[R] H. P. Rosenthal, *On relatively disjoint families of measures with some applications to Banach space theory*, Studia Math. 37 (1970), 13–36.
[S] V. A. Shestakov, *Complex interpolation in Banach spaces of measurable functions*, Vestnik Leningrad. Univ. 19 (1974), 64–68 (in Russian); English transl.: Vestnik Leningrad Univ. Math. 7 (1979).

Faculty of Mathematics and Computer Science
A. Mickiewicz University
Matejki 48/49
60-769 Poznań, Poland
and
Mathematical Institute
Polish Academy of Sciences
Poznań Branch
Matejki 48/49
60-769 Poznań, Poland
E-mail: mastylo@math.amu.edu.pl

Faculty of Mathematics
Voronezh University
Universitetskaya pl., 1
394693 Voronezh, Russia
E-mail: vio@func.vsu.ru

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On convergence for the square root of the Poisson kernel in symmetric spaces of rank 1

by

JAN-OLAV RÖNNING (Skövde)

Abstract. Let $P(z,\beta)$ be the Poisson kernel in the unit disk $\mathbb{U}$, and let $P_\lambda f(z) = \int_{\partial\mathbb{U}} P(z,\varphi)^{1/2+\lambda} f(\varphi)\,d\varphi$ be the λ-Poisson integral of f, where $f\in L^p(\partial\mathbb{U})$. We let $\mathcal{P}_\lambda f$ be the normalization $P_\lambda f/P_\lambda 1$. If $\lambda>0$, we know that the best (regular) regions where $\mathcal{P}_\lambda f$ converges to f for a.a. points on $\partial\mathbb{U}$ are of nontangential type.

If $\lambda = 0$ the situation is different. In a previous paper, we proved a result concerning the convergence of $\mathcal{P}_0 f$ toward f in an L^p weakly tangential region, if $f\in L^p(\partial\mathbb{U})$ and $p>1$. In the present paper we will extend the result to symmetric spaces X of rank 1. Let f be an L^p function on the maximal distinguished boundary K/M of X. Then $\mathcal{P}_0 f(x)$ will converge to $f(kM)$ as x tends to kM in an L^p weakly tangential region, for a.a. $kM\in K/M$.

1. Introduction. Let $X = G/K$ be a Riemannian symmetric space of noncompact type and of rank 1. (The notation is explained in Section 2.) On X, we consider the λ-Poisson operator

$$P_\lambda f(g\cdot o) = \int_{K/M} f(kM)P^{\lambda+\varrho}(kM,g)\,dkM,$$

where $P(kM,g)$ is the Poisson kernel of G/K, $f\in L^p(K/M)$, and $\lambda+\varrho\in\mathfrak{a}$. We know that $P_\lambda f$ satisfies the equation

$$\Delta P_\lambda f = (|\lambda|^2 - |\varrho|^2)P_\lambda f,$$

where Δ is the Laplace–Beltrami operator on X.

If $\lambda\ge 0$, it is known that $P_\lambda f(g)$ does not necessarily converge to $f(kM)$ as g tends to kM. To obtain convergence, we need to consider the normalization $\mathcal{P}_\lambda f = P_\lambda f/P_\lambda 1$. We know that $\mathcal{P}_\lambda f$ converges admissibly to f a.e. on the boundary if $f\in L^p$, $p\ge 1$. In a previous paper, [JOR], we proved that if X is the hyperbolic unit disk $\mathbb{U}$ and $\lambda=0$, we have convergence in a larger region, which we call an L^p *weakly tangential region* $(1\le p<\infty)$.

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These regions increase with p, and if $p = 1$, they are the same as the weakly tangential region considered in [Sjö83, Sjö84, Sjö84a, Sjö88].

In [Sjö88] Sjögren extends his previous L^1 result on $\mathbb{U}$ to general symmetric spaces X of rank 1. In X, the weakly tangential convergence regions are defined as

$$\{n_1 \exp(tH_0)n \cdot x \in X : x \in D,\ n \in B(Ct^q),\ t > c\},$$

where D is a compact subset of X, $q = 1/(2\langle \varrho, H_0\rangle)$, and $B(Ct^q)$ are balls in $\overline{N}$, to be defined in Section 2. It is the factor n which makes these regions larger than the ordinary admissible convergence regions.

