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Hardy type inequalities for two-parameter Vilenkin–Fourier coefficients

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Abstract. Our main result is a Hardy type inequality with respect to the two-parameter Vilenkin system

$$(*)\qquad \Big(\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}|\widehat{f}(k,j)|^p(kj)^{p-2}\Big)^{1/p}\le C_p\|f\|_{H^p_{**}}\qquad(1/2<p\le 2)$$

where f belongs to the Hardy space $H^p_{**}(G_m\times G_s)$ defined by means of a maximal function. This inequality is extended to $p>2$ if the Vilenkin–Fourier coefficients of f form a monotone sequence. We show that the converse of $(*)$ also holds for all $p>0$ under the monotonicity assumption.

1. Introduction. The Hardy inequality, i.e. the estimate of type $(*)$ for $p=1$ was proved in trigonometric Fourier analysis by Hardy and Littlewood [9] and Coifman and Weiss [6]. The analogous statement for one-parameter Vilenkin systems of bounded type is due to Ladhawala [10] and Chao [5]. For systems of unbounded type Fridli and Simon [7] showed the inequality in the case $p=1$. Their result was later generalized to $1/2<p\le 2$ by Simon and Weisz [16]. In the two-parameter case Weisz [19], [20] proved that $(*)$ holds without any condition on the system, but with $H^p_{**}(G_m\times G_s)$ replaced by the Hardy space defined by the conditional quadratic variation.

For a two-parameter Vilenkin system we define a sequence of σ-algebras and consider the martingales with respect to this sequence. We introduce the Hardy spaces $H^p_{**}(G_m\times G_s)$, $H^p_{*}(G_m\times G_s)$ and $H^p_{\sigma}(G_m\times G_s)$ $(0<p<\infty)$ which contain all martingales f for which the L^p norms of the maximal functions f^{**}, f^* and of the conditional quadratic variation $\sigma(f)$

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are bounded, respectively. In the one-parameter case the space $H^p_{**}(G_m)$ and the maximal function f^{**} were introduced and investigated by Simon [15].

First we establish the results that will be used later. In this connection we refer to the books by Schipp, Wade, Simon and Pál [14] and Weisz [20] (see also Simon and Weisz [16]).

In Section 3 we give the relation between the three martingale Hardy spaces and the atomic decomposition of $H^p_{**}(G_m \times G_s)$ and $H^p_\sigma(G_m \times G_s)$. A result about the boundedness of an operator from H^p to L^p is also formulated.

The main results are obtained in Section 4. The inequality (∗) will be shown for an arbitrary two-parameter Vilenkin system. The analogous statement for BMO-spaces is proved by the known duality argument. The basic idea of our investigations for $0 < p \le 1$ is the atomic description of the Hardy spaces. We prove that it is enough to verify the so-called strong boundedness of the left side in (∗) for rectangle p-atoms. For $1 < p \le 2$ we get the inequality by interpolation. Next (∗) is extended to $p > 2$ and to functions having monotone Vilenkin–Fourier coefficients, i.e. assuming that the real and imaginary parts of the Vilenkin–Fourier coefficients are non-increasing. Finally, under the same conditions we give a converse-like version of (∗) for Vilenkin systems under certain growth conditions on the sequences m and s.

2. Preliminaries and notations. First of all we introduce the most important definitions and notations for two-parameter Vilenkin systems.

Let $m = (m_0, m_1, \ldots, m_k, \ldots)$ be a sequence of natural numbers with $m_k \ge 2$ $(k \in \mathbb{N} := \{0, 1, \ldots\})$. For all $k \in \mathbb{N}$ we denote by Z_{m_k} the m_kth discrete cyclic group represented by $\{0, 1, \ldots, m_k - 1\}$. The complete direct product G_m of the Z_{m_k}'s is a compact Abelian group with a normalized Haar measure. The elements of G_m are sequences of the form $(x_0, x_1, \ldots, x_k, \ldots)$, where $x_k \in Z_{m_k}$ for every $k \in \mathbb{N}$ and the topology of G_m is completely determined by the *simple intervals*, i.e. by the sets

$$I_n(0) := \{(x_0, x_1, \ldots, x_k, \ldots) \in G_m : x_j = 0 \ (j = 0, \ldots, n-1)\}$$

$(0 \ne n \in \mathbb{N},\ I_0(0) := G_m)$. Let $I_n(x) := x + I_n(0)$ $(n \in \mathbb{N})$ and

$$I_n(x, k) := \{(y_0, y_1, \ldots) \in I_n(x) : y_n = k\}$$

$(x \in G_m,\ k \in Z_{m_n},\ n \in \mathbb{N})$.

The concept of intervals in G_m was introduced in Simon and Weisz [16] (see also Simon [15]) as follows. If $\mathcal{I} = \{n \in \mathbb{N} : k \le n \le t\}$ $(k, t \in \mathbb{N}, k < t)$ is a set of indices and $[\alpha]$ denotes the integer part of a real number α then let

$$d_0(\mathcal{I}) := \{n \in \mathbb{N} : k \le n \le [(k+t)/2]\},$$
$$d_1(\mathcal{I}) := \{n \in \mathbb{N} : [(k+t)/2] < n \le t\}.$$

Every set of the form

$$\mathcal{U} = d_{u_k}(d_{u_{k-1}}(\ldots d_{u_1}(Z_{m_n})\ldots)),$$

where $u_i = 0$ or 1, $i = 1, \ldots, k$, is said to be a *dyadic subset* of Z_{m_n}. Let N_n denote the smallest integer for which there exists a sequence $u_1, \ldots, u_{N_n}$ of 0's and 1's such that the dyadic set $d_{u_{N_n}}(d_{u_{N_{n-1}}}(\ldots d_{u_1}(Z_{m_n})\ldots))$ has only one element. Actually, $N_n = [\log_2 m_n]$. We define a sequence of σ-algebras as follows:

$$\mathcal{F}^0_n := \mathcal{F}_n := \sigma(\{I_n(x) : x \in G_m\})$$

and

$$\mathcal{F}^k_n := \sigma\Big(\Big\{I = \bigcup_{l \in d_{u_k}(d_{u_{k-1}}(\ldots d_{u_1}(Z_{m_n})\ldots))} I_n(x, l) : u_i = 0 \text{ or } 1, 1 \le i \le k\Big\}\Big), \tag{1}$$

where $n \in \mathbb{N}$, $0 < k \le N_n - 1$ and $\sigma(\mathcal{H})$ denotes the σ-algebra generated by an arbitrary set system $\mathcal{H}$. Set $\mathcal{F}^{-1}_n := \mathcal{F}^{N_{n-1}-1}_{n-1}$, $\mathcal{F}^{N_n}_n := \mathcal{F}_{n+1}$ and $\mathcal{F}^{-1}_0 := \mathcal{F}_{-1} := \mathcal{F}_0$.

