Hardy type inequalities
for two-parameter Vilenkin–Fourier coefficients

by

PÉTER SIMON and FERENC WEISSZ (Budapest)

Abstract. Our main result is a Hardy type inequality with respect to the two-parameter Vilenkin system

\[ \left( \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |\hat{f}(k,j)|^p (kj)^{p-2} \right)^{1/p} \leq C_p \|f\|_{H^p_L} \quad (1/2 < p \leq 2) \]

where \( f \) belongs to the Hardy space \( H^p_L(G_m \times G_a) \) defined by means of a maximal function. This inequality is extended to \( p > 2 \) if the Vilenkin–Fourier coefficients of \( f \) form a monotone sequence. We show that the converse of (\( \ast \)) also holds for all \( p > 0 \) under the monotonicity assumption.

1. Introduction. The Hardy inequality, i.e. the estimate of type (\( \ast \)) for \( p = 1 \) was proved in trigonometric Fourier analysis by Hardy and Littlewood [9] and Coifman and Weiss [6]. The analogous statement for one-parameter Vilenkin systems of bounded type is due to Ladhawala [10] and Chao [5]. For systems of unbounded type Frizli and Simon [7] showed the inequality in the case \( p = 1 \). Their result was later generalized to \( 1/2 < p \leq 2 \) by Simon and Weiss [16]. In the two-parameter case Weiss [19], [20] proved that (\( \ast \)) holds without any condition on the system, but with \( H^p_L(G_m \times G_a) \) replaced by the Hardy space defined by the conditional quadratic variation.

For a two-parameter Vilenkin system we define a sequence of \( \sigma \)-algebras and consider the martingales with respect to this sequence. We introduce the Hardy spaces \( H^p_L(G_m \times G_a) \), \( H^p_L(G_m \times G_a) \) and \( H^p_L(G_m \times G_a) \) (\( 0 < p < \infty \)) which contain all martingales \( f \) for which the \( L^p \) norms of the maximal functions \( f^{**}, f^* \) and of the conditional quadratic variation \( \sigma(f) \)

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are bounded, respectively. In the one-parameter case the space $H^{p_0}_x(G_m)$ and
the maximal function $f^{**}$ were introduced and investigated by Simon [15].

First we establish the results that will be used later. In this connection we refer to the books by Schipp, Wade, Simon and Pál [14] and Weiss [20] (see also Simon and Weiss [16]).

In Section 3 we give the relation between the three martingale Hardy spaces and the atomic decomposition of $H^p_x(G_m \times G_x)$ and $H^p_x(G_m \times G_x)$. A result about the boundedness of an operator from $H^p$ to $L^p$ is also formulated.

The main results are obtained in Section 4. The inequality (*) will be shown for an arbitrary two-parameter Vilenkin system. The analogous statement for BMO-spaces is proved by the known duality argument. The basic idea of our investigations for $0 < p \leq 1$ is the atomic description of the Hardy spaces. We prove that it is enough to verify the so-called strong boundedness of the left side in (*) for rectangle $p$-atoms. For $1 < p \leq 2$ we get the inequality by interpolation. Next (*) is extended to $p > 2$ and to functions having monotone Vilenkin–Fourier coefficients, i.e. assuming that the real and imaginary parts of the Vilenkin–Fourier coefficients are non-increasing.

Finally, under the same conditions we give a converse-like version of (*) for Vilenkin systems under certain growth conditions on the sequences $m$ and $s$.

2. Preliminaries and notations. First of all we introduce the most important definitions and notations for two-parameter Vilenkin systems.

Let $m = (m_0, m_1, \ldots, m_k, \ldots)$ be a sequence of natural numbers with $m_k \geq 2$ ($k \in \mathbb{N} := \{0, 1, \ldots\}$). For all $k \in \mathbb{N}$ we denote by $Z_{mk}$ the $m_k$th discrete cyclic group represented by $\{0, 1, \ldots, m_k - 1\}$. The complete direct product $G_m$ of the $Z_{mk}$'s is a compact Abelian group with a normalized Haar measure. The elements of $G_m$ are sequences of the form $(x_0, x_1, \ldots, x_k, \ldots)$, where $x_k \in Z_{mk}$ for every $k \in \mathbb{N}$ and the topology of $G_m$ is completely determined by the simple intervals, i.e. by the sets

$I_n(0) := \{(x_0, x_1, \ldots, x_k, \ldots) \in G_m : x_j = 0 \ (j = 0, 1, \ldots, n - 1)\}$

$(0 \neq n \in \mathbb{N}, \ I_0(0) := G_m)$. Let $I_n(x) := x + I_n(0)$ ($n \in \mathbb{N}$) and

$I_n(x, k) := \{(y_0, y_1, \ldots, y_n) \in I_n(x) : y_n = k\}$

$(x \in G_m, k \in Z_{mk}, n \in \mathbb{N})$.

The concept of intervals in $G_m$ was introduced in Simon and Weiss [16] (see also Simon [15]) as follows. If $I = \{n \in \mathbb{N} : k \leq n \leq t\} (k, t \in \mathbb{N}, k < t)$ is a set of indices and $[a]$ denotes the integer part of a real number $a$ then let

$d_0(I) := \{n \in \mathbb{N} : k \leq n \leq \lfloor (k + t)/2 \rfloor\}$

$d_1(I) := \{n \in \mathbb{N} : \lfloor (k + t)/2 \rfloor < n \leq t\}$.