We would like to have an intuitive geometric view of the weakly tangential regions. If we only consider admissible convergence regions, that is, ignore the factor n in the above expression, we can consider the region as a tube along the curve $n_1 \exp tH_0$. This is because for each fixed value t_0, we dilate $n_1 \exp t_0 H_0$ with a fixed set D. (Observe that the fact that we approach a boundary point when t tends to infinity gives us the possibility to identify this boundary point with the asymptotic direction of $n_1 \exp tH_0$, which gives one unique boundary point for each tube.) If we now multiply $n_1 \exp tH_0$ with an arbitrary point $n \in B(Ct^q)$, we make a dilation of $n_1 \exp tH_0$ in certain directions determined by $\overline{N}$. The fact that the radius of B increases to infinity with t ensures that this enlargement of the admissible regions does not produce another (wider) tube. Instead, we get a region which increases in width as t tends to infinity.

For $f \in L^p$, $1 < p < \infty$, we will now extend this result, to obtain convergence of $\mathcal{P}_0 f$ to f in the L^p weakly tangential convergence regions

$$\{n_1 \exp(tH_0)n \cdot x \in X : x \in D,\ n \in B(Ct^{pq}),\ t > c\}.$$

The reader should observe that the difference between these regions and the weakly tangential regions is that the radii of the balls are now Ct^{pq} instead of Ct^q. As in the bidisk, these regions are strictly increasing with p, so the result is an extension of the L^1 result of Sjögren.

The structure of this paper is as follows: In Section 2, we explain the notation and give the necessary structure theory of symmetric spaces, including a definition of the λ-Poisson integrals. In Section 3, we state and prove the convergence result. We prove this result by establishing the usual maximal function estimate. The proof of this estimate is rather technical, and rests heavily on (a generalization of) a lemma given by Sjögren in [Sjö83]. The main point is that we split the kernel into small pieces, which gives us control of the corresponding operators.

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2. Preliminaries about symmetric spaces. Let X be a Riemannian symmetric space of noncompact type, with symmetry s_p with respect to a fixed point $p \in X$. Let $\sigma(g) = s_p \circ g \circ s_p^{-1}$, and let $\theta = d\sigma$ be the Cartan involution. Let $\widetilde{K}$ denote the isotropy group of p, which is a subgroup of the isometry group of X. It is well known that we can write X as $X = G/K$, where G is the identity component of the isometry group of X, and $K = \widetilde{K} \cap G$. We write gx (and occasionally $g \cdot x$) for the action of an element of G on X, and we let $o = eK$. Then o is the point p in this representation of X.

Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{k}$ that of K. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition of $\mathfrak{g}$, where $\mathfrak{p}$ is a linear subspace of $\mathfrak{g}$. Let $\mathfrak{a}$ be a maximal abelian subalgebra in $\mathfrak{p}$ and A the corresponding connected Lie group. By means of the Killing form $\langle \cdot, \cdot \rangle$, we can identify $\mathfrak{a}$ with its dual. Then the roots (that is, eigenvalues $\lambda(H)$ of the operator $\mathrm{ad}(H)$) are in $\mathfrak{a}$.

Let $\mathfrak{a}_+ \subset \mathfrak{a}$ be the positive Weyl chamber, that is, a component of the subset of $\mathfrak{a}$ where none of the roots is zero. In the case of a rank 1 space, this is a set of type $\{tH_0 : t > 0\}$ for some fixed but arbitrary $H_0 \in \mathfrak{a}$. A root is called *positive* if it is positive on $\mathfrak{a}_+$. We let $\varrho \in \mathfrak{a}_+$ denote the half-sum of the positive roots, counted with multiplicity.

Let $\mathfrak{g}_\lambda$ be the root subspace (i.e., eigenspace of $\mathrm{ad}(H)$) corresponding to the root λ. Let $\mathfrak{n} = \bigoplus \mathfrak{g}_\lambda$, where the sum is taken over the positive roots, and let N be the corresponding connected Lie group. We let $\overline{N}$ be the image of N under the Cartan involution. We can now make an Iwasawa decomposition KAN (or, equivalently, $\overline{N}AK$, $KA\overline{N}$, etc.) of G. Thus an element $g \in G$ can be written uniquely as $g = k(g)\exp(H(g))n(g)$, with $k(g) \in K$, $H(g) \in \mathfrak{a}$, and $n(g) \in N$.

