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On the relation between complex and real methods of interpolation

by

MIECZYSŁAW MASTYŁO (Poznań) and
VLADIMIR I. OVCHINNIKOV (Voronezh)

Abstract. We study those compatible couples of Banach spaces for which the complex method interpolation spaces are also described by the K-method of interpolation. As an application we present counter-examples to Cwikel's conjecture that all interpolation spaces of a Banach couple are described by the K-method whenever all complex interpolation spaces have this property.

0. Introduction. One of the most fundamental problems in the theory of interpolation spaces is the description of all interpolation spaces with respect to a given compatible couple of Banach spaces. In almost all known cases interpolation spaces are K-monotone or, equivalently, are described by the K-method, which means that the norms of such spaces depend monotonically on the K-functional of the couple. Moreover, at the early stages of interpolation theory there was a conjecture that such a description is possible in general. However, it was soon shown (see, for example, [Cw1], [OD]) that there exist couples for which the complex method interpolation spaces are not all described by the K-method. For other examples we refer to [M2] and [MX]. In [Cw2] (see also [CN]) it was conjectured that all interpolation spaces with respect to a Gagliardo complete couple are described by the K-method whenever all lower complex method interpolation spaces are K-monotone.

The main purpose of this paper is to present several examples for which Cwikel's conjecture fails. In particular, it is shown that interpolation with respect to the couple $\overline{X} = \{l_\infty(l_\infty), l_\infty(c_0(2^{-n}))\}$ is not described by the K-method, while the upper complex method interpolation space $[\overline{X}]^\theta$ coincides

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with the extreme real interpolation space $\overline{X}_{\theta,\infty}$ for every $0 < \theta < 1$.

We also construct a couple of weighted sequence Banach lattices with the Fatou property for which all complex interpolation spaces are described by the K-method but for which there exists an interpolation space which is not K-monotone. The spaces of this Banach couple can be used as parameters of the real method. Thus the property of being a counter-example to Cwikel's conjecture can be lifted to couples of any nature, for instance to couples of rearrangement invariant spaces.

It is well known (see, for example, [DKO], [BK]) that every interpolation space is described by the K-method if and only if the interpolation orbit of any element from the sum of the spaces of the Banach couple coincides with the corresponding K-orbit. We prove that for a quite large class of couples of Banach lattices of two-sided sequences the interpolation orbits coincide with the K-orbits for all elements having a quasi-power K-functional.

We note that the counter-examples to Cwikel's conjecture were independently found by both authors of the paper. We are very grateful to Michael Cwikel for his kind suggestion which inspired the authors to prepare this joint paper.

1. Preliminaries and notation. Our notation and terminology is standard and we refer to [BL], [BK] and [O2]. For the reader's convenience, we give some definitions and results that will be used later.

Let $\overline{X} = \{X_0, X_1\}$ be a Banach couple and let X be an intermediate space with respect to $\overline{X}$. We denote by X° the closure of $\Delta(\overline{X}) := X_0 \cap X_1$ in X. The couple $\overline{X}$ is *regular* if $\Delta(\overline{X})$ is dense in X_j, $j = 0, 1$.

Let $\overline{X} = \{X_0, X_1\}$ and $\overline{Y} = \{Y_0, Y_1\}$ be couples of Banach spaces and let $L(\overline{X}, \overline{Y})$ be the Banach space of all linear operators $T : \overline{X} \to \overline{Y}$ (meaning, as usual, that $T : X_0 + X_1 \to Y_0 + Y_1$ is linear and $T : X_j \to Y_j$ boundedly for $j = 0, 1$) equipped with the norm

$$\|T\|_{\overline{X}\to\overline{Y}} := \max\{\|T\|_{X_0\to Y_0}, \|T\|_{X_1\to Y_1}\}.$$

The elements $x \in \Sigma(\overline{X}) := X_0 + X_1$ and $y \in \Sigma(\overline{Y})$ are said to be *orbitally equivalent* with respect to the couples $\overline{X}$ and $\overline{Y}$ if there exist linear operators $T : \overline{X} \to \overline{Y}$ and $S : \overline{Y} \to \overline{X}$ such that $Tx = y$ and $Sy = x$. The couple $\overline{X}$ is a *partial retract* of $\overline{Y}$ if every $x \in \Sigma(\overline{X})$ is orbitally equivalent to some $y \in \Sigma(\overline{Y})$.

Let $\overline{A}$ be any fixed Banach couple and let $a \in A_0 + A_1$. We recall (see e.g. [O2]) that the *interpolation orbit* $\mathrm{Orb}(a, \overline{A} \to \overline{X})$ of a in the couple $\overline{X}$ is the Banach space consisting of all elements of the form $x = Ta$ for some $T : \overline{A} \to \overline{X}$, equipped with the norm

$$\|x\| := \inf\{\|T\|_{\overline{A}\to\overline{X}} : x = Ta\}.$$

So the orbital equivalence of $x \in \Sigma(\overline{X})$ and $y \in \Sigma(\overline{Y})$ with respect to $\overline{X}$ and $\overline{Y}$ means that $x \in \mathrm{Orb}(y, \overline{Y} \to \overline{X})$ and $y \in \mathrm{Orb}(x, \overline{X} \to \overline{Y})$.

The *K-functional* is defined on $\Sigma(\overline{X})$ by

$$K(t, x; \overline{X}) := \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : x = x_0 + x_1\}, \quad t > 0.$$

The *K-orbit* of the element $0 \neq a \in A_0 + A_1$ in the couple $\overline{X}$ is the Banach space $KO(a, \overline{A} \to \overline{X})$ which coincides with the so-called *Marcinkiewicz space* $\overline{X}_{\varphi,\infty}$ determined by the concave function $\varphi = K(\cdot, a; \overline{X})$, i.e., the space of all $x \in X_0 + X_1$ such that

$$\|x\|_{\varphi,\infty} := \sup\{K(t, x; \overline{X})/\varphi(t) : t > 0\} < \infty.$$

The Banach spaces X and Y, intermediate with respect to $\overline{X}$ and $\overline{Y}$ respectively, are said to be *relative K-monotone* whenever $x \in X$ and $y \in \Sigma(\overline{Y})$ with

$$K(t, y; \overline{Y}) \le K(t, x; \overline{X})$$

for all $t > 0$ imply that $y \in Y$, i.e., $KO(x, \overline{X} \to \overline{Y}) \hookrightarrow Y$ for any $x \in X$.

Obviously, relative K-monotone spaces are relative interpolation spaces. If all relative interpolation spaces with respect to $\overline{X}$ and $\overline{Y}$ are relative K-monotone, then we say that interpolation is *described by the K-method* (or equivalently that $\overline{X}$ and $\overline{Y}$ are *relative Calderón couples*). If $\overline{X} = \overline{Y}$, then $\overline{X}$ is said to be a *Calderón couple*.

