Non-reflexive pentagon subspace lattices

by

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Dedicated to Paul R. Halmos in celebration of his 80th birthday

Abstract. On a complex separable (necessarily infinite-dimensional) Hilbert space $H$, any three subspaces $K, L$ and $M$ satisfying $K \cap M = \{0\}, K \vee L = H$ and $L \subseteq M$ give rise to what has been called by Halmos [4, 5] a pentagon subspace lattice $\mathcal{P} = \{(0), K, L, M, H\}$. Then $n = \dim M \cap L$ is called the gap-dimension of $\mathcal{P}$. Examples are given to show that, if $n < \infty$, the order-interval $[L, M]_{\text{Lat Alg}} \mathcal{P} = \{N \in \text{Lat Alg} \mathcal{P} : L \subseteq N \subseteq M\}$ in $\text{Lat Alg} \mathcal{P}$ can be either (i) a nest with $n + 1$ elements, or (ii) an atomic Boolean algebra with $n$ atoms, or (iii) the set of all subspaces of $H$ between $L$ and $M$. For $n > 1$, since $\text{Lat Alg} \mathcal{P} = \mathcal{P} \cup [L, M]_{\text{Lat Alg}} \mathcal{P}$, all such examples of pentagons are non-reflexive, the examples in case (iii) extremely so.

1. Introduction. On a complex separable Hilbert space $H$ any three (closed) subspaces $K, L$ and $M$ satisfying $K \cap M = \{0\}, K \vee L = H$ and $L \subseteq M$ give rise to what has been called by Halmos [4, 5] a pentagon subspace lattice $\mathcal{P} = \{(0), K, L, M, H\}$. Here inclusion is the partial order and a labelled Hasse diagram of $\mathcal{P}$ is given in Figure 1.

![Diagram of a pentagon subspace lattice](image)

Fig. 1

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THEOREM 1 (Foiaş [3]). Let $A$ be a positive operator on a Hilbert space $H$. If the function $\varphi : [0, ||A||] \to \mathbb{R}$ is non-negative, non-decreasing, continuous and concave, then the range of $\varphi(A)$ is invariant under every operator on $H$ which leaves the range of $A$ invariant.

2. Preliminaries. Throughout what follows, $H$ will denote a complex separable infinite-dimensional Hilbert space which we will usually identify with $l^2$. The terms “operator” and “subspace” will mean bounded linear mapping of $H$ into itself, and closed linear manifold of $H$, respectively. We use $"\cup"$ to denote closed linear span and also use $\langle e, f, g, \ldots, h \rangle$ to denote the subspace spanned by the vectors $e, f, g, \ldots, h$. The set of operators on $H$ is denoted by $B(H)$. If $T \in B(H)$, then $\mathcal{R}(T)$ denotes the range of $T$ and $G(T) = \{ (x, Tx) : x \in H \}$ the graph of $T$. If $C$ is a collection of subspaces of $H$, then $C$ denotes the set of operators on $H$ which leave every member of $C$ invariant. If $F$ is a collection of operators on $H$, then $\text{Lat } F$ denotes the set of subspaces of $H$ which are invariant under every member of $F$. Clearly, $H \subseteq \text{Lat } F$. If $C = \text{Lat } F$, then $C$ is called reflexive. Every reflexive collection $C$ of subspaces contains $H$ and is closed under the formation of arbitrary intersections and arbitrary closed linear spans. Any collection of subspaces satisfying the latter conditions (and not necessarily reflexive) is called a sublattice on $H$.

Let $L$ be an abstract complete lattice with greatest element $1$ and least element $0$. The usual conventions $\bigvee \emptyset = 0$, $\bigwedge \emptyset = 1$ were adopted. An element $a \in L$ is an atom if $0 < b < a$ and $b \in L$ implies that $b = 0$ or $a$. If every element of $L$ is the join of the atoms that it contains, $L$ is called atomic. If $L$ is totally-ordered it is called a nest. If, for every $c \in L$, there exists $c' \in L$ such that $c < c' < 1$, $c \wedge c' = 0$, then $L$ is complemented. If $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ and its dual hold identically in $L$, then $L$ is called distributive. If $L$ is complemented, distributive and atomic it is called an atomic Boolean algebra.

For every $u, v \in L$ we let $[u, v]_L$ denote the set $\{ w \in L : u \leq w \leq v \}$. Then $[u, v]_L$ is a complete sublattice of $L$, that is, it is closed under the formation of arbitrary (non-empty) meets and joins.

Two abstract complete lattices $L_1$ and $L_2$ are called isomorphic if there exists a bijection $\psi : L_1 \to L_2$ satisfying $a \leq b$ (in $L_1$) if and only if $\psi(a) \leq \psi(b)$ (in $L_2$). Such a map $\psi$ is then called a lattice-isomorphism.