Every set of the form

$U = d_{u_k}(d_{u_{k-1}}(\ldots d_{u_1}(Z_{m_n} \ldots)))$,

where $u_i = 0$ or $1, i = 1, \ldots, k$, is said to be a dyadic subset of $Z_{m_n}$. Let $N_n$ denote the smallest integer for which there exists a sequence $v_1, \ldots, v_{N_n}$ of $0$'s and $1$'s such that the dyadic set $d_{u_k}(d_{u_{k-1}}(\ldots d_{u_1}(Z_{m_n}) \ldots))$ has only one element. Actually, $N_n = \lfloor \log_2 m_n \rfloor$. We define a sequence of $\sigma$-algebras as follows:

$\mathcal{F}_n^0 := \mathcal{F}_n := \sigma(\{I_n(x) : x \in G_m\})$

and

$\mathcal{F}_n^k := \sigma(\{I_n(x, i) : I = d_{u_k}(d_{u_{k-1}}(\ldots d_{u_1}(Z_{m_n}) \ldots)) : u_i = 0 \text{ or } 1, 1 \leq i \leq k\})$,

where $n \in \mathbb{N}, 0 < k \leq N_n - 1$ and $\sigma(\mathcal{H})$ denotes the $\sigma$-algebra generated by an arbitrary set system $\mathcal{H}$. Set $\mathcal{F}^{-1}_n := \bigcup_{n \leq -1} \mathcal{F}_n^{N_n - 1}$, $\mathcal{F}_0^N := \mathcal{F}_n$ and $\mathcal{F}_0^0 := \mathcal{F}_n := \mathcal{F}_0$.

The atoms of the $\sigma$-algebras $\mathcal{F}_n^k$ ($n \in \mathbb{N}, 0 \leq k \leq N_n - 1$) are called intervals. It is clear that every interval $I$ is of the form

$I = \bigcup_{i \in U} I_n(x, i)$,

where $U$ is a dyadic subset of $Z_{m_n}$ ($n \in \mathbb{N}$). The measure $|I|$ of the interval $I$ is evidently $\gamma M^{-1}_{n+1}$, where $\gamma$ denotes the cardinality of $U$ and

$M_{n+1} := \prod_{j=0}^{n} m_j$.

Of course, $I_k(y)$ is an interval for all $y \in G_m, k \in \mathbb{N}$. For each interval $I$ there is a unique sequence $I^0, \ldots, I^p$ of intervals (for some $n \in \mathbb{N}$) such that $I^k \in \mathcal{F}_0^k$ implies $I^{k+1} \in \mathcal{F}_0^{k+1} (n \in \mathbb{N}, 0 \leq l \leq N_n - 1)$ and, moreover,

$G_m = I^0 \supset I^{p-1} \supset \ldots \supset I^0 = I$;

$\frac{1}{4} \leq |I^k| \leq \frac{3}{2}$.

Also,

$\frac{|I^k|}{|I|} = \frac{3}{4}$

for two intervals $I$ and $I'$ with $I \in \mathcal{F}_n^k$ and $I' \in \mathcal{F}_n^{k-1} (n \in \mathbb{N}, 0 \leq k \leq N_n - 1)$.
Now let another sequence \( s = (s_0, s_1, \ldots, s_k, \ldots) \) of natural numbers be given with the same properties as \( m \), i.e. \( s_k \geq 2 \) \((k \in \mathbb{N})\), and consider the group \( G_s \). Then all the above objects can also be defined in \( G_s \); in particular, the simple intervals and the \( \sigma \)-algebras \((1)\), for which we will use the notation \( J_n(y) \) and \( G^*_n \) \((n \in \mathbb{N}, -1 \leq k \leq \lfloor \log_2 s_n \rfloor, y \in G_s)\), respectively.

The direct product \( G := G_m \times G_s \) is then also a compact Abelian group with a normalized Haar measure.

Let \((j, u), (l, v)\) be admissible indices, i.e. \((j, u), (l, v) \in \mathbb{N}^2 \) and \( u \leq \lfloor \log_2 m_j \rfloor - 1, v \leq \lfloor \log_2 s_l \rfloor - 1 \), and

\[
\mathcal{F}^i_{j,u} := \sigma(\mathcal{F}^i_{j,l} \times \mathcal{F}^i_{j,l})
\]

The sequence \( \mathcal{F} = (\mathcal{F}^i_{j,u}) \) is non-decreasing, more exactly, \( \mathcal{F}^i_{j,u} \subset \mathcal{F}^i_{j,u} \) if \( j \leq j, l \leq l \), and if \( j = j \) or \( l = l \) then \( u \leq u \) or \( v \leq v \), respectively.

The conditional expectation operator relative to \( \mathcal{F}^i_{j,u} \) is denoted by \( E_{j,u}^i \).

We are going to consider martingales with respect to \( \mathcal{F} \). An integrable sequence \( f = (f^i_{j,u}) \) is said to be a martingale if

(i) \( f^i_{j,u} \) is \( \mathcal{F}^i_{j,u} \)-measurable for all admissible indices \((j, u), (l, v) \in \mathbb{N}^2 \);

(ii) \( E^{i+1}_{j,u}(f^i_{j,u}) = f^i_{j,u} \) for all admissible indices \((j, u), (l, v) \in \mathbb{N}^2 \) and \((j, u), (l, v) \in \mathbb{N}^2 \) such that \( j \leq j, l \leq l \), and if \( j = j \) or \( l = l \) then \( u \leq u \) or \( v \leq v \), respectively.

Furthermore, let \( \mathcal{F}_{n,k} := \mathcal{F}^0_{0,0} \) and \( f_{n,k} := f^0_{0,k} \) \((n, k \in \mathbb{N})\). We will assume that \( f_{0,n} = f_{0,n} = 0 \) \((n, k \in \mathbb{N})\). Of course, the theorems to be proved later hold without this condition.

The atoms of the \( \sigma \)-algebras \( \mathcal{F}^i_{j,u} \) (resp. \( \mathcal{F}_{j,u} \)) are called rectangles (resp. simple rectangles). Note that the sequence \( \mathcal{F} \) is regular (for the definition see Weisz [20]).

