**Références**

[1] I. P. Cornfeld, S. V. Fomin and Y. G. Sinai, *Ergodic Theory*, Springer, 1982.
[2] M. Lemańczyk, F. Parreau and J.-P. Thouvenot, *On the disjointness problem for Gaussian automorphisms*, preprint.
[3] M. Lemańczyk and F. Parreau, *Gaussian automorphisms whose self-joinings are Gaussian*, preprint.
[4] M. Lemańczyk and J. de Sam Lazaro, *Spectral analysis of certain compact factors for Gaussian dynamical systems*, Israel J. Math. (1996), à paraître.
[5] L. H. Loomis, *An Introduction to Abstract Harmonic Analysis*, Van Nostrand, Princeton, 1953.
[6] D. Newton, *On Gaussian processes with simple spectrum*, Z. Wahrsch. Verw. Gebiete 5 (1966), 207–209.
[7] D. Newton and W. Parry, *On a factor automorphism of a normal dynamical system*, Ann. Math. Statist. 37 (1966), 1528–1533.
[8] W. Parry, *Generators in Ergodic Theory*, Benjamin, New York, 1969.
[9] T. de la Rue, *Entropie d'un système dynamique gaussien : cas d'une action de* $\mathbb{Z}^d$, C. R. Acad. Sci. Paris Sér. I 317 (1993), 191–194.
[10] J. P. Thouvenot, *Some properties and applications of joinings in ergodic theory*, in: Ergodic Theory and its Connections with Harmonic Analysis, London Math. Soc. Lecture Note Ser. 205, Cambridge Univ. Press, 1995, 207–235.
[11] —, *Utilisation des processus gaussiens en théorie ergodique*, preprint.

Institute of Mathematics
Technical University of Wrocław
Wybrzeże Wyspiańskiego 27
50-370 Wrocław, Pologne
E-mail: iwanik@banach.im.pwr.wroc.pl

Department of Mathematics and Computer Science
Nicolas Copernicus University
Chopina 12/18
87-100 Toruń, Pologne
E-mail: mlem@mat.uni.torun.pl

Analyse et Modèles Stochastiques
URA CNRS 1378
Université de Rouen
76821 Mont-Saint-Aignan Cedex, France
E-mail: delarue@univ-rouen.fr
lazaro@univ-rouen.fr

*Received June 12, 1996*
*Revised version July 1, 1996*
(3690)

STUDIA MATHEMATICA 125 (3) (1997)

# Conical measures and properties of a vector measure determined by its range

by

L. RODRÍGUEZ-PIAZZA and M. C. ROMERO-MORENO (Sevilla)

**Abstract.** We characterize some properties of a vector measure in terms of its associated Kluvánek conical measure. These characterizations are used to prove that the range of a vector measure determines these properties. So we give new proofs of the fact that the range determines the total variation, the $\sigma$-finiteness of the variation and the Bochner derivability, and we show that it also determines the $(p,q)$-summing and $p$-nuclear norm of the integration operator. Finally, we show that Pettis derivability is not determined by the range and study when every measure having the same range of a given measure has a Pettis derivative.

**1. Introduction.** In [R1], answering a question in [AD], it was proved that the range of a vector measure determines its total variation; that is, if two measures with values in a Banach space have the same range, or even just ranges with the same closed convex hulls, then they have the same total variation. Later, in [R2], it was proved that the range also determines the Bochner derivability and the $\sigma$-finiteness of the variation. This suggests that these and other properties of a vector measure may depend on some structure only depending on the range. In this way we will use the conical measure associated with a vector measure introduced by I. Kluvánek in [K].

The symmetrization of the associated conical measure depends only on the range of the vector measure. If a property of a vector measure has a characterization in terms of the conical measure which is invariant under symmetrization, this property is determined by the range. In Section 2 we provide such a characterization for the total variation, the $\sigma$-finiteness of the variation, and the $(p,q)$-summing and $p$-nuclear norms of the integration operator.

1991 *Mathematics Subject Classification*: 46G10, 28B05, 47D50.
*Key words and phrases*: vector measures, range, conical measures, operator ideal norms, Pettis integral.
Research supported in part by DGICYT grant #PB93-0926.

In Section 3 we study the Pettis derivability of the measure. We first give an example showing that Pettis derivability is not determined by the range. We also establish that, when we are dealing with a measure of bounded variation, the associated conical measure is "localized" by a certain positive measure, unique under certain restrictions, whose symmetrization is determined by the range. We characterize in terms of this localization when every measure having this range has a Pettis derivative and when there exists at least one measure having this range with a Pettis derivative.

Throughout the paper our Banach space and vector measure terminology will follow [DU]. $X$ will be a (real) Banach space, $X^*$ will be its dual space and $B_X$ will stand for the closed unit ball of $X$. For us a *vector measure* will be a countably additive function $F:\Sigma\to X$ defined on a measurable space $(\Omega,\Sigma)$ with values in the Banach space $X$. $|F|$ will be the variation of $F$ which is a positive measure, and by $\|F\|$ we will denote the total variation of $F$, that is, $\|F\|=|F|(\Omega)$. Despite the fact that the variation may not be bounded there always exists a finite positive measure $\lambda$ satisfying $\lim_{\lambda(A)\to 0}F(A)=0$. Such a $\lambda$ is called a *control measure* for $F$.

We will consider the Bartle integral with respect to $F$ [DU, p. 6]. It can be viewed as an operator $I_F: L^\infty(\lambda)\to X$ defined on a simple function $f=\sum_{n=1}^N a_n\chi_{A_n}$ as

$$I_F(f)=\int f\,dF=\sum_{n=1}^N a_nF(A_n).$$

We know that $I_F$ is a weak*-weak continuous operator when $L^\infty(\lambda)$ is considered as the dual of $L^1(\lambda)$. Given $f\in L^\infty(\lambda)$ the vector measure $fF$ with density $f$ with respect to $F$ is defined by

$$(fF)(A)=\int_A f\,dF=\int f\chi_A dF\quad\text{for every } A\in\Sigma.$$

For the variation of $fF$ we have $|fF|(A)=\int_A|f|\,d|F|$ for every $A\in\Sigma$ (see [L, Th. 4.2]).

The range of $F$ will be denoted by $\operatorname{rg}F$, that is, $\operatorname{rg}F=\{F(A):A\in\Sigma\}$. This is a relatively weakly compact set in $X$ having $\frac12F(\Omega)$ as center of symmetry, and whose closed convex hull is given by $\overline{\mathrm{co}}(\operatorname{rg}F)=\{I_F(f):0\le f\le 1\}$ (see [DU, p. 263]). Usually we will be considering another vector measure $G:(\Omega',\Sigma')\to X$ perhaps defined on a different $\sigma$-algebra such that $\overline{\mathrm{co}}(\operatorname{rg}G)$ is a translate of $\overline{\mathrm{co}}(\operatorname{rg}F)$. As the ranges of $F$ and $G$ are relatively weakly compact and symmetric, this condition is equivalent to the identity $\overline{\mathrm{co}}(\operatorname{rg}F)-\overline{\mathrm{co}}(\operatorname{rg}F)=\overline{\mathrm{co}}(\operatorname{rg}G)-\overline{\mathrm{co}}(\operatorname{rg}G)$.

**2. Properties of a vector measure in terms of its associated conical measure.** In [K], Kluvánek introduced the conical measure associated with a vector measure to study the closed convex hull of the range of a vector measure. We will characterize some properties of a vector measure in terms of this conical measure. Let us recall some basic facts and definitions about conical measures in this context.