The atoms of the σ-algebras $\mathcal{F}^k_n$ $(n \in \mathbb{N},\ 0 \le k \le N_n - 1)$ are called *intervals*. It is clear that every interval I is of the form

$$I = \bigcup_{l \in \mathcal{U}} I_n(x, l),$$

where $\mathcal{U}$ is a dyadic subset of Z_{m_n} $(n \in \mathbb{N})$. The measure $|I|$ of the interval I is evidently γM^{-1}_{n+1}, where γ denotes the cardinality of $\mathcal{U}$ and

$$M_{n+1} := \prod_{j=0}^{n} m_j.$$

Of course, $I_k(y)$ is an interval for all $y \in G_m$, $k \in \mathbb{N}$. For each interval I there is a unique sequence $I^0, \ldots, I^\nu$ of intervals (for some $\nu \in \mathbb{N}$) such that $I^k \in \mathcal{F}^l_n$ implies $I^{k+1} \in \mathcal{F}^{l-1}_n$ $(n \in \mathbb{N},\ 0 \le l \le N_n - 1)$ and, moreover,

$$\text{(i)}\quad G_m = I^\nu \supset I^{\nu-1} \supset \ldots \supset I^0 = I; \qquad \text{(ii)}\quad \frac{1}{4} \le \frac{|I^k|}{|I^{k+1}|} \le \frac{3}{5}. \tag{2}$$

Also,

$$\frac{|I|}{|I'|} \le \frac{3}{4} \tag{3}$$

for two intervals I and I' with $I \in \mathcal{F}^k_n$ and $I' \in \mathcal{F}^{k-1}_n$ $(n \in \mathbb{N},\ 0 \le k \le N_n - 1)$.

Now let another sequence $s = (s_0, s_1, \dots, s_k, \dots)$ of natural numbers be given with the same properties as m, i.e. $s_k \geq 2$ $(k \in \mathbb{N})$, and consider the group G_s. Then all the above objects can also be defined in G_s; in particular, the simple intervals and the σ-algebras (1), for which we will use the notation $J_n(y)$ and $\mathcal{G}_n^k$ $(n \in \mathbb{N}, -1 \leq k \leq [\log_2 s_n],\ y \in G_s)$, respectively.

The direct product $G := G_m \times G_s$ is then also a compact Abelian group with a normalized Haar measure.

Let $(j,u),(l,v)$ be admissible indices, i.e. $(j,u),(l,v) \in \mathbb{N}^2$ and $u \leq [\log_2 m_j] - 1,\ v \leq [\log_2 s_l] - 1)$, and

$$\mathcal{F}_{j,u}^{l,v} := \sigma(\mathcal{F}_j^u \times \mathcal{G}_l^v).$$

The sequence $\mathcal{F} = (\mathcal{F}_{j,u}^{l,v})$ is non-decreasing, more exactly, $\mathcal{F}_{j,u}^{l,v} \subset \mathcal{F}_{\widetilde{j},\widetilde{u}}^{\widetilde{l},\widetilde{v}}$ if $j \leq \widetilde{j},\ l \leq \widetilde{l}$, and if $j = \widetilde{j}$ or $l = \widetilde{l}$ then $u \leq \widetilde{u}$ or $v \leq \widetilde{v}$, respectively.

The conditional expectation operator relative to $\mathcal{F}_{j,u}^{l,v}$ is denoted by $E_{j,u}^{l,v}$.

We are going to consider martingales with respect to $\mathcal{F}$. An integrable sequence $f = (f_{j,u}^{l,v})$ is said to be a *martingale* if

(i) $f_{j,u}^{l,v}$ is $\mathcal{F}_{j,u}^{l,v}$-measurable for all admissible indices $(j,u),(l,v) \in \mathbb{N}^2$;

(ii) $E_{j,u}^{l,v}(f_{\widetilde{j},\widetilde{u}}^{\widetilde{l},\widetilde{v}}) = f_{j,u}^{l,v}$ for all admissible indices $(j,u),(l,v) \in \mathbb{N}^2$ and $(\widetilde{j},\widetilde{u}),(\widetilde{l},\widetilde{v}) \in \mathbb{N}^2$ such that $j \leq \widetilde{j},\ l \leq \widetilde{l}$, and if $j = \widetilde{j}$ or $l = \widetilde{l}$ then $u \leq \widetilde{u}$ or $v \leq \widetilde{v}$, respectively.

Furthermore, let $\mathcal{F}_{n,k} := \mathcal{F}_{n,k}^{0,0}$ and $f_{n,k} := f_{n,k}^{0,0}$ $(n,k \in \mathbb{N})$. We will assume that $f_{n,0} = f_{0,n} = 0$ $(n \in \mathbb{N})$. Of course, the theorems to be proved later hold without this condition.

The atoms of the σ-algebras $\mathcal{F}_{j,u}^{l,v}$ (resp. $\mathcal{F}_{j,u}$) are called *rectangles* (resp. *simple rectangles*). Note that the sequence $\mathcal{F}$ is regular (for the definition see Weisz [20]).

It is well known [14] that the characters of G_m form a complete orthonormal system in $L^1(G_m)$ (the so-called *Vilenkin system*). If

$$r_n(x) := \exp \frac{2\pi i x_n}{m_n}$$

$(n \in \mathbb{N},\ x = (x_0, x_1, \dots) \in G_m,\ i := \sqrt{-1})$, then r_n's and their finite products are evidently characters. These products can be ordered in Paley's sense, which means the following enumeration. We write each $n \in \mathbb{N}$ uniquely in the form

$$n = \sum_{k=0}^{\infty} n_k M_k,$$

where $M_0 := 1$, M_k $(k \geq 1)$ are defined above and $n_k \in Z_{m_k}$ $(k \in \mathbb{N})$. It can easily be seen that the characters of G_m are nothing else but the functions

$$\Psi_n := \prod_{k=0}^{\infty} r_k^{n_k}.$$

If we replace G_m by G_s, then we write ϱ_j instead of r_n and Φ_j instead of Ψ_n, respectively. So

$$\varrho_j(y) := \exp \frac{2\pi i y_j}{s_j}$$

$(y = (y_0, y_1, \dots) \in G_s)$, $\Phi_j := \prod_{k=0}^{\infty} \varrho_k^{j_k}$, where $j \in \mathbb{N}$, $j = \sum_{k=0}^{\infty} j_k P_k$ $(j_k \in Z_{s_k})$ and $P_0 := 1$, $P_k := \prod_{l=0}^{k-1} s_l$ $(k \geq 1)$.