It is well known [14] that the characters of \( G_m \) form a complete orthonormal system in \( L^1(G_m) \) (the so-called Vilenkin system). If

\[
r_n(x) := \exp \frac{2\pi i n x}{m_n} \quad (n \in \mathbb{N}, x = (x_0, x_1, \ldots) \in G_m, i := \sqrt{-1}),
\]

then \( r_n \)'s and their finite products are evidently characters. These products can be ordered in Paley's sense, which means the following enumeration. We write each \( n \in \mathbb{N} \) uniquely in the form

\[
n = \sum_{k=0}^{\infty} n_k M_k,
\]

where \( M_0 := 1, M_k \) \((k \geq 1)\) are defined above and \( n_k \in \mathbb{Z}_{m_k} \) \((k \in \mathbb{N})\). It can easily be seen that the characters of \( G_m \) are nothing else but the functions

\[
\psi_n := \prod_{k=0}^{\infty} r_k^{n_k}.
\]

If we replace \( G_m \) by \( G_s \), then we write \( r_n \) instead of \( r_n \) and \( \Phi_j \) instead of \( \psi_n \), respectively. So

\[
\psi_j(y) := \exp \frac{2\pi i j y_j}{d_j} \quad (y = (y_0, y_1, \ldots) \in G_s), \quad \Phi_j := \prod_{k=0}^{\infty} \Phi_k^{j_k}, \quad j \in \mathbb{N}, \quad j = \sum_{k=0}^{\infty} j_k p_k \quad (j_k \in \mathbb{Z}_{p_k}) \quad \text{and} \quad p_0 := 1, \quad p_k := \prod_{j=0}^{k-1} s_j \quad (k \geq 1).
\]

The two-parameter Vilenkin system is defined to consist of the Kronecker products of the functions \( \Psi_j \) and \( \Phi_k \), i.e. for \((j, k) \in \mathbb{N}^2 \) let

\[
\Psi_{j,k}(x, y) := \psi_j(x) \Phi_k(y) \quad ((x, y) \in G).
\]

The Fourier coefficients of a function \( f \in L^1(G) \) with respect to the system \( \Psi_{j,k} \) are denoted by \( \hat{f}(j, k) \), i.e.

\[
\hat{f}(j, k) := \int \overline{\Psi_{j,k}} \quad ((j, k) \in \mathbb{N}^2).
\]

(The bar stands here for complex conjugation.) This definition can be extended to martingales in the usual way (see Weisz [20]).

\( G_p > 0 \) will denote a constant depending only on \( p \), although not always the same in different occurrences.

3. Martingale Hardy spaces. The Hardy spaces \( H^p_m(G) \) and \( H^p_s(G) \) \((0 < p < \infty)\) will be defined by the maximal functions \( f^{**} \) and \( f^* \) of the martingale \( f = (f^i_{j,u}) \) and \( (j, u), (l, v) \in \mathbb{N}^2, u \leq \lfloor \log_2 m_j \rfloor - 1, v \leq \lfloor \log_2 s_l \rfloor - 1 \):

\[
f^{**} := \sup_{j, n, u, v} |f^i_{j,u}|, \quad f^* := \sup_{n, k} |f_{n,k}|.
\]

Furthermore, let \( \sigma(f) \) (the conditional quadratic variation of \( f \)) be defined by

\[
\sigma(f) := \left( \sum_{n,k \in \mathbb{N}} E_{n-1,k-1} |f_{n,k} - f_{n-1,k} - f_{n,k-1} + f_{n-1,k-1}|^2 \right)^{1/2}.
\]

We remark that in case \( f \in L^1(G) \) the maximal functions \( f^{**} \) and \( f^* \) can also be given by

\[
f^{**}(x, y) = \sup_{I, J} (|I| \cdot |J|)^{-1} \left| \int_I f \right|, \quad f^*(x, y) = \sup_{I, J} \int_{I \times J} f \quad (x, y) \in G.
\]

where \((x, y) \in G \) and the first supremum is taken over all intervals \( I \subset G_m \) and \( J \subset G_s \) such that \((x, y) \in I \times J \).
Denote by $H^2_p(G)$, $H^2_p(G)$, $H^2_p(G)$ the spaces of martingales for which

$$\|f\|_{H^2_p} := \|f^*\|_p < \infty, \quad \|f\|_{H^2_p} := \|f^*\|_p < \infty, \quad \|f\|_{H^2_p} := \|\sigma(f)\|_p < \infty,$$

respectively.

It is well known that the atomic characterization plays an important role in the theory of Hardy spaces. For such a description of $H^2_p(G)$ ($0 < p \leq 1$) we give first the concept of atoms. Namely, a function $a \in L^2(G)$ is a $p$-atom if

(i) $\text{supp } a \subset F$ for an open set $F \subset G$;

(ii) $|a|_2 \leq |F|^{1/2-1/p}$, where $|F|$ is the Haar measure of $F$;

(iii) $a$ can be further decomposed into the sum $a = \sum R a_R$ satisfying

(a) $\text{supp } a_R \subset R$ for a rectangle $R \subset F$;

(b) for all $R$ and $(x,y) \in G$, $\int_G a_R(x,y) \, dx = \int_G a_R(x,y) \, dy = 0$;

(c) $\left( \sum_R |a_R|^2 \right)^{1/2} \leq |F|^{1/2-1/p}$.

If the rectangles $R$ are all simple then $a$ is said to be a simple $p$-atom. Furthermore, if $a \in L^2(G)$ satisfies (i) with a rectangle $F$ (resp. with a simple rectangle $F$), (ii) and (iii) then $a$ is called a rectangle $p$-atom (resp. a simple rectangle $p$-atom).

Now, we can give the atomic characterization of $H^2_p(G)$ as follows.

**Theorem 1** (Weisz [18]). A martingale $f = (f^*)_{j,u}$, $(j,u) \in \mathbb{N}^2$, $u \leq |\log m_j| - 1, v \leq |\log s_i| - 1$ is in $H^2_p(G)$ ($0 < p \leq 1$) if and only if there exist a sequence $(a_k, k \in \mathbb{N})$ of $p$-atoms and a sequence $(\alpha_k, k \in \mathbb{N})$ of real numbers such that $\sum_{k=0}^{\infty} |\alpha_k|^p < \infty$ and

$$\sum_{k=0}^{\infty} \mu_k E_{j,u} f^*_j \alpha_k = f^*_{j,u}$$

for all $(j,u), (j,u) \in \mathbb{N}^2$, $u \leq |\log m_j| - 1, v \leq |\log s_i| - 1$. Moreover, the following equivalence of norms holds:

$$\|f\|_{H^2_p} \sim \inf \left( \sum_{k=0}^{\infty} |\alpha_k|^p \right)^{1/p},$$

where the infimum is taken over all decompositions of $f$ of the form (4).