Given a Banach space $X$, $h(X)$ will stand for the smallest vector lattice of functions on $X$ with respect to the pointwise order and the linear operations that contains $X^*$. Every element of $h(X)$ can be written in the form

$$z^*=\bigvee_{i=1}^n x_i^*-\bigvee_{i=n+1}^m x_i^*,\tag{1}$$

where $x_i^*\in X^*$ for $i=1,\dots,m$, and $\vee$ (resp. $\wedge$) denotes the least upper (resp. greatest lower) bound in a lattice; which in this case is a pointwise maximum (resp. minimum) of functions.

A *conical measure* on $X$ is a positive linear functional on $h(X)$. The set of all conical measures over $X$ will be denoted $M^+(X)$. It is a complete lattice with respect to the order $v\le u$ iff $v(z^*)\le u(z^*)$ for every $z^*\in h(X)$, $z^*\ge 0$. We refer to [C, Sections 38–40] for these and more facts about conical measures. In particular, we have the Riesz decomposition [C, 10.5]: given $u_1$, $u_2$ and $v$ in $M^+(X)$ such that $v\le u_1+u_2$, there exist $v_1$ and $v_2$ in $M^+(X)$ with $v=v_1+v_2$, $v_1\le u_1$ and $v_2\le u_2$.

Given a conical measure in $X$, the *symmetrization* of $u$ is defined by $u^s=\frac12(u+\check u)$ where $\check u(z^*)=u(z^*\circ\sigma)$ with $\sigma(x)=-x$ for every $x\in X$. The *resultant* of a conical measure $u$, if it exists, is defined as the vector $r(u)$ in $X$ satisfying $u(x^*)=x^*(r(u))$ for all $x^*\in X^*$. If $r(v)$ exists for every $0\le v\le u$, the *zonoform* associated with $u$ is the set in $X$, $K_u=\{r(v):v\in M^+(X),\ 0\le v\le u\}$.

Let us now recall the construction of the Kluvánek conical measure. If $F:\Sigma\to X$ is a vector measure and $\lambda$ a control measure for $F$, it defines a linear map from $X^*$ to $L^1(\lambda)$ sending each $x^*$ to the Radon–Nikodym derivative $f_{x^*}=d(x^*\circ F)/d\lambda$. This map can be extended to a lattice homomorphism $\Phi_F$ from $h(X)$ to $L^1(\lambda)$ such that for every $z^*\in h(X)$ of the form (1),

$$\Phi_F(z^*)=f_{z^*}=\bigvee_{i=1}^n f_{x_i^*}-\bigvee_{i=n+1}^m f_{x_i^*}.$$

The *conical measure $u_F$ associated with $F$* is defined by

$$u_F(z^*)=\int_\Omega \Phi_F(z^*)\,d\lambda\quad\text{for every } z^*\in h(X).$$

Observe that the above definition does not depend on the control measure $\lambda$; in fact, the original definition of Kluvánek in [K] used the lattice structure of the set of real-valued measures defined on $\Sigma$ instead of that of $L^1(\lambda)$.

Kluvánek proved that the closed convex hull of the range of $F$ coincides with the zonoform associated with $u_F$, that is, $\overline{\text{co}}(\text{rg}\,F) = K_{u_F}$. In order to obtain this result he used the following proposition whose statement has been adapted for future reference. We refer to [K] for its proof.

PROPOSITION 2.1. *Let $F$ be a vector measure defined on $(\Omega, \Sigma)$ and $u_F$ its associated conical measure. If $v$ is a conical measure with $0 \le v \le u_F$, then there exists a measurable function $f : \Omega \to [0,1]$ such that $v = u_{fF}$.*

The range of a vector measure does not determine the associated conical measure; that is, there are examples of vector measures $F$ and $G$, even in finite dimension, with $\text{rg}\,F = \text{rg}\,G$ but $u_F \ne u_G$. However, the range determines the symmetrization of the conical measure. This is due to the identity $K_{u_F} + K_{\check{u}_F} = K_{u_F+\check{u}_F}$ (using Riesz decomposition), which implies

$$\overline{\text{co}}(\text{rg}\,F) - \overline{\text{co}}(\text{rg}\,F) = K_{u_F} + K_{\check{u}_F} = K_{u_F+\check{u}_F} = 2K_{u_F^s},$$

and the fact proved by Choquet that $K_u = K_v$ implies $u = v$ whenever $u$ and $v$ are symmetric conical measures (see [C, Vol. III, p. 52]). So we can state:

THEOREM 2.2. *Let $F$ and $G$ be two $X$-valued vector measures. Then $\overline{\text{co}}(\text{rg}\,F)$ is a translate of $\overline{\text{co}}(\text{rg}\,G)$ if and only if $u_F^s = u_G^s$.*

This theorem will be the key to proving properties determined by the range of a vector measure. We will characterize some properties of vector measures in terms of the associated conical measure. These characterizations turn out to be invariant under symmetrization, so these properties are determined by the range thanks to Theorem 2.2. We start the study with the total variation.

THEOREM 2.3. *A vector measure $F$ has finite variation if and only if there exists a constant $M > 0$ such that for every finite set $\{x_i^*\}_{i=1}^n \subseteq B_{X^*}$,*

$$u_F\Big(\bigvee_{i=1}^n |x_i^*|\Big) \le M. \tag{2}$$

*In this case, the total variation of $F$ is the infimum of all constants $M$ satisfying* (2).

Proof. Suppose that (2) holds. Given a partition of $\Omega$ into disjoint measurable sets $A_1, \dots, A_n$ choose $x_1^*, \dots, x_n^*$ such that $\|x_j^*\| = 1$ and $\langle x_j^*, F(A_j)\rangle = \|F(A_j)\|$. If $\lambda$ is a control measure for $F$ and $f_{x_j^*} = d(x_j^* \circ F)/d\lambda$, then



$$\sum_{j=1}^n \|F(A_j)\| = \sum_{j=1}^n \langle x_j^*, F(A_j)\rangle = \sum_{j=1}^n \int_{A_j} f_{x_j^*}\,d\lambda \le \int \sup_{1\le j\le n} |f_{x_j^*}|\,d\lambda = u_F\Big(\bigvee_{j=1}^n |x_j^*|\Big) \le M.$$

Consequently, $\|F\| \le M$.

Now suppose that $\|F\| < \infty$. Then given $x_1^*, \dots, x_n^*$ in $B_{X^*}$, one can choose disjoint measurable sets $\{A_j\}_{j=1}^n$ such that $|f_{x_j^*}|(\omega) = \sup_{k=1,\dots,n} |f_{x_k^*}|(\omega)$ for every $\omega \in A_j$. Then

$$u_F\Big(\bigvee_{j=1}^n |x_j^*|\Big) = \sum_{j=1}^n \int_{A_j} |f_{x_j^*}|\,d\lambda = \sum_{j=1}^n |x_j^* \circ F|(A_j) \le \sum_{j=1}^n |F|(A_j) \le \|F\|. \ \blacksquare$$

Let us agree that a conical measure $u$ has *bounded variation* if it satisfies condition (2) from the last theorem. We have the following result characterizing vector measures of $\sigma$-finite variation.

THEOREM 2.4. *Let $F$ be a vector measure with values in a Banach space $X$ and $u_F$ the conical measure associated with $F$. The following properties are equivalent*:

(a) *$F$ has $\sigma$-finite variation.*

(b) *There exists a sequence $\{u_n\}_n$ of conical measures of bounded variation such that $u_F(z^*) = \sum_{n=1}^\infty u_n(z^*)$ for every $z^* \in h(X)$.*

(c) *There exists a conical measure $v \le u_F$ of bounded variation such that for every $w \le u_F$, $w > 0$, we have $v \wedge w > 0$.*

Proof. (a)⇒(b). If $F$ has $\sigma$-finite variation, then $\Omega = \bigcup_{n=1}^\infty A_n$ (pairwise disjoint) with $|F|(A_n) < \infty$. If $u_n = u_{\chi_{A_n}F}$, it is clear that the sequence $\{u_n\}_n$ works.

