_n) \subset L_1(\mu)^*$.

By Lemma 1, the sequence $(g_n)_n$ contains a subsequence $(g_{m_k})_k$ which splits into a disjoint sequence $(u_k)_k$ and a weakly convergent sequence $(v_k)_k$:

$$g_{m_k} = u_k + v_k.$$

Now we can select an increasing sequence of positive integers $k_1 < k_2 < \dots$ and a sequence of real numbers $\alpha_n \ge 0$ with $\sum_{i=k_n+1}^{k_{n+1}} \alpha_i = 1$ for all n, such that the sequence $z_n := \sum_{i=k_n+1}^{k_{n+1}} \alpha_i v_i$ is norm convergent. Let n_0 be a positive integer such that $\|z_n - z_m\| < \varepsilon/8$ and $\|T(z_n - z_m)\| < r/4$ for all $n, m \ge n_0$.

Take the disjoint sequence $y_n := \sum_{i=k_n+1}^{k_{n+1}} \alpha_i u_i$, and the sequences $x_n := y_n + z_n$, $h_n := f_{m_{k_n+1}}$. Note that $h_i(x_j) > \varepsilon$ for $i \le j$ and $h_i(x_j) = 0$ for $j < i$, hence $\|x_n - x_m\| > \varepsilon$ when $n \ne m$. Thus, for $n \ge n_0$, we have

$$\|y_{2n} - y_{2n+1}\| \ge \|x_{2n} - x_{2n+1}\| - \|z_{2n} - z_{2n+1}\| > (7/8)\varepsilon. \tag{2}$$

On the other hand, $\|T(g_n)+y\| < r/4$ implies that $\|T(x_{2n} - x_{2n+1})\| < r/2$. So, for $n \ge n_0$, we have

$$\|T(y_{2n} - y_{2n+1})\| \le \|T(x_{2n} - x_{2n+1})\| + \|T(z_{2n} - z_{2n+1})\| < (3/4)r.$$

By inequality (2), we have $\lambda_n := \|y_{2n} - y_{2n+1}\|^{-1} \le 8(7\varepsilon)^{-1} < 4/3$. Write $w_n := \lambda_n(y_{2n} - y_{2n+1})$. Thus we have obtained a normalized disjoint sequence (w_n) satisfying $\|T(w_n)\| < r$, in contradiction with inequality (1). ∎

Remark. We have already proved in Corollary 9 that if $T \in \mathcal{B}(L_1(\mu), Y)$ is tauberian then T^{**} is tauberian. We used there supertauberian operators but, after Proposition 11, it is possible to give a direct proof of this fact. Actually, if $T \in \mathcal{B}(X, Y)$ is strongly tauberian then T^{**} is also strongly tauberian. The proof, given in [23], is essentially as follows:

If $T \in \mathcal{B}(X, Y)$ is a strongly tauberian operator, then $\widetilde{T} : X^{**}/X \to Y^{**}/Y$ is an isomorphism. But for any Banach space X, we can identify canonically $(X^{**}/X)^{**}$ with X^{4*}/X^{**}, and $\widetilde{T}^{**}$ with $\widetilde{T^{**}} : X^{4*}/X^{**} \to Y^{4*}/Y^{**}$. Therefore, $\widetilde{T^{**}}$ is an isomorphism, and consequently, T^{**} is strongly tauberian. Repeating the process, we conclude that T^{2n*} is strongly tauberian for all $n \in \mathbb{N}$.

Finally in this section we characterize tauberian operators $T : L_1(\mu) \to Y$ by their action over the dyadic tree of $L_1(\mu)$. Our proofs can be adapted to any separable $L_1(\mu)$ space with μ a purely non-atomic finite measure, but in this case $L_1(\mu)$ is isomorphic to $L_1[0,1]$.