If we replace $H^2_p(G)$ by $H^2_p(G)$ and the $p$-atoms by simple $p$-atoms, then the same theorem holds with the restriction $l = v = 0$.

The Hardy spaces $H^2_p(G)$, $H^2_p(G)$ and $H^2_p(G)$ are proper subspaces of $L^1(G)$. Furthermore, the following theorem holds.

**Theorem 2.** We have $H^2_p(G) \sim H^2_p(G) \sim L^p(G)$ for $1 < p < \infty$ and

$$\|f\|_{H^2_p} \lesssim \|f\|_{H^2_p} \lesssim C_p \|f\|_{H^2_p} \quad (0 < p \leq 2),$$

$$\|f\|_{H^2_p} \lesssim C_p \|f\|_{H^2_p} \sim \|f\|_{H^2_p} \quad (2 \leq p < \infty).$$

If $m$ (or $s$) is unbounded, then the converse inequalities are not true. Furthermore, if $m, s$ are bounded then $H^2_p(G) \sim H^2_p(G) \sim H^2_p(G)$ for all $0 < p < \infty$, where $\sim$ denotes equivalence of spaces and norms.

Observe that the second inequality of (5) follows from Theorem 1 and from the fact that every simple $p$-atom is a $p$-atom. The other inequalities are known or trivial (see Brossard [1], [2], Cairoli [4], Métrats [11] or Weiss [20]).

For each interval $I \subset G_m$, the interval $I^r$ ($r \in \mathbb{N}$) is defined in (2) (for $r > 0$ let $I^r := G_m$). For an interval $J \subset G_n$, $J^r$ ($r \in \mathbb{N}$) can be defined analogously. If $R := I \times J$ is a rectangle then set $R^r := I^r \times J^r$.

Let $\Omega$ be an arbitrary non-empty set and $A$ be a $\sigma$-algebra on it. For each interval $I$ we define $\overline{I}$ in $A$ such that $I \subset \overline{I}$ implies $I \subset \overline{I}$. For a rectangle $R = I \times J$ let $\overline{R} = I \times J$. If $F \subset G$ is open then set

$$\overline{F} := \bigcup_{R \in \mathbb{F}} \overline{R}.$$

It is clear that, for open sets, $F_1 \subset F_2$ implies $\overline{F}_1 \subset \overline{F}_2$. We consider the measure space $(\Omega^2, \sigma(A \times A), \eta)$ and the corresponding real $L^p(\Omega^2) := L^p(\Omega^2, \sigma(A \times A), \eta)$ space.

Although $H^2_p(G)$ cannot be decomposed into rectangle $p$-atoms (see Weiss [20]), the following theorem holds.

**Theorem 3.** Suppose that $0 < p \leq 1$ and an operator $T$ which maps the set of martingales into the collection of $\sigma(A \times A)$-measurable functions is sublinear. Furthermore, assume that with a constant $C > 0$,

$$\eta(\overline{F}) \leq C|F| \quad \text{for all open sets } F \subset G,$$

and assume that there exists $\delta > 0$ such that for every rectangle $p$-atom $a$ supported on the rectangle $R$ and for every $r \in \mathbb{N}$ one has

$$\int_{\Omega^2} |Ta|^p \, d\eta \leq C_2 2^{-\delta r}. \quad (7)$$

If $T$ is bounded from $L^2(G)$ to $L^2(\Omega^2)$ then

$$\|Tf\|_{L^p(\Omega^2)} \leq C_p \|f\|_{H^2_p} \quad (f \in H^2_p(G)).$$

We omit the proof because it is similar to that of Theorem 1 in Weiss [22].

4. **Hardy type inequalities.** First we prove a Hardy type inequality for the space $H^2_p(G)$ ($1/2 < p \leq 2$). This is the two-dimensional analogue of the inequality

$$\left( \sum_{k=1}^{\infty} |f(k)|^p \right)^{1/p} \lesssim C_p \|f\|_{H^2_p}.$$
proved by Simon and Weiss [16]. (See also Hardy and Littlewood [9] and Coifman and Weiss [6] in the classical case for trigonometric Fourier coefficients, and Ladhawala [10], Chao [5] and Fridli and Simon [7] for Vilenkin systems.) The inequality just mentioned was proved by Weiss [19], [20] in the two-parameter case, but for $H^s_2(G)$ instead of $H^s_1(G)$ ($0 < p \leq 2$). In view of Theorem 2, the next theorem generalizes this result. Furthermore, we point out that Theorem 4 holds also for $L^p(G)$ ($1 < p \leq 2$), which was not contained in Weisz’s theorem if $m$ or $s$ is unbounded.

**Theorem 4.** Suppose that $1/2 < p \leq 2$. Then there exists a constant $C_p > 0$ such that

$$
\left( \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{j=1}^{\infty} \frac{|f(j,k)|^p}{k^2 - 2 - p} \right)^{1/p} \leq C_p \|f\|_{H^s_2}.
$$

for all $f \in H^s_2(G)$.

**Proof.** Suppose that $1/2 < p \leq 1$. We are going to apply Theorem 3. Set $\Omega := \mathbb{P} := \mathbb{N} \setminus \{0\}$ and let us introduce on $\mathbb{P}^2$ the measure $\eta(n,m) := 1/(n^2m^2)$. If

$$
Tf(n,k) := nk\overline{f}(n,k) \quad (n,k \in \mathbb{P}),
$$

then it follows by Parseval’s formula that $T$ is bounded from $L^2(\mathbb{P})$ to $L^2(\mathbb{P}^2)$.

For an interval $I$ let $\overline{I}$ be the set $\{k \in \mathbb{P} : k > |I|^{-1}\}$. Obviously, $I \subset \overline{I}$ implies $\overline{I} \subset \overline{I}$.