If $A \in B(H)$ is positive injective and non-invertible and $M$ is a non-zero finite-dimensional subspace of $H$ satisfying $M \cap \mathcal{R}(A) = \{ 0 \}$ then it is easily verified that

$$P(A; M) = \{ (0), G(-A), G(A), G(A) + M, H \}$$

is a pentagon subspace lattice on $H \oplus H$ with gap-dimension equal to $\dim M$.
(note that $G(T)^{-1} = \{(T^* x, x) : x \in H\}$, for every $T \in B(H)$). Also,
\[
\text{Alg } \mathcal{P}(A; M) = \left\{ \begin{pmatrix} X + ZA & Z \\ AZA & Y + AZ \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} : X, Y, Z \in B(H), \ Y \in \mathcal{A}(A; M) \text{ and } YA = AX \right\},
\]
where, by definition,
\[
\mathcal{A}(A; M) = \{ T \in B(H) : T \mathcal{R}.(A) \subseteq \mathcal{R}(A) \text{ and } TM \subseteq M \}.
\]
The latter notation is suggested by the use of $\mathcal{A}$ in [7] to denote the set of those operators on $H$ which leave $\mathcal{R}(A)$ invariant. Of course, $\mathcal{A}(A; M) \subseteq \mathcal{A}(A)$, and if $T \in \mathcal{A}(A)$, then it follows from the range inclusion theorem of R. G. Douglas (see [1]) that $TA = AW$, for some operator $W \in B(H)$. Since $A$ is injective this operator $W$ is uniquely determined. The set of operators $\mathcal{A}(A; M)$ is a unital algebra and we denote by $\mathcal{A}(A; M) \upharpoonright M$ the unital algebra of operators on $M$ obtained by restricting each element of $\mathcal{A}(A; M)$ to $M$. The following proposition simplifies our constructions.

**Proposition 1.** Let $A \in B(H)$ be positive, injective and non-invertible and let $M$ be a non-zero finite-dimensional subspace of $H$ satisfying $M \cap \mathcal{R}(A) = 0$. Let $\mathcal{P}(A; M)$ be the pentagon subspace lattice on $H \oplus H$ given by $\mathcal{P}(A; M) = \{ 0, \mathcal{G}(A), \mathcal{G}(A) \oplus (0) \oplus M, \mathcal{H} \}$. Then
\[
\text{Lat Alg } \mathcal{P}(A; M) = \mathcal{P}(A; M) \cup \{ \mathcal{G}(A) \oplus (0) \oplus N : N \in \text{ Lat } \mathcal{A}(A; M) \upharpoonright M \}.
\]
Moreover, the mapping $\psi : N \mapsto \mathcal{G}(A) \oplus (0) \oplus N$ is a lattice-isomorphism of Lat $\mathcal{A}(A; M) \upharpoonright M$ onto the interval $\mathcal{G}(A), \mathcal{G}(A) \oplus (0) \oplus M \} \text{ Lat Alg } \mathcal{P}(A; M)$ of Lat Alg $\mathcal{P}(A; M)$.

**Proof.** As remarked earlier,
\[
\text{Lat Alg } \mathcal{P}(A; M) = \mathcal{P}(A; M) \cup \{ \mathcal{G}(A), \mathcal{G}(A) \oplus (0) \oplus M \} \text{ Lat Alg } \mathcal{P}(A; M).
\]
Using this, Lat Alg $\mathcal{P}(A; M)$ will have the required form if

(a) $\mathcal{G}(A) \oplus (0) \oplus N \in \text{ Lat Alg } \mathcal{P}(A; M)$, for every $N \in \text{ Lat } \mathcal{A}(A; M) \upharpoonright M$,

and

(b) if $L \in \text{ Lat Alg } \mathcal{P}(A; M)$ and $\mathcal{G}(A) \subseteq L \subseteq \mathcal{G}(A) \oplus (0) \oplus M$, then $L = \mathcal{G}(A) \oplus (0) \oplus N$, for some $N \in \text{ Lat } \mathcal{A}(A; M) \upharpoonright M$.

Let $N \in \text{ Lat } \mathcal{A}(A; M) \upharpoonright M$. Then $N \subseteq M$ and $N \in \text{ Lat } \mathcal{A}(A; M)$. For every $u \in H$ and $v \in N$ and every operator $\begin{pmatrix} X + ZA & Z \\ AZA & Y + AZ \end{pmatrix}$ of Alg $\mathcal{P}(A; M)$ it can be readily verified that
\[
\begin{pmatrix} X + ZA & Z \\ AZA & Y + AZ \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ Aw + v \end{pmatrix},
\]
for some $w \in H$. Since $YN \subseteq N$, it follows that $\mathcal{G}(A) \oplus (0) \oplus N \in \text{ Lat Alg } \mathcal{P}(A; M)$.

Next, let $L$ be a subspace of $H \oplus H$ satisfying $\mathcal{G}(A) \subseteq L \subseteq \mathcal{G}(A) \oplus (0) \oplus M$. Put $N = \{ x \in H : (0, x) \in L \}$. Then $N$ is a subspace of $H$, $N \subseteq M$ and $L = \mathcal{G}(A) \oplus (0) \oplus N$. If additionally $L \in \text{ Lat Alg } \mathcal{P}(A; M)$ then $N \in \text{ Lat } \mathcal{A}(A; M) \upharpoonright M$. For, let $x \in N$ and $Y \in \mathcal{A}(A; M)$. Since $L \in \text{ Lat Alg } \mathcal{P}(A; M)$,
\[
\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \begin{pmatrix} 0 \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ Yx \end{pmatrix} \in L,
\]
where $YA = AX$, and so $Yx \in N$.