The *two-parameter Vilenkin system* is defined to consist of the Kronecker products of the functions Ψ_j and Φ_k, i.e. for $(j,k) \in \mathbb{N}^2$ let

$$\Psi_{j,k}(x,y) := \Psi_j(x)\Phi_k(y) \qquad ((x,y) \in G).$$

The Fourier coefficients of a function $f \in L^1(G)$ with respect to the system $(\Psi_{j,k})$ are denoted by $\widehat{f}(j,k)$, i.e.

$$\widehat{f}(j,k) := \int f \overline{\Psi}_{j,k} \qquad ((j,k) \in \mathbb{N}^2).$$

(The bar stands here for complex conjugation.) This definition can be extended to martingales in the usual way (see Weisz [20]).

$C_p > 0$ will denote a constant depending only on p, although not always the same in different occurrences.

3. Martingale Hardy spaces. The Hardy spaces $H_{**}^p(G)$ and $H_*^p(G)$ $(0 < p < \infty)$ will be defined by the maximal functions f^{**} and f^* of the martingale $f = (f_{j,u}^{l,v} : (j,u),(l,v) \in \mathbb{N}^2,\ u \leq [\log_2 m_j] - 1),\ v \leq [\log_2 s_l] - 1)$:

$$f^{**} := \sup_{j,u,l,v} |f_{j,u}^{l,v}|, \qquad f^* := \sup_{n,k} |f_{n,k}|.$$

Furthermore, let $\sigma(f)$ (the conditional quadratic variation of f) be defined by

$$\sigma(f) := \Big(\sum_{n,k \in \mathbb{N}} E_{n-1,k-1} |f_{n,k} - f_{n-1,k} - f_{n,k-1} + f_{n-1,k-1}|^2 \Big)^{1/2}.$$

We remark that in case $f \in L^1(G)$ the maximal functions f^{**} and f^* can also be given by

$$f^{**}(x,y) = \sup_{I,J} (|I| \cdot |J|)^{-1} \Big| \int_{I \times J} f \Big|, \qquad f^*(x,y) = \sup_{k,l} M_k P_l \Big| \int_{I_k(x) \times J_l(y)} f \Big|,$$

where $(x,y) \in G$ and the first supremum is taken over all intervals $I \subset G_m$ and $J \subset G_s$ such that $(x,y) \in I \times J$.

Denote by $H^p_{**}(G), H^p_*(G), H^p_\sigma(G)$ the spaces of martingales for which

$$\|f\|_{H^p_{**}} := \|f^{**}\|_p < \infty, \quad \|f\|_{H^p_*} := \|f^*\|_p < \infty, \quad \|f\|_{H^p_\sigma} := \|\sigma(f)\|_p < \infty,$$

respectively.

It is well known that the atomic characterization plays an important role in the theory of Hardy spaces. For such a description of $H^p_{**}(G)$ $(0 < p \le 1)$ we give first the concept of atoms. Namely, a function $a \in L^2(G)$ is a *p-atom* if

(i) $\operatorname{supp} a \subset F$ for an open set $F \subset G$;
(ii) $\|a\|_2 \le |F|^{1/2-1/p}$, where $|F|$ is the Haar measure of F;
(iii) a can be further decomposed into the sum $a = \sum_R a_R$ satisfying
  (α) $\operatorname{supp} a_R \subset R$ for a rectangle $R \subset F$;
  (β) for all R and $(x,y) \in G$, $\int_{G_m} a_R(x,y)\,dx = \int_{G_s} a_R(x,y)\,dy = 0$;
  (γ) $(\sum_R \|a_R\|_2^2)^{1/2} \le |F|^{1/2-1/p}$.

If the rectangles R are all simple then a is said to be a *simple p-atom*. Furthermore, if $a \in L^2(G)$ satisfies (i) with a rectangle F (resp. with a simple rectangle F), (ii) and (β) then a is called a *rectangle p-atom* (resp. a *simple rectangle p-atom*).

Now, we can give the atomic characterization of $H^p_{**}(G)$ as follows.

THEOREM 1 (Weisz [18]). *A martingale* $f = (f^{l,v}_{j,u}; (j,u),(l,v) \in \mathbb{N}^2, u \le [\log_2 m_j] - 1,\ v \le [\log_2 s_l] - 1)$ *is in* $H^p_{**}(G)$ $(0 < p \le 1)$ *if and only if there exist a sequence* $(a^k, k \in \mathbb{N})$ *of p-atoms and a sequence* $(\mu_k, k \in \mathbb{N})$ *of real numbers such that* $\sum_{k=0}^\infty |\mu_k|^p < \infty$ *and*

$$\sum_{k=0}^{\infty} \mu_k E^{l,v}_{j,u} a^k = f^{l,v}_{j,u} \tag{4}$$

for all $(j,u),(l,v) \in \mathbb{N}^2$, $u \le [\log_2 m_j] - 1, v \le [\log_2 s_l] - 1$. *Moreover, the following equivalence of norms holds*:

$$\|f\|_{H^p_{**}} \sim \inf \Big(\sum_{k=0}^{\infty} |\mu_k|^p\Big)^{1/p},$$

where the infimum is taken over all decompositions of f of the form (4).

If we replace $H^p_{**}(G)$ *by* $H^p_\sigma(G)$ *and the p-atoms by simple p-atoms, then the same theorem holds with the restriction* $l = v = 0$.

The Hardy spaces $H^p_*(G)$, $H^p_{**}(G)$ and $H^p_\sigma(G)$ are proper subspaces of $L^1(G)$. Furthermore, the following theorem holds.

THEOREM 2. *We have* $H^p_{**}(G) \sim H^p_*(G) \sim L^p(G)$ *for* $1 < p < \infty$ *and*

$$\begin{aligned} &\|f\|_{H^p_*} \le \|f\|_{H^p_{**}} \le C_p \|f\|_{H^p_\sigma} && (0 < p \le 2),\\ &\|f\|_{H^p_\sigma} \le C_p \|f\|_{H^p_*} \sim \|f\|_{H^p_{**}} && (2 \le p < \infty). \end{aligned} \tag{5}$$

If m (or s) is unbounded, then the converse inequalities are not true. Furthermore, if m, s are bounded then $H^p_{**}(G) \sim H^p_*(G) \sim H^p_\sigma(G)$ *for all* $0 < p < \infty$, *where $\sim$ denotes equivalence of spaces and norms.*

Observe that the second inequality of (5) follows from Theorem 1 and from the fact that every simple p-atom is a p-atom. The other inequalities are known or trivial (see Brossard [1], [2], Cairoli [4], Métraux [11] or Weisz [20]).