Let M be the centralizer of A in K. The (maximal distinguished) boundary of X is K/M. The Bruhat map $n \to k(n)M$ is a diffeomorphism of $\overline{N}$ onto almost all of K/M, with respect to the normalized invariant measure dkM in K/M.

Let H vary in $\mathfrak{a}$. If $\langle \alpha, H\rangle$ tends to infinity, for all positive roots α, we say that H *tends to infinity*. This gives a natural meaning to the expression "large H". In the case of a rank 1 space, this of course just means that t is large. We say that a function $\psi : \mathfrak{a} \to \mathbb{R}$ is *increasing* if $H - H' \in \mathfrak{a}_+$ implies $\psi(H) \geq \psi(H')$. We say that $g = k\exp(H) \cdot o$ (or $g = n_1 \exp(H) \cdot o$ in another notation) tends to $kM \in K/M$ if $H \to \infty$. We let $n^H = \exp(H)n\exp(-H)$, where $n \in \overline{N}$ and $H \in \mathfrak{a}$.

For $\lambda \in \mathfrak{a}$, the *λ-Poisson integral* of $f \in L^p(K/M)$, $1 \le p \le \infty$, is

$$P_\lambda f(g) = P_\lambda f(g\cdot o) = \int_{K/M} f(kM) e^{-\langle \lambda+\varrho, H(g^{-1}k)\rangle}\, dkM, \qquad g \in G.$$

As a function in X, we know that $P_\lambda f$ is an eigenfunction of the Laplace–Beltrami operator. In fact, it is a joint eigenfunction of all G-invariant differential operators on X. Hence $P_\lambda f(g)$ satisfies Harnack's inequality:

$$P_\lambda f(g\cdot x) \sim P_\lambda f(g\cdot x'),$$

where x and x' stay in a compact subset of X, and the inequalities are uniform in g.

In order to get convergence to f in any meaning when we approach the boundary K/M, we need a normalization of $P_\lambda f$. This is done by dividing $P_\lambda f$ by $P_\lambda 1$, which gives us the *normalized Poisson integral*

$$\mathcal{P}_\lambda f(g) = P_\lambda f(g)/P_\lambda 1(g).$$

We can transform the Poisson integral $P_\lambda f$ to an integral defined on $\overline{N}$. If $g\cdot o = n_1 a_1 \cdot o$, with $n_1 \in \overline{N}$, $a_1 \in A$, we have

$$P_\lambda f(n_1 a_1) = \int_{\overline{N}} f(k(n)M) e^{-\langle \lambda+\varrho, H((n_1a_1)^{-1}n)\rangle} e^{\langle \lambda-\varrho, H(n)\rangle} dn,$$

where dn is the Haar measure in $\overline{N}$. (See [Sjö84, pp. 49–50] for references.) We will replace $f(k(n)M)$ by $f(n)$, to simplify the notation, and work only with functions on $\overline{N}$.

To work with $\mathcal{P}_\lambda f(n_1a_1)$, we need estimates of $P_\lambda 1(n_1a_1)$. For $\lambda = 0$, which is the case we are working with here, these are given in other works of Sjögren ([Sjö83, Sjö84, Sjö84a, Sjö88]) in terms of the function

$$\psi(H) = e^{\langle \varrho, H\rangle} P_0 1(\exp H).$$

If $H = tH_0$, as is the case for spaces of rank 1, then $\psi(H)$ is bounded if $t \le 0$, and $\psi(H) \sim t$ if $t > 0$.

We also need the fact that $P_\lambda 1$ is biinvariant, i.e., both left and right K-invariant.