First we prove condition (6) with $C = 12$. Let $F \subset G$ be an open set. It is easy to see that there exist finitely many rectangles $R_k = I_k \times J_k \subset F$, $k = 1, \ldots, K$ ($K \in \mathbb{N}$), such that $|J_k|$ strictly decreases and $|I_k|$ strictly increases and

$$
F = \bigcup_{k=1}^{K} R_k.
$$

The inequality

$$
\sum_{k=1}^{n-1} \frac{1}{k^2} \leq \frac{2}{n} - \frac{2}{l} \quad (n \geq 1)
$$

implies that

$$
\frac{1}{2} \eta(F) = \frac{1}{2} \sum_{(n,l) \in F} \frac{1}{(nl)^2} \leq \sum_{k=1}^{K} |J_k|(|I_k| - |I_{k-1}|)
$$

where $|I_0| := 0$. We will show that

$$
\sum_{k=1}^{K} |J_k|(|I_k| - |I_{k-1}|) \leq 6|F|.
$$

Let $H_1$ and $H_2$ be two disjoint non-empty subsets of $\mathbb{P}$ for which $\{1, \ldots, K\} = H_1 \cup H_2$. For each $k \in H_i$ ($i = 1, 2$) define

$$
k_{H_i} := \max\{l \in H_i \cup \{0\} : l < k\}.
$$

Observe that

$$
\sum_{k=1}^{K} |J_k|(|I_k| - |I_{k-1}|) = \sum_{k \in H_1} |J_k|(|I_k| - |I_{k-1}|) + \sum_{k \in H_2} |J_k|(|I_k| - |I_{k-1}|)
$$

$$
\leq \sum_{k \in H_1} |J_k|(|I_k| - |I_{k_{H_1}}|) + \sum_{k \in H_2} |J_k|(|I_k| - |I_{k_{H_2}}|)
$$

because $|I_k|$ is increasing. It is clear that if we consider the lengths of the atoms of a fixed $\sigma$-algebra $\mathcal{G}_n^\perp$, then we get one or two numbers. So, by the preceding inequality we can suppose that there are no two different intervals $J_k$ and $J_l$ ($1 \leq k, l \leq K$) which are atoms of the same $\sigma$-algebra $\mathcal{G}_n^\perp$. In the same way, we can also assume that if $J_k$ is an atom of $\mathcal{G}_n^{\perp 1}$, then $J_{k+1}$ is not an atom of $\mathcal{G}_n^\perp$ and of $\mathcal{G}_n^{\perp 1}$. Under these conditions we show that

$$
\sum_{k=1}^{K} |J_k|(|I_k| - |I_{k-1}|) \leq |F|,
$$

which proves (6). Let $A := \bigcup_{k=1}^{K} R_k$. Then $A \subset F$ and, of course, $|A| \leq |F|$. Hence it is enough to prove that

$$
\sum_{k=1}^{K} |J_k|(|I_k| - |I_{k-1}|) \leq |A|.
$$

Choose sets $B_1^k$ and $D_1^k$ ($i = 1, 2; k = 1, \ldots, K$) such that $|B_1^k| = |D_1^k| = |I_k|$ and $|B_2^k| = |D_2^k| = |J_k|$. We also assume that, for a fixed $i$, two different $B_j^k$ sets are always disjoint or one contains the other. We suppose the same for the sets $D_k$. Set $B_k := B_1^k \times B_2^k$, $D_k := D_1^k \times D_2^k$, $B := \bigcup_{k=1}^{K} B_k$ and $D := \bigcup_{k=1}^{K} D_k$. Moreover, suppose that the intersection of two arbitrary sets $B_{k_1}$ and $B_{k_2}$ is non-empty. Then it can easily be verified that

$$
\sum_{k=1}^{K} |J_k|(|I_k| - |I_{k-1}|) = |B|.
$$

By induction on $K$ we shall see that $D$ has minimal measure if and only if the intersection of arbitrary two sets $D_{k_1}$ and $D_{k_2}$ is non-empty. For $K = 1$ or $K = 2$ it is trivial. Let $1 \leq l < K$ be the minimal index for which
$D_{i+1} \cap D_i = \emptyset$. If there is no such index then the intersection of arbitrary two sets $D_{h_{2}}$ and $D_{b_{2}}$ is non-empty. Notice that

$$|D_{i+1} \cup \bigcup_{k=1}^{i-1} D_k| = |B_i - \bigcup_{k=1}^{i-1} B_k|.$$ 

Since $D_{i+1}$ and $D_i$ are disjoint and, by (3),

$$\sum_{k=i+2}^{K} |D_{k}^{2}| = \sum_{k=i+2}^{K} |J_{k}| \leq \sum_{k=i}^{\infty} \left( \frac{3}{4} \right)^{k} |J_{i+1}| < |J_{i+1}| = |D_{i+1}^{2}| = |B_{i+1}^{2}|,$$

we can conclude that

$$|D_{i} - \bigcup_{k=1}^{K} D_{k}| > |B_{i} - \bigcup_{k=1}^{K} B_{k}|.$$ 

But, by the induction hypothesis we have

$$\left| \bigcup_{k=1}^{K} D_{k} \right| \geq \left| \bigcup_{k=1}^{K} B_{k} \right|,$$

and consequently, $|D| > |B|$. Thus we have shown (9) and the proof of the condition (6) is complete.