(b)⇒(c). Let $u_F = \sum_{n=1}^\infty u_n = \lim_n v_n$ where $v_n = \sum_{j=1}^n u_j$. By hypothesis, there exist constants $M_n < \infty$ such that $v_n(\bigvee_{i=1}^k |x_i^*|) \le M_n$ for every finite set $\{x_i^*\}_{i=1}^k \subseteq B_{X^*}$. If $0 < w \le u_F$, there exist $n$ such that $w \wedge v_n > 0$ because $w = w \wedge u_F = w \wedge (\sup_n v_n) = \sup_n (w \wedge v_n)$. If we take a sequence $(\alpha_n)_n$ of positive numbers such that $\sum_n \alpha_n \le 1$ and $\sum_n \alpha_n M_n < \infty$, then $v = \sum_n \alpha_n v_n$ satisfies $v \le u_F$ and $v$ is of bounded variation. If $w \wedge v = 0$, then $w \wedge (\alpha_n v_n) = 0$ for every $n$, so $w \wedge v_n = 0$, which is a contradiction.

(c)⇒(a). Suppose that $0 \le v \le u_F$ with $v$ of bounded variation such that for all $w \le u_F$ with $w > 0$ we have $w \wedge v > 0$. If $\lambda$ is a control measure for $F$, we are going to prove that if $A \in \Sigma$ with $\lambda(A) > 0$, then there exists $B \subseteq A$ with $0 < \lambda(B)$ and $|F|(B) < \infty$. As an easy consequence of the Exhaustion Lemma [DU, p. 70], this implies that there exists a sequence $(A_n)$ of pairwise disjoint members of $\Sigma$ such that $\Omega = \bigcup_{n=1}^\infty A_n$ and $|F|(A_n) < \infty$.

Take $w = u_{\chi_A F}$; we can suppose $|F|(A) > 0$ and so $w > 0$. Then $0 < w \wedge v \le w = u_{\chi_A F}$ and, again by Proposition 2.1 this time applied to the vector measure $\chi_A F$, there exists $h$ measurable with $0 \le h \le 1$ such that $w \wedge v = u_{h\chi_A F}$. As $0 < u_{h\chi_A F} \le v$, we conclude by Theorem 2.3 that $h\chi_A F$ has finite variation, and it is easy to obtain $B \subseteq A$ with $0 < |F|(B) < \infty$. ∎

COROLLARY 2.5. *Let $F$ and $G$ be two vector measures such that $\overline{\mathrm{co}}(\mathrm{rg}\, F)$ is a translate of $\overline{\mathrm{co}}(\mathrm{rg}\, G)$. Then:*

(i) $\|F\| = \|G\|$.

(ii) *$F$ has $\sigma$-finite variation if and only if $G$ has $\sigma$-finite variation.*

Proof. (i) is an easy consequence of Theorem 2.2 and the fact that for every $\{x_i^*\}_{i=1}^n \subseteq B_{X^*}$ we have $u_F(\bigvee_{i=1}^n |x_i^*|) = u_F^s(\bigvee_{i=1}^n |x_i^*|)$.

To show (ii) we are going to prove that condition (c) from the last theorem is invariant under symmetrization. First suppose that there exists $v \le u_F$ of bounded variation such that $w \wedge v > 0$ for all $w \le u_F$, $w > 0$. Take $v^s = \frac{1}{2}(v + \check{v}) \le u_F^s$, which obviously has bounded variation. If $w \le u_F^s$ and $w > 0$ then by the Riesz decomposition, $w = w_1 + w_2$ with $w_1 \le u_F/2$ and $w_2 \le \check{u}_F/2$. Either $w_1 > 0$ or $w_2 > 0$. Suppose that $w_1 > 0$. Then

$$0 < \frac{1}{2}(w_1 \wedge v) \le \frac{1}{2}(w_1 \wedge (v + \check{v})) = \frac{w_1}{2} \wedge v^s \le w_1 \wedge v^s \le w \wedge v^s.$$

If $w_2 > 0$, we also obtain $w \wedge v^s > 0$ because in this case $w_2 \wedge \check{v} > 0$. Conversely, suppose that there exists $v \le u_F^s$ of bounded variation such that $0 \le w \le u_F^s$ implies $w \wedge v > 0$. If $v_1 = v \wedge u_F$, then for every $w > 0$ with $w \le u_F$, we have $w \wedge v_1 = w \wedge (u_F \wedge v) = w \wedge v \ge \frac{w}{2} \wedge v > 0$. Therefore $w \wedge v_1 > 0$ and $u_F$ satisfies (c) of the last theorem. ∎

Another way of looking at the case of finite variation is that $\|F\| < \infty$ if and only if the integration operator $I_F$ is 1-summing and in this case $\pi_1(I_F) = \|F\|$ [DU, p. 162]. This result suggests proving an analogous result for the $(p,q)$-summing norm and the characterization in terms of the conical measure. We refer to [DJT] for properties of $(p,q)$-summing operators.

If $1 \le q, p < \infty$, it is known (see [DJT, p. 330]) that an operator $T : L^\infty(\lambda) \to X$ is $(p,q)$-summing if and only if there exists a constant $M < \infty$ such that the inequality

$$\Big(\sum_{i=1}^n \|Tf_i\|^p\Big)^{1/p} \le M \Big\| \Big(\sum_{i=1}^n |f_i|^q\Big)^{1/q} \Big\|_\infty \tag{3}$$

holds for every positive integer $n$ and for all $f_1, \dots, f_n \in L^\infty(\lambda)$. The least such $M$ is $\pi_{(p,q)}(T)$, the *$(p,q)$-summing norm* of $T$. In the following theorem we characterize the $(p,q)$-summing norm of the integration operator $I_F$ in terms of the conical measure $u_F$. Let $p'$ and $q'$ be the conjugate exponents



of $p$ and $q$ respectively; that is, $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$, and let $\ell_p^n$ denote as usual $\mathbb{R}^n$ with the $\ell_p$ norm.

THEOREM 2.6. *The operator $I_F$ is $(p,q)$-summing if and only if there exists a constant $M < \infty$ such that*

$$u_F\Big(\bigvee_{j=1}^m \Big|\sum_{i=1}^n a_i^j x_i^*\Big|\Big) \le M \tag{4}$$

*for every scalar matrix $((a_i^j)_{i=1}^n)_{j=1}^m$ such that $\sum_{i=1}^n |a_i^j|^q \le 1$ for every $j$, and every family $x_1^*, \dots, x_n^*$ in $X^*$ with $(\|x_i^*\|)_{i=1}^n \in B_{\ell_{p'}^n}$. In this case, $\pi_{(p,q)}(I_F) = \inf M$ such that (4) holds.*

Proof. $I_F$ is $(p,q)$-summing if and only if there exists a constant $K > 0$ such that, for every finite sequence $x_1^*, \dots, x_n^*$ in $X^*$,

$$\Big\|\Big(\sum_{i=1}^n |f_{x_i^*}|^{q'}\Big)^{1/q'}\Big\|_1 \le K\Big(\sum_{i=1}^n \|x_i^*\|^{p'}\Big)^{1/p'}.$$