For each interval $I \subset G_m$, the interval I^r $(r \in \mathbb{N})$ is defined in (2) (for $r > \nu$ let $I^r := G_m$). For an interval $J \subset G_s$, J^r $(r \in \mathbb{N})$ can be defined analogously. If $R := I \times J$ is a rectangle then set $R^r := I^r \times J^r$.

Let Ω be an arbitrary non-empty set and $\mathcal{A}$ be a σ-algebra on it. For each interval I we define $\overline{I} \in \mathcal{A}$ such that $I \subset I_*$ implies $\overline{I} \subset \overline{I}_*$. For a rectangle $R = I \times J$ let $\overline{R} = \overline{I} \times \overline{J}$. If $F \subset G$ is open then set

$$\overline{F} := \bigcup_{R \subset F} \overline{R}.$$

It is clear that, for open sets, $F_1 \subset F_2$ implies $\overline{F}_1 \subset \overline{F}_2$. We consider the measure space $(\Omega^2, \sigma(\mathcal{A} \times \mathcal{A}), \eta)$ and the corresponding real $L^p(\Omega^2) := L^p(\Omega^2, \sigma(\mathcal{A} \times \mathcal{A}), \eta)$ space.

Although $H^p_{**}(G)$ cannot be decomposed into rectangle p-atoms (see Weisz [20]), the following theorem holds.

THEOREM 3. *Suppose that* $0 < p \le 1$ *and an operator T which maps the set of martingales into the collection of* $\sigma(\mathcal{A} \times \mathcal{A})$*-measurable functions is sublinear. Furthermore, assume that with a constant* $C > 0$,

$$\eta(\overline{F}) \le C|F| \quad \text{for all open sets } F \subset G, \tag{6}$$

and assume that there exists $\delta > 0$ such that for every rectangle p-atom a supported on the rectangle R and for every $r \in \mathbb{N}$ one has

$$\int_{\Omega^2 \setminus \overline{R^r}} |Ta|^p \, d\eta \le C_p 2^{-\delta r}. \tag{7}$$

If T is bounded from $L^2(G)$ to $L^2(\Omega^2)$ then

$$\|Tf\|_{L^p(\Omega^2)} \le C_p \|f\|_{H^p_{**}} \quad (f \in H^p_{**}(G)).$$

We omit the proof because it is similar to that of Theorem 1 in Weisz [22].

4. Hardy type inequalities. First we prove a Hardy type inequality for the space $H^p_{**}(G)$ $(1/2 < p \le 2)$. This is the two-dimensional analogue of the inequality

$$\Big(\sum_{k=1}^{\infty} \frac{|\widehat{f}(k)|^p}{k^{2-p}}\Big)^{1/p} \le C_p \|f\|_{H^p_{**}}$$

proved by Simon and Weisz [16]. (See also Hardy and Littlewood [9] and Coifman and Weiss [6] in the classical case for trigonometric Fourier coefficients, and Ladhawala [10], Chao [5] and Fridli and Simon [7] for Vilenkin systems.) The inequality just mentioned was proved by Weisz [19], [20] in the two-parameter case, but for $H_\sigma^p(G)$ instead of $H_{**}^p(G)$ $(0 < p \le 2)$. In view of Theorem 2, the next theorem generalizes this result. Furthermore, we point out that Theorem 4 holds also for $L^p(G)$ $(1 < p \le 2)$, which was not contained in Weisz's theorem if m or s is unbounded.

THEOREM 4. *Suppose that* $1/2 < p \le 2$. *Then there exists a constant* $C_p > 0$ *such that*

$$\left(\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{|\widehat{f}(k,j)|^p}{(kj)^{2-p}}\right)^{1/p} \le C_p\|f\|_{H_{**}^p}$$

for all $f \in H_{**}^p(G)$.

Proof. Suppose that $1/2 < p \le 1$. We are going to apply Theorem 3. Set $\Omega := \mathbb{P} := \mathbb{N}\setminus\{0\}$ and let us introduce on $\mathbb{P}^2$ the measure $\eta(n,m) := 1/(n^2m^2)$. If

$$Tf(n,k) := nk\widehat{f}(n,k) \quad (n,k \in \mathbb{P}),$$

then it follows by Parseval's formula that T is bounded from $L^2(G)$ to $L^2(\mathbb{P}^2)$.

For an interval I let $\overline{I}$ be the set $\{k \in \mathbb{P} : k > |I|^{-1}\}$. Obviously, $I \subset I_*$ implies $\overline{I} \subset \overline{I}_*$.

First we prove condition (6) with $C = 12$. Let $F \subset G$ be an open set. It is easy to see that there exist finitely many rectangles $R_k = I_k \times J_k \subset F$, $k = 1, \dots, K$ $(K \in \mathbb{N})$, such that $|J_k|$ strictly decreases and $|I_k|$ strictly increases and

$$\overline{F} = \bigcup_{k=1}^{K} \overline{R}_k.$$

The inequality

$$\sum_{k=n}^{l-1}\frac{1}{k^2} \le \frac{2}{n} - \frac{2}{l} \quad (n \ge 1)$$

implies that

$$\frac{1}{2}\eta(\overline{F}) = \frac{1}{2}\sum_{(n,l)\in\overline{F}}\frac{1}{(nl)^2} \le \sum_{k=1}^{K}|J_k|(|I_k| - |I_{k-1}|)$$

where $|I_0| := 0$. We will show that

$$\sum_{k=1}^{K}|J_k|(|I_k| - |I_{k-1}|) \le 6|F|. \tag{8}$$

Let H_1 and H_2 be two disjoint non-empty subsets of $\mathbb{P}$ for which $\{1, \dots, K\} = H_1 \cup H_2$. For each $k \in H_i$ $(i = 1,2)$ define

$$k_{H_i}^- := \max\{l \in H_i \cup \{0\} : l < k\}.$$

Observe that

$$\begin{aligned}\sum_{k=1}^{K}|J_k|(|I_k| - |I_{k-1}|) &= \sum_{k\in H_1}|J_k|(|I_k| - |I_{k-1}|) + \sum_{k\in H_2}|J_k|(|I_k| - |I_{k-1}|)\\ &\le \sum_{k\in H_1}|J_k|(|I_k| - |I_{k_{H_1}^-}|) + \sum_{k\in H_2}|J_k|(|I_k| - |I_{k_{H_2}^-}|)\end{aligned}$$