It is now easy to verify that the mapping $\psi$, as described in the statement of the proposition, has the required property.

The preceding proposition shows that the extent of the non-triviality of the set of invariant subspaces of the finite-dimensional algebra $\mathcal{A}(A; M) \upharpoonright M$ is a measure of the non-reflexivity of $\mathcal{P}(A; M)$.

The remainder of this note will be devoted to showing, with $n = \dim M$, that $\text{ Lat } \mathcal{A}(A; M) \upharpoonright M$ can be either (i) a nest with $n + 1$ elements, or (ii) an atomic Boolean algebra with $n$ atoms, or (iii) the lattice of all subspaces of $M$. In the third case, the last part of the proof of the preceding proposition then shows that $\{ [G(A), G(A) \oplus (0) \oplus M] \text{ Lat Alg } \mathcal{P}(A; M) \}$ consists of all those subspaces of $H \oplus H$ lying between $G(A)$ and $G(A) \oplus (0) \oplus M$. For obvious reasons we take $n \geq 2$.

### 3. The examples

In what follows we take $H = l^2$ for simplicity and identify each operator on $H$ with its matrix relative to the usual orthonormal basis.

(i) Nest case. Let $n \in \mathbb{Z}^+$, $n \geq 2$, and let $0 < a < 1$. Let $A \in B(H)$ be the diagonal matrix $A = \text{diag}(1, a, a^2, \ldots)$. Then $\| A \| = 1$ and $A$ is positive, injective and non-invertible (even compact). Clearly,
\[
\mathcal{R}(A) \subseteq \mathcal{R}(A^{(n-1)/n}) \subseteq \ldots \subseteq \mathcal{R}(A^{1/n}) \subseteq \mathcal{R}(A^1).
\]
In fact, the inclusions are strict. This follows from [1, Theorem 2.1] or more directly as follows. Define vectors $e_j$, $1 \leq j \leq n$, by $e_j = (a^{(n-j+1)/n} - a^{-j}/n) x_j$. Then $e_n \notin \mathcal{R}(A^{1/n})$ and $e_j = A^{(n-j)/n} e_n$, $1 \leq j \leq n - 1$, from which it follows that $e_j \in \mathcal{R}(A^{(n-j)/n}) \setminus \mathcal{R}(A^{(n-j+1)/n})$, $1 \leq j \leq n - 1$. Put $M = \{ e_1, \ldots, e_n \}$. A proof by induction shows that, if $1 \leq j \leq n$ and $\sum_{i=n-j+1}^{n} \alpha_i e_i \in \mathcal{R}(A^{1/n})$ where the $\alpha_i$ are scalars, then $\alpha_n - 1 \leq j \leq n$. (Begin the proof of the inductive step by observing that $\mathcal{R}(A^{(n-j+1)/n}) \subseteq \mathcal{R}(A^{1/n})$ and $e_n \in \mathcal{R}(A^{1/n})$.) It now follows that $M \cap \mathcal{R}(A) = \{ 0 \}$, that $\{ e_1, \ldots, e_n \}$ is linearly independent and that $M \cap \mathcal{R}(A^{1/n}) = \{ e_1, \ldots, e_{n-j} \}$, $1 \leq j \leq n - 1$. Thus we have a pentagon $P(A; M)$ on $H \oplus H$ with gap-dimension $n$. We show that Lat $\mathcal{A}(A; M) \upharpoonright M$ is the nest $\{ N_j : 0 \leq j \leq n \}$ where $N_0 = \{ 0 \}$ and $N_j = \{ e_1, \ldots, e_j \}$, $1 \leq j \leq n$. 


If $T \in \mathcal{A}(A; M)$ then, by Theorem 1, $T$ leaves $N_j = M \cap \mathcal{R}(A^{(n-j)/n})$ invariant, for every $1 \leq j \leq n-1$. Thus \( \{N_j : 0 \leq j \leq n\} \subseteq \text{Lat} \mathcal{A}(A; M)|_{M} \).

To prove the reverse inclusion it is enough to show that there is an element $J \in \mathcal{A}(A; M)$ such that the matrix of $J|_{M}$ relative to the basis \( \{e_1, \ldots, e_n\} \) of $M$ is the elementary Jordan matrix

$$
\begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix},
$$

since it is well-known (and easily proved) that $J|_{M}$ will then have only the obvious invariant subspaces, namely \( \{N_j : 0 \leq j \leq n\} \). Let $S^*$ be the backward shift operator on $H$. Then $S^*A = aAS^*$, so $S^*\mathcal{R}(A) \subseteq \mathcal{R}(A)$. It follows that, for any scalars $\gamma_i$, $0 \leq i \leq n-1$, the operator $J = A^{1/n} \sum_{i=0}^{n-1} \gamma_i (S^*)^i$ also leaves $\mathcal{R}(A)$ invariant. Note that, for $1 \leq j \leq n$, $S^*e_j = a(a^{j-1}+1)\gamma_{j-1}E_j$, so $J e_j = \sum_{i=0}^{n-1} \gamma_i (a^{(n-j+1)/n})^i e_j$. Since $A^{1/n} e_j = e_{j-1}$, $2 \leq j \leq n$, let us choose scalars $\gamma_i$, $0 \leq i \leq n-1$, such that $\sum_{i=0}^{n-1} \gamma_i a^i = 0$ and $\sum_{i=0}^{n-1} \gamma_i (a^{(n-j+1)/n})^i = 1$, $2 \leq j \leq n$. This choice is possible since the $n \times n$ Vandermonde determinant