The *dyadic tree* on $L_1[0,1]$ consists of the functions

$$\chi_k^n := 2^n \chi_{((k-1)/2^n, k/2^n)}, \quad n = 0, 1, 2, \dots;\ 1 \le k \le 2^n.$$

The intervals $((k-1)/2^n, k/2^n)$ are called *dyadic*. Any operator $T \in \mathcal{B}(L_1[0,1], Y)$ is determined by the image of the dyadic tree: If for every $f \in L_1[0,1]$ we define

$$P_n(f) := \sum_{k=1}^{2^n} 2^{-n} \Big(\int_0^1 \chi_k^n f\, dx\Big) \chi_k^n,$$

then $\|f - P_n f\|$ tends to 0 as n tends to infinity. In particular, the dyadic tree generates a dense subspace of $L_1[0,1]$. Moreover, we have

$$Tf := \lim_n \sum_{k=1}^{2^n} 2^{-n} \Big(\int_0^1 \chi_k^n f\, dx\Big) T\chi_k^n.$$

We refer to [5] for the details. We need the following elementary lemma.

LEMMA 12. *If $T \in \mathcal{T}(X,Y)$ and (x_n) is a sequence in X equivalent to the unit vector basis of ℓ_1, then there exists an $n_0 \in \mathbb{N}$ such that the restriction of T to the linear closed span of $\{x_n : n \geq n_0\}$ is an isomorphism.*

Proof. By [9, Theorem 4.2], the restriction of T to the closed space generated by $\{x_n : n \in \mathbb{N}\}$ is upper semi-Fredholm, and then the result follows. ∎

Recall that the set of all simple functions is dense in $L_1[0,1]$.

THEOREM 13. *An operator $T \in \mathcal{B}(L_1[0,1], Y)$ is tauberian if and only if for every sequence (x_n) in the dyadic tree of $L_1[0,1]$ equivalent to the unit vector basis of ℓ_1, there exists $n_0 \in \mathbb{N}$ such that $(Tx_n)_{n \geq n_0}$ is equivalent to the unit vector basis of ℓ_1.*

Proof. Let $T \in \mathcal{B}(L_1[0,1], Y)$ be a tauberian operator and let (x_n) be a sequence contained in the dyadic tree in $L_1[0,1]$ and equivalent to the unit vector basis of ℓ_1. By Lemma 12, there is a positive integer n_0 such that the restriction of T to the subspace generated by $\{x_n : n \geq n_0\}$ is an isomorphism.

Assume now that T is not tauberian. By Theorem 2 there is a disjoint normalized sequence $(f_n) \subset L_1[0,1]$ such that $\lim_n \|Tf_n\| = 0$, and we may suppose that every f_n is a simple function. For each measurable set A and $\varepsilon > 0$ there are disjoint dyadic intervals $I_1, \ldots, I_n$ such that $\mu((\bigcup_{l=1}^n I_l) \triangle A) < \varepsilon$, where $\triangle$ stands for the symmetric difference. It follows that for every f_n there are two positive integers k_n, m_n with $k_1 < k_2 < \ldots$, a finite collection of disjoint dyadic intervals $I_1^n, \ldots, I_{m_n}^n$ of length 2^{-k_n} and scalars $\beta_1^n, \ldots, \beta_{m_n}^n$ such that if

$$g_n := \sum_{l=1}^{m_n} 2^{k_n} \beta_l^n \chi_{I_l^n}$$

then $\|f_n - g_n\| < 1/n$, $\|g_n\| = 1$ and $\mu(D(f_n) \triangle D(g_n)) < 1/n$ for all $n \in \mathbb{N}$. Thus $\lim_n \|Tg_n\| = 0$. Passing to a subsequence if necessary, we can assume that $\mu(D(g_{n+1})) \leq (1/8)2^{-k_n}$ for all n. Therefore the sets

$$A_n := \bigcup_{k=n+1}^{\infty} \bigcup_{l=1}^{m_k} I_l^k$$

satisfy

$$\mu(A_n) \leq \sum_{p=n+1}^{\infty} \frac{1}{8} 2^{-k_{p-1}} \leq \frac{1}{4} 2^{-k_n}, \tag{3}$$

and taking $J_l^n := I_l^n \setminus A_n$ for $l = 1, \ldots, m_n$, we obtain $\mu(J_l^n) \geq (3/4)2^{-k_n}$.