And $\pi_{(p,q)}(I_F) = \inf K$ such that the last inequality holds. To see this, suppose first that $I_F$ is $(p,q)$-summing. Since $(L^1(\ell_{q'}^n))^* = L^\infty(\ell_q^n)$, we have

$$\begin{aligned}
&\Big\|\Big(\sum_{i=1}^n |f_{x_i^*}|^{q'}\Big)^{1/q'}\Big\|_1 \\
&\quad = \Big\|\Big(\sum_{i=1}^n |I_F^* x_i^*|^{q'}\Big)^{1/q'}\Big\|_1 = \sup\Big\{\sum_{i=1}^n |\langle I_F^* x_i^*, g_i\rangle| : 1 \ge \Big\|\sum_{i=1}^n |g_i|^q\Big\|_\infty\Big\} \\
&\quad \le \sup\Big\{\Big(\sum_{i=1}^n \|x_i^*\|^{p'}\Big)^{1/p'}\Big(\sum_{i=1}^n \|I_F g_i\|^p\Big)^{1/p} : 1 \ge \Big\|\sum_{i=1}^n |g_i|^q\Big\|_\infty\Big\} \\
&\quad \le \pi_{(p,q)}(I_F)\Big(\sum_{i=1}^n \|x_i^*\|^{p'}\Big)^{1/p'}.
\end{aligned}$$

The proof of the reverse implication is similar. To finish it is enough to note that

$$\Big\|\Big(\sum_{i=1}^n |f_{x_i^*}|^{q'}\Big)^{1/q'}\Big\|_1 = \sup\Big\{u_F\Big(\bigvee_{j=1}^m \Big|\sum_{i=1}^n a_i^j x_i^*\Big|\Big) : \sum_{i=1}^n |a_i^j|^q \le 1\Big\}. \ ∎$$

Remark. Since $L^\infty(\lambda)$ is a $\mathcal{C}(K)$ space, the $p$-summing norm of $I_F$ coincides with its $p$-integral norm $i_p(I_F)$ [DJT, Corollary 5.8]. Therefore the previous theorem, with $p = q$, is a characterization of the $p$-integral norm of $I_F$.

The Bochner differentiability of a measure is another property that can be expressed in terms of operator ideal norms of the integration operator;

in fact, $F$ is Bochner differentiable with respect to its variation if and only if $I_F$ is nuclear. In the following theorem we characterize, in terms of the conical measure $u_F$, when $I_F$ is $p$-nuclear, and its $p$-nuclear norm $\nu_p(I_F)$. See [DJT] for the definition of $\nu_p$ and recall that the nuclear operators are just the 1-nuclear operators.

Let us introduce some notation. If $C$ is a bounded absolutely convex closed subset of $X$, let $X_C$ be the linear subspace generated by $C$, that is, $X_C = \{\lambda x : x \in C,\ \lambda > 0\}$. Provided with the Minkowski functional $\|\cdot\|_C$ of $C$, $X_C$ becomes a Banach space such that we have the continuous inclusion $X_C \to X$. For every $x^* \in X^*$ set $\|x^*\|_C = \sup_{x\in C}|x^*(x)|$; it is an easy consequence of the Hahn–Banach Theorem that for every $x \in X$ we have $\|x\|_C = \sup\{|x^*(x)| : x^* \in X^*,\ \|x^*\|_C \le 1\}$, where we understand $\|x\|_C = \infty$ whenever $x \notin X_C$. If $T$ is an operator such that $\operatorname{rg} T \subseteq X_C$, we can consider $T$ as having values in $X_C$ and in this case we denote it by $T^C$.

THEOREM 2.7. *The operator $I_F$ is $p$-nuclear if and only if there exists an absolutely convex compact set $K \subseteq B_X$ and a constant $M < \infty$ such that*

$$u_F\Big(\bigvee_{j=1}^{m}\Big|\sum_{i=1}^{n} a_i^j x_i^*\Big|\Big) \le M \tag{5}$$

*for every scalar matrix $((a_i^j)_{i=1}^n)_{j=1}^m$ such that $\sum_{i=1}^n |a_i^j|^p \le 1$ for every $j$, and every family $x_1^*,\dots,x_n^*$ in $X^*$ with $(\|x_i^*\|_K)_{i=1}^n \in B_{\ell_{p'}^n}$. Moreover, $\nu_p(I_F) = \inf M$ such that* (5) *holds for some $K \subseteq B_X$.*

Proof. With the above notation, it was proved in [R2, Theorem 5.2] that if $I_F$ is nuclear, then there exists an absolutely convex compact subset $K$ of $X$ such that $\operatorname{rg} F$ is contained in $X_K$ and $F$ has finite variation when considered as an $X_K$-valued measure. Following the same steps one can see that if $T : Y \to X$ is a $p$-nuclear operator, then there exists an absolutely convex compact set $K \subset B_X$ with $T(Y) \subset X_K$, and $T^K$ being $p$-integral. Moreover, for every $\varepsilon > 0$, $K$ can be chosen so that $i_p(T^K) \le \nu_p(T) + \varepsilon$. The converse is true: if we have $T(Y) \subset X_K$ and $T^K$ is $p$-integral for a certain compact $K \subset B_X$, then $T$ is the composition of $T^K$, a $p$-integral operator, and a compact operator, the inclusion $X_K \to X$. So $T$ is $p$-nuclear and $\nu_p(T) \le i_p(T^K)\|X_K \to X\| \le i_p(T^K)$ [DJT, Theorem 5.27].

When (5) is satisfied we have, for $\|x^*\|_K \le 1$ and $g \in L^\infty(\lambda)$,

$$|\langle I_F g, x^*\rangle| = \Big|\int g I_F^* x^*\,d\lambda\Big| \le \|g\|_\infty u_F(|x^*|) \le M\|g\|_\infty.$$

So $I_F$ is $X_K$-valued. Now, following the proof of Theorem 2.6, it is easy to see that (5) is equivalent to $I_F^K$ being $p$-integral with $i_p(I_F^K) = \inf M$, $M$ admissible in (5). The considerations above imply that $\nu_p(I_F) = \inf\{i_p(I_F^K)\}$, where the infimum extends over all compact $K \subset B_X$ for which $I_F^K$ is $p$-integral. The theorem follows. ∎

Remark. In fact, once we know that $I_F$ is $p$-nuclear, we do not have to worry about characterizing the exact value of $\nu_p(I_F)$, since we have characterized $i_p(I_F)$ in Theorem 2.6 and $\nu_p(I_F) = i_p(I_F)$ when $I_F$ is $p$-nuclear. This is a consequence of the facts that the finite rank operators are dense in the space of $p$-nuclear operators and that the dual of $L^\infty(\lambda)$ has the metric approximation property. For $p = 1$, it is proved in [P, p. 132] that, in this situation, the nuclear and integral norms concide for finite rank operators. The argument there can be adapted for general $p$, but we have not found a precise reference.

As conditions (4) and (5) are invariant under symmetrization, we obtain

COROLLARY 2.8. *If $F$ and $G$ are two vector measures such that $\overline{\operatorname{co}}(\operatorname{rg} F)$ is a translate of $\overline{\operatorname{co}}(\operatorname{rg} G)$, then:*

(1) *$I_F$ is $(p,q)$-summing if and only if $I_G$ is, and $\pi_{(p,q)}(I_F) = \pi_{(p,q)}(I_G)$.*

(2) *$I_F$ is $p$-nuclear if and only if $I_G$ is, and $\nu_p(I_F) = \nu_p(I_G)$. In particular, $F$ has a Bochner derivative if and only if $G$ does.*

**3. Pettis derivability.** In this section we will study the existence of a Pettis derivative of a vector measure in terms of the conical measure; more concretely, this will be done in terms of a localization of the conical measure which will be given in Theorem 3.2.

Let us recall that a vector measure $F : (\Omega,\Sigma) \to X$ is *Pettis differentiable* with respect to a control measure $\mu$ if there exists a function $\varphi : \Omega \to X$ such that $x^*\varphi \in L^1(\mu)$ for every $x^* \in X^*$, and we have

$$\langle x^*, F(A)\rangle = \int_A x^*\varphi\,d\mu \quad \text{for every } x^* \in X^* \text{ and every } A \in \Sigma.$$

With the help of the Radon–Nikodym theorem for positive measures one can see that if a vector measure is Pettis differentiable with respect to some control measure then it is Pettis differentiable with respect to any control measure. A Pettis differentiable measure turns out to have $\sigma$-finite variation (see [M]).