because $|I_k|$ is increasing. It is clear that if we consider the lengths of the atoms of a fixed σ-algebra $\mathcal{G}_n^l$, then we get one or two numbers. So, by the preceding inequality we can suppose that there are no two different intervals J_k and J_i $(1 \le k, i \le K)$ which are atoms of the same σ-algebra $\mathcal{G}_n^l$. In the same way, we can also assume that if J_k is an atom of $\mathcal{G}_n^{l-1}$, then J_{k+1} is not an atom of $\mathcal{G}_n^l$ and of $\mathcal{G}_n^{l+1}$. Under these conditions we show that

$$\sum_{k=1}^{K}|J_k|(|I_k| - |I_{k-1}|) \le |F|,$$

which proves (6). Let $A := \bigcup_{k=1}^{K} R_k$. Then $A \subset F$ and, of course, $|A| \le |F|$. Hence it is enough to prove that

$$\sum_{k=1}^{K}|J_k|(|I_k| - |I_{k-1}|) \le |A|. \tag{9}$$

Choose sets B_k^i and D_k^i $(i = 1,2; k = 1, \dots, K)$ such that $|B_k^1| = |D_k^1| = |I_k|$ and $|B_k^2| = |D_k^2| = |J_k|$. We also assume that, for a fixed i, two different B_k^i sets are always disjoint or one contains the other. We suppose the same for the sets D_k^i. Set $B_k := B_k^1 \times B_k^2$, $D_k := D_k^1 \times D_k^2$, $B := \bigcup_{k=1}^{K} B_k$ and $D := \bigcup_{k=1}^{K} D_k$. Moreover, suppose that the intersection of two arbitrary sets B_{k_1} and B_{k_2} is non-empty. Then it can easily be verified that

$$\sum_{k=1}^{K}|J_k|(|I_k| - |I_{k-1}|) = |B|.$$

By induction on K we shall see that D has minimal measure if and only if the intersection of arbitrary two sets D_{k_1} and D_{k_2} is non-empty. For $K = 1$ or $K = 2$ it is trivial. Let $1 \le l < K$ be the minimal index for which

$D_{l+1} \cap D_l = \emptyset$. If there is no such index then the intersection of arbitrary two sets D_{k_1} and D_{k_2} is non-empty. Notice that

$$\Big|D_l - \bigcup_{k=1}^{l-1} D_k\Big| = \Big|B_l - \bigcup_{k=1}^{l-1} B_k\Big|.$$

Since D_{l+1} and D_l are disjoint and, by (3),

$$\sum_{k=l+2}^{K} |D_k^2| = \sum_{k=l+2}^{K} |J_k| \le \sum_{k=3}^{\infty} \left(\frac{3}{4}\right)^k |J_{l+1}| < |J_{l+1}| = |D_{l+1}^2| = |B_{l+1}^2|,$$

we can conclude that

$$\Big|D_l - \bigcup_{\substack{k=1\\k\ne l}}^{K} D_k\Big| > \Big|B_l - \bigcup_{\substack{k=1\\k\ne l}}^{K} B_k\Big|.$$

But, by the induction hypothesis we have

$$\Big|\bigcup_{\substack{k=1\\k\ne l}}^{K} D_k\Big| \ge \Big|\bigcup_{\substack{k=1\\k\ne l}}^{K} B_k\Big|,$$

and consequently, $|D| > |B|$. Thus we have shown (9) and so the proof of the condition (6) is complete.

Now we have to check the inequality (7). To this end let a be an arbitrary rectangle p-atom such that $\operatorname{supp} a \subset R = I \times J$, $|I| = \alpha M_{n+1}^{-1}$, $|J| = \beta M_{w+1}^{-1}$, $0 < \alpha < m_n$, $0 < \beta < s_w$ (see the definition of p-atoms). Then

$$\begin{aligned}\mathbb{P} \setminus \overline{R^r} = ((\mathbb{P} \setminus \overline{I^r}) \times \overline{J}) \cup ((\mathbb{P} \setminus \overline{I^r}) \times (\mathbb{P} \setminus \overline{J}))\\ \cup (\overline{I} \times (\mathbb{P} \setminus \overline{J^r})) \cup ((\mathbb{P} \setminus \overline{I}) \times (\mathbb{P} \setminus \overline{J^r})).\end{aligned}$$

In the proof of (7) we integrate over these four sets. We begin with the first one:

$$\int_{(\mathbb{P}\setminus\overline{I^r})\times\overline{J}} |Ta|^p \, d\eta = \sum_{k \le |I^r|^{-1}} \sum_{l > |J|^{-1}} |\widehat{a}(k,l)|^p (kl)^{p-2}.$$

First observe that by the definition of the rectangle atom,

$$\widehat{a}(k,l) = \int_I \int_J a(x,y) \overline{\Psi}_{k,l}(x,y) \, dx \, dy = 0$$

if either Ψ_k is constant on I or Φ_l is constant on J. This is so if $k < M_n$ or $l < P_w$. Furthermore, for $k = jM_n + u$ and $l = tP_w + v$ ($j = 1, \ldots, m_n - 1$, $u = 0, \ldots, M_n - 1$, $t = 1, \ldots, s_w - 1$, $v = 0, \ldots, P_w - 1$) we get

$$\begin{aligned}|\widehat{a}(k,l)| &= \Big|\int_I\int_J a(x,y) r_n^{-j}(x) \overline{\Psi}_u(x) \varrho_w^{-t}(y) \overline{\Phi}_v(y) \, dx \, dy\Big|\\ &= \Big|\int_I\int_J a(x,y) r_n^{-j}(x) \varrho_w^{-t}(y) \, dx \, dy\Big| = |\widehat{a}(jM_n, tP_w)|.\end{aligned}$$