We show that the sequence $2^{k_1}\chi_{I_1^1}, \ldots, 2^{k_1}\chi_{I_{m_1}^1}; 2^{k_2}\chi_{I_1^2}, \ldots, 2^{k_2}\chi_{I_{m_2}^2}; \ldots$ is equivalent to the unit vector basis of ℓ_1. Given a sequence $((a_l^n)_{l=1}^{m_n})_n$ of

scalars we have

$$\sum_{n=1}^{\infty} \sum_{l=1}^{m_n} |a_l^n| = \sum_{n=1}^{\infty} \sum_{l=1}^{m_n} |a_l^n| \int_0^1 2^{k_n} \chi_{I_l^n}\, d\mu \leq \frac{4}{3} \sum_{n=1}^{\infty} \sum_{l=1}^{m_n} |a_l^n| \int_0^1 2^{k_n} \chi_{J_l^n}\, d\mu,$$

and as the functions $\chi_{J_l^n}$ are by construction disjointly supported for all n and l, we have

$$\begin{aligned} \frac{4}{3} \sum_{n=1}^{\infty} \sum_{l=1}^{m_n} |a_l^n| \int_0^1 2^{k_n} \chi_{J_l^n}\, d\mu & \\ &= \frac{4}{3} \int_0^1 \Big| \sum_{n=1}^{\infty} \sum_{l=1}^{m_n} 2^{k_n} a_l^n \chi_{J_l^n} \Big|\, d\mu \\ &\leq \frac{4}{3} \int_0^1 \Big| \sum_{n=1}^{\infty} \sum_{l=1}^{m_n} 2^{k_n} a_l^n \chi_{I_l^n} \Big|\, d\mu + \frac{4}{3} \int_0^1 \Big| \sum_{n=1}^{\infty} \sum_{l=1}^{m_n} 2^{k_n} a_l^n \chi_{A_n} \Big|\, d\mu. \end{aligned}$$

Now, it follows from (3) that

$$\frac{4}{3} \int_0^1 \Big| \sum_{n=1}^{\infty} \sum_{l=1}^{m_n} 2^{k_n} a_l^n \chi_{A_n} \Big|\, d\mu \leq \frac{1}{3} \sum_{n=1}^{\infty} \sum_{l=1}^{m_n} |a_l^n|,$$

and we obtain

$$\frac{1}{2} \sum_{n=1}^{\infty} \sum_{l=1}^{m_n} |a_l^n| \leq \int_0^1 \Big| \sum_{n=1}^{\infty} \sum_{l=1}^{m_n} 2^{k_n} a_l^n \chi_{I_l^n} \Big|\, d\mu \leq \sum_{n=1}^{\infty} \sum_{l=1}^{m_n} |a_l^n|.$$

We have just shown that the sequence $((2^{k_n}\chi_{I_l^n})_{l=1}^{m_n})_n$ contained in the dyadic tree of $L_1[0,1]$ is equivalent to the unit vector basis of ℓ_1. However, as every Tg_n is an absolutely convex combination taken in $\{T2^{k_n}\chi_{I_1^n}, \ldots \ldots, T2^{k_n}\chi_{I_{m_n}^n}\}$ and $\lim_n \|Tg_n\| = 0$, we see that $((T2^{k_n}\chi_{I_l^n})_{l=1}^{m_n})_{n \geq n_0}$ is not equivalent to the unit vector basis of ℓ_1, and the proof is finished. ∎

3. The perturbation class of $\mathcal{T}(L_1(\mu), Y)$. For a Banach space A and a subset $S \subset A$, Lebow and Schechter [15] define the *perturbation class of S in A* as follows:

$$P(S) := \{a \in A : a + s \in S \text{ for all } s \in S\}.$$

We say that $C \subset A$ is an *admissible class* for S if $C \subset P(S)$. Here we study the perturbation class of $\mathcal{T}(L_1(\mu), Y)$ in $\mathcal{B}(L_1(\mu), Y)$.