Clearly, interpolation with respect to $\overline{X}$ and $\overline{Y}$ is described by the K-method if and only if for every $0 \neq x \in X_0 + X_1$ the interpolation orbit $\mathrm{Orb}(x, \overline{X} \to \overline{Y})$ coincides with the corresponding K-orbit $KO(x, \overline{X} \to \overline{Y})$. Thus $\overline{X}$ and $\overline{Y}$ are relative Calderón couples provided for any $x \in \Sigma(\overline{X})$ and $y \in \Sigma(\overline{Y})$ satisfying $K(t, y; \overline{Y}) \le K(t, x; \overline{X})$ for all $t > 0$ there exists an operator $T : \overline{X} \to \overline{Y}$ such that $Tx = y$. If there exists a constant λ, independent of x and y, such that $\|T\|_{\overline{X}\to\overline{Y}} \le \lambda$, then we say that $\overline{X}$ and $\overline{Y}$ are *relative uniform Calderón* couples. If $\overline{X} = \overline{Y}$, then $\overline{X}$ is said to be a *uniform Calderón couple*.

We remark that $\overline{X}$ and $\{l_\infty, l_\infty(2^{-n})\}$ are relative uniform Calderón couples for any Banach couple $\overline{X}$ (see [CP], [Pe]).

It is well known that if $\overline{X}$ and $\overline{Y}$ are relative Calderón couples, then the Gagliardo (relative) completion $\overline{Y}^c := \{Y_0^c, Y_1^c\}$ coincides with $\overline{Y}$ (see for example [BK]). Recall that if X is an intermediate space with respect to $\overline{X}$ then its *Gagliardo completion* X^c is the Banach space of all limits in $X_0 + X_1$ of sequences that are bounded in X.

In what follows for a given Banach lattice and weight ω we denote by $E(\omega)$ the weighted Banach lattice equipped with the natural norm $\|x\|_{E(\omega)} = \|x\omega\|_E$.

The couples $\{l_p, l_p(2^{-n})\}$, $1 \le p \le \infty$, and $\{c_0, c_0(2^{-n})\}$ are denoted, as usual, by $\overline{l}_p$ and $\overline{c}_0$, respectively.

If E is a Banach lattice of sequences on $\mathbb{Z}$ and A is a subset of $\mathbb{Z}$, then the restriction of E to A is denoted by $E|_A$. If $A = \{n \in \mathbb{Z} : n \geq 0\}$, we simply write E^+ instead of $E|_A$.

We note that the assumption that $\overline{X}$ is a uniform Calderón couple is a rather strong condition. In fact, from the K-divisibility theorem of Brudnyĭ and Krugljak [BK] it follows that every interpolation space X with respect to such a couple is a K-space. This means that for some intermediate space E with respect to $\{l_\infty, l_\infty(2^{-n})\}$ we have $X = (X_0, X_1)_E$, where the *K-space* $\overline{X}_E := (X_0, X_1)_E$ consists of all $x \in X_0 + X_1$ with $\{K(2^n, x; \overline{X})\}_{n\in\mathbb{Z}} \in E$. The norm in $\overline{X}_E$ is defined by

$$\|x\|_{\overline{X}_E} := \|\{K(2^n, x; \overline{X})\}\|_E.$$

When $E = l_q(2^{-n\theta})$, $1 \leq q \leq \infty$, we recover the Lions–Peetre scale $\overline{X}_{\theta,q}$. Note that X_0^c and X_1^c are special cases of K-spaces obtained by choosing $E = l_\infty$ and $E = l_\infty(2^{-n})$, respectively.

2. Marcinkiewicz spaces constructed by the complex method.

In this section we study K-monotone spaces. As an application we present examples of non-Calderón couples for which upper complex method interpolation spaces are Marcinkiewicz spaces. Thus these couples are counterexamples to Cwikel's conjecture.

It should be mentioned that a description of Banach couples for which every upper complex method interpolation space is equal to the Marcinkiewicz space is unknown. Clearly, the couple $\{l_\infty, l_\infty(2^{-n})\}$ is among such couples, as well as any of its partial retracts. In [CM1] it is shown that such equalities for couples of Banach function spaces of the form $\{E, L_\infty\}$, or for any couple of weighted spaces $\{E(\omega_0), E(\omega_1)\}$, with E satisfying the Fatou property, imply that those couples are partial retracts of $\{l_\infty, l_\infty(2^{-n})\}$. As we shall see below, there exist Calderón couples of sequence lattices whose upper complex method interpolation spaces are Marcinkiewicz spaces, but which are not partial retracts of $\{l_\infty, l_\infty(2^{-n})\}$.

Let us consider partial retracts of $\{l_\infty, l_\infty(2^{-n})\}$. For completeness we include here a new proof of the following theorem on partial retracts of $\{l_\infty, l_\infty(2^{-n})\}$, which was proved in [CM2] with the help of some factorization results for weakly compact operators. Our proof is based on one-sided interpolation of compact operators by the real method (see [Cw3]).

Recall that a Banach couple is said to be *non-trivial* if $\Delta(\overline{X})$ is not a closed subspace of $\Sigma(\overline{X})$.

THEOREM 1. *Let $\overline{X} = \{X_0, X_1\}$ be a non-trivial Banach couple which is a partial retract of $\overline{l}_\infty = \{l_\infty, l_\infty(2^{-n})\}$. Then both X_0 and X_1 contain subspaces isomorphic to l_∞.*

Proof. For non-trivial couples $\{A_0, A_1\}$ one has $(A_0, A_1)^\circ_{\theta,\infty} \neq (A_0, A_1)_{\theta,\infty}$ for any $\theta \in (0,1)$ (see [CM2]). Therefore, for any $\theta \in (0,1)$ there exists $x = x_\theta \in \overline{X}_{\theta,\infty} \setminus \overline{X}$ and operators $S : \overline{X} \to \overline{l}_\infty$ and $T : \overline{l}_\infty \to \overline{X}$ satisfying $TSx = x$.

Without loss of generality we suppose that X_0 does not contain a subspace isomorphic to l_∞. Then by Rosenthal's result [R], $T : l_\infty \to X_0$ is a weakly compact operator. Thus its restriction to c_0 is a compact operator by the well known facts that c_0 has the Dunford–Pettis property and does not contain a copy of l_1. By applying one-sided interpolation of compact operators by the real method (see [Cw3]) we find that

$$T : (c_0, c_0(2^{-n}))^\circ_{\theta,\infty} \to (X_0, X_1)^\circ_{\theta,\infty}$$

is compact. This implies that

$$T : (l_\infty, l_\infty(2^{-n}))_{\theta,\infty} \to (X_0, X_1)^\circ_{\theta,\infty},$$

because

$$(l_\infty, l_\infty(2^{-n}))_{\theta,\infty} = l_\infty(2^{-n\theta}) = (c_0, c_0(2^{-n}))_{\theta,\infty} = ((c_0, c_0(2^{-n}))^\circ_{\theta,\infty})^c.$$

Interpolation yields

$$x = STx \in (X_0, X_1)^\circ_{\theta,\infty}.$$

This contradiction completes the proof.

To state the next result on partial retracts of $\{l_\infty, l_\infty(2^{-n})\}$ we need the following class $\mathcal{X}$ of couples. A Banach couple $\overline{X}$ belongs to $\mathcal{X}$ if there exists a constant $C > 0$ such that for any $s > 0$ we can find $x_s \in X_0 + X_1$ satisfying

$$\min(1, t/s) \leq K(t, x_s; \overline{X}) \leq C \min(1, t/s) \tag{1}$$

for all $t > 0$.