$$
\begin{vmatrix}
1 & 1 & 1 & \cdots & 1 \\
(a^{1/n})^2 & (a^{2/n})^2 & (a^{3/n})^2 & \cdots & a^2 \\
(a^{2/n})^2 & (a^{3/n})^2 & \cdots & \cdots & a^2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
(a^{n/n})^2 & (a^{(n+1)/n})^2 & (a^{(n+2)/n})^2 & \cdots & a^{n-1}
\end{vmatrix} = \prod_{j=1}^{n} (a^{j/n} - a^{j/n})
$$

is non-zero. With this choice $J e_1 = 0$ and $J e_j = e_{j-1}$, $2 \leq j \leq n$. Thus $J \in \mathcal{A}(A; M)$ and the matrix of $J|_{M}$ relative to \( \{e_1, \ldots, e_n\} \) in elementary Jordan.

(II) Boolean algebra case. For the remaining examples we need the following lemma.

**Lemma 1.** Let $s \in \mathbb{Z}^+$ and, for every $1 \leq m \leq s$, let \( (b^{(m)}_{j})_{j=1}^{\infty} \) be a strictly decreasing sequence of positive real numbers converging to zero. Then there exists a strictly decreasing sequence \( (t^{(m)}_{j})_{j=1}^{\infty} \) of positive real numbers converging to zero, with $t_1 = 1$, such that each of the piecewise linear functions $\varphi_m : [0,1] \to \mathbb{R}$, $1 \leq m \leq s$, given by $\varphi_m(0) = 0$ and, for every $j \geq 1$,

$$
\varphi_m(x) = b^{(m)}_j + \left( \frac{b^{(m)}_{j+1} - b^{(m)}_j}{t_j - t_{j+1}} \right) (x - t_j), \quad t_{j+1} \leq x \leq t_j,
$$

is non-negative, strictly increasing, continuous and concave.

**Proof.** For every $j \geq 2$, \( (b^{(m)}_{j} - b^{(m)}_{j+1})/(t_j - t_{j+1}) > 0 \), for every $1 \leq m \leq s$. For each $j \geq 2$ choose $\delta_j > 0$ such that

1. \( \delta_j \leq \min \{ \frac{b^{(m)}_{j} - b^{(m)}_{j+1}}{b^{(m)}_{j+1} - b^{(m)}_{j}} \} \),
2. \( \sum_{j=2}^{\infty} (\prod_{k=2}^{j} \delta_k) \) converges.

Define \( (\Delta_j)_{j=1}^{\infty} \) by $\Delta_1 = 1/(1 + \sum_{j=2}^{\infty} (\prod_{k=2}^{j} \delta_k))$, $\Delta_j = (\prod_{k=2}^{j} \delta_k) \Delta_{j-1}$, $j \geq 2$. Define \( (t^{(m)}_{j})_{j=1}^{\infty} \) by $t_1 = 1$, $t_j = 1 - \sum_{k=1}^{j-1} \Delta_k$, $j \geq 2$. Note that \( \sum_{j=1}^{\infty} \Delta_j = 1 \), so $t_j = \sum_{k=1}^{j} \Delta_k$, for every $j \geq 1$. Clearly, \( (t^{(m)}_{j})_{j=1}^{\infty} \) is a strictly decreasing sequence of positive real numbers converging to zero. For each $1 \leq m \leq s$ define the piecewise linear function $\varphi_m : [0,1] \to \mathbb{R}$ as in the statement of the lemma. Then each $\varphi_m$ is non-negative, strictly increasing and continuous on $[0,1]$. Note that (as far as gradients of line segments are concerned), for $1 \leq m \leq s$,

$$
\frac{b^{(m)}_{j} - b^{(m)}_{j+1}}{t_j - t_{j+1}} \geq \frac{b^{(m)}_{j} - b^{(m)}_{j+1}}{t_{j-1} - t_{j}}, \quad j \geq 2.
$$

For, if $1 \leq m \leq s$ and $j \geq 2$ we have

$$
\delta_j = \frac{\Delta_j}{\Delta_{j-1}} = \frac{t_j - t_{j+1}}{t_{j-1} - t_{j}} \leq \frac{b^{(m)}_{j} - b^{(m)}_{j+1}}{b^{(m)}_{j-1} - b^{(m)}_{j}}.
$$

That each $\varphi_m$ is concave now follows from [2, Theorem 2.3, p. 351] (applied to $-\varphi_m$).