It is not difficult to see that the class $\mathrm{WCo}(X,Y)$ of all weakly compact operators is an admissible class for $\mathcal{T}(X,Y)$ (cf. [25]). Moreover, a broader admissible class for $\mathcal{T}(X,Y)$ can be introduced as follows. An operator $T \in \mathcal{B}(X,Y)$ is said to be *R-strictly singular* if for any operator L into

X such that TL is tauberian, L is weakly compact [7]. The perturbation class $P(\mathcal{T}(X,Y))$ is not well known in general. However, for $X = L_1(\mu)$, we find that $P(\mathcal{T}(L_1(\mu),Y))$ coincides with the class $\mathrm{Ro}(L_1(\mu),Y)$ of all weakly precompact operators. Recall that $T \in \mathcal{B}(X,Y)$ is said to be a *weakly precompact operator* if for every bounded sequence $(x_n) \subset X$, (Tx_n) contains a weakly Cauchy subsequence.

The following result makes sense only in the case when $\mathcal{T}(L_1(\mu),Y)$ is non-empty; equivalently, Y contains a subspace isomorphic to $L_1(\mu)$, as a consequence of Corollary 5.

PROPOSITION 14. *Let Y be a Banach space such that $\mathcal{T}(L_1(\mu),Y) \neq \emptyset$. An operator $K \in \mathcal{B}(L_1(\mu),Y)$ is weakly precompact if and only if for every $T \in \mathcal{T}(L_1(\mu),Y)$, the operator $T+K$ is tauberian.*

Proof. Let $T \in \mathcal{B}(L_1(\mu),Y)$ be a tauberian operator and assume that $T+K$ is not tauberian. By Theorem 2, there is a normalized disjoint sequence $(x_n) \subset L_1(\mu)$ such that $\lim_n \|(T+K)x_n\| = 0$. Since (x_n) is equivalent to the unit vector basis of ℓ_1, by Lemma 12 there is $n_0 \in \mathbb{N}$ such that $(Tx_n)_{n=n_0}^{\infty}$ is also equivalent to the unit vector basis of ℓ_1. As $\lim_n (T+K)x_n = 0$, a perturbation argument for basic sequences [16, Proposition 1.a.9] allows us to conclude that (Kx_n) has a subsequence equivalent to the unit vector basis of ℓ_1, and so K is not weakly precompact.

For the converse, let $K \in \mathcal{B}(L_1(\mu),Y)$ and suppose K is not weakly precompact. Thus, by Rosenthal's ℓ_1-theorem [4, p. 201], there is a bounded sequence $(g_n) \subset L_1(\mu)$ such that (Kg_n) is equivalent to the unit vector basis of ℓ_1. Applying the Kadec–Pełczyński Lemma (Lemma 1) as in the proof of (4)⇒(1) in Theorem 2, we may assume that (g_n) is a normalized disjoint sequence, and so the subspace M generated by (g_n) is isomorphic to ℓ_1 and complemented in $L_1(\mu)$. Write $L_1(\mu) = M \oplus H$ and $\widetilde{M} := K(M)$. Note that $K|_M$ is an isomorphism, and H is isomorphic to $L_1(\mu)$ because $L_1(\mu)$ is primary [6].

Our goal is to obtain a tauberian operator $T \in \mathcal{B}(L_1(\mu),Y)$ for which the kernel $N(T+K)$ is not reflexive, which leads to $K \notin P(\mathcal{T}(L_1(\mu),Y))$.

By hypothesis Y contains a closed subspace L isomorphic to $L_1(\mu)$. One of the following cases should happen: (a) $\widetilde{M}+L$ is closed and $\widetilde{M} \cap L$ is finite-dimensional; (b) $\widetilde{M} \cap L$ is infinite-dimensional; (c) $\widetilde{M}+L$ is not closed and $\widetilde{M} \cap L$ is finite-dimensional.

(a) Passing to a finite-codimensional subspace, we can take $\widetilde{M} \cap L = \{0\}$. Since there is an isomorphism $U : H \to L$, we have an isomorphism $T : L_1(\mu) = M \oplus H \to \widetilde{M} \oplus L \subset Y$ given by $T(x,y) := -Kx + Uy$. Then $T \in \mathcal{B}(L_1(\mu),Y)$ is tauberian and $N(T+K)$ is not reflexive because it contains M, which is isomorphic to ℓ_1.