We also need another class $\mathcal{X}_0$ of couples $\overline{X}$ for which there exist $t_0 > 0$ and $C > 0$ such that for any $x \in X_0 + X_1$ and $t \geq t_0$,

$$K(t, x; \overline{X}) = K(t_0, x; \overline{X}),$$

and for any $s \in (0, t_0]$ we can find $x_s \in X_0 + X_1$ satisfying (1).

Examples of couples in $\mathcal{X}$ are $\overline{l}_p$, $\overline{c}_0$ and couples of Lebesgue spaces $\{L_{p_0}(\mathbb{R}), L_{p_1}(\mathbb{R})\}$ with $p_0 \neq p_1$. For more examples of couples in $\mathcal{X}$ or $\mathcal{X}_0$ we refer to [M2] and [MX].

The following result is motivated by considerations in [MX], where K-monotone spaces with respect to the couple $l_\infty(\overline{X}) := \{l_\infty(X_0), l_\infty(X_1)\}$ were studied. Here, for a Banach space X, $l_\infty(X)$ denotes the Banach space of X-valued two-sided sequences with the natural norm

$$\|\{x_n\}\| := \sup_{n\in\mathbb{Z}} \|x_n\|_X.$$

LEMMA 1. *Assume* $\{X_0, X_1\}$ *is a Gagliardo complete couple in* $\mathcal{X}$. *If* $\{l_\infty(X_0), l_\infty(X_1)\}$ *and* $\{Y_0, Y_1\}$ *are relative Calderón couples, then* $\{Y_0, Y_1\}$ *is a partial retract of* $\{l_\infty, l_\infty(2^{-n})\}$.

Proof. Fix $0 \neq y \in Y_0 + Y_1$. By the concavity of $\varphi = K(\cdot, y; \overline{Y})$, for any $t > 0$ we have

$$\sup_{n\in\mathbb{Z}} \varphi(2^n)\min(1, t/2^n) \le \varphi(t) \le 2\sup_{n\in\mathbb{Z}} \varphi(2^n)\min(1, t/2^n).$$

Now choose a sequence $\{x_n\}$ in $X_0 + X_1$ such that

$$\min(1, t/2^n) \le K(t, x_n; \overline{X}) \le C\min(1, t/2^n)$$

for any $t > 0$ and $n \in \mathbb{Z}$, where C is a constant depending only on $\overline{X}$. Thus

$$K(t, y; \overline{Y}) \le \sup_{n\in\mathbb{Z}} K(t, s_n x_n; \overline{X}) \le K(t, \{s_n x_n\}; l_\infty(\overline{X})),$$

where $\{s_n\} = \{2\varphi(2^n)\}$. Since $l_\infty(\overline{X})$ and $\overline{Y}$ are relative Calderón couples, there exists an operator

$$T : \{l_\infty(X_0), l_\infty(X_1)\} \to \{Y_0, Y_1\}$$

such that $T(\{s_n x_n\}) = y$. Consider the operator U defined by $U\xi = \{\xi_n x_n\}$ for $\xi = \{\xi_n\} \in l_\infty + l_\infty(2^{-n})$. By the Gagliardo completeness of $\overline{X}$ we have

$$\|x_n\|_{X_j} = \lim_{t\to\infty} K(t, x_n; \overline{X})/t^{jn} \le C, \quad n \in \mathbb{Z}, \ j = 0, 1.$$

Hence U maps the couple $\{l_\infty, l_\infty(2^{-n})\}$ boundedly into $\{l_\infty(X_0), l_\infty(X_1)\}$.

Now since $\overline{Y}$ and $\overline{l}_\infty$ are relative Calderón couples (see [CP]) and

$$K(t, \{s_n\}; \overline{l}_\infty) \asymp \sup_{n\in\mathbb{Z}} K(2^n, y; \overline{Y})\min(1, t/2^n) \asymp K(t, y; \overline{Y}),$$

there exists an operator $S : \{Y_0, Y_1\} \to \{l_\infty, l_\infty(2^{-n})\}$ such that $Sy = \{s_n\}$. Hence we have

$$TUSy = TU(\{s_n\}) = T(\{s_n x_n\}) = y.$$

This implies that $y \in Y_0 + Y_1$ is orbitally equivalent to $\{s_n\} \in l_\infty + l_\infty(2^{-n})$. Thus $\overline{Y}$ is a partial retract of $\overline{l}_\infty$. The proof is complete.

COROLLARY 1. *Let* $\overline{X} = \{X_0, X_1\}$ *be a couple in* $\mathcal{X}$ *or in* $\mathcal{X}_0$. *Then* $l_\infty(\overline{X})$ *is a Calderón couple if and only if* $\overline{X}$ *is a partial retract of* $\overline{l}_\infty$.

Proof. If $l_\infty(\overline{X}) = \{l_\infty(X_0), l_\infty(X_1)\}$ is a Calderón couple, then clearly $l_\infty(\overline{X})$ and $\overline{X}$ are relative Calderón couples. Hence $\overline{X}$ is Gagliardo complete. Thus if $\overline{X} \in \mathcal{X}$, Lemma 1 applies. In the case $\overline{X} = \overline{Y} \in \mathcal{X}_0$ it can be shown, as in the proof of Lemma 1, that $\overline{X}$ is a partial retract of $\{l_\infty, l_\infty(2^{-n})\}$. The proof of the converse is obvious, since $\{l_\infty, l_\infty(2^{-n})\}$ is a Calderón couple by [Pe] and $l_\infty(\overline{X})$ is a partial retract of $\{l_\infty, l_\infty(2^{-n})\}$ whenever $\overline{X}$ is a partial retract of $\{l_\infty, l_\infty(2^{-n})\}$.

We now turn to the construction of Banach couples for which Cwikel's conjecture fails. In fact, we are going to show a bit more. We shall see that even the stronger assumption that the scale $[\overline{X}]^\theta$ coincides with the scale $\overline{X}_{\theta,\infty}$ does not imply that $\overline{X}$ is a Calderón couple.

Note that if $[X_0, X_1]^\theta = (X_0, X_1)_{\theta,\infty}$ the same is true for the couple $\{l_\infty(X_0), l_\infty(X_1)\}$, but the latter is not Calderón if $\overline{X} \in \mathcal{X}$ is not a partial retract of $\{l_\infty, l_\infty(2^{-n})\}$. By Bergh's result [B] that $[A_0, A_1]_\theta = ([A_0, A_1]^\theta)^\circ$ isometrically for any complex Banach couple $\{A_0, A_1\}$, we conclude that

$$[l_\infty(X_0), l_\infty(X_1)]_\theta = (l_\infty(X_0), l_\infty(X_1))^\circ_{\theta,\infty}$$

for every $0 < \theta < 1$. This means that all complex interpolation spaces with respect to $\{l_\infty(X_0), l_\infty(X_1)\}$ are described by the K-method. So we obtain the following.

COROLLARY 2. *Let* $\{X_0, X_1\} \in \mathcal{X}$ *be a couple of complex Banach spaces such that* $[X_0, X_1]^\theta = (X_0, X_1)_{\theta,\infty}$ *for every* $0 < \theta < 1$ *and* $\overline{X}$ *is not a partial retract of* $\overline{l}_\infty$. *Then* $\{l_\infty(X_0), l_\infty(X_1)\}$ *is a counter-example to Cwikel's conjecture.*

The conditions of the corollary turn out to be consistent.

