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Departamento de Análisis Matemático
Facultad de Ciencias Matemáticas
Universidad Complutense
Madrid, 28040, Spain
E-mail: daniel@sunam1.mat.ucm.es

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Non-reflexive pentagon subspace lattices

by

M. S. LAMBROU (Iraklion) and
W. E. LONGSTAFF (Nedlands, Western Australia)

Dedicated to Paul R. Halmos
in celebration of his 80th birthday

Abstract. On a complex separable (necessarily infinite-dimensional) Hilbert space H any three subspaces K, L and M satisfying $K \cap M = (0)$, $K \vee L = H$ and $L \subset M$ give rise to what has been called by Halmos [4, 5] a *pentagon subspace lattice* $\mathcal{P} = \{(0), K, L, M, H\}$. Then $n = \dim M \ominus L$ is called the *gap-dimension* of $\mathcal{P}$. Examples are given to show that, if $n < \infty$, the order-interval $[L, M]_{\operatorname{Lat}\operatorname{Alg}\mathcal{P}} = \{N \in \operatorname{Lat}\operatorname{Alg}\mathcal{P} : L \subseteq N \subseteq M\}$ in $\operatorname{Lat}\operatorname{Alg}\mathcal{P}$ can be either (i) a nest with $n+1$ elements, or (ii) an atomic Boolean algebra with n atoms, or (iii) the set of all subspaces of H between L and M. For $n > 1$, since $\operatorname{Lat}\operatorname{Alg}\mathcal{P} = \mathcal{P} \cup [L, M]_{\operatorname{Lat}\operatorname{Alg}\mathcal{P}}$, all such examples of pentagons are non-reflexive, the examples in case (iii) extremely so.

1. Introduction. On a complex separable Hilbert space H any three (closed) subspaces K, L and M satisfying $K \cap M = (0)$, $K \vee L = H$ and $L \subset M$ give rise to what has been called by Halmos [4, 5] a *pentagon subspace lattice* $\mathcal{P} = \{(0), K, L, M, H\}$. Here inclusion is the partial order and a labelled Hasse diagram of $\mathcal{P}$ is given in Figure 1.

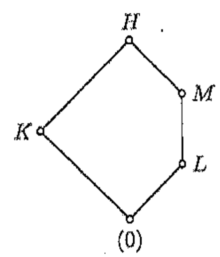

Fig. 1

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(This poset is non-modular, so the underlying space H is necessarily infinite-dimensional.) Call $n = \dim M \ominus L$ the *gap-dimension* of $\mathcal{P}$. In [5] Halmos showed that every pentagon $\mathcal{P}$ with gap-dimension one is reflexive in the sense that $\mathcal{P} = \operatorname{Lat}\operatorname{Alg}\mathcal{P}$. Also, he asked whether every pentagon is reflexive. The answer was shown to be negative in [6] where a non-reflexive pentagon with gap-dimension 2 is exhibited (though its Lat Alg has not yet been fully determined). More examples of non-reflexive pentagons are given below, including some which are "extremely" non-reflexive. In each example $\operatorname{Lat}\operatorname{Alg}\mathcal{P}$ is fully determined.

For any pentagon $\mathcal{P}$, as in Figure 1, if $\mathcal{F}$ denotes the set of rank one operators of $\operatorname{Alg}\mathcal{P}$ then $\operatorname{Lat}\mathcal{F} = \mathcal{P} \cup [L, M]$, where $[L, M] = \{N : N \text{ is a subspace of } H \text{ and } L \subseteq N \subseteq M\}$. Consequently, $\operatorname{Lat}\operatorname{Alg}\mathcal{P} = \mathcal{P} \cup [L, M]_{\operatorname{Lat}\operatorname{Alg}\mathcal{P}}$, where

$$[L, M]_{\operatorname{Lat}\operatorname{Alg}\mathcal{P}} = \{N : N \in \operatorname{Lat}\operatorname{Alg}\mathcal{P} \text{ and } L \subseteq N \subseteq M\}.$$

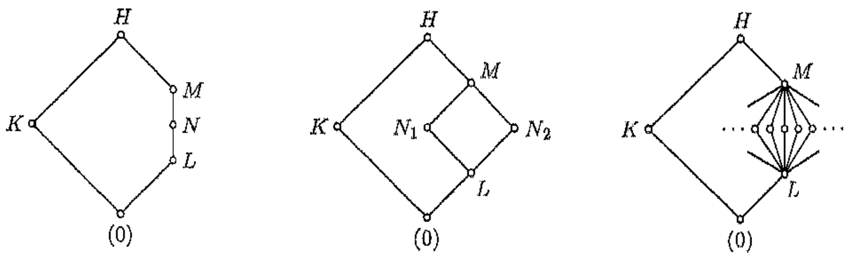
Of course, $[L, M]_{\operatorname{Lat}\operatorname{Alg}\mathcal{P}}$ is a complete lattice (partially ordered by inclusion). Examples are given showing that, for a pentagon subspace lattice $\mathcal{P}$ with gap-dimension $n \in \mathbb{Z}^+$, it is possible to have

(i) $[L, M]_{\operatorname{Lat}\operatorname{Alg}\mathcal{P}}$ a nest with $n + 1$ elements, or

(ii) $[L, M]_{\operatorname{Lat}\operatorname{Alg}\mathcal{P}}$ an atomic Boolean algebra with n atoms, or

(iii) $[L, M]_{\operatorname{Lat}\operatorname{Alg}\mathcal{P}} = [L, M]$.