(b) Since $\widetilde{M}$ is isomorphic to ℓ_1, we see that $\widetilde{M} \cap L$ is non-reflexive. Applying again the Kadec–Pełczyński Lemma, we can find a subspace $N_1 \subset \widetilde{M} \cap L$ isomorphic to ℓ_1 and complemented in L. Now, putting $M_1 := (K|_M)^{-1}(N_1)$, we find that M_1 is complemented in $L_1(\mu)$. So we can find closed subspaces $E \subset L$ and $H \subset L_1(\mu)$ such that $L_1(\mu) = M_1 \oplus H$ and $L = N_1 \oplus E$. Since $L_1(\mu)$ is primary, E and H are isomorphic to $L_1(\mu)$. Therefore, taking a bijective isomorphism $V : H \to E$, we can define a tauberian operator $T : L_1(\mu) = M_1 \oplus H \to N_1 \oplus E$ by $T(x,y) := -Kx + Vy$. Note that $N(T+K)$ is not reflexive because M_1 is contained in it.

(c) As in case (a) we can assume $\widetilde{M} \cap L = \{0\}$. Let $\imath : M \to L_1(\mu)$ be the natural embedding, and consider the quotient operator $q : Y \to Y/L$. Since $\widetilde{M}+L$ is not closed, the operator $q \circ K \circ \imath$ is not upper semi-Fredholm, and so there is a nuclear operator $K_1 : M \to Y/L$ such that the kernel $N(q \circ K \circ \imath + K_1)$ is infinite-dimensional [15, Lemma 4.3]. Note that K_1 can be written as $K_1 = q \circ K_2$, where $K_2 : M \to Y$ is a nuclear operator. Take the compact operator $Q : M \oplus H \to Y$ given by $Q(x,y) := K_2(x)$. Then $M \cap N(q \circ (K+Q))$ is infinite-dimensional, which implies that $(K+Q)(M) \cap L$ is also infinite-dimensional. Repeating the argument of part (b) we obtain a tauberian operator $T : L_1(\mu) \to Y$ such that $N(T+K+Q)$ is not reflexive. Since $T+Q$ is tauberian, the proof is finished. ■

Remark. In some cases $\mathrm{Ro}(X,Y)$ is not an admissible class for $\mathcal{T}(X,Y)$. For instance, let J denote the classical quasireflexive James space, and let c_0 be the space of all null sequences. The natural inclusion $\imath \in \mathcal{B}(J,c_0)$ is tauberian [7] and weakly precompact, but the null operator $0 \in \mathcal{B}(J,c_0)$ is not tauberian.

Herman [10] calls an operator $T \in \mathcal{B}(X,Y)$ *almost weakly compact* if given a closed subspace $H \subset X$ such that $T|_H$ is an isomorphism, H is reflexive.

PROPOSITION 15. *For an operator $T \in \mathcal{B}(L_1(\mu),Y)$, the following statements are equivalent*:

(1) *T is weakly precompact*;
(2) *T is R-strictly singular*;
(3) *T is almost weakly compact.*

Proof. (1)⇒(2). Suppose T is weakly precompact, and let $L \in \mathcal{B}(Z,L_1(\mu))$ be an operator such that TL is tauberian. Since the class of weakly precompact operators is an operator ideal, TL is weakly precompact as well. Moreover, for every compact operator $K \in \mathcal{B}(Z,Y)$, $N(TL+K)$ is reflexive. Thus, by [9, Theorem 2] we see that Z contains no copy of ℓ_1, and since $L_1(\mu)$ is weakly sequentially complete, the operator L is weakly compact.

(2)⇒(3). If $T|_H$ is an isomorphism, by hypothesis the embedding $\imath : H \to L_1(\mu)$ is weakly compact. Hence H is reflexive.

(3)⇒(1). If T is not weakly precompact, applying Rosenthal's ℓ_1-theorem [4, p. 201], we obtain a sequence $(f_n) \subset L_1(\mu)$ equivalent to the unit vector basis of ℓ_1 such that the restriction of T to the closed linear span of (f_n) is an isomorphism. ∎

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