THEOREM 2. *The Gagliardo completion of the couple* $\{c_0, c_0(2^{-n})\}$ *is a Calderón couple which is not a partial retract of* $\{l_\infty, l_\infty(2^{-n})\}$.

Proof. We first observe that for any operator $T : \{l_\infty, l_\infty(2^{-n})\} \to \{l_\infty, l_\infty(2^{-n})\}$ the restriction of T to $\Delta(\overline{c}_0)$ maps $\Delta(\overline{c}_0)$, equipped with the norm of $c_0(2^{-jn})$, continuously into $l_\infty(2^{-jn}) \cap c_0(\min(1, 2^{-n}))$, $j = 0, 1$. Now, by the definition of Gagliardo completion, it is easy to check that

$$c_0(2^{-jn})^c = l_\infty(2^{-jn}) \cap c_0(\min(1, 2^{-n})), \quad j = 0, 1, \tag{2}$$

with equivalent norms. This implies, by the regularity of $\overline{c}_0$, that $T : \overline{c}_0 \to \overline{c}_0^c$. In consequence,

$$T : \{c_0^c, c_0(2^{-n})^c\} \to \{c_0^c, c_0(2^{-n})^c\}.$$

Hence, as $\overline{l}_\infty$ is a Calderón couple (see [Pe]) and the equivalences

$$\sup_{n\in\mathbb{Z}} |\xi_n|\min(1, t/2^n) \asymp K(t, \xi; \overline{l}_\infty) \asymp K(t, \xi; \overline{c}_0) = K(t, \xi; \overline{c}_0^c)$$

hold for every $\xi = \{\xi_n\} \in c_0 + c_0(2^{-n})$, we conclude that $\overline{c}_0^c$ is a Calderón couple.

In order to show the second part of the theorem, we first observe that, by (2), the natural map P defined by

$$P\{\xi_n\} = \{\ldots, 0, \xi_0, \ldots, \xi_n, \ldots\}$$

is a continuous projection of the couple $\overline{c}_0^c = \{c_0^c, c_0(2^{-n})^c\}$ onto the couple $\{l_\infty^+, c_0^+(2^{-n})\}$. In particular, this implies that $\{l_\infty^+, c_0^+(2^{-n})\}$ is a partial retract of $\overline{c}_0^c$. Thus, if $\overline{c}_0^c$ were a partial retract of $\{l_\infty, l_\infty(2^{-n})\}$, then clearly $\{l_\infty^+, c_0^+(2^{-n})\}$ would also be a partial retract of $\{l_\infty, l_\infty(2^{-n})\}$. But $c_0^+(2^{-n})$ is separable, so we arrive at a contradiction with Theorem 1. This completes the proof.

So we see that the couple $\{l_\infty(c_0^c), l_\infty(c_0(2^{-n})^c)\}$ constitutes a counterexample to Cwikel's conjecture.

Remark. Notice that the Gagliardo completion of $\overline{c}_0 = \{c_0, c_0(2^{-n})\}$ is a K-closed subcouple of $\{l_\infty, l_\infty(2^{-n})\}$ in the sense of Pisier [Pi]. Clearly, by (2), it follows that $\overline{c}_0$ is a K-closed subcouple of $\{E_0, E_1\} = \overline{c}_0^c$ and that the second Köthe dual E_j'' is equal to $l_\infty(2^{-nj})$, $j = 0, 1$. Thus $\{E_0, E_1\}$ is a Calderón couple of lattices without the Fatou property such that any upper complex method interpolation space between E_0 and E_1 coincides with the Marcinkiewicz space but the couple is not a partial retract of $\{l_\infty, l_\infty(2^{-n})\}$.

3. Parameters of the real method. We recall that a Banach lattice F of sequences is said to be a *parameter of the real method* if $\Delta(\overline{l}_\infty) \subset F \subset \Sigma(\overline{l}_1)$ and $T : F \to F$ for any $T : \overline{l}_1 \to \overline{l}_\infty$. It is easily seen that F is a parameter of the real method if and only if the *Calderón operator* $\mathcal{P}$ defined by

$$\mathcal{P}(\xi)_n := \sum_{k=-\infty}^{\infty} \min(1, 2^{n-k})\xi_k$$

is bounded in F. For example, if E is any translation invariant Banach lattice of two-sided sequences (i.e., $\|\{\xi_{n-k}\}_n\|_E = \|\{\xi_n\}_n\|_E$ for all $k \in \mathbb{Z}$), then $F = E(2^{-n\theta})$ is a parameter of the real method for all $0 < \theta < 1$. Indeed (see [A] for more general results),

$$\|\{\mathcal{P}(\xi)_n\}\|_{E(2^{-n\theta})} = \Big\|\Big\{\sum_k \min(1, 2^k)\xi_{n-k}\Big\}\Big\|_{E(2^{-n\theta})} \le C\|\xi\|_{E(2^{-n\theta})}$$

for all $\xi = \{\xi_n\} \in E(2^{-n\theta})$, where $C = \sum_{k=-\infty}^{\infty} \min(1, 2^k)2^{-k\theta} < \infty$.

It is worthwhile to note that reiteration for spaces $\overline{X}_{E_0}, \overline{X}_{E_1}$, where E_0 and E_1 are parameters, takes place in the most general form (see [DO], [BK]). Namely, for any interpolation functor $\mathcal{F}$ we have

$$\mathcal{F}(\overline{X}_{E_0}, \overline{X}_{E_1}) = \overline{X}_{\mathcal{F}(E_0, E_1)}.$$

In this section we present a construction of non-Calderón couples of parameters of the real method having the Fatou property such that all complex method interpolation spaces are described by the K-method.

First we observe that interpolation from $\{l_\infty, l_\infty(2^{-n})\}$ to $\{l_1, l_1(2^{-n})\}$ is not described by the K-method, which follows for instance from Theorem 1. This can also be deduced from the description of the interpolation orbit of any element $a \in l_\infty + l_\infty(2^{-n})$ in the couple $\{l_1, l_1(2^{-n})\}$, which is equal to

the space $l_1(\varphi^*(2^{-n}))$ (see [O1], [O2]), where $\varphi(t) = K(t, a; \{l_\infty, l_\infty(2^{-n})\})$ and $\varphi^*(t) = 1/\varphi(1/t)$. So if we take $a_\theta = \{2^{n\theta}\}$ with $0 < \theta < 1$, then

$$\mathrm{Orb}(a_\theta; \{l_\infty, l_\infty(2^{-n})\} \to \{l_1, l_1(2^{-n})\}) = l_1(2^{-n\theta}),$$

while the K-orbit of a_θ is $l_\infty(2^{-n\theta})$. Hence the K-orbit is not equal to the orbit, and thus interpolation from $\{l_\infty, l_\infty(2^{-n})\}$ to $\{l_1, l_1(2^{-n})\}$ is not described by the K-method.

Let us consider analogous couples of sequence spaces with non-trivial weights $\alpha = \{\alpha_n\}_{n\in\mathbb{N}}$, $\beta = \{\beta_n\}_{n\in\mathbb{N}}$, and consider interpolation from the couple $\{l_\infty, l_\infty(\alpha)\}$ to $\{l_1, l_1(\beta)\}$.