In this section we will consider only vector measures of bounded variation. The case of $\sigma$-finite variation can be reduced to this one by using Corollary 2.2 of [R2]. If $F$ has $\sigma$-finite variation and $\overline{\operatorname{co}}(\operatorname{rg} G)$ is a translate of $\overline{\operatorname{co}}(\operatorname{rg} F)$, then this result allows us to decompose $F = \sum \varphi_n F$ and $G = \sum \psi_n G$, where $\varphi_n F$ and $\psi_n G$ have bounded variation and $\overline{\operatorname{co}}(\operatorname{rg} \psi_n G)$ is a translate of $\overline{\operatorname{co}}(\operatorname{rg} \varphi_n F)$, for every $n$. In particular, the fact of having a strongly measurable Pettis derivative is determined by the range since $F$ has such a derivative if and only if each $\varphi_n F$ is Bochner derivable, and we know from Corollary 2.8 that the range determines the existence of a Bochner derivative.

Unfortunately, Pettis derivability is not determined by the range, as shown in the following example which is a remark on a construction of Fremlin and Talagrand [FT]. Recall that a finite measure space $(\Omega, \Sigma, \mu)$ is called *perfect* if for every measurable function $f : \Omega \to \mathbb{R}$ and for every $A \subset \mathbb{R}$ such that $f^{-1}(A) \in \Sigma$, there exists a Borel set $B$ in $\mathbb{R}$ such that $B \subset A$ and $\mu(f^{-1}(A)) = \mu(f^{-1}(B))$. Then the same is true if we replace $\mathbb{R}$ by any separable metric space. An example of a perfect measure is a Radon measure. See [T, 1-3] for these facts.

EXAMPLE 3.1. *There exist two vector measures of bounded variation with values in $\ell_\infty$ having the same ranges such that one of them is Pettis derivable and the other is not.*

Proof. In [FT] an example is given of a Pettis integrable function whose indefinite integral has a range which is not relatively norm-compact. Let us describe roughly this example (see [T, 4-2-5 and 13-2-1]). Consider the compact set $\Omega = \{0,1\}^{\mathbb{N}}$, the $\sigma$-algebra $\mathcal{B}$ of Borel sets of $\Omega$ and the probability $\mu$ on $\mathcal{B}$ which is the product of the probability $\frac{1}{2}(\delta_0 + \delta_1)$ taken on each factor $\{0,1\}$.

There exist a $\sigma$-algebra $\Sigma$ on $\Omega$ containing $\mathcal{B}$, a probability $\overline{\mu}$ on $\Sigma$ extending $\mu$ and a bounded function $\varphi : \Omega \to \ell_\infty$ which is Pettis integrable with respect to $\overline{\mu}$ such that if $F$ is the indefinite integral of $\varphi$, then $\operatorname{rg} F$ is not relatively norm compact. $F$ has bounded variation since $\varphi$ is bounded and $\overline{\mu}$ is a probability.

Moreover, every set in $\Sigma$ is $\overline{\mu}$-equivalent to a set in $\mathcal{B}$; that is, for every set $M \in \Sigma$ there exists $B \in \mathcal{B}$ such that $\overline{\mu}(B \triangle M) = 0$; $\overline{\mu}$ being a control measure for $F$, this implies that $F(M) = F(B)$. So, if we let $G$ be the restriction of $F$ to $\mathcal{B}$, we have $\operatorname{rg} F = \operatorname{rg} G$.

Now, $F$ is Pettis derivable but $G$ is not. Indeed, the variation of $G$ is a finite measure on the Borel $\sigma$-algebra of a compact metric space and so it is a Radon measure; hence $(\Omega, \Sigma, |G|)$ is a perfect measure space and, by a result due to Stegall [T, 4-1-6], if $G$ were Pettis derivable with respect to $|G|$, then its range would be relatively compact. ■

Remark. In the references given for the previous example, the function $\varphi$ is defined on $\Omega \times \Omega$. This is not relevant, for there is a homeomorphism from $\Omega \times \Omega$ to $\Omega$ sending $\mu \otimes \mu$ to $\mu$ and the above description is the example in [T, 4-2-5] via this homeomorphism.

The last example also shows that we cannot characterize Pettis derivability in terms of the conical measure. Both vector measures in this example have the same associated conical measure. But in terms of this conical mea-



sure we can characterize the existence of at least one Pettis derivable vector measure having the same range and when every measure having the same range has a Pettis derivative. All these characterizations will be done in terms of a localization of the conical measure. We will use the term "localization of a conical measure" in a more general sense than in [C, 30.4]. By a *localization* of $u$ we will understand a positive measure $\mu$ such that $u(z^*) = \int z^* \, d\mu$ for every $z^* \in h(X)$. It is not always possible to find a localization $\mu$ defined on $X$ for a conical measure of bounded variation. But this is possible if we look for $\mu$ defined on $X^{**}$. Observe that every element of $h(X)$ can be extended (univocally) as a continuous function on $X^{**}$ for the weak* topology.

Recall that the smallest $\sigma$-algebra on a topological space $T$ making measurable every real-valued continuous function is called the *Baire $\sigma$-algebra* of $T$. If we consider the weak topology in $X$ or the weak* topology in $X^{**}$, the Baire $\sigma$-algebras, denoted respectively by $\mathrm{Ba}(X, w)$ and $\mathrm{Ba}(X^{**}, w^*)$, turn out to be the $\sigma$-algebras generated respectively in $X$ and in $X^{**}$ by the functions in $X^*$ [T, 2-2-4].

If $F$ has bounded variation the integration operator defined on $L^\infty(|F|)$ can be extended to $L^1(|F|)$ as a norm one operator, so there exists a weak* density for $F$ (see [T, 7-1-2]); that is, there exists a map $\varphi : \Omega \to B_{X^{**}}$ such that

$$x^* \circ \varphi = \frac{d(x^* \circ F)}{d|F|} \quad |F|\text{-a.e., for every } x^* \in X^*. \tag{6}$$

As we can assume that $(\Omega, \Sigma, |F|)$ is a complete measure space (if not one can consider its completion), this function $\varphi$ is measurable from $(\Omega, \Sigma)$ to $(X^{**}, \mathrm{Ba}(X^{**}, w^*))$. In the following theorem we use this function to establish the existence of a localization of the conical measure and we give some conditions for its uniqueness.

If $\mu$ is a measure on $\mathrm{Ba}(X^{**}, w^*)$, then $\mu^s$ will denote its symmetrization defined as $\mu^s(A) = \frac{1}{2}(\mu(A) + \mu(-A))$ for every $A \in \mathrm{Ba}(X^{**}, w^*)$. Let us remark that in general the unit ball of $X^{**}$ is not Baire measurable for $w^*$. This is the reason why we have to consider the outer measure $\mu_F^*(B_{X^{**}})$ in the following theorem.

THEOREM 3.2. *Let $F$ be an $X$-valued measure of bounded variation. There exist a unique positive measure $\mu_F$ and an $X$-valued vector measure $R_F$ defined on $\mathrm{Ba}(X^{**}, w^*)$ such that:*

(a) $\|F\| = \mu_F(X^{**}) = \mu_F^*(B_{X^{**}})$.