Observe that, by (2), $|I^r|^{-1} \le (3/5)^r |I|^{-1}$. Hence

$$\begin{aligned}\int_{(\mathbb{P}\setminus\overline{I^r})\times\overline{J}} |Ta|^p \, d\eta &\le \sum_{M_n \le k \le |I^r|^{-1}} \sum_{l > |J|^{-1}} |\widehat{a}(k,l)|^p (kl)^{p-2}\\ &\le \sum_{M_n \le k \le (3/5)^r |I|^{-1}} \sum_{l > |J|^{-1}} |\widehat{a}(k,l)|^p (kl)^{p-2}\\ &\le \sum_{j=1}^{m_n^{(r)}} \sum_{v=0}^{M_n - 1} \sum_{l > |J|^{-1}} |\widehat{a}(jM_n + v, l)|^p (jM_n + v)^{p-2} l^{p-2}\\ &= \sum_{j=1}^{m_n^{(r)}} \sum_{v=0}^{M_n - 1} \sum_{l > |J|^{-1}} |\widehat{a}(jM_n, l)|^p (jM_n + v)^{p-2} l^{p-2}\\ &\le M_n^{p-1} \sum_{j=1}^{m_n^{(r)}} j^{p-2} \sum_{l > |J|^{-1}} |\hat{a}(jM_n, l)|^p l^{p-2},\end{aligned}$$

where $m_n^{(r)} := [(3/5)^r / (|I| M_n)]$. If $j = 1, \ldots, m_n^{(r)}$ and $l > |J|^{-1}$, then for all $c \in Z_{m_n}$ it follows (see (β) in the definition of the p-atom) that

$$\begin{aligned}|\widehat{a}(jM_n, l)|^p &= \Big|\int_I\int_J a(x,y) r_n^{-j}(x) \overline{\Phi}_l(y) \, dx \, dy\Big|^p\\ &= \Big|\int_J \Big(\int_I a(x,y)(e^{-2\pi i j x_n / m_n} - e^{-2\pi i j c / m_n}) \, dx\Big) \overline{\Phi}_l(y) \, dx \, dy\Big|^p\\ &\le \Big(\int_I |e^{-2\pi i j x_n / m_n} - e^{-2\pi i j c / m_n}| \cdot \Big|\int_J a(x,y) \overline{\Phi}_l(y) \, dy\Big| \, dx\Big)^p\\ &\le C_p j^p m_n^{-p} \Big(\int_I |x_n - c| \Big|\int_J a(x,y) \overline{\Phi}_l(y) \, dy\Big| \, dx\Big)^p.\end{aligned}$$

It can evidently be assumed that $|x_n - c| \le |I| M_{n+1} = \alpha$, and thus

$$\begin{aligned}|\widehat{a}(jM_n, l)|^p &\le C_p j^p m_n^{-p} \Big(|I| M_{n+1} \int_I \Big|\int_J a(x,y) \overline{\Phi}_l(y) \, dy\Big| \, dx\Big)^p\\ &= C_p (j|I| M_n)^p \Big(\int_I \Big|\int_J a(x,y) \overline{\Phi}_l(y) \, dy\Big| \, dx\Big)^p.\end{aligned}$$

From this we get

$$\int_{(\mathbb{P}\setminus\overline{I^r})\times\overline{J}} |Ta|^p\,d\eta$$

$$\le C_p \sum_{l>|J|^{-1}} M_n^{p-1} \sum_{j=1}^{m_n^{(r)}} (lj)^{p-2} (jM_n|I|)^p \Big(\int_I \Big|\int_J a(x,y)\overline{\Phi}_l(y)\,dy\Big|\,dx\Big)^p$$

$$= C_p |I|^p M_n^{2p-1} \sum_{l>|J|^{-1}} \sum_{j=1}^{m_n^{(r)}} l^{p-2} j^{2p-2} \Big(\int_I \Big|\int_J a(x,y)\overline{\Phi}_l(y)\,dy\Big|\,dx\Big)^p$$

$$\le C_p |I|^p (M_n m_n^{(r)})^{2p-1} \sum_{l>|J|^{-1}} l^{p-2} \Big(\int_I \Big|\int_J a(x,y)\overline{\Phi}_l(y)\,dy\Big|\,dx\Big)^p.$$

We denote the last sum by S. Applying Hölder's inequality we obtain

$$S \le \Big(\sum_{l>|J|^{-1}} l^{-2}\Big)^{1-p/2} \Big(\sum_{l>|J|^{-1}} \Big(\int_I \Big|\int_J a(x,y)\overline{\Phi}_l(y)\,dy\Big|\,dx\Big)^2\Big)^{p/2}$$

$$\le C_p |J|^{1-p/2} \Big(\sum_{l>|J|^{-1}} \Big(\int_I \Big|\int_J a(x,y)\overline{\Phi}_l(y)\,dy\Big|\,dx\Big)^2\Big)^{p/2}.$$

Again by Hölder's and Parseval's inequalities and by (ii) this yields

$$S \le C_p |J|^{1-p/2} \Big(\int_I |I| \sum_{l>|J|^{-1}} \Big|\int_J a(x,y)\overline{\Phi}_l(y)\,dy\Big|^2\,dx\Big)^{p/2}$$

$$\le C_p |J|^{1-p/2} |I|^{p/2} \Big(\int_I\int_J |a(x,y)|^2\,dx\,dy\Big)^{p/2} \le C_p |I|^{p-1}.$$

Summarizing the above estimations we conclude that

$$\int_{(\mathbb{P}\setminus\overline{I^r})\times\overline{J}} |Ta|^p\,d\eta \le C_p (|I| M_n m_n^{(r)})^{2p-1} \le C_p (3/5)^{r(2p-1)} \le C_p 2^{-r\delta} \tag{10}$$

if $0<\delta\le(2p-1)\log_2 5/3$.

The integral over $(\mathbb{P}\setminus\overline{I^r})\times(\mathbb{P}\setminus J)$ can be estimated as follows:

$$\int_{(\mathbb{P}\setminus\overline{I^r})\times(\mathbb{P}\setminus J)} |Ta|^p\,d\eta \le \sum_{k\le(3/5)^r|I|^{-1}} \sum_{l\le|J|^{-1}} |\widehat{a}(k,l)|^p (kl)^{p-2}$$

$$\le \sum_{j=1}^{m_n^{(r)}} \sum_{v=0}^{M_n-1} \sum_{t\le(|J|P_w)^{-1}} \sum_{z=0}^{P_w-1} \frac{|\widehat{a}(jM_n+v,tP_w+z)|^p}{(jM_n+v)^{2-p}(tP_w+z)^{2-p}}$$

$$\le (M_nP_w)^{p-1} \sum_{j=1}^{m_n^{(r)}} j^{p-2} \sum_{t\le(|J|P_w|)^{-1}} t^{p-2} |\widehat{a}(jM_n,tP_w)|^p.$$

For all $c\in Z_{m_n}$ and $d\in Z_{s_w}$ we get

$$|\widehat{a}(jM_n,tP_w)|^p = \Big|\int_I\int_J a(x,y)(e^{-2\pi i j x_n/m_n} - e^{-2\pi i j c/m_n})$$

$$\times (e^{-2\pi i t y_w/s_w} - e^{-2\pi i t d/s_w})\,dx\,dy\Big|^p$$

$$\le C_p \Big(\int_I\int_J |a(x,y)| \frac{jt|x_n-c|\cdot|y_w-d|}{m_n s_w}\,dx\,dy\Big)^p.$$