LEMMA 2. *Interpolation from $\{l_\infty, l_\infty(\alpha)\}$ to $\{l_1, l_1(\beta)\}$ is not described by the K-method for any unbounded weights.*

Proof. Without loss of generality we can assume that $\alpha_n \ge \alpha_0 > 0$, so that $l_\infty(\alpha) \subset l_\infty$. The unboundedness of the weight sequences α and β implies that $l_\infty(\alpha) \ne l_\infty$ and $l_1(\beta) \ne l_1$.

Take any $a \notin l_\infty(\alpha)$ in the closure of $l_\infty(\alpha)$ in l_∞. Then $a \in c_0$ and hence the function $\varphi(t) = K(t, a; \{l_\infty, l_\infty(\alpha)\})$ satisfies $\varphi(t) \to 0$ as $t \to 0$ and $\varphi(t)/t \to 0$ as $t \to \infty$.

Since the interpolation orbit of a with respect to all linear operators mapping $\{l_\infty, l_\infty(\alpha)\}$ into $\{l_1, l_1(\beta)\}$ is $l_1(\varphi^*(\beta))$, we have to show that

$$l_1(\varphi^*(\beta)) \ne (l_1, l_1(\beta))_{\varphi,\infty}.$$

To prove this we construct a sequence ξ such that

$$K(t, \xi; \{l_1, l_1(\beta)\}) \le C\varphi(t)$$

for all $t > 0$, and

$$\sum_{n=0}^{\infty} |\xi_n|\varphi^*(\beta_n) = \infty.$$

It is easily seen that there exists a subsequence β_{n_j} such that $\beta_{n_j} \le \beta_{n_{j+1}}$ and

$$\min\left(\frac{\varphi(\beta_{n_j}^{-1})}{\varphi(\beta_{n_{j+1}}^{-1})}, \frac{\beta_{n_{j+1}}\varphi(\beta_{n_j}^{-1})}{\beta_{n_j}\varphi(\beta_{n_{j+1}}^{-1})}\right) \ge 2 \tag{3}$$

for all $j \ge 1$. If we now take $\xi_n = 0$ for $n \ne n_j$ and $\xi_n = \varphi(1/\beta_n)$ for $n = n_j$, then

$$K(t, \xi; \{l_1, l_1(\beta)\}) = \sum_{j=1}^{\infty} \varphi(1/\beta_{n_j}) \min(1, t\beta_{n_j}).$$

Obviously the sequence $\{\beta_{n_j}^{-1}\}$ is uniformly sparse for φ (see [J]) and therefore

$$K(\beta_{n_j}^{-1}, \xi; \{l_1, l_1(\beta)\}) \asymp \varphi(\beta_{n_j}^{-1})$$

for all j (see [O2], Lemma 4.2.3). Since $K(t,\xi;\{l_1,l_1(\beta)\})$ is linear on every interval $[\beta_{n_j}^{-1},\beta_{n_{j+1}}^{-1}]$ and φ is concave, we deduce that for any $t>0$,

$$K(t,\xi;\{l_1,l_1(\beta)\})\le C\varphi(t).$$

Since

$$\sum_{n=0}^{\infty}|\xi_n|\varphi^*(\beta_n)=\sum_{j=1}^{\infty}\varphi(\beta_{n_j}^{-1})/\varphi(\beta_{n_j}^{-1})=\infty,$$

the lemma is proved.

In what follows we will construct a translation invariant Banach sequence space E on $\mathbb{Z}$ such that for some infinite subset $A\subset\mathbb{Z}$ the restriction of the space E to the set A is equal to the restriction of l_∞ to that subset, and the restriction to another infinite subset $B\subset\mathbb{Z}$ is equal to the restriction of l_1 to B. In other words, we will construct a translation invariant Banach sequence space which is extremely non-rearrangement invariant.

First, let us consider any Banach lattice G of sequences $\{\xi_k\}_{k\in\mathbb{N}}$ which has the same property, i.e., $G|_A=l_1|_A$ and $G|_B=l_\infty|_B$ for some disjoint infinite sets A and B. For instance, we can take for A the set of odd numbers and for B the set of even numbers and define the norm on G by

$$\|\xi\|_G=\sup_k|\xi_{2k}|+\sum_{k=0}^{\infty}|\xi_{2k+1}|.$$

Now, take any sequence $\{\xi_n\}_{n\in\mathbb{Z}}$ and consider the functional

$$\|\xi\|_E=\sup_{m\in\mathbb{Z}}\|\{\xi_{m-2^k}\}_k\|_G.$$

It is easily seen that it is translation invariant, and the space $E=\{\xi:\|\xi\|_E<\infty\}$ is a translation invariant Banach lattice with the Fatou property.

Let us show that the restriction of E to the set $C=\{-2^k:k\in\mathbb{N}\}$ is equal to G. Indeed, the intersection of C with any $C+m$ $(m\ne0)$ has at most one point. If there were two points, then

$$2^{k_1}=2^{l_1}-m,\qquad 2^{k_2}=2^{l_2}-m$$

for some natural numbers k_1,k_2,l_1,l_2, which implies $2^{k_1}-2^{k_2}=2^{l_1}-2^{l_2}$ and hence $k_1=l_1$ if $k_1>k_2$ or $k_2=l_2$ if $k_2>k_1$. Thus, $m=0$.

Therefore for any ξ with $\operatorname{supp}\xi\subset C$ we have

$$\sup_{m\ne0}\|\{\xi_{m-2^k}\}_k\|_G=\|\xi\|_{l_\infty},$$

and

$$\sup_{m\in\mathbb{Z}}\|\{\xi_{m-2^k}\}_k\|_G=\|\{\xi_{-2^k}\}\|_G.$$

This implies that for some infinite subset A of C we have $E|_A=l_\infty|_A$ and for some disjoint infinite subset B we have $E|_B=l_1|_B$ with equal norms.

THEOREM 3. *Let $a\ne1$ be a positive number. Then $\{E,E(a^{-n})\}$ is a non-Calderón couple for which any complex interpolation space is described by the K-method.*

Proof. If interpolation for $\{E,E(a^{-n})\}$ were described by the K-method, then interpolation from $\{E|_A,E(a^{-n})|_A\}$ to $\{E|_B,E(a^{-n})|_B\}$ would also be described by the K-method. But this yields that interpolation from $\{l_\infty,l_\infty(\alpha)\}$ to $\{l_1,l_1(\beta)\}$ is described by the K-method for some unbounded weights α and β, which is impossible by Lemma 2. Thus, $\{E,E(a^{-n})\}$ is not a Calderón couple.