(b) *For every $z^* \in h(X)$,*

$$u_F(z^*) = \int_{X^{**}} z^*(x^{**}) \, d\mu_F(x^{**}).$$

(c) *For every* $x^* \in X^*$ *and every* $A \in \mathrm{Ba}(X^{**}, w^*)$,

$$\langle x^*, R_F(A)\rangle = \int_A x^*(x^{**})\,d\mu_F(x^{**}).$$

(d) $|R_F| = \mu_F$ *and* $\overline{\mathrm{co}}(\mathrm{rg}\,F) = \overline{\mathrm{co}}(\mathrm{rg}\,R_F)$.

*Moreover, if* $G$ *is another* $X$*-valued measure of bounded variation, then* $\overline{\mathrm{co}}(\mathrm{rg}\,F)$ *is a translate of* $\overline{\mathrm{co}}(\mathrm{rg}\,G)$ *if and only if* $\mu_F^{\mathrm{s}} = \mu_G^{\mathrm{s}}$.

Proof. The existence of $\mu_F$ could be proved using a result of E. Thomas [Th, Theorem 16] about localization of conical measures; but for our purposes it will be more interesting to use a weak* density $\varphi$ with values in $B_{X^{**}}$. We can define the measure $\mu_F$ as the image measure of $|F|$ by $\varphi$: $\mu_F(A) = |F|(\varphi^{-1}(A))$ for every $A \in \mathrm{Ba}(X^{**}, w^*)$. Clearly, as $\varphi$ is $B_{X^{**}}$-valued we have (a). To prove (b), considering $|F|$ as a control measure for $F$, take $z^* \in h(X)$ as in (1). Then we have, thanks to (6),

$$u_F(z^*) = \int\Big(\bigvee_{i=1}^{n} x_i^*\circ\varphi - \bigvee_{i=n+1}^{m} x_i^*\circ\varphi\Big)\,d|F| = \int z^*\circ\varphi\,d|F| = \int z^*\,d\mu_F.$$

Taking the image measure by the same weak* density $\varphi$ we can transfer not only the variation of $F$ but even the vector measure itself. So define the vector measure $R_F$ as $R_F(A) = F(\varphi^{-1}(A))$ for every $A \in \mathrm{Ba}(X^{**}, w^*)$. It is clear that

$$\langle x^*, R_F(A)\rangle = \int_A x^*\,d\mu_F \quad \text{for every } A \in \mathrm{Ba}(X^{**}, w^*) \text{ and } x^* \in X^*.$$

Therefore we can view $R_F$ as the indefinite weak* integral of the identity map in $X^{**}$ with respect to $\mu_F$. To prove (d), for every $x^* \in X^*$ we have, for $A_{x^*} = \{x^* \ge 0\}$,

$$\begin{aligned}\sup\{x^*(F(A)) : A \in \Sigma\} &= \int_{\{x^*\varphi\ge 0\}} x^*\varphi\,d|F| = \int_{A_{x^*}} x^*\,d\mu_F\\ &= \sup\{x^*(R_F(A)) : A \in \mathrm{Ba}(X^{**}, w^*)\}.\end{aligned}$$

We also have $|R_F| = \mu_F$. Indeed, for every $A \in \mathrm{Ba}(X^{**}, w^*)$, we have

$$\|R_F(A)\| = \|F(\varphi^{-1}(A))\| \le |F|(\varphi^{-1}(A)) = \mu_F(A),$$

implying $|R_F|(A) \le \mu_F(A)$ for every $A$. We know that $\overline{\mathrm{co}}(\mathrm{rg}\,F) = \overline{\mathrm{co}}(\mathrm{rg}\,R_F)$ and so, by Corollary 2.5, $|R_F|(X^{**}) = \|F\| = \mu_F(X^{**})$. Then $|R_F|$ and $\mu_F$ coincide in a total set; since $\mu_F$ is always greater they must be equal.

Next we prove the uniqueness of $\mu_F$. Suppose $\mu_1$ and $\mu_2$ are two measures satisfying (a) and (b). Let $\mathcal{B}$ be the Baire $\sigma$-algebra of $B_{X^{**}}$ for the weak* topology; this is the trace in $B_{X^{**}}$ of the $\sigma$-algebra $\mathrm{Ba}(X^{**}, w^*)$; that is, for every $A \in \mathcal{B}$, there exists $A' \in \mathrm{Ba}(X^{**}, w^*)$ such that $A = A' \cap B_{X^{**}}$. As $B_{X^{**}}$ has full outer measure for $\mu_1$ and $\mu_2$, if we put $\nu_j(A) = \mu_j(A')$, $j = 1, 2$, we have two well-defined measures in $\mathcal{B}$ that still satisfy for every $z^* \in h(X)$,

$$u_F(z^*) = \int_{B_{X^{**}}} z^*\,d\nu_1 = \int_{B_{X^{**}}} z^*\,d\nu_2. \tag{7}$$

If we show $\nu_1 = \nu_2$, we will have $\mu_1 = \mu_2$. To see that $\nu_1 = \nu_2$, it is enough to prove that $\int f\,d\nu_1 = \int f\,d\nu_2$ for every continuous function on $B_{X^{**}}$, and we have to check this only on a dense set in $\mathcal{C}(B_{X^{**}})$. By (7) and the dominated convergence theorem, it is clear that $\int f\,d\nu_1 = \int f\,d\nu_2$ for every continuous function $f$ that is the $(\nu_1+\nu_2)$-almost everywhere limit of a uniformly bounded sequence in $h(X)$. The set $E$ of all those functions is a vector sublattice of $\mathcal{C}(B_{X^{**}})$ which contains $h(X)$, in particular, it separates the points of $B_{X^{**}}$. If we prove that $E$ contains the constants, then by the Stone–Weierstrass theorem for lattices, it will be dense in $\mathcal{C}(B_{X^{**}})$. Recall that $\nu_1(B_{X^{**}}) = \nu_2(B_{X^{**}}) = \|F\|$ and use Theorem 2.3 to obtain, for every positive $\varepsilon$, a finite set $x_1^*, \ldots, x_k^*$ in the unit ball of $X^*$ such that $u_F(\bigvee_{j=1}^k |x_j^*|) \ge \|F\| - \varepsilon$. This gives us a function $z^*$ in $h(X)$ such $0 \le z^* \le 1$ in $B_{X^{**}}$ and $\int z^*\,d\nu_j \ge \int 1\,d\nu_j - \varepsilon$. We can then produce an increasing sequence $(z_n^*)$ in $h(X)$ such that $0 \le z_n^* \le 1$ in $B_{X^{**}}$ for every $n$, and

$$\lim_{n\to\infty}\int z_n^*\,d(\nu_1+\nu_2) = \int 1\,d(\nu_1+\nu_2).$$

This implies that the constant function 1 belongs to $E$, finishing the proof of the uniqueness of $\mu_F$.

To prove the last assertion, it is clear that $u_F^{\mathrm{s}}(z^*) = \int z^*\,d\mu_F^{\mathrm{s}}$ for every $z^* \in h(X)$. Then $\mu_F^{\mathrm{s}} = \mu_G^{\mathrm{s}}$ implies $u_F^{\mathrm{s}} = u_G^{\mathrm{s}}$ and it is enough to use Theorem 2.2 to deduce that $\overline{\mathrm{co}}(\mathrm{rg}\,F)$ is a translate of $\overline{\mathrm{co}}(\mathrm{rg}\,G)$. For the reverse implication, the same theorem will give us $u_F^{\mathrm{s}} = u_G^{\mathrm{s}}$; to see that this implies $\mu_F^{\mathrm{s}} = \mu_G^{\mathrm{s}}$ use the same arguments as in the proof of the uniqueness of $\mu_F$. ■

Remark. The measure $\mu_F$ has already been used to study properties of weak densities and representation of operators. For example, Edgar [E] proved that $F$ has a Bochner derivative if and only if there exists a separable subspace $H$ of $X$ such that $\mu_F^*(H) = \|F\|$. As a consequence of this characterization we obtain a new proof of the fact that Bochner derivability is determined by the range, since, $H$ being a symmetric set, the condition $\mu_F^*(H) = \|F\|$ still holds if we replace $\mu_F$ by $\mu_F^{\mathrm{s}}$.