Let $a_1 = \exp(H)$, $H \in \mathfrak{a}_+$, and let $n_1 = kan \in KAN$. We first observe that if n_1 stays in a compact set in $\overline{N}$, so do k, a, and n. Thus, n^{-H} stays bounded for large values of H. We can write $n_1\exp(H) = k\exp(H)an^{-H}$. Because of Harnack's inequality and the biinvariance of $P_\lambda 1$, we then have

$$P_0 1(n_1 \exp(H)) = P_0 1(k\exp(H)an^{-H}) \sim P_0 1(\exp(H)) = \psi(H) e^{-\langle \varrho, H\rangle}.$$

Let H_0 be a fixed element in $\mathfrak{a}_+$. We can find a function $|\cdot| : \overline{N} \to \mathbb{R}_+$ which satisfies the following conditions: $|\cdot|$ is a smooth function outside e

and vanishes only at e. It also satisfies $|n^{-1}| = |n|$ and

$$|n^{tH_0}| = e^{-t}|n|, \qquad t \in \mathbb{R}.$$

The function $|\cdot|$ is called a *smooth homogeneous gauge* on $\overline{N}$. The gauge satisfies the following (quasinorm) inequality:

$$|nn'| \le C(|n| + |n'|).$$

Let $B(r)$ stand for the ball $\{n : |n| \le r\}$. Then

$$|B(r)| \text{ is proportional to } r^{2\langle \varrho, H_0\rangle},$$

where $|B(r)|$ denotes the Haar measure of $B(r)$.

If X is of rank 1, then N is a step two nilpotent Liegroup and its Lie algebra $\mathfrak{n}$ can be written in the form $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$. If $n = \exp(X+Y)$, $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_{2\alpha}$, then an appropriate gauge is given by

$$|n| = (|X|^4 + |Y|^2)^{1/4},$$

provided $\alpha(tH_0) = t$ (see [Kor85]).

Let

$$P(n) = e^{-2\langle \varrho, H(n)\rangle}$$

be the Poisson kernel in $\overline{N}$. Given the gauge above, we have the following result (see [He78, Theorem IX.3.8, p. 414] or [DR92, p. 239]) for a rank 1 space:

$$P(n) \sim (1+|n|)^{-4\langle \varrho, H_0\rangle}.$$

This will be the estimate of the Poisson kernel we use most in this paper.

Finally, we have the notational convention that C, C', C'' and c denote various positive constants, which need not be equal even in the same formula. Hopefully, the context will clarify which constants are equal.

3. Convergence in rank 1 spaces. We begin by stating a generalization of Lemma 1 in [JOR] to a general symmetric space:

LEMMA 3.1. *Let $L \subset \overline{N}$ be a compact set, and let $p > 1$ be given. Assume that the sublinear operators $(T_k)_{k=1}^\infty$ are defined on $L^p(L)$, that they take values which are nonnegative measurable functions on L, and satisfy the following conditions for some $C_1 < \infty$:*

(a) *Each T_k is of weak type (p,p) with constant at most C_1.*

(b) *Each T_k, $k = 1, 2\ldots$, is given by $T_k f(n) \equiv \sup_{i\in I_k} |f| * K_i(n)$, where I_k is an index set and the kernels K_i satisfy $\operatorname{supp} K_i \subset B(\gamma_i)$.*

(c) $\int K_i^*(n)\, dn \le C_1$.

Here, for $i \in I_k$, $K_i^*(n) = \sup_{n' \in B(\gamma_{k+N'})} K_i(nn')$, for some natural number N', and the positive numbers γ_k satisfy the following condition: γ_k is de-

creasing and $\gamma_k^{1/\alpha} < \gamma_{k-C}$ *for some* $C(\alpha) > 0$ *and all fixed* $\alpha > 0$. *Here* α *will depend on the group* $\overline{N}$.

Then the operator $Tf \equiv \sup_k T_k f$ *is of weak type* (p,p) *with constant only depending on* L, N' *and* C_1.

The proof is almost identical to that in [JOR], so there is no need to give it here.

We now state the main theorem of this paper.

THEOREM 3.2. *Let* $\operatorname{rank} X = 1$. *Let* H_0 *be a fixed element in* $\mathfrak{a}_+$ *and let* $q = 1/(2\langle \varrho, H_0\rangle)$. *Let* $f \in L^p(K/M)$, $p \geq 1$. *Then, for almost all* $n_1 \in \overline{N}$,

$$\mathcal{P}_0 f(n_1 \exp(tH_0)n' \cdot x) \to f(k(n_1)M)$$

as $t \to \infty$, x *stays in a compact subset of* X, *and* n' *stays in the ball* $B(Ct^{pq})$ *for some fixed constant* C.