Consider now the complex method interpolation spaces for the couple $\{E,E(a^{-n})\}$. They are easily described by using the Calderón construction for lattices (see [Ca]). By [S], the space $[E,E(a^{-n})]_\theta$ is equal to the closure of $E\cap E(a^{-n})$ in

$$E^{1-\theta}E(a^{-n})^\theta=E(a^{-n\theta}).$$

Since $l_\infty\cap l_\infty(a^{-n})\subset l_1(a^{-n\theta})$, we conclude that the closure of $E\cap E(a^{-n})$ in $E(a^{-n\theta})$ is $E_0(a^{-n\theta})$, where E_0 denotes the regular part of E, i.e., the closure of the set of sequences with finite supports. Therefore

$$[E,E(a^{-n})]_\theta=E_0(a^{-n\theta}).$$

Clearly, E_0 is also a translation invariant Banach lattice, hence $E_0(a^{-n\theta})$ is invariant with respect to the operator of convolution with the sequence $\{\min(1,a^{-n})\}_{n\in\mathbb{Z}}$. Thus for the real method functor $\mathcal{F}(X_0,X_1):=\{x:\{K(a^n,x;\overline{X})\}\in E(a^{-n\theta})\}$ we have

$$\mathcal{F}(l_1,l_1(a^{-n}))=\mathcal{F}(l_\infty,l_\infty(a^{-n}))=E_0(a^{-n\theta}).$$

Combining this with the continuous inclusions $l_1\hookrightarrow E\hookrightarrow l_\infty$ we have

$$\mathcal{F}(E,E(a^{-n}))=E_0(a^{-n\theta}).$$

Hence

$$[E,E(a^{-n})]_\theta=E_0(a^{-n\theta})=\mathcal{F}(E,E(a^{-n})).$$

Thus, all complex method interpolation spaces between E and $E(2^{-n})$ are described by the K-method, yet $\{E,E(2^{-n})\}$ is not a Calderón couple. The theorem is proved.

Any couple $\{E(2^{-n\theta_0}),E(2^{-n\theta_1})\}$ with different real numbers θ_0 and θ_1 is obviously isomorphic to $\{E,E(a^{-n})\}$. If we take $0<\theta_0<\theta_1<1$, then we can use the spaces of the couple $\{E(2^{-n\theta_0}),E(2^{-n\theta_1})\}$ as parameters of the real method and generalize the previous example.

THEOREM 4. *Let $\overline{X}=\{X_0,X_1\}$ be any Banach couple in $\mathcal{X}$. Then every couple of the form $\{\overline{X}_{E(2^{-n\theta_0})},\overline{X}_{E(2^{-n\theta_1})}\}$ fails to be a Calderón couple, yet any complex interpolation space of this couple is described by the K-method.*

This statement follows immediately from the following result, if we apply the reiteration theorem for parameters of the real method.

THEOREM 5. *Let $\{E_0, E_1\}$ be a couple of real parameters. Then for any Banach couple $\overline{X} \in \mathcal{X}$ the couple $\{E_0, E_1\}$ is a partial retract of $\{\overline{X}_{E_0}, \overline{X}_{E_1}\}$.*

Proof. Notice that from the proof of Theorem 4.5.7 in [BK] it follows that any couple $\overline{X} \in \mathcal{X}$ is *Conv$_0$-abundant*, i.e., for every quasi-concave function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\varphi(t) \to 0$ as $t \to 0$ and $\varphi(t)/t \to 0$ as $t \to \infty$ there exists $x \in (X_0 + X_1)^\circ$ with $\varphi(t) \asymp K(t, x; \overline{X})$.

Now take $\xi \in E_0 + E_1$. Then $\xi \in l_1 + l_1(2^{-n})$, and hence $K(t, \xi; \overline{l}_1) \to 0$ as $t \to 0$ and $K(t, \xi; \overline{l}_1)/t \to 0$ as $t \to \infty$. Since $\overline{X}$ belongs to $\mathcal{X}$, there exists $x \in X_0 + X_1$ such that

$$K(t, x; \overline{X}) \asymp K(t, \xi; \overline{l}_1).$$

Let us show that ξ is orbitally equivalent to $x \in \overline{X}_{E_0} + \overline{X}_{E_1}$. Denote by a_x the sequence $\{K(2^n, x; \overline{X})\}$, which is equivalent to $a_\xi = \{K(2^n, \xi; \overline{l}_1)\}$. Note first that a_ξ and ξ are orbitally equivalent with respect to $\{E_0, E_1\}$. Indeed, $|\xi| \le \mathcal{P}(|\xi|)$ and $\{\mathcal{P}(|\xi|)_n\} \asymp a_\xi$, where $\mathcal{P}$ is the Calderón operator, and $\mathcal{P} : \{E_0, E_1\} \to \{E_0, E_1\}$ since E_0 and E_1 are parameters of the real method. Now, by Theorem 7.3.1 of [O2], x and a_x are orbitally equivalent with respect to $\{E_0, E_1\}$ and $\{\overline{X}_{E_0}, \overline{X}_{E_1}\}$. The proof is complete.

It can also be shown that analogous results are true for ordered $\mathcal{X}_0$ couples. So we are able to construct counter-examples to Cwikel's conjecture with couples of rearrangement invariant spaces of functions on the interval $[0,1]$ or $[0,\infty)$. To see this, it suffices to apply Theorem 5 for the couples $\overline{X} = \{L_1, L_\infty\}$. The next result gives other examples of couples of rearrangement invariant spaces for which Cwikel's conjecture fails.

THEOREM 6. *For any $1 < p < \infty$ there exists a rearrangement invariant space X on $[0,\infty)$ such that the Boyd indices of X satisfy $p_X = q_X = p$ and $\{L_1, X\}$ is not a Calderón couple, yet any complex method interpolation space for this couple is described by the K-method.*

Proof. In [M1] it is shown that if X is a rearrangement invariant space on $[0,\infty)$ such that $1 < p_X = q_X < \infty$, then the Calderón spaces $(L_1)^{1-\theta} X^\theta$ are described by the K-method for any $0 < \theta < 1$.

Now consider the weighted Banach sequence lattice $E(w^n)$, where $1 < w < 2$ and E is any translation invariant Banach sequence lattice on $\mathbb{Z}$ such that E^+ is a symmetric sequence space which is not an l_p-space. Then, by [K], the space X of measurable functions f on $[0,\infty)$ such that

$$\|f\|_X := \|\{2^{-n} K(2^n, f; \{L_1, L_\infty\})\}\|_{E(w^n)} < \infty$$

is a rearrangement invariant space on $[0,\infty)$ for which

$$p_X = q_X = (\log_2 w)^{-1},$$

and $\{L_1, X\}$ is not a Calderón couple. The proof is finished by taking $w = 2^{1/p}$.

4. K-orbits. In this section we study interpolation orbits in couples of the form $\{E_0, E_1(2^{-n})\}$, where $\{E_0, E_1\}$ is any couple of translation invariant Banach lattices of two-sided sequences. We will consider orbits of all elements for which the K-functionals are in the class $\mathcal{P}^{+-}$ of *quasi-power* functions. In what follows a quasi-concave function φ is said to be of class $\mathcal{P}^{+-}$ if the *dilation indices* of φ defined by

$$\alpha_\varphi := \lim_{t\to 0} \frac{\ln s_\varphi(t)}{\ln t}, \quad \beta_\varphi := \lim_{t\to\infty} \frac{\ln s_\varphi(t)}{\ln t}$$

are non-trivial, i.e., $0 < \alpha_\varphi \le \beta_\varphi < 1$. Here $s_\varphi(t) = \sup_{u>0} s(tu)/s(u)$ for $t > 0$.