Now we have to check the inequality (7). To this end let $a$ be an arbitrary rectangle $p$-atom such that $\text{supp} \ a \subset R = I \times J$, $|I| = \alpha M_{n+1}$, $|J| = \beta M_{n+1}$, $0 < \alpha < \epsilon_{n}$, $0 < \beta < \epsilon_{n}$ (see the definition of $p$-atoms). Then

$$P \setminus \overline{P} = ((P \setminus \overline{T}) \times J) \cup ((P \setminus \overline{T}) \times (P \setminus \overline{J}))$$

$$= (I \times (P \setminus \overline{J})) \cup ((P \setminus \overline{T}) \times (P \setminus \overline{J})).$$

In the proof of (7) we integrate over these four sets. We begin with the first one:

$$\int_{(P \setminus \overline{I}) \times J} |Ta|^{p} \, d\eta = \sum_{k \leq |I^{*}|^{-1}} \sum_{l > |I^{*}|^{-1}} |\tilde{a}(k, l)|^{p} |kl|^{p-2}.$$ 

First observe that by the definition of the rectangle atom,

$$\tilde{a}(k, l) = \int_{I \times J} a(x, y) \overline{w}_{k,l}(x, y) \, dx \, dy = 0$$

if either $\Psi_{k}$ is constant on $I$ or $\Phi_{l}$ is constant on $J$. This is so if $k < M_{n}$ or $l < F_{w}$. Furthermore, for $k = jM_{n} + u$ and $l = tP_{w} + v$ ($j = 1, \ldots, m_{n} - 1$,

$$u = 0, \ldots, M_{n} - 1, \ t = 1, \ldots, s_{w} - 1, \ v = 0, \ldots, P_{w} - 1$) we get

$$|\tilde{a}(k, l)|^{p} = \left| \sum_{j=1}^{M_{n}} \sum_{t=1}^{s_{w}} \sum_{v=0}^{P_{w}-1} a(x, y) w_{-j}(x) \overline{w}_{t,v}(y) \, dx \, dy \right|^{p} \leq \sum_{j=1}^{M_{n}} \sum_{t=1}^{s_{w}} \sum_{v=0}^{P_{w}-1} |a(x, y)|^{p} (jM_{n} + u)^{p-2} |kl|^{p-2}.$$ 

Observe that, by (2), $|I^{*}|^{-1} \leq (3/5)|I|^{-1}$. Hence

$$\int_{(P \setminus \overline{I}) \times J} |Ta|^{p} \, d\eta \leq \sum_{M_{n} \leq |I^{*}|^{-1}} \sum_{j > |I^{*}|^{-1}} \sum_{t > |I^{*}|^{-1}} |\tilde{a}(k, l)|^{p} (kl)^{p-2}$$

$$\leq \sum_{M_{n} \leq |I^{*}|^{-1}} \sum_{j > |I^{*}|^{-1}} \sum_{t > |I^{*}|^{-1}} |\tilde{a}(k, l)|^{p} (jM_{n} + u)^{p-2} |kl|^{p-2}$$

$$\leq \sum_{j=1}^{M_{n}} \sum_{t=1}^{s_{w}} \sum_{v=0}^{P_{w}-1} |\tilde{a}(jM_{n} + u, l)|^{p} (jM_{n} + v)^{p-2} |kl|^{p-2}$$

$$\leq \sum_{j=1}^{M_{n}} \sum_{t=1}^{s_{w}} \sum_{v=0}^{P_{w}-1} |\tilde{a}(jM_{n} + u, l)|^{p} (jM_{n} + v)^{p-2} |kl|^{p-2},$$

where $m_{n}^{(r)} := (3/5)^{p} |(I^{*})^{r}|$. If $j = 1, \ldots, m_{n}^{(r)}$ and $l > |I^{*}|^{-1}$, then for all $c \in Z_{m_{n}}$ it follows (see (8) in the definition of the $p$-atom) that

$$|\tilde{a}(jM_{n} + u, l)|^{p} \leq \sum_{l} \int_{J} a(x, y) w_{-j}(x) \overline{w}_{t,v}(y) \, dx \, dy \right|^{p}$$

$$\leq \sum_{l} \left( \int_{J} \left| a(x, y) \left( e^{-2\pi i \alpha x / m_{n} - e^{-2\pi i \beta y / m_{n}} } \right) \overline{w}_{t,v}(y) \, dy \right| \right)^{p}$$

$$\leq C_{p} \int_{J} |x_{n} - c| \int_{J} a(x, y) \overline{w}_{t,v}(y) \, dy \, dx \right|^{p}.$$ 

This can evidently be assumed that $|x_{n} - c| \leq |I|/M_{n+1} = \alpha$, and thus

$$|\tilde{a}(jM_{n} + u, l)|^{p} \leq C_{p} |I|/M_{n+1} \sum_{l} \int_{J} \left| a(x, y) \overline{w}_{t,v}(y) \, dy \right| \right|^{p}$$

$$= C_{p} |I|/M_{n+1} \sum_{l} \int_{J} \left| a(x, y) \overline{w}_{t,v}(y) \, dy \right| \right|^{p}.$$
From this we get
\[
\int_{(P \setminus \overline{P}) \times J} |Ta|^p \, d\eta 
\leq C_p \sum_{l > |J|^{-1}} M^p_{x_l} \sum_{j=1}^{m^{(c)}} (d_j)^{p-2} (jM_{|J|})^p \left( \int_{I} \left( \int_{J} a(x, y) \overline{f}_l(y) \, dy \right)^2 \, dx \right)^p
\]
\[
= C_p |J|^p M^{2p-1}_{x_1} \sum_{l > |J|^{-1}} \sum_{j=1}^{m^{(c)}} \left( \int_{I} \left( \int_{J} a(x, y) \overline{f}_l(y) \, dy \right)^2 \, dx \right)^p
\]
\[
\leq C_p |J|^p (M_{x_1} m^{(c)}_x)^{2p-1} \sum_{l > |J|^{-1}} \sum_{j=1}^{m^{(c)}} \left( \int_{I} \left( \int_{J} a(x, y) \overline{f}_l(y) \, dy \right)^2 \, dx \right)^p.
\]

We denote the last sum by \( S \). Applying Hölder’s inequality we obtain
\[
S \leq \left( \sum_{l > |J|^{-1}} i^{-2} \right)^{1-p/2} \left( \sum_{l > |J|^{-1}} \left( \int_{I} \left( \int_{J} a(x, y) \overline{f}_l(y) \, dy \right)^2 \, dx \right)^{p/2} \right)^{p/2}
\]
\[
\leq C_p |J|^{1-p/2} \left( \sum_{l > |J|^{-1}} \left( \int_{I} \left( \int_{J} a(x, y) \overline{f}_l(y) \, dy \right)^2 \, dx \right)^{p/2} \right)^{p/2}.
\]