With the same methods as in [Sjö84] we see that we can assume f to have compact support L. Harnack's inequality implies that it is enough to investigate the convergence of $\mathcal{P}_0 f(n_1 \exp(tH_0)n' \cdot o)$ to the function $f(k(n_1)M)$ for a.a. $n_1 \in \overline{N}$. This will follow from standard methods if we can prove the following theorem:

THEOREM 3.3. *For any* C', *the operator*

$$M_p f(n_1) = \sup_{t,n' : t \geq C, \, n' \in B(C't^{pq})} \mathcal{P}_0|f|(n_1 \exp(tH_0)n')$$

is of weak type (p,p), $p \geq 1$.

Proof. For simplicity, we assume that $C' = 1$ and $f \geq 0$, which is no restriction.

For $p = 1$, the proof is given in [Sjö88]. Let $p > 1$. Because

$$\langle \alpha, H\rangle \sim |H| = \langle H, H\rangle^{1/2}$$

for large H, Proposition 3.3 in [Sjö88] and the definition of $M_p f$ imply that it is enough to consider the operator

$$M'_p f(n_1) = \sup_{t,n' : t > C', \, |n'| \leq t^{pq}} t^{-1} e^{2\langle \varrho, tH_0\rangle} \int f(n) \widetilde{P}^{1/2}((n^{-1} n_1 (n')^{tH_0})^{-tH_0}) \, dn$$

where $\widetilde{F} = F(n^{-1})$ and $P(n) = e^{-2\langle \varrho, H(n)\rangle}$ is the Poisson kernel in $\overline{N}$. In order to get the desired estimates, we split $\overline{N}$ in two parts for each t:

$$N_1 = \{n \in \overline{N} : |n^{-tH_0}| \geq Ct^{pq}\} \quad \text{and} \quad N_2 = \{n \in \overline{N} : |n^{-tH_0}| < Ct^{pq}\},$$

with $C \geq 2$. We will now show that the operators M_p^1 and M_p^2, given by the restrictions of the integral in the definition of $M'_p f$ to these sets, are of weak type (p,p).

Consider first $M_p^1 f$. We see that if $n \in N_1$, then $|n^{-tH_0} n'| \sim |n^{-tH_0}|$. If we make the partition

$$\{n : 2^k t^{pq} \leq |n^{-tH_0}| \leq 2^{k+1} t^{pq}\}, \qquad k = 1, 2, \ldots,$$

we easily see that we can estimate $M_p^1 f$ by $\sup f * (I + Q)$, where the sup is taken over the same set as in the definition of $M'_p f$,

$$I = t^{-1} e^{2\langle \varrho, tH_0\rangle} \sum_{k=1}^{M(p,q,t)} (1 + 2^k t^{pq})^{-2\langle \varrho, H_0\rangle} \chi_{|n^{-tH_0}| \leq 2^{k+1} t^{pq}},$$

$$Q = t^{-1} e^{2\langle \varrho, tH_0\rangle} (1 + |n^{-tH_0}|)^{-2\langle \varrho, H_0\rangle} \chi_{e^{2t} t^{pq} \leq |n^{-tH_0}|},$$

and

$$M(p,q,t) = O(\log(e^t t^{pq^2})) = O(t).$$

The Haar measure of the sets $\{n : |n^{-tH_0}| \leq 2^{k+1} t^{pq}\}$ is proportional to $2^{(k+1)2\langle \varrho, H_0\rangle} e^{-2\langle \varrho, tH_0\rangle} t^p$, and thus we have

$$I \leq C' t^{-1} \sum_{k=1}^{M(p,q,t)} \frac{1}{2^{(k+1)2\langle \varrho, H_0\rangle} e^{-2\langle \varrho, tH_0\rangle} t^p} \chi_{|n| \leq 2^{k+1} e^{-t} t^{pq}}$$

$$= C' t^{-1} \sum_{k=1}^{M(p,q,t)} \frac{1}{|\{n : |n| \leq 2^{k+1} e^{-t} t^{pq}\}|} \chi_{|n| \leq 2^{k+1} e^{-t} t^{pq}}.$$

This gives

$$\sup_t f * I(n_1) \leq C' \frac{1}{t} \sum_{k=1}^{O(t)} Mf(n_1) \leq CMf(n_1).$$

We also have

$$Q \leq t^{-1} e^{2\langle \varrho, tH_0\rangle} (1 + |n^{-tH_0}|)^{-2\langle \varrho, H_0\rangle},$$

which is the standard estimate for the kernel of $\mathcal{P}_0$ with the usual admissible convergence regions. The corresponding maximal function of this estimate is of weak type (p,p), $1 \leq p \leq \infty$, a fact which has been well known for many years.