Note that by [KPS, p. 75], $\varphi \in \mathcal{P}^{+-}$ if and only if

$$\int_0^\infty \min(1, t/s)\varphi(s)\, \frac{ds}{s} \asymp \varphi(t).$$

Remark. If φ is a quasi-concave function, then for any $\varepsilon > 0$ there exists a constant $C = C(\varepsilon)$ such that

$$s_\varphi(t) \le C \max(t^{\alpha_\varphi - \varepsilon}, t^{\beta_\varphi + \varepsilon}). \tag{4}$$

Before presenting the main result of this section, we shall need some estimates of the K-functional for spaces obtained by the Lions–Peetre construction. In the following lemma we simplify Holmstedt's formula for the K-functional.

LEMMA 3. *Let $\overline{X} = \{X_0, X_1\}$ be a Banach couple and $x \in X_0 + X_1$ be such that $K(t, x; \overline{X})$ is a quasi-power function. Then for any θ_0 and θ_1 such that $0 < \theta_0 < \alpha_K \le \beta_K < \theta_1 < 1$ we have*

$$K(t, x; \{\overline{X}_{\theta_0,q_0}, \overline{X}_{\theta_1,q_1}\}) \asymp t^{-\theta_0/(\theta_1-\theta_0)} K(t^{1/(\theta_1-\theta_0)}, x; \overline{X})$$

for all $1 \le q_0, q_1 \le \infty$.

Proof. Since $0 < \theta_0 < \alpha_K \le \beta_K < \theta_1 < 1$, the function

$$\varphi(t) := t^{-\theta_0/(\theta_1-\theta_0)} K(t^{1/(\theta_1-\theta_0)}, x; \overline{X})$$

belongs to $\mathcal{P}^{+-}$. Thus, from Holmstedt's formula [H]

$$K(t, x; \overline{X}_{\theta_0,1}, \overline{X}_{\theta_1,1}) \asymp \int_0^\infty K(s, x; \overline{X}) \min(s^{-\theta_0}, ts^{-\theta_1})\, \frac{ds}{s}$$

it follows that the latter integral is equivalent to $\varphi(t)$.

Similarly, using the equivalence (see [H])

$$K(t,x;\{\overline{X}_{\theta_0,\infty},\overline{X}_{\theta_1,\infty}\})\asymp \sup_{s>0} K(s,x;\overline{X})\min(s^{-\theta_0},ts^{-\theta_1})$$

and conditions on the dilation indices of $K(\cdot,x;\overline{X})$, one can easily check that the latter supremum is equivalent to

$$K(s,x;\overline{X})\min(s^{-\theta_0},ts^{-\theta_1})$$

when $s^{-\theta_0}=ts^{-\theta_1}$. Hence,

$$K(t,x;\{\overline{X}_{\theta_0,\infty},\overline{X}_{\theta_1,\infty}\})\asymp t^{-\theta_0/(\theta_1-\theta_0)}K(t^{1/(\theta_1-\theta_0)},x;\overline{X}).$$

Now, by the well known continuous inclusions (cf. [BL])

$$\overline{X}_{\theta,1}\hookrightarrow \overline{X}_{\theta,q}\hookrightarrow \overline{X}_{\theta,\infty},$$

we obtain

$$\begin{aligned}K(t,x;\{\overline{X}_{\theta_0,1},\overline{X}_{\theta_1,1}\})&\asymp K(t,x;\{\overline{X}_{\theta_0,q_0},\overline{X}_{\theta_1,q_1}\})\\ &\asymp K(t,x;\{\overline{X}_{\theta_0,\infty},\overline{X}_{\theta_1,\infty}\}),\end{aligned}$$

which yields the required equivalence.

LEMMA 4. *Let $x\in \overline{X}_{\theta_0,q_0}+\overline{X}_{\theta_1,q_1}$ be such that*

$$\varphi(t)=K(t,x;\{\overline{X}_{\theta_0,q_0},\overline{X}_{\theta_1,q_1}\})$$

is a quasi-power function. Then

$$\varphi(t)\asymp t^{-\theta_0/(\theta_1-\theta_0)}K(t^{1/(\theta_1-\theta_0)},x;\overline{X}),$$

and $\theta_0<\alpha_K\le\beta_K<\theta_1$.

Proof. Again, as in the proof of Lemma 3, the non-triviality of the dilation indices of φ implies that we can reduce the general case to $q_0=q_1=\infty$. Thus, we have

$$\begin{aligned}(5)\qquad \varphi(t)&\asymp \sup_{s>0}K(s,x;\overline{X})\min(s^{-\theta_0},ts^{-\theta_1})\\ &=\sup_{u>0}u^{-\theta_0/(\theta_1-\theta_0)}K(u^{1/(\theta_1-\theta_0)},x;\overline{X})\min(1,t/u).\end{aligned}$$

Then, for some quasi-exponential sequence $\{u_n\}$ (i.e., a sequence equivalent to some geometric progression)

$$\varphi(u_n)\asymp u_n^{-\theta_0/(\theta_1-\theta_0)}K(u_n^{1/(\theta_1-\theta_0)},x;\overline{X}).$$

So we have to show that if

$$\varphi(t)\asymp\sup_{s>0}A(s)\min(1,t/s),$$

then we can find a quasi-exponential sequence $\{s_n\}$ such that $A(s_n)\asymp\varphi(s_n)$. First choose $\{s_n\}$ such that $A(s_n)$ increases, $A(s_n)/s_n$ decreases and

$$\varphi(t)\asymp\sup_n A(s_n)\min(1,t/s_n).$$

According to Lemma 4.2.1 of [O2] the sequence $\{s_n\}$ will be a semi-filling for φ, which means that there exists $C>0$ such that for any $t>0$ there exists n with

$$\varphi(t)\le C\varphi(s_n),\qquad \varphi(t)/t\le C\varphi(s_n)/s_n.$$

If we take $t=2^k$, then

$$(6)\qquad \varphi(2^k)\le C\varphi(s_k),\qquad \varphi(2^k)/2^k\le C\varphi(s_k)/s_k.$$

Since $\varphi\in\mathcal{P}^{+-}$, it follows by (4) that there exist $C_1>0$ and $0<\alpha\le\beta<1$ such that

$$\varphi(s)\le C_1\max((s/t)^\alpha,(s/t)^\beta)\varphi(t)$$

for every $s,t>0$. Therefore, if $\varphi(t)\le C\varphi(s)$, then $t\le C_2 s$ for a suitable $C_2>0$. Thus (6) implies that $s_k\asymp 2^k$. Consequently, we obtain

$$K(v_n,x;\overline{X})\asymp v_n^{\theta_0}\varphi(v_n^{\theta_1-\theta_0}),$$

where $v_n=u_n^{1/(\theta_1-\theta_0)}$ is also quasi-exponential. Hence,

$$K(t,x;\overline{X})\asymp t^{\theta_0}\varphi(t^{\theta_1-\theta_0}).$$

One can easily show that $\theta_0<\alpha_K\le\beta_K<\theta_1$, and this completes the proof.

We now state the main theorem of this section about interpolation orbits in the couple $\{E_0,E_1(2^{-n})\}$, where E_0 and E_1 are translation invariant Banach sequence spaces.