Let c,d be selected such that $|x_n-c|\le|I|M_{n+1}$ and $|y_w-d|\le|J|P_{w+1}$. Then

$$|\widehat{a}(jM_n,tP_w)|^p \le C_p \frac{(jt|I|\cdot|J|M_{n+1}P_{w+1})^p}{(m_ns_w)^p} \Big(\int_I\int_J |a(x,y)|\,dx\,dy\Big)^p$$

$$\le C_p (jt|I|\cdot|J|M_nP_w)^p (|I|\cdot|J|)^{p/2} \|a\|_2^p$$

$$\le C_p (jtM_nP_w)^q (|I|\cdot|J|)^{2p-1}.$$

This implies that

$$\int_{(\mathbb{P}\setminus\overline{I^r})\times(\mathbb{P}\setminus J)} |Ta|^p\,d\eta \tag{11}$$

$$\le C_p (|I|\cdot|J|M_nP_w)^{2p-1} \sum_{j=1}^{m_n^{(r)}} j^{2p-2} \sum_{t\le(|J|P_w)^{-1}} t^{2p-2}$$

$$\le C_p (|I|\cdot|J|M_nP_w)^{2p-1} (m_n^{(r)}(|J|P_w)^{-1})^{2p-1}$$

$$\le C_p (3/5)^{r(2p-1)} \le C_p 2^{-r\delta},$$

whenever $0<\delta\le(2p-1)\log_2 5/3$.

The integral over the third and fourth sets can be estimated analogously. Taking into account (10) and (11) we have thus proved condition (7) as well as the theorem for $1/2<p\le1$. For $1<p\le2$ we get the theorem by interpolation (see Weisz [19], [20]). ∎

We can also formulate the dual inequalities to Theorem 4. With the help of stopping times the $\mathrm{BMO}(G)$ space is defined in Weisz [20] and it is proved there that the dual of $H^1_{**}(G)$ is $\mathrm{BMO}(G)$. By the usual duality argument (cf. Weisz [21]) we can verify

COROLLARY 1. *If* $nk|a_{n,k}|$ $(n,k\geq 1)$ *are uniformly bounded real numbers then*

$$\Big\|\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}a_{n,k}\Psi_{n,k}\Big\|_{BMO}\leq C\sup_{n,k\geq 1}nk|a_{n,k}|,$$

where $C>0$ *is a constant.*

Again by the duality argument we get (cf. Weisz [20], Theorem 6.10, and also Simon and Weisz [16])

COROLLARY 2. *If* $2\leq q<\infty$ *and* $(a_{n,k};n,k\geq 1)$ *is a sequence of complex numbers such that*

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{|a_{n,k}|^q}{(nk)^{2-q}}<\infty$$

then

$$\Big\|\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}a_{n,k}\Psi_{n,k}\Big\|_q\leq C_q\Big(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{|a_{n,k}|^q}{(nk)^{2-q}}\Big)^{1/q}.$$

Theorem 4 can be extended to $p>2$ under suitable conditions, e.g. if the Vilenkin–Fourier coefficients are partially non-increasing. We will say that the sequence $\widehat{f}(j,k)$ $(j,k\in\mathbb{N})$ is *partially non-increasing* if

$$\operatorname{Re}(\widehat{f}(j,k)-\widehat{f}(j+1,k)-\widehat{f}(j,k+1)+\widehat{f}(j+1,k+1))\geq 0,$$
$$\operatorname{Im}(\widehat{f}(j,k)-\widehat{f}(j+1,k)-\widehat{f}(j,k+1)+\widehat{f}(j+1,k+1))\geq 0$$

$(j,k\in\mathbb{N})$. The proof is similar to that of Theorem 3 of [16], and therefore it will be omitted.

THEOREM 5. *Let* $2<p<\infty$ *and* $\lambda_n:=\max\{m_{n-1},m_n\}$, $\mu_n:=\max\{s_{n-1},s_n\}$ $(n\in\mathbb{N},\ m_{-1}:=m_0,\ s_{-1}:=s_0)$. *Then*

$$\Big(\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}(\lambda_n\mu_j)^{1-p}\sum_{k=M_n}^{M_{n+1}-1}\sum_{l=P_j}^{P_{j+1}-1}\frac{|\widehat{f}(k,l)|^p}{(kl)^{2-p}}\Big)^{1/p}\leq C_p\|f\|_p$$

for all functions $f\in L^p(G)$ *having partially non-increasing Vilenkin–Fourier coefficients.*

We remark that Theorem 5 remains true also for f having λ-blocwise monotone Vilenkin–Fourier coefficients (for the details see Simon and Weisz [16]). Furthermore, if m and p are bounded, then Theorem 5 leads to

$$\Big(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}|\widehat{f}(j,k)|^p(jk)^{p-2}\Big)^{1/p}\leq C_p\|f\|_p,$$

which was verified by Weisz [19].

A converse-like version of Theorem 4 is known for one-parameter Vilenkin systems under certain conditions on the sequence m (in particular also for some unbounded m's). Now we give the analogue of this result for two-parameter Vilenkin systems. Let $\varphi:[0,\infty)\to[0,\infty)$ be a non-decreasing function such that for all $\mu>0$ the growth condition

$$\limsup_{x\to\infty}\frac{\varphi(x)}{e^{\mu x}}<\infty$$

holds. For example, a simple calculation shows that the functions $\varphi(x):=e^{x^\delta}$ $(0\leq\delta<1)$ and $\varphi(x):=x^\sigma$ $(\sigma\geq 0)$ satisfy this condition. Notice that if

$$m_n=O(\varphi(n)),\qquad s_n=O(\varphi(n))\qquad(n\to\infty),\tag{12}$$

then it is not hard to see that the estimates

$$m_{\nu_k}\leq C_\mu k^\mu,\qquad s_{\nu_k}\leq C_\mu k^\mu\qquad(k=1,2,\ldots)$$

are valid for indices satisfying $M_{\nu_k}\leq k<M_{\nu_k+1}$ $(k\in\mathbb{N})$ and $P_{\nu_k}\leq k<P_{\nu_k+1}$ $(k\in\mathbb{N})$, respectively (see Simon and Weisz [16]).

The proof of the following theorem can be performed in a similar way to the one-dimensional case (see Simon and Weisz [16] and also Weisz [20]), and thus again we leave out the details.