Below, several sequences will be defined in equally-sized “blocks” of terms and sometimes these blocks will consist of equally-sized “sub-blocks”. In general, if $p \in \mathbb{Z}^+$ then the terms of a sequence \( (\xi^{(m)}_{j})_{j=1}^{\infty} \) in its $p$th block are taken to be the terms \( (\xi^{(m)}_{j})_{j=1+s(p-1)}^{n+p-1} \), where $r \in \mathbb{Z}^+$ is the (common) size of each block. Additionally, if $s \in \mathbb{Z}^+$ divides $r$, then each block consists of $t = r/s$ sub-blocks and the terms of the $q$th sub-block of the $p$th block, where $1 \leq q \leq t$, are \( (\xi^{(m)}_{j})_{j=1+s(q-1)+r(p-1)}^{r(p)} \). Also, if $i, n \in \mathbb{Z}^+$ we define

$$
[i]_n = \begin{cases} 
1 & \text{if } i \leq n, \\
0 & \text{if } i > n.
\end{cases}
$$

Let $n \in \mathbb{Z}^+$, $n \geq 2$, and let $0 < a < 1$. Let \( (r^{(m)}_{k})_{k=1}^{\infty} \) be the sequence $r_0 = (n+1)(k(k-1))$, $k \geq 1$. Consider the $n \times n$ array
For each \(1 \leq m \leq n\), let \((s^{(m)}_j)_{j=1}^{\infty}\) be the sequence of natural numbers which has the \(m\)th row of this array as the terms in its \(k\)th block (so each block has size \(n\)). Then \((s^{(m)}_j)_{j=1}^{\infty}\) is strictly increasing. For each \(1 \leq m \leq n\) put \((b^{(m)}_j)_{j=1}^{\infty} = (a^{(m)}_j)_{j=1}^{\infty}\). Then \((b^{(m)}_j)_{j=1}^{\infty}\) is a strictly decreasing sequence of positive real numbers. Also, \((b^{(m)}_j)_{j=1}^{\infty}\) is square-summable, so \(b^{(m)}_j \to 0\) as \(j \to \infty\). On \(H = \mathbb{R}^2\) define operators \(B_m, 1 \leq m \leq n\), by \(B_m = \text{diag}(b^{(m)}_1, b^{(m)}_2, \ldots)\). Next, define sequences \((f_m)_{m=1}^{\infty}\), \(1 \leq m \leq n\), each of them in blocks of size \(n(n-1)\) with each block consisting of \(n-1\) sub-blocks of size \(n\), by specifying that the terms of \(f_m\) in the \(q\)th sub-block of the \(p\)th block (where \(p \geq 1\) and \(1 \leq q \leq n-1\)) be given by

\[
0, \ldots, 0, b^{(q+m)_p}_{p}(0, 1, \ldots, 0)
\]

where \([q + m]_n\) is as defined on p. 193, where \(v_{p,q} = (q-1) + n(n-1)(p-1)\) and where the subscript on the non-zero term is purely positional. Thus the \(q\)th block of \(n(n-1)\) terms of \(f_m\) is the \(m\)th row of the array

\[
\begin{array}{cccccc}
0 & b^{(2)}_1 & 0 & \ldots & 0 & 0 \\
0 & 0 & b^{(2)}_1 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & \ldots & b^{(2)}_1 & 0 \\
0 & 0 & \ldots & \ldots & 0 & b^{(n)}_1 \\
0 & 0 & \ldots & \ldots & 0 & 0 \\
\end{array}
\]

where the subscripts are purely positional and the sub-blocks have been separated for the sake of clarity. The sum, \(f\), say, of the \(f_m\), \(1 \leq m \leq n\), occurs in blocks of size \(n\), its terms in the \(k\)th block \((k \geq 1)\) being

\[
b^{(1)}_{1, m-1}, b^{(2)}_{2, m-1}, \ldots, b^{(n)}_{n, m-1}
\]

or, more precisely,

\[
a^{(m)}_{1, 0}, a^{(m)}_{1, 1}, \ldots, a^{(m)}_{n, n-1}, a^{(m)}_{n, n}
\]

It follows that \(f \in L^2\), so \(f_m \in L^2\), for every \(1 \leq m \leq n\). For \(1 \leq m \leq n\), the non-zero terms of \((f_m^{(j)} / b^{(m)}_j)_{j=1}^{\infty}\) are, in order, simply \(a, a^2, a^3, \ldots\) so \((f_m^{(j)} / b^{(m)}_j)_{j=1}^{\infty}\) is in \(L^2\) and \(f_m \in \mathcal{R}(B_m)\).

Put \(M = (f_1, \ldots, f_n)\). Then \(M \cap \mathcal{R}(B_m) = \{f_m\}, 1 \leq m \leq n\). For \((\xi_j)_{j=1}^{\infty} \in L^2\) and \(B_m((\xi_j)_{j=1}^{\infty}) = (b^{(m)}_j \xi_j)_{j=1}^{\infty} = \sum_{j=1}^{\infty} \beta_j f_j\), with \(\beta_j \in \mathbb{C}\), \(1 \leq j \leq n\). For every \(1 \leq i \leq n\), \(i \neq m\), there is a unique \(1 \leq k \leq n-1\) such that \([q+k]_n = m\). If \(i \neq m\) and \(p \geq 1\) put \(j_{p,q} = m + v_{p,q}\).