These inequalities shows that M_p^1 is dominated by a sum of two weak type (p,p) operators and thus is of weak type (p,p).

To show that M_p^2 is of weak type (p,p) is more difficult. Most of the difficulties come from the fact that in N_2 we cannot get rid of n' in the same easy way as in N_1. Instead, we will make a partition of N_2 and apply the lemma to the corresponding operators. We have

$$M_p^2 f(n_1)$$

$$= \sup_{t,n' : t > C', |n'| \le t^{pq}} t^{-1} e^{2\langle \varrho, tH_0\rangle} f * \widetilde{P}_H^{1/2} \chi_{|n^{-tH_0}| \le Ct^{pq}} (n_1 (n')^{tH_0})$$

$$= \sup_{t,n' : t > C', |n'| \le t^{pq}} t^{-1} e^{2\langle \varrho, tH_0\rangle} f * \widetilde{P}_H^{1/2} \chi_{|n| \le Ce^{-t}t^{pq}} (n_1 (n')^{tH_0})$$

$$\sim \sup_{\substack{t,n' :\\ t > C', |n'| \le t^{pq}}} t^{-1} e^{2\langle \varrho, tH_0\rangle} f * (1 + |n^{-tH_0}|)^{-2\langle \varrho, H_0\rangle} \chi_{|n| \le Ce^{-t}t^{pq}} (n_1 (n')^{tH_0})$$

$$= \sup_{t,n' : t > C', |n'| \le e^{-t}t^{pq}} t^{-1} e^{2\langle \varrho, tH_0\rangle} f * (1 + e^t |n|)^{-2\langle \varrho, H_0\rangle} \chi_{|n| \le Ce^{-t}t^{pq}} (n_1 n').$$

In the last equality, we made the transformation $(n')^{tH_0} \to n'$. We discretize the last estimate by considering the partition

$$A_k = \{n : 2^{k-1} t^q e^{-t} \le |n| \le 2^k t^q e^{-t}\}, \qquad k = 0, 1, \ldots, N(p, q, t),$$

where $N(p, q, t) = O(\log t)$. This gives

$$M_p^2 f(n_1) \le C \sup_{t,n' : t > C', |n'| \le e^{-t}t^{pq}} t^{-1} e^{2\langle \varrho, tH_0\rangle} \times \sum_{k=0}^{N(p,q,t)} f * (1 + 2^{k-1} t^q)^{-2\langle \varrho, H_0\rangle} \chi_{|n| \le C'' 2^k e^{-t} t^q} (n_1 n')$$

$$\le C \sup_{t,n' : t > C', |n'| \le e^{-t}t^{pq}} t^{-1} \times \sum_{k=0}^{N(p,q,t)} f * \frac{1}{2^{(k-1)2\langle \varrho, H_0\rangle} t e^{-2\langle \varrho, tH_0\rangle}} \chi_{|n(n')^{-1}| \le C'' 2^k e^{-t} t^q} (n_1).$$

Let $B_j = \{t : 2^j \le t \le 2^{j+1}\}$, $j \ge N_0$, for some N_0 large enough. If $t \in B_j$, then $N(p, q, t) \le C(j + 1)$ and

$$M_p^2 f(n_1) \le C \sup_{j \ge N_0} \sup_{t,n' : t \in B_j, |n'| \le 2^{jq(p-1)} e^{-t} t^q} \frac{1}{2^j} \times \sum_{k=0}^{C(j+1)} f * \frac{1}{2^{(k-1)2\langle \varrho, H_0\rangle} t e^{-2\langle \varrho, tH_0\rangle}} \chi_{|n(n')^{-1}| \le C'' 2^k e^{-t} t^q} (n_1)$$

$$\equiv C \sup_j T_j f(n_1).$$

Now we will use the lemma. It is obvious that $T_j f$ is a measurable, nonnegative function on L for $f \in L^p$. We have