Again by Hölder’s and Parseval’s inequalities and by (ii) this yields
\[
S \leq C_p |J|^{1-p/2} \left( \sum_{l > |J|^{-1}} \left( \int_{I} \left( \int_{J} a(x, y) \overline{f}_l(y) \, dy \right)^2 \, dx \right)^{p/2} \right)^{p/2} \leq C_p |J|^{p-1}.
\]

Summarizing the above estimations we conclude that
\[
\int_{(P \setminus \overline{P}) \times (P \setminus J)} |Ta|^p \, d\eta \leq C_p (|J| M_{x_1} m^{(c)}_x)^{2p-1}
\]
\[
\leq C_p (3/5)^{(2p-1)} \leq C_p 2^{-r \delta}
\]
if \( 0 < \delta \leq (2p - 1) \log_2 5/3 \).

The integral over \((P \setminus \overline{P}) \times (P \setminus J)\) can be estimated as follows:
\[
\int_{(P \setminus \overline{P}) \times (P \setminus J)} |Ta|^p \, d\eta \leq \sum_{k \leq (3/5)|J|^{-1}} \sum_{l \leq |J|^{-1}} |\widehat{a}(k, l)| \Psi(k) \Psi(l) F_p \leq C_p (|J| M_{x_1} m^{(c)}_x)^{2p-1} \sum_{j=1}^{m^{(c)}_x} \sum_{v=0}^{M_{x_1} - 1} \sum_{z=0}^{P_{x_1} - 1} |a(jM_{x_1} + v, t_{P_{x_1}} + z)|^p
\]
\[
\leq C_p (|J| M_{x_1} m^{(c)}_x)^{2p-1} \sum_{j=1}^{m^{(c)}_x} \sum_{v=0}^{M_{x_1} - 1} \sum_{z=0}^{P_{x_1} - 1} |a(jM_{x_1} + v, t_{P_{x_1}} + z)|^p
\]
\[
\leq C_p (3/5)^{(2p-1)} \leq C_p 2^{-r \delta}.
\]


For all \( c \in Z_{x_1} \) and \( d \in Z_{x_1} \) we get
\[
|\widehat{a}(jM_{x_1}, tP_{x_1})|^p = \left( \int_{I} \left( \int_{J} a(x, y) (e^{-2\pi i j y / m_{x_1}} - e^{-2\pi i j x / m_{x_1}}) \right) \, dy \right)^p \right)^p
\]
\[
\leq C_p \left( \int_{I} \left( \int_{J} |a(x, y)| \, dy \right)^p \, dx \right)^p.
\]

Let \( c, d \) be selected such that \( |x_n - c| \leq |J| M_{x_1} + 1 \) and \( |y_n - d| \leq |J| P_{x_1} + 1 \). Then
\[
|\widehat{a}(jM_{x_1}, tP_{x_1})|^p \leq C_p \frac{(jt |x_n - c| + |J| M_{x_1} + 1)}{m_{x_1}^p} \left( \int_{I} \left( \int_{J} |a(x, y)| \, dy \right)^p \, dx \right)^p
\]
\[
\leq C_p (jt |x_n - c| + |J| M_{x_1} + 1) \leq C_p (jt |x_n - c| + |J| P_{x_1} + 1) \leq C_p (jt M_{x_1} P_{x_1}) \left( |J| \cdot |J|^p / p \right)^{1/2} \left\| a \right\|_p^2
\]
\[
\leq C_p (jt M_{x_1} P_{x_1}) \left( |J| \cdot |J|^p / p \right)^{1/2} \left\| a \right\|_p^2.
\]

This implies that
\[
|\widehat{a}(jM_{x_1}, tP_{x_1})|^p \leq C_p \frac{(jt |x_n - c| + |J| M_{x_1} + 1)}{m_{x_1}^p} \left( \int_{I} \left( \int_{J} |a(x, y)| \, dy \right)^p \, dx \right)^p
\]
\[
\leq C_p \left( \int_{I} \left( \int_{J} |a(x, y)| \right)^p \, dx \right)^p.
\]

The integral over the third and fourth sets can be estimated analogously. Taking into account (10) and (11) we have thus proved condition (7) as well as the theorem for \( 1/2 < p \leq 1 \). For \( 1 < p \leq 2 \) we get the theorem by interpolation (see Weisz [19], [20]).

We can also formulate the dual inequalities to Theorem 4. With the help of stopping times the BMO\( (G) \) space is defined in Weisz [20] and it is proved there that the dual of \( H^1_{c} (G) \) is BMO\( (G) \). By the usual duality argument (cf. Weisz [21]) we can verify
COROLLARY 1. If \( nk|a_{n,k}| (n, k \geq 1) \) are uniformly bounded real numbers then
\[
\left\| \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} a_{n,k} \psi_{n,k} \right\|_{BMO} \leq C \sup_{n,k \geq 1} nk|a_{n,k}|,
\]
where \( C > 0 \) is a constant.

Again by the duality argument we get (cf. Weiss [20], Theorem 6.10, and also Simon and Weiss [16])

COROLLARY 2. If \( 2 \leq q < \infty \) and \( (a_{n,k}, n, k \geq 1) \) is a sequence of complex numbers such that
\[
\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{|a_{n,k}|^q}{(nk)^{2-q}} < \infty
\]
then
\[
\left\| \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} a_{n,k} \psi_{n,k} \right\| \leq C \left( \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{|a_{n,k}|^q}{(nk)^{2-q}} \right)^{1/q}.
\]

Theorem 4 can be extended to \( p > 2 \) under suitable conditions, e.g., if the Vilenkin–Fourier coefficients are partially non-increasing. We will say that the sequence \( \hat{f}(j,k) (j, k \in \mathbb{N}) \) is partially non-increasing if
\[
\text{Re}(\hat{f}(j,k) - \hat{f}(j+1,k)) \geq 0, \quad \text{Im}(\hat{f}(j,k) - \hat{f}(j+1,k)) \geq 0
\]
for \( j, k \in \mathbb{N} \). The proof is similar to that of Theorem 3 of [16], and therefore it will be omitted.