Then the \((p,q)\)th term of \(\sum_{j=1}^{n} \beta_j f_j\) is simply \(\beta_j f^{(p)}_{j_{p,q}} = \beta_j b^{(m)}_{j_{p,q}}\), so \(\beta_j f^{(p)}_{j_{p,q}} = b^{(m)}_{j_{p,q}}\), which gives \(\beta_j = \delta_{p,q}\). Since the latter is true for every \(p \geq 1\) and \(q \leq n-1\), we have \(\beta_j = 0\).

Now let \((t_j)_{j=1}^{\infty}\) be the sequence arising from \((b^{(m)}_j)_{j=1}^{\infty} : 1 \leq m \leq n\) as in Lemma 1 and let \(\varphi_m : [0, 1] \to \mathbb{R}, 1 \leq m \leq n\), be the associated concave functions. Let \(A \in B(H)\) be the operator \(A = \text{diag}(t_1, t_2, \ldots)\). Then \(\|A\| = t_1 = 1\) and \(A\) is positive, injective and non-invertible (even compact). Also, \(B_m = \varphi_m(A), 1 \leq m \leq n\). It is clear that \(\{f_1, \ldots, f_n\}\) is linearly independent. By Theorem 1, for every \(1 \leq m \leq n\), \(\mathcal{R}(B_m)\) is an invariant operator range of \(\mathcal{A}(A)\), so, since every non-zero invariant operator range of \(\mathcal{A}(A)\) contains \(\mathcal{R}(A)\) (cf. [7]), we have \(\mathcal{R}(A) \subseteq \mathcal{R}(B_m)\). It follows that \(M \cap \mathcal{R}(A) = \{0\}\), so we have a pentagon \(P(A; M)\) on \(H \otimes H\) with gap-dimension \(n\). We show that \(\mathcal{A}(A; M)\) is the atomic Boolean algebra with atoms \(\{f_j : 1 \leq j \leq n\} \in \mathcal{A}(A; M)\) consists precisely of those operators on \(M\) whose matrix relative to the basis \(\{f_1, \ldots, f_n\}\) is diagonal.

For every \(1 \leq m \leq n\) let \(A_m \in B(H)\) be the operator whose matrix (relative to the usual orthonormal basis of \(L^2\)) is diagonal, the diagonal sequence having a one exactly where \(f_m\) has a non-zero term, with zeroes elsewhere. Then \(A_m A = A A_m, 0 \leq A_m \in \mathcal{A}(A), 1 \leq m \leq n\). Also, \(A_m f_i = b^{(m)}_i f_m\), \(1 \leq i, m \leq n\). Hence, for each \(1 \leq m \leq n\), \(A_m \in \mathcal{A}(A; M)\) and the matrix of \(A_m\) relative to the basis \(\{f_1, \ldots, f_n\}\) is diagonal with nth diagonal entry 1 and 0 otherwise. It follows that \(\mathcal{A}(A; M)\) contains every operator whose matrix is diagonal relative to \(\{f_1, \ldots, f_n\}\). For the reverse inclusion, let \(T \in \mathcal{A}(A; M)\) and let \(1 \leq m \leq n\). Then, by Theorem 1, \(T \mathcal{R}(B_m) \subseteq \mathcal{R}(B_m)\), so \(T\) leaves \(M \cap \mathcal{R}(B_m) = \{f_m\}\) invariant. Hence \(T f_m = \lambda_m f_m\), for some \(\lambda_m \in \mathbb{C}\), so the matrix of \(T f_m\) relative to \(\{f_1, \ldots, f_n\}\) is diagonal.
(III) Extreme case. Let \( n \in \mathbb{Z}^+, n \geq 2 \), and let \( 0 < a < 1 \). Let \((r_k)_{k=1}^\infty\), \((b_j^{(m)})_{j=1}^\infty\), and \(B_m : 1 \leq m \leq n\) be as defined in our discussion of the preceding case. Define sequences \( g_m = (b_j^{(m)})_{j=1}^\infty, 1 \leq m \leq n\), each of them in blocks of size \( n^2 \) with each block consisting of \( n \) sub-blocks of size \( n \); by specifying that the terms of \( g_m \) in the \( q \)th sub-block of the \( p \)th block where \( p \geq 1 \) and \( 1 \leq q \leq n-1 \) be given by

\[
0, \ldots, 0, b_{(q+m)n}, 0, \ldots, 0
\]

where \( [q+m]_n \) is as before, where \( w_{p,n} = n(q-1) + n^2(p-1) \) and where the subscript on the non-zero term is purely positional. Also, we specify that the terms of \( g_m, 1 \leq m \leq n \), in the \( n \)th sub-block of the \( p \)th block be given by

\[
b_{(1+nw_{p,n})}^{(m-1)} b_{(2+nw_{p,n})}^{(m-1)} \ldots b_{(n+nw_{p,n})}^{(m-1)} 0, b_{(m+1+nw_{p,n})}^{(m-1)} \ldots b_{(n+nw_{p,n})}^{(m-1)}
\]

where \( w_{p,n} = n(n-1) + n^2(p-1) \). Thus the \( p \)th block of \( n^2 \) terms of \( g_m \) is the \( m \)th row of the array