$$\frac{1}{2^{(k-1)2\langle \varrho, H_0\rangle} t e^{-2\langle \varrho, tH_0\rangle}} \chi_{|n(n')^{-1}| \le C'' 2^k e^{-t} t^q} \le \frac{2^{jp}}{2^{(k-1)2\langle \varrho, H_0\rangle} e^{-2\langle \varrho, tH_0\rangle}} \times \frac{1}{2^{jp} t} \chi_{|n(n')^{-1}| \le C' 2^{jpq} e^{-t} t^q},$$

which implies

$$T_j = \frac{1}{2^j} \sum_{k=0}^{C(j+1)} T_{jk}$$

with

$$T_{jk} \equiv \sup_{\substack{t,n' :\\ t \in B_j, |n'| \le 2^{jq(p-1)} e^{-t} t^q}} f * \frac{1}{2^{(k-1)2\langle \varrho, H_0\rangle} t e^{-2\langle \varrho, tH_0\rangle}} \chi_{|n(n')^{-1}| \le C'' 2^k e^{-t} t^q} (n_1)$$

$$\le \frac{2^{jp}}{2^{(k-1)2\langle \varrho, H_0\rangle}} \sup_{\substack{t,n' :\\ t \in B_j, |n'| \le 2^{jq(p-1)} e^{-t} t^q}} f * \frac{e^{2\langle \varrho, tH_0\rangle}}{2^{jp} t} \chi_{|n(n')^{-1}| \le C' 2^{jpq} e^{-t} t^q} (n_1)$$

$$\le \frac{2^{jp}}{2^{(k-1)2\langle \varrho, H_0\rangle}} \sup_{t \in B_j} f * \frac{e^{2\langle \varrho, tH_0\rangle}}{2^{jp} t} \chi_{|n| \le C 2^{jpq} e^{-t} t^q} (n_1)$$

$$\le C \frac{2^{jp}}{2^{(k-1)2\langle \varrho, H_0\rangle}} M f(n_1).$$

This calculation shows that T_{jk} is of weak type $(1,1)$, with constant $C 2^{jp} / 2^{(k-1)2\langle \varrho, H_0\rangle}$. It is easy to see that T_{jk} is bounded on L^∞, uniformly in j and k, so that Marcinkiewicz' interpolation theorem gives

$$\|T_{jk}\|_r \le C(p, r) \left(\frac{2^{jp}}{2^{(k-1)2\langle \varrho, H_0\rangle}} \right)^{1/r}.$$

This implies

$$\|T_j\|_r \le C'(p, r) 2^{j(p/r - 1)} \le C'(p, r) \qquad \text{if } r \ge p.$$

Thus condition (a) in the lemma is satisfied.

In order to deal with condition (b), we define

$$K_{t,n'}(n) \equiv \sum_{k=0}^{N(p,q,t)} \frac{1}{2^j} \cdot \frac{e^{2\langle \varrho, tH_0\rangle}}{2^{(k-1)2\langle \varrho, H_0\rangle} t} \chi_{|n(n')^{-1}| \le C'' 2^k e^{-t} t^q}.$$

It is easy to see that

$$T_j f(n_1)$$
$$= \sup_{t,n' : t\in B_j,\, |n'|\le e^{-t}t^{pq}} f * K_{t,n'}(n_1)$$
$$\le C \sup_{\substack{t,n' :\\ t\in B_j,\, |n'|\le 2^{jq(p-1)}e^{-t}t^q}} \sum_{k=0}^{Cj} \frac{1}{2^j} f * \frac{e^{2\langle \varrho, tH_0\rangle}}{2^{(k-1)2\langle \varrho, H_0\rangle}t} \chi_{|n(n')^{-1}|\le C'' 2^k e^{-t}t^q}(n_1)$$
$$\le C \sup_{t\in B_j} \sum_{k=0}^{Cj} \frac{1}{2^j} f * \frac{e^{2\langle \varrho, tH_0\rangle}}{2^{(k-1)2\langle \varrho, H_0\rangle}t} \chi_{|n|\le C(2^k + 2^{jq(p-1)})e^{-t}t^q}(n_1).$$

Let
$$I_j = \{(t,n') : |n'| \le 2^{jq(p-1)} e^{-t} t^q,\ t \in B_j\}$$
$$\subset \{(t,n') : |n'| \le 2^{jq(p-1)} e^{-2^j} t^q,\ t \in B_j\}$$
$$\subset \{(t,n') : |n'| \le C 2^{jqp} e^{-2^j} t^q,\ t \in B_j\}.$$