THEOREM 7. *Let E_0 and E_1 be any translation invariant Banach lattices of two-sided sequences. Then the interpolation orbit in $\{E_0,E_1(2^{-n})\}$ of any element $x\in E_0+E_1(2^{-n})$ such that $\varphi=K(\cdot,x;\{E_0,E_1(2^{-n})\})$ is a quasi-power function coincides with its K-orbit, both being equal to $l_\infty(\varphi^\star(2^{-n}))$.*

Proof. Let $\overline{X}=\{E_0,E_1(2^{-n})\}$. Since $E_0\hookrightarrow l_\infty$, $E_1(2^{-n})\hookrightarrow l_\infty(2^{-n})$ and

$$\mathrm{Orb}(x,\overline{X}\to\overline{X})\hookrightarrow(X_0,X_1)_{\varphi,\infty},$$

it follows from the easily verified equality $(l_\infty,l_\infty(2^{-n}))_{\varphi,\infty}=l_\infty(\varphi^\star(2^{-n}))$ that

$$\mathrm{Orb}(x,\overline{X}\to\overline{X})\hookrightarrow l_\infty(\varphi^\star(2^{-n})).$$

Thus, we need to show the reverse inclusion. To do this let x_φ denote the sequence $\{\varphi(2^n)\}_{n\in\mathbb{Z}}$. It is easily seen that $|x|\le Cx_\varphi$ for some universal constant $C>0$ which does not depend on x. We also have

$$K(t,x;\overline{X})\prec K(t,x;\{l_1,l_1(2^{-n})\})\prec K(t,x_\varphi;\{l_1,l_1(2^{-n})\}).$$

Since φ is a quasi-power, we have

$$K(t,x_\varphi;\{l_1,l_1(2^{-n})\})\asymp K(t,x_\varphi;\{l_\infty,l_\infty(2^{-n})\})\asymp\varphi(t).$$

Combining these estimates we have

$$(7)\qquad K(t,x;\{l_1,l_1(2^{-n})\}) \asymp K(t,x_\varphi;\{l_1,l_1(2^{-n})\}) \asymp \varphi(t).$$

To finish the proof, we need only show that $x_\varphi \in \mathrm{Orb}(x,\overline{X}\to\overline{X})$, since that implies

$$l_\infty(\varphi^\star(2^{-n})) \hookrightarrow \mathrm{Orb}(x,\overline{X}\to\overline{X}).$$

To this end, we extrapolate the couple $\{E_0,E_1(2^{-n})\}$, i.e., consider the couple $\overline{A}=\{l_\infty(2^n),l_\infty(2^{-2n})\}$ for which we have

$$(8)\qquad \begin{aligned} E_0 &= (l_\infty(2^n),l_\infty(2^{-2n}))_{F_0},\\ E_1(2^{-n}) &= (l_\infty(2^n),l_\infty(2^{-2n}))_{F_1}, \end{aligned}$$

where $F_0=E_0(2^{-n/3})$, $F_1=E_1(2^{-2n/3})$, and

$$(9)\qquad \begin{aligned} l_1 &= (l_\infty(2^n),l_\infty(2^{-2n}))_{1/3,1},\\ l_1(2^{-n}) &= (l_\infty(2^n),l_\infty(2^{-2n}))_{2/3,1}. \end{aligned}$$

In view of (7), (9) and Lemma 4 we get

$$\varphi(t) \asymp t^{-\theta_0/(\theta_1-\theta_0)}K(t^{1/(\theta_1-\theta_0)},x;\overline{A})$$

and

$$\varphi(t) \asymp t^{-\theta_0/(\theta_1-\theta_0)}K(t^{1/(\theta_1-\theta_0)},x_\varphi;\overline{A}).$$

Hence

$$K(t,x;\overline{A}) \asymp K(t,x_\varphi;\overline{A}).$$

Since $\overline{A}$ is a Calderón couple (see [Pe]), there exists a linear operator $T:\overline{A}\to\overline{A}$ such that $T(x)=x_\varphi$. Then, by (8), $T:\{E_0,E_1(2^{-n})\}\to\{E_0,E_1(2^{-n})\}$, and thus

$$x_\varphi \in \mathrm{Orb}(x,\{E_0,E_1(2^{-n})\}\to\{E_0,E_1(2^{-n})\}).$$

This completes the proof.

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Faculty of Mathematics and Computer Science
A. Mickiewicz University
Matejki 48/49
60-769 Poznań, Poland
and
Mathematical Institute
Polish Academy of Sciences
Poznań Branch
Matejki 48/49
60-769 Poznań, Poland
E-mail: mastylo@math.amu.edu.pl

Faculty of Mathematics
Voronezh University
Universitetskaya pl., 1
394693 Voronezh, Russia
E-mail: vio@func.vsu.ru

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On convergence for the square root of the Poisson kernel in symmetric spaces of rank 1

by

JAN-OLAV RÖNNING (Skövde)

Abstract. Let $P(z,\beta)$ be the Poisson kernel in the unit disk $\mathbb{U}$, and let $P_\lambda f(z) = \int_{\partial\mathbb{U}} P(z,\varphi)^{1/2+\lambda} f(\varphi)\,d\varphi$ be the λ-Poisson integral of f, where $f \in L^p(\partial\mathbb{U})$. We let $\mathcal{P}_\lambda f$ be the normalization $P_\lambda f/P_\lambda 1$. If $\lambda > 0$, we know that the best (regular) regions where $\mathcal{P}_\lambda f$ converges to f for a.a. points on $\partial\mathbb{U}$ are of nontangential type.

If $\lambda = 0$ the situation is different. In a previous paper, we proved a result concerning the convergence of $\mathcal{P}_0 f$ toward f in an L^p weakly tangential region, if $f \in L^p(\partial\mathbb{U})$ and $p > 1$. In the present paper we will extend the result to symmetric spaces X of rank 1. Let f be an L^p function on the maximal distinguished boundary K/M of X. Then $\mathcal{P}_0 f(x)$ will converge to $f(kM)$ as x tends to kM in an L^p weakly tangential region, for a.a. $kM \in K/M$.

1. Introduction. Let $X = G/K$ be a Riemannian symmetric space of noncompact type and of rank 1. (The notation is explained in Section 2.) On X, we consider the λ-Poisson operator

$$P_\lambda f(g\cdot o) = \int_{K/M} f(kM) P^{\lambda+\varrho}(kM,g)\,dkM,$$

where $P(kM,g)$ is the Poisson kernel of G/K, $f \in L^p(K/M)$, and $\lambda+\varrho \in \mathfrak{a}$. We know that $P_\lambda f$ satisfies the equation

$$\Delta P_\lambda f = (|\lambda|^2 - |\varrho|^2) P_\lambda f,$$

where Δ is the Laplace–Beltrami operator on X.

If $\lambda \geq 0$, it is known that $P_\lambda f(g)$ does not necessarily converge to $f(kM)$ as g tends to kM. To obtain convergence, we need to consider the normalization $\mathcal{P}_\lambda f = P_\lambda f/P_\lambda 1$. We know that $\mathcal{P}_\lambda f$ converges admissibly to f a.e. on the boundary if $f \in L^p$, $p \geq 1$. In a previous paper, [JOR], we proved that if X is the hyperbolic unit disk $\mathbb{U}$ and $\lambda = 0$, we have convergence in a larger region, which we call an L^p *weakly tangential region* ($1 \leq p < \infty$).

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