\[
\begin{array}{cccccccc}
0 & b_2^{(2)} & 0 & \ldots & 0 & 0 & 0 & b_2^{(3)} & 0 & \ldots & 0 \\
0 & 0 & b_3^{(2)} & 0 & \ldots & 0 & 0 & 0 & 0 & b_3^{(4)} & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & b_n^{(2)} & b_n^{(3)} & 0 & 0 & 0 & 0 & \ldots & 0 \\
b_1^{(2)} & 0 & 0 & \ldots & 0 & 0 & b_2^{(2)} & 0 & \ldots & 0 & \ldots & \ldots & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & b_n^{(n-2)} & 0 & b_1^{(2)} & b_2^{(2)} & \ldots & 0 & \ldots & \ldots & \ldots \\
0 & \ldots & 0 & b_n^{(n-1)} & 0 & b_1^{(2)} & b_2^{(2)} & \ldots & b_n^{(n-1)} & 0 & \ldots & \ldots & \ldots \\
\end{array}
\]

where the subscripts are purely positional and the sub-blocks have once again, been separated for clarity. A similar proof to that given to that in the discussion of the preceding case (begun by adding the \( g_m \)'s) shows that \( g_m \in \mathbb{F}, 1 \leq m \leq n \). Also, for every \( 1 \leq m \leq n \), the non-zero terms of \((b_j^{(m)}/b_j^{(m)})_{j=1}^\infty\) are, in order simply

\[
a, a^2, \ldots, a^{n-1}, a^n, a^{n+1}, a^{n+2}, \ldots, a^{2n-1},
\]

so, since the latter sequence is square-summable, \( g_m \in \mathcal{R}(B_m) \).

Put \( M = (g_1, \ldots, g_n) \). Again with proof as before, \( M \cap \mathcal{R}(B_m) = (g_m) \), \( 1 \leq m \leq n \) (take \( j_{p,n} = m + w_{p,n} \), this time).

Assume that \( n \geq 3 \) (the case \( n = 2 \) will be considered separately). For \( n+1 \leq m \leq 2n-1 \) define the sequence \((b_j^{(m)})_{j=1}^\infty\), in blocks of size \( n^2 \) with each block consisting of \( n \) sub-blocks of size \( n \), by specifying that the terms of \((b_j^{(m)})_{j=1}^\infty\) in the \( p \)th block (where \( p \geq 1 \) be given by the \((m-n)\)th row of the array

\[
\begin{array}{cccccccc}
b_2^{(2)} & b_2^{(1)} & 0 & \ldots & 0 & 0 & 0 & b_2^{(3)} & 0 & \ldots & 0 \\
b_2^{(2)} & b_2^{(1)} & 0 & \ldots & 0 & 0 & 0 & 0 & b_2^{(4)} & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
b_2^{(2)} & b_2^{(1)} & 0 & \ldots & 0 & b_n^{(2)} & b_n^{(3)} & 0 & 0 & 0 & \ldots & 0 \\
b_2^{(2)} & b_2^{(1)} & 0 & \ldots & 0 & b_n^{(2)} & b_n^{(3)} & \ldots & b_n^{(n-1)} & 0 & \ldots & \ldots & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}
\]

where the subscripts are purely positional and the sub-blocks have been clearly separated. Then \((b_j^{(m)})_{j=1}^\infty, n+1 \leq m \leq 2n-1\), is a strictly decreasing sequence of positive real numbers converging to zero. Define operators \( B_m, n+1 \leq m \leq 2n-1 \), on \( H \) by \( B_m = \text{diag}(b_1^{(m)}, b_2^{(m)}, \ldots) \). Then \( g_m \notin \mathcal{R}(B_m) \), for every \( 1 \leq m \leq n-1 \), since the \( j \)th term of \( g_m \) equals \( b_j^{(m+n)} \) for infinitely many values of \( j \) (consider the \( r \)th sub-block (size \( n \)) in each block (size \( n^2 \))). Also, \( g_m - g_{m+1} \in \mathcal{R}(B_{m+n}) \), for every \( 1 \leq m \leq n-1 \). For, if \( 1 \leq m \leq n-1 \), the \( p \)th block (\( p \geq 1 \) of \( g_m - g_{m+1} \) is

\[
\begin{array}{cccccccc}
0 & \ldots & 0 & b_{m+1}^{(m+1)} - b_{m+3}^{(m+3)} & 0 & \ldots & 0 & 0 & 0 & b_{m+2}^{(m+2)} - b_{m+3}^{(m+3)} & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \ldots & 0 & b_n^{(n-1)} - b_{n}^{(n)} & -b_1^{(1)} & 0 & \ldots & 0 & b_{n}^{(n)} & -b_1^{(1)} & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \ldots & 0 & 0 & b_n^{(m+n-1)} - b_{n}^{(m+n)} & 0 & \ldots & 0 & 0 & 0 & b_n^{(m+n)} & b_{n}^{(m+n)} & 0 & \ldots & 0 \\
\end{array}
\]

where the subscripts are purely positional and the superscripts give the positions in each sub-block. Dividing by the corresponding terms of \((b_j^{(m+n)})_{j=1}^\infty\) we find that the absolute values of non-zero terms arising in the \( p \)th block
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are, respectively,
\[ a^{k_r}, a^{k_r+1}, a^{k_r+2}, \ldots, a^{k_r+n}, \]
where \( k_r = (p - 1)n + 1 \). Since \( (a_k^{k_r})_{k_r=1}^\infty \in \mathbb{R}^2 \), \( g_m - g_{m+1} \in \mathcal{R}(B_m) \).