If we consider another weak* density $\psi$, even if it takes values off $B_{X^{**}}$, it will produce the same image measure $\mu_F$ since we will have $x^*\circ\varphi = x^*\circ\psi$ $|F|$-almost everywhere on $\Omega$ for every $x^* \in X^*$, and this implies that $\varphi^{-1}(A)$ and $\psi^{-1}(A)$ are $|F|$-equivalent sets for every $A \in \mathrm{Ba}(X^{**}, w^*)$. If there exists a Pettis derivative of $F$, one can use it as such a function $\psi$. We use this fact in the following proposition.

Proposition 3.3. *Let $F$ be an $X$-valued measure of bounded variation. There exists a vector measure $G$ with a Pettis derivative such that $\overline{\text{co}}(\text{rg}\, G)$ is a translate of $\overline{\text{co}}(\text{rg}\, F)$ if and only if $\mu_F^*(X) = \|F\|$.*

Proof. Suppose $G$ is Pettis derivable and $\overline{\text{co}}(\text{rg}\, G)$ is a translate of $\overline{\text{co}}(\text{rg}\, F)$. Then if $\psi$ is a Pettis derivative of $G$ with respect to $|G|$, as mentioned before, one can use $\psi$ to define $\mu_G$ and then $\psi^{-1}(A)$ is a total set for every $A \in \text{Ba}(X^{**}, w^*)$ containing $X$. So $\mu_G^*(X) = \|G\| = \|F\|$. Since $X$ is a symmetric set and $\mu_F^{\text{s}} = \mu_G^{\text{s}}$ we also get $\mu_F^*(X) = \|F\|$.

For the converse implication, if $X$ has full outer measure for $\mu_F$, then as $\text{Ba}(X, w)$ is the trace in $X$ of $\text{Ba}(X^{**}, w^*)$, putting $G(A \cap X) = R_F(A)$ and $\nu(A \cap X) = \mu_F(A)$ for every $A \in \text{Ba}(X^{**}, w^*)$, we define a vector measure $G$ and a positive measure $\nu$ on $\text{Ba}(X, w)$. Since $G$ has the same range as $R_F$, we have $\overline{\text{co}}(\text{rg}\, G) = \overline{\text{co}}(\text{rg}\, F)$. It is easy to see that $\nu = |G|$ since $|R_F| = \mu_F$. Finally, the identity map in $X$ is a Pettis derivative of $G$ with respect to $\nu$ because

$$\int_{A\cap X} x^*\, d\nu = \int_X \chi_A x^*\, d\nu = \int_{X^{**}} \chi_A x^*\, d\mu_F = \langle x^*, R_F(A)\rangle = \langle x^*, G(A \cap X)\rangle$$

for every $A \in \text{Ba}(X^{**}, w^*)$ and every $x^* \in X^*$. ∎

Proposition 3.4. *Let $F$ be an $X$-valued measure of bounded variation. The following properties are equivalent*:

(a) *Every vector measure $G$ such that $\overline{\text{co}}(\text{rg}\, G)$ is a translate of $\overline{\text{co}}(\text{rg}\, F)$ has a Pettis derivative.*

(b) *$R_F$ has a Pettis derivative.*

(c) *There exists a function $\psi : X^{**} \to X$ such that, for every $x^* \in X^*$, $x^* \circ \psi = x^*$ $\mu_F$-almost everywhere in $X^{**}$.*

Proof. Of course (a) implies (b) because $\overline{\text{co}}(\text{rg}\, F) = \overline{\text{co}}(\text{rg}\, R_F)$. It is also easy to see that (b) and (c) are equivalent: a Pettis derivative of $R_F$ with respect to its variation $\mu_F$ is exactly a function $\psi$ as in (c) since, for every $x^* \in X^*$, we have $x^* = d(x^* \circ R_F)/d\mu_F$.

To prove that (c) implies (a), we first prove that (c) implies $F$ is Pettis derivable. Let $\varphi : \Omega \to X^{**}$ be the weak* density used to define $\mu_F$ and $R_F$. Then $\psi \circ \varphi$ is a Pettis derivative of $F$ with respect to $|F|$ since $x^* \circ \psi = x^*$ $\mu_F$-almost everywhere in $X^{**}$ implies

$$x^* \circ \psi \circ \varphi = x^* \circ \varphi = \frac{d(x^* \circ F)}{d|F|} \quad |F|\text{-a.e., for every } x^* \in X^*.$$

The only thing we have to prove is that (c) for $\mu_F$ implies (c) for $\mu_G$ when $\overline{\text{co}}(\text{rg}\, G)$ is a translate of $\overline{\text{co}}(\text{rg}\, F)$. By Theorem 3.2 we know that $\mu_F^{\text{s}} = \mu_G^{\text{s}}$; as every set of measure zero for $\mu_G^{\text{s}}$ has measure zero for $\mu_G$ it is enough to prove that (c) for $\mu_F$ implies (c) for $\mu_F^{\text{s}}$. Let $h : X^{**} \to [0, 2]$ be the Radon–Nikodym derivative of $\mu_F$ with respect to $\mu_F^{\text{s}}$. Let $B = \{h = 0\}$, $C = B \setminus (B \cap -B)$, and define $\psi_1(x^{**}) = \psi(x^{**})$ if $x^{**} \in X^{**} \setminus C$, and $\psi_1(x^{**}) = -\psi(-x^{**})$ if $x^{**} \in C$. For $x^* \in X^*$, we have $x^* \circ \psi_1 = x^*$ $\mu_F^{\text{s}}$-almost everywhere since $\mu_F$ and $\mu_F^{\text{s}}$ are equivalent measures in $X \setminus C$, and they are exactly the same measure on $-C$. ∎



In Example 3.1 the measure space in which the derivable vector measure is defined is not perfect. One can think that the existence of non-perfect measure spaces, with large $\sigma$-algebras, is the reason for the Pettis derivability not being determined by the range. It raises the following problem: if we restrict our attention to vector measures defined on perfect measure spaces (vector measures having a perfect control measure), does the range determine the Pettis derivability? Let us remark that $\mu_F$ is a perfect measure since it is the restriction to the Baire $\sigma$-algebra of a Radon measure defined on the Borel sets of $(X^{**}, w^*)$ supported by $B_{X^{**}}$: the Radon measure representing the linear functional $f \mapsto \int f\, d\mu_F$ on $\mathcal{C}(B_{X^{**}})$. The next proposition is a partial answer to the above question; it implies, for instance, that if the Banach space is $\ell_\infty$, then the answer is yes.

Proposition 3.5. *Let $(\Omega, \Sigma, \mu)$ be a finite perfect measure space, $\varphi : \Omega \to X$ a Pettis integrable function and $F$ its indefinite integral. Suppose $F$ has bounded variation and there exists a sequence $\{x_n^*\}$ in $X^*$ separating the points of the linear span of $\varphi(\Omega)$. Then every measure $G$ such that $\overline{\text{co}}(\text{rg}\, G)$ is a translate of $\overline{\text{co}}(\text{rg}\, F)$ is Pettis derivable.*

Proof. We can reduce the proof to the case $\mu = |F|$, since $|F|$ is still a perfect measure and the Pettis derivative of $F$ with respect to $|F|$ is the product of a scalar function and $\varphi$. In the case $\mu = |F|$ we can use $\varphi$ to define $\mu_F$. Using Proposition 3.4 we only have to construct a function $\psi : X^{**} \to X$ satisfying the statement (c) of that proposition.