Letting N' be the constant (of our choice and independent of j) in the definition of $K^*_{t,n'}(n)$, and letting $\gamma'_j = C2^{jpq}e^{-2^j}$ we see that, for $j > J_0(N')$, γ'_j and I_j satisfy the conditions on γ_k and I_k in the lemma, where $J_0(N')$ depends only on N'. If we modify γ'_j in a suitable way for $j \le J_0(N')$, we get a sequence $\{\gamma_j\}$ which, together with I_j, satisfies the conditions of the lemma for all positive integers j. For $t \in B_j$ we have

$$K^*_{t,n'}(n) \equiv \sup_{m : |m|\le \gamma_{j+N'}} K_{t,n'}(nm)$$
$$= \sup_{m : |m|\le \gamma_{j+N'}} \sum_{k=0}^{N(p,q,t)} \frac{1}{2^j} \cdot \frac{e^{2\langle \varrho, tH_0\rangle}}{t 2^{(k-1)2\langle \varrho, H_0\rangle}} \chi_{|nm(n')^{-1}|\le C'' 2^k e^{-t}t^q}.$$

We have, for $t \in B_j$,

$$\int K^*_{t,n'}(n)\,dn$$
$$= e^{2\langle \varrho, tH_0\rangle} \int \sup_{m : |m|\le \gamma_{j+N'}} \sum_{k=0}^{N(p,q,t)} \frac{1}{2^j} \cdot \frac{1}{t 2^{(k-1)2\langle \varrho, H_0\rangle}} \chi_{|nm(n')^{-1}|\le C'' 2^k e^{-t}t^q}\, dn$$
$$\le \frac{e^{2\langle \varrho, tH_0\rangle}}{t} \int \sum_{k=0}^{N(p,q,t)} \frac{1}{2^j} \cdot \frac{1}{2^{(k-1)2\langle \varrho, H_0\rangle}} \chi_{|n|\le C''(2^k e^{-t}t^q + \gamma_{j+N'})}\, dn$$
$$= \frac{e^{2\langle \varrho, tH_0\rangle}}{t} \sum_{k=0}^{N(p,q,t)} \frac{1}{2^j} \int \frac{1}{2^{(k-1)2\langle \varrho, H_0\rangle}} \chi_{|n|\le C''(2^k e^{-t}t^q + \gamma_{j+N'})}\, dn \le C(p,q,t).$$

In the first inequality, we got rid of n' because of the fact that dn is a translation invariant Haar measure. It follows directly from the definition of

γ_j and the size of $N(p,q,t)$ that $C(p,q,t)$ is bounded with respect to t. Thus all conditions in the lemma are satisfied, and an application of it shows that M^2_p is of weak type (p,p) if $p > 1$. The theorem is proved. ∎

References

[He78] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Pure Appl. Math. 80, Academic Press, 1978.

[DR92] E. Damek and F. Ricci, *Harmonic analysis on solvable extensions of H-type groups*, J. Geom. Anal. 2 (1992), 213–248.

[Kor85] A. Korányi, *Geometric properties of Heisenberg-type groups*, Adv. Math. 56 (1985), 28–38.

[JOR] J.-O. Rönning, *Convergence for square roots of Poisson kernels in weakly tangential regions*, Math. Scand., to appear.

[Sjö83] P. Sjögren, *Fatou theorems and maximal functions for eigenfunctions of the Laplace–Beltrami operator in a bidisk*, J. Reine Angew. Math. 345 (1983), 93–110.

[Sjö84] —, *A Fatou theorem for eigenfunctions of the Laplace–Beltrami operator in a symmetric space*, Duke Math. J. 51 (1984), 47–56.

[Sjö84a] —, *Une remarque sur la convergence des fonctions propres du Laplacian à valeur propre critique*, in: Lecture Notes in Math. 1096, Springer, 1984, 544–548.

[Sjö88] —, *Convergence for the square root of the Poisson kernel*, Pacific J. Math. 131 (1988), 361–391.

Institutionen för Naturvetenskap
Högskolan i Skövde
Box 408
541 28 Skövde, Sweden
E-mail: jan-olav@inv.his.se

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