This time let \((t_j)_{j=1}^\infty\) be the sequence arising from \((t_j)_{j=1}^m \in \mathbb{R}, 1 \leq m \leq 2n - 1\) as in Lemma 1 and let \( \varphi_m : [0, 1] \to \mathbb{R}, 1 \leq m \leq 2n - 1\), be the associated concave functions. Let \( A \in \mathcal{B}(H) \) be the operator \( A = \text{diag}(t_1, t_2, \ldots) \). Then \( \|A\| = t_1 = 1 \) and \( A \) is positive, injective and non-invertible (even compact). Also, \( B_m = \varphi_m(A), 1 \leq m \leq 2n - 1\). Clearly, \( \{g_1, \ldots, g_n\} \) is linearly independent and by Theorem 1, \( \mathcal{R}(B_m) \) is an invariant operator range of \( \mathcal{A}(A) \), so \( \mathcal{R}(A) \subseteq \mathcal{R}(B_m) \), \( 1 \leq m \leq 2n - 1\). Since \( M \cap \mathcal{R}(B_m) = (g_m) \), \( 1 \leq m \leq n \), it follows that \( M \cap \mathcal{R}(A) = (0) \). Once again we have a pentagon \( \mathcal{P}(A; M) \) on \( H \oplus H \) with gap-dimension \( n \). We show that \( \mathcal{A}(A; M)_{|M} = \mathcal{C}I \) (then, of course, \( \mathcal{L} \mathcal{A}(A; M)_{|M} = \mathcal{C}I \)) is the set of all subspaces of \( M \).

Let \( T \in \mathcal{A}(A; M) \). As in the discussion of the preceding case, for every \( 1 \leq m \leq n \), \( T \) leaves \( M \cap \mathcal{R}(B_m) = (g_m) \) invariant, so \( Tg_m = \lambda_m g_m \) for some \( \lambda_m \in \mathbb{C} \). Let \( 1 \leq m \leq n - 1 \). By Theorem 1, \( T \) also leaves \( \mathcal{R}(B_m) \) invariant, so since \( g_m = g_{m+1} \in \mathcal{R}(B_{m+1}) \), we have \( T(g_m - g_{m+1}) = \lambda_m g_m - \lambda_{m+1} g_{m+1} \in \mathcal{R}(B_{m+1}) \).

Hence
\[ (\lambda_m - \lambda_{m+1})g_m = \lambda_m g_m - \lambda_{m+1} g_{m+1}, \]
But \( g_m \notin \mathcal{R}(B_{m+1}) \), so \( \lambda_m = \lambda_{m+1} \). Hence \( \lambda_1 = \lambda_2 = \cdots = \lambda_m, \) and \( T^m \notin \mathcal{C}I \).

Finally, consider the case where \( n = 2 \). The required example can be found by considering the case where \( n = 3 \). Our discussion of the latter case provides \( A \in \mathcal{B}(H) \) positive, injective and non-invertible with \( \|A\| = 1 \) and operator ranges \( \mathcal{R}(B_m), 1 \leq m \leq 5 \), each invariant under every member of \( \mathcal{A}(A) \) together with linearly independent vectors \( g_1, g_2, g_3 \) of \( H \) satisfying
\[ \langle g_1, g_2, g_3 \rangle \cap \mathcal{R}(A) = (0), \]
\[ \langle g_1, g_2, g_3 \rangle \cap \mathcal{R}(B_m) = (g_m), \quad m = 1, 2, 3, \]
\[ g_1 - g_2 \in \mathcal{R}(B_4), \quad g_1 \notin \mathcal{R}(B_4), \]
\[ g_2 - g_3 \in \mathcal{R}(B_5), \quad g_2 \notin \mathcal{R}(B_5). \]
Hence, with \( M = \langle g_1, g_2 \rangle \), we have \( \dim M = 2 \), \( M \cap \mathcal{R}(A) = (0), \) \( M \cap \mathcal{R}(B_m) = (g_m) \) for \( m = 1, 2 \), and \( g_1 - g_2 \in \mathcal{R}(B_4), g_1 \notin \mathcal{R}(B_4) \). With this \( M \), the pentagon \( \mathcal{P}(A; M) \) is the required example for the case where \( n = 2 \).

4. Conclusion. Since \( \text{Lat} \mathcal{F} \) is reflexive for any collection \( \mathcal{F} \) of operators on \( H \) (clearly, \( \text{Lat} \mathcal{F} \subseteq \text{Lat} \mathcal{F} \)), our examples when \( n = 2 \) show that there exist subspace lattices isomorphic to those given in Figures 2(a) and 2(b) which are reflexive and have \( \dim M \oplus L = 2 \). Non-reflexive subspace lattices of these two lattice types can also be found.

References


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