THEOREM 5. Let \( 2 < p < \infty \) and \( \lambda_n := \max\{m_n-1, m_n\}, \mu_n := \max\{s_n-1, s_n\} \) \((n \in \mathbb{N}, m_n := m_0, s_n := s_0)\). Then
\[
\left( \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} (\lambda_n \mu_j)^{1-p} \sum_{k=M_n}^{M_{n+1}-1} \sum_{l=F_j}^{F_{j+1}-1} |\hat{f}(k,l)|^p \frac{(kd)^{2-p}}{(kd)^{2-p}} \right)^{1/p} \leq C_p \|f\|_p
\]
for all functions \( f \in L^p(G) \) having partially non-increasing Vilenkin–Fourier coefficients.

We remark that Theorem 5 remains true also for \( f \) having \( \lambda \)-blockwise monotone Vilenkin–Fourier coefficients (for the details see Simon and Weiss [16]). Furthermore, if \( m \) and \( p \) are bounded, then Theorem 5 leads to
\[
\left( \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |\hat{f}(j,k)|^p \right)^{2/p} \leq C_p \|f\|_{L^p(G)}
\]
which was verified by Weiss [19].

A converse-like version of Theorem 4 is known for one-parameter Vilenkin systems under certain conditions on the sequence \( m \) (in particular also for some unbounded \( m \)'s). Now we give the analogue of this result for two-parameter Vilenkin systems. Let \( \varphi : [0, \infty) \rightarrow [0, \infty) \) be a non-decreasing function such that for all \( \mu > 0 \) the growth condition
\[
\limsup_{x \to \infty} \frac{\varphi(x)}{e^{\mu x}} < \infty
\]
holds. For example, a simple calculation shows that the functions \( \varphi(x) := e^{x^\mu} (0 \leq \delta < 1) \) and \( \varphi(x) := x^x (x \geq 0) \) satisfy this condition. Notice that if
\[
m_n = O(\varphi(n)), \quad s_n = O(\varphi(n)) \quad (n \to \infty),
\]
then it is not hard to see that the estimates
\[
m_{\nu k} \leq C_{\mu} k^{\alpha} \quad \text{and} \quad s_{\nu k} \leq C_{\mu} k^{\alpha} \quad (k = 1, 2, \ldots)
\]
are valid for indices satisfying \( M_{\nu k} \leq k < M_{\nu k+1} \) \((k \in \mathbb{N})\) and \( P_{\nu k} \leq k < P_{\nu k+1} \) \((k \in \mathbb{N})\), respectively (see Simon and Weiss [16]).

The proof of the following theorem can be performed in a similar way to the one-dimensional case (see Simon and Weiss [16] and also Weiss [20]), and thus again we leave out the details.

THEOREM 6. Assume that the sequences \( m, s \) satisfy the growth condition (12). Then for all \( p > 0 \) and \( 0 < \nu < 1 \) there exists a constant \( C_{p,\nu} > 0 \) depending only on \( p \) and \( \nu \) such that
\[
\left\| \sup_{n,l} \sum_{j=1}^{n} \sum_{k=1}^{l} |\hat{f}(j,k)\psi_{j,k}| \right\|_p \leq C_{p,\nu} \left( \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |\hat{f}(j,k)|^p \right)^{2/p}
\]
for all \( f \in L^p(G) \) having partially non-increasing Vilenkin–Fourier coefficients.

References

Quelques remarques sur les facteurs des systèmes dynamiques gaussiens

par

A. IWANIK (Wrocław), M. LEMAŃCZYK (Toruń),
T. DE LA RUE (Rouen) et J. DE SAM LAZARO (Rouen)

Abstract. We study the factors of Gaussian dynamical systems which are generated by functions depending only on a finite number of coordinates. As an application, we show that for Gaussian automorphisms with simple spectrum, the partition \((X_0 \leq 0), (X_0 > 0)\) is generating.

Introduction. On se place dans le cadre d'un système dynamique \((\Omega, \mathcal{A}, \mu, T)\), que l'on suppose gaussien : il existe un processus gaussien réel centré \((X_p)_{p \in \mathbb{Z}}\) qui engendre \(\mathcal{A}\), avec \(X_p = X_0 \circ T^p\) pour tout entier \(p\). La loi d'un tel processus, et donc toutes les propriétés du système dynamique qu'il engendre, est entièrement déterminée par la donnée de ses covariances, qui s'écrit

\[
\langle X_p, X_q \rangle_{L^2(\mu)} = \mathbb{E}[X_p X_q] = \int_{[-\pi, \pi]} e^{i(p-q)t} \, d\sigma(t),
\]

où \(\sigma\) est une mesure finie symétrique sur \([-\pi, \pi]\), appelée mesure spectrale du système. Un tel système est construit canoniquement en prenant \(\Omega = \mathbb{R}, X_p\) étant la projection sur la première coordonnée, \(T\) le décalage des coordonnées et \(\mu\) la probabilité sur \(\mathbb{R}\) qui donne au processus \((X_p)\) la loi voulue. On pourra toujours supposer dans la suite que le modèle utilisé est celui-ci. On suppose aussi le système ergodique, ce qui équivaut à

\[
\forall t \in [-\pi, \pi], \quad \sigma(t) = 0.
\]

Pour une présentation détaillée de ces systèmes, on peut par exemple consulter [1].

On s'intéresse ici aux facteurs d'un tel système, c'est-à-dire aux sous-tribus \(\mathcal{F}\) de \(\mathcal{A}\) qui sont \(T\)-invariantes. Rappelons que chaque facteur \(\mathcal{F}\) de \(T\) définit un système dynamique noté \(T_{\mathcal{F}}\) sur l'espace \(\Omega_{\mathcal{F}}\), obtenu à partir de \(\Omega\) en identifiant les points \(\omega\) et \(\omega'\) tels que \(\forall F \in \mathcal{F}, 1_{F}(\omega) = 1_{F}(\omega')\).