Define $T : X^{**} \to \mathbb{R}^{\mathbb{N}}$ by $Tx^{**} = (\langle x_n^*, x^{**}\rangle)_{n=1}^{\infty}$. Then $T$ is continuous if we consider on $X^{**}$ the weak* topology and on $\mathbb{R}^{\mathbb{N}}$ the product topology. The map $T \circ \varphi$ is measurable from the perfect measure space $(\Omega, \Sigma, |F|)$ to the metric separable space $\mathbb{R}^{\mathbb{N}}$. So given the set $A = T \circ \varphi(\Omega)$, as $\Omega = (T \circ \varphi)^{-1}(A)$, there exists a Borel set $B$ in $\mathbb{R}^{\mathbb{N}}$ such that $B \subset A$ and $|F|((T \circ \varphi)^{-1}(B)) = |F|(\Omega) = \|F\|$.

If $B_0 = T^{-1}(B)$, then $B_0 \in \text{Ba}(X^{**}, w^*)$, $\mu_F(B_0) = |F|(\varphi^{-1}(B_0)) = \|F\|$, and $Tx^{**} \in T \circ \varphi(\Omega)$ for every $x^{**} \in B_0$. So $X^{**} \setminus B_0$ is a negligible set and for every $x^{**} \in B_0$, there exists $y \in \varphi(\Omega) \subset X$ such that $Tx^{**} = Ty$. This $y$ is necessarily unique since the sequence $\{x_n^*\}$ separates the points of $\varphi(\Omega)$, and we define $\psi(x^{**}) = y$. If $x^{**} \in X^{**} \setminus B_0$ we define $\psi(x^{**}) = 0$. Let us check that $\psi$ satisfies (c) of Proposition 3.4. Pick $x_0^*$ in $X^*$. In order to see that $x_0^* \circ \psi = x_0^*$, $\mu_F$-almost everywhere in $X^{**}$, define $S : X^{**} \to \mathbb{R}^{\mathbb{N}}$ by $Sx^{**} = (\langle x_{n-1}^*, x^{**}\rangle)_{n=1}^{\infty}$. Using the same arguments as before we deduce the existence of a set $B_1 \in \mathrm{Ba}(X^{**}, w^*)$ such that $X^{**} \setminus B_1$ is negligible and for every $x^{**} \in B_1$, there exists a point $y \in \varphi(\Omega)$ such that $Sx^{**} = Sy$, which implies $Tx^{**} = Ty$ and $x_0^*(x^{**}) = x_0^*(y)$. If in addition $x^{**} \in B_0$, then $y = \psi(x^{**})$ and $x_0^*(x^{**}) = x_0^*(y)$, so we have proved $x_0^* \circ \psi(x^{**}) = x_0^*(x^{**})$ for every $x^{**} \in B_0 \cap B_1$. As $X^{**} \setminus (B_1 \cap B_0)$ is $\mu_F$-negligible we have finished. ■

## References


[AD] R. Anantharaman and J. Diestel, *Sequences in the range of a vector measure*, Comment. Math. Prace Mat. 30 (1991), 221–235.

[C] C. H. Choquet, *Lectures on Analysis, Vols. I, II, III*, Benjamin, New York, 1969.

[DJT] J. Diestel, H. Jarchow and A. Tonge, *Absolutely Summing Operators*, Cambridge Stud. Adv. Math. 43, Cambridge Univ. Press, 1995.

[DU] J. Diestel and J. J. Uhl, Jr., *Vector Measures*, Math. Surveys 15, Amer. Math. Soc. Providence, R.I., 1977.

[E] G. Edgar, *Measurability in a Banach space*, Indiana Univ. Math. J. 26 (1977), 663–677.

[FT] D. Fremlin and M. Talagrand, *A decomposition theorem for additive set functions and applications to Pettis integral and ergodic means*, Math. Z. 168 (1979), 117–142.

[K] I. Kluvánek, *Characterization of the closed convex hull of the range of a vector measure*, J. Funct. Anal. 21 (1976), 316–329.

[L] D. R. Lewis, *On integrability and summability in vector spaces*, Illinois J. Math. 16 (1972), 294–307.

[M] K. Musiał, *The weak Radon–Nikodym property in Banach spaces*, Studia Math. 64 (1979), 151–174.

[P] A. Pietsch, *Operator Ideals*, North-Holland, Amsterdam, 1980.

[R1] L. Rodríguez-Piazza, *The range of a vector measure determines its total variation*, Proc. Amer. Math. Soc. 111 (1991), 205–214.

[R2] —, *Derivability, variation and range of a vector measure*, Studia Math. 112 (1995), 165–187.

[T] M. Talagrand, *Pettis integral and measure theory*, Mem. Amer. Math. Soc. 307 (1984).

[Th] E. Thomas, *Integral representations in convex cones*, Groningen University Report ZW-7703 (1977).



Departamento de Análisis Matemático
Facultad de Matemáticas
Universidad de Sevilla
Aptdo. 1160
Sevilla 41080, Spain
E-mail: piazza@cica.es
mcromero@cica.es


*Received November 6, 1996* (3771)

STUDIA MATHEMATICA 125 (3) (1997)

# Best constants and asymptotics of Marcinkiewicz–Zygmund inequalities

by

ANDREAS DEFANT (Oldenburg) and MARIUS JUNGE (Kiel)


**Abstract.** We determine the set of all triples $1 \le p, q, r \le \infty$ for which the so-called *Marcinkiewicz–Zygmund inequality* is satisfied: There exists a constant $c \ge 0$ such that for each bounded linear operator $T : L_q(\mu) \to L_p(\nu)$, each $n \in \mathbb{N}$ and functions $f_1, \ldots, f_n \in L_q(\mu)$,

$$\left(\int \left(\sum_{k=1}^n |Tf_k|^r\right)^{p/r} d\nu\right)^{1/p} \le c\|T\| \left(\int \left(\sum_{k=1}^n |f_k|^r\right)^{q/r} d\mu\right)^{1/q}.$$

This type of inequality includes as special cases well-known inequalities of Paley, Marcinkiewicz, Zygmund, Grothendieck, and Kwapień. If such a Marcinkiewicz–Zygmund inequality holds for a given triple $(p, q, r)$, then we calculate the best constant $c \ge 0$ (with the only exception: the important case $1 \le p < r = 2 < q \le \infty$); if such an inequality does not hold, then we give asymptotically optimal estimates for the graduation of these constants in $n$. Two problems of Gasch and Maligranda from [9] are solved; as a by-product we obtain best constants of several important inequalities from the theory of summing operators.


**0. Introduction.** Fix a triple $(p, q, r)$ of scalars with $1 \le p, q, r \le \infty$. We call—for the purpose of this paper—an inequality of the following type a *Marcinkiewicz–Zygmund inequality*: There is a constant $c \ge 0$ (depending on $p, q$ and $r$ only) such that for each (linear and continuous) operator $T : L_q(\mu) \to L_p(\nu)$ ($\mu$ and $\nu$ arbitrary measures) and arbitrarily many functions $f_1, \ldots, f_n \in L_q(\mu)$,

$$\text{(MZ)} \qquad \left(\int \left(\sum_{k=1}^n |Tf_k|^r\right)^{p/r} d\nu\right)^{1/p} \le c\|T\| \left(\int \left(\sum_{k=1}^n |f_k|^r\right)^{q/r} d\mu\right)^{1/q}.$$

By a density and closed graph argument it is equivalent to say that each operator $T : L_q(\mu) \to L_p(\nu)$ allows an $\ell_r$-valued extension, i.e. there is an operator

$$T^{\ell_r} : L_q(\mu, \ell_r) \to L_p(\nu, \ell_r)$$

1991 *Mathematics Subject Classification*: Primary 46B07, 47B10; Secondary 42B25.