THEOREM 6. *Assume that the sequences* m,s *satisfy the growth condition* (12). *Then for all* $p>0$ *and* $0<\nu<1$ *there exists a constant* $C_{p,\nu}>0$ *depending only on* p *and* ν *such that*

$$\Big\|\sup_{n,l}\Big|\sum_{j=1}^{n}\sum_{k=1}^{l}\widehat{f}(j,k)\Psi_{j,k}\Big|\Big\|_p\leq C_{p,\nu}\Big(\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|\widehat{f}(j,k)|^p(jk)^{p+\nu-2}\Big)^{1/p}$$

for all $f\in L^p(G)$ *having partially non-increasing Vilenkin–Fourier coefficients.*

References

[1] J. Brossard, *Comparaison des "normes" L^p du processus croissant et de la variable maximale pour les martingales régulières à deux indices. Théorème local correspondant*, Ann. Probab. 8 (1980), 1183–1188.

[2] —, *Régularité des martingales à deux indices et inégalités de normes*, in: Processus Aléatoires à Deux Indices, Lecture Notes in Math. 863, Springer, Berlin, 1981, 91–121.

[3] D. L. Burkholder, *Distribution function inequalities for martingales*, Ann. Probab. 1 (1973), 19–42.

[4] R. Cairoli, *Une inégalité pour martingales à indices multiples et ses applications*, in: Séminaire de Probabilités V, Lecture Notes in Math. 124, Springer, Berlin, (1970), 1–27.

[5] J. A. Chao, *Hardy spaces on regular martingales*, in: Martingale Theory in Harmonic Analysis and Banach Spaces, Lecture Notes in Math. 939, Springer, Berlin, 1982, 18–28.

[6] R. R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. 83 (1977), 569–645.

[7] S. Fridli and P. Simon, *On the Dirichlet kernels and a Hardy space with respect to the Vilenkin system*, Acta Math. Hungar. 45 (1985), 223–234.

[8] A. M. Garsia, *Martingale Inequalities. Seminar Notes on Recent Progress*, Math. Lecture Notes Series, Benjamin, New York, 1973.

[9] G. H. Hardy and J. E. Littlewood, *Some new properties of Fourier constants*, J. London Math. Soc. 6 (1931), 3–9.

[10] N. R. Ladhawala, *Absolute summability of Walsh–Fourier series*, Pacific J. Math. 65 (1976), 103–108.

[11] C. Métraux, *Quelques inégalités pour martingales à paramètre bidimensionnel*, in: Séminaire de Probabilités XII, Lecture Notes in Math. 649, Springer, Berlin, 1978, 170–179.

[12] F. Móricz, *On double cosine, sine and Walsh series with monotone coefficients*, Proc. Amer. Math. Soc. 109 (1990), 417–425.

[13] —, *On Walsh series with coefficients tending monotonically to zero*, Acta Math. Acad. Sci. Hungar. 38 (1981), 183–189.

[14] F. Schipp, W. R. Wade, P. Simon and J. Pál, *Walsh Series: An Introduction to Dyadic Harmonic Analysis*, Adam Hilger, Bristol, 1990.

[15] P. Simon, *Investigations with respect to the Vilenkin system*, Ann. Univ. Sci. Budapest Eötvös Sect. Math. 28 (1985), 87–101.

[16] P. Simon and F. Weisz, *Hardy–Littlewood type inequalities for Vilenkin–Fourier coefficients*, Anal. Math., to appear.

[17] N. Y. Vilenkin, *On a class of complete orthonormal systems*, Izv. Akad. Nauk SSSR Ser. Mat. 11 (1947), 363–400 (in Russian); English transl.: Amer. Math. Soc. Transl. 28 (1963), 1–35.

[18] F. Weisz, *Hardy spaces and Cesàro means of two-dimensional Fourier series*, in: Approximation Theory and Function Series (Budapest, 1995), Bolyai Soc. Math. Stud. 5, Budapest, 1996, 353–367.

[19] —, *Inequalities relative to two-parameter Vilenkin–Fourier coefficients*, Studia Math. 99 (1991), 221–233.

[20] —, *Martingale Hardy Spaces and their Applications in Fourier-Analysis*, Lecture Notes in Math. 1568, Springer, Berlin, 1994.

[21] —, *Strong convergence theorems for two-parameter Walsh–Fourier and trigonometric-Fourier series*, Studia Math. 117 (1996), 173–194.

[22] —, *Two-parameter Hardy–Littlewood inequalities*, ibid. 118 (1996), 175–184.

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Quelques remarques sur les facteurs des systèmes dynamiques gaussiens

par

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Abstract. We study the factors of Gaussian dynamical systems which are generated by functions depending only on a finite number of coordinates. As an application, we show that for Gaussian automorphisms with simple spectrum, the partition $\{(X_0 \le 0), (X_0 > 0)\}$ is generating.

Introduction. On se place dans le cadre d'un système dynamique $(\Omega, \mathscr{A}, \mu, T)$, que l'on suppose gaussien : il existe un processus gaussien réel centré $(X_p)_{p\in\mathbb{Z}}$ qui engendre $\mathscr{A}$, avec $X_p = X_0 \circ T^p$ pour tout entier p. La loi d'un tel processus, et donc toutes les propriétés du système dynamique qu'il engendre, est entièrement déterminée par la donnée de ses covariances, qui s'écrivent

$$(1) \qquad \langle X_p, X_q\rangle_{L^2(\mu)} = \mathbb{E}[X_p X_q] = \int\limits_{[-\pi,\pi]} e^{i(p-q)t}\, d\sigma(t),$$

où σ est une mesure finie symétrique sur $[-\pi,\pi]$, appelée *mesure spectrale* du système. Un tel système est construit canoniquement en prenant $\Omega = \mathbb{R}^{\mathbb{Z}}$, X_p étant la projection sur la $p$ième coordonnée, T le décalage des coordonnées et μ la probabilité sur $\mathbb{R}^{\mathbb{Z}}$ qui donne au processus (X_p) la loi voulue. On pourra toujours supposer dans la suite que le modèle utilisé est celui-ci. On suppose aussi le système ergodique, ce qui équivaut à

$$\forall t \in [-\pi,\pi], \quad \sigma(\{t\}) = 0.$$

Pour une présentation détaillée de ces systèmes, on peut par exemple consulter [1].

On s'intéresse ici aux *facteurs* d'un tel système, c'est-à-dire aux sous-tribus $\mathscr{F}$ de $\mathscr{A}$ qui sont T-invariantes. Rappelons que chaque facteur $\mathscr{F}$ de T définit un système dynamique noté $T_{\mathscr{F}}$ sur l'espace $\Omega_{\mathscr{F}}$, obtenu à partir de Ω en identifiant les points ω et ω' tels que $\forall F \in \mathscr{F}$, $\mathbf{1}_F(\omega) = \mathbf{1}_F(\omega')$.

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