For $n > 1$, all such pentagons are non-reflexive, the examples in case (iii) extremely so. For $n = 2$, Figures 2(a) and 2(b) are labelled Hasse diagrams of $\operatorname{Lat}\operatorname{Alg}\mathcal{P}$ corresponding to cases (i) and (ii), respectively, and Figure 2(c) is a schematic representation of $\operatorname{Lat}\operatorname{Alg}\mathcal{P}$ corresponding to case (iii).



(a) (b) (c)

Fig. 2

The following result due to Foiaş [3] (see also [7, 8]) lies at the heart of our constructions.

THEOREM 1 (Foiaş [3]). *Let A be a positive operator on a Hilbert space H. If the function $\varphi : [0, \|A\|] \to \mathbb{R}$ is non-negative, non-decreasing, continuous and concave, then the range of $\varphi(A)$ is invariant under every operator on H which leaves the range of A invariant.*

2. Preliminaries. Throughout what follows, H will denote a complex separable infinite-dimensional Hilbert space which we will usually identify with l^2. The terms "operator" and "subspace" will mean bounded linear mapping of H into itself, and closed linear manifold of H, respectively. We use "$\vee$" to denote closed linear span and also use $\langle e, f, g, \ldots, h\rangle$ to denote the subspace spanned by the vectors $e, f, g, \ldots, h$. The set of operators on H is denoted by $\mathcal{B}(H)$. If $T \in \mathcal{B}(H)$, then $\mathcal{R}(T)$ denotes the range of T and $G(T) = \{(x, Tx) : x \in H\}$ the graph of T. If $\mathcal{L}$ is a collection of subspaces of H, then $\operatorname{Alg}\mathcal{L}$ denotes the set of operators on H which leave every member of $\mathcal{L}$ invariant. If $\mathcal{F}$ is a collection of operators on H, then $\operatorname{Lat}\mathcal{F}$ denotes the set of subspaces of H which are invariant under every member of $\mathcal{F}$. Clearly, $\mathcal{L} \subseteq \operatorname{Lat}\operatorname{Alg}\mathcal{L}$. If $\mathcal{L} = \operatorname{Lat}\operatorname{Alg}\mathcal{L}$, then $\mathcal{L}$ is called *reflexive*. Every reflexive collection $\mathcal{L}$ of subspaces contains (0) and H and is closed under the formation of arbitrary intersections and arbitrary closed linear spans. Any collection of subspaces satisfying the latter conditions (and not necessarily reflexive) is called a *subspace lattice on H*.

Let L be an abstract complete lattice with greatest element 1 and least element 0. The usual conventions $\bigvee \emptyset = 0$, $\bigwedge \emptyset = 1$ are adopted. An element $a \in L$ is an *atom* if $0 \le b \le a$ and $b \in L$ implies that $b = 0$ or a. If every element of L is the join of the atoms that it contains, L is called *atomic*. If L is totally-ordered it is called a *nest*. If, for every $c \in L$, there exists $c' \in L$ such that $c \vee c' = 1$, $c \wedge c' = 0$, then L is *complemented*. If $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and its dual hold identically in L, then L is called *distributive*. If L is complemented, distributive and atomic it is called an *atomic Boolean algebra*.

For every $u, v \in L$ we let $[u, v]_L$ denote the set $[u, v]_L = \{w \in L : u \le w \le v\}$. Then $[u, v]_L$ is a complete sublattice of L, that is, it is closed under the formation of arbitrary (non-empty) meets and joins.

Two abstract complete lattices L_1 and L_2 are called *isomorphic* if there exists a bijection $\psi : L_1 \to L_2$ satisfying $a \le b$ (in L_1) if and only if $\psi(a) \le \psi(b)$ (in L_2). Such a map ψ is then called a *lattice-isomorphism*.

If $A \in \mathcal{B}(H)$ is positive injective and non-invertible and M is a non-zero finite-dimensional subspace of H satisfying $M \cap \mathcal{R}(A) = (0)$ then it is easily verified that

$$\mathcal{P}(A; M) = \{(0), G(-A), G(A), G(A) + (0) \oplus M, H \oplus H\}$$

is a pentagon subspace lattice on $H \oplus H$ with gap-dimension equal to $\dim M$

(note that $G(T)^\perp = \{(-T^*x, x) : x \in H\}$, for every $T \in \mathcal{B}(H)$). Also,

$$\operatorname{Alg}\mathcal{P}(A;M) = \left\{ \begin{pmatrix} X+ZA & Z \\ AZA & Y+AZ \end{pmatrix} : \begin{matrix} X,Y,Z \in \mathcal{B}(H), \\ Y \in \mathscr{A}(A;M) \text{ and } YA = AX \end{matrix} \right\},$$

where, by definition,

$$\mathscr{A}(A;M) = \{T \in \mathcal{B}(H) : T\mathcal{R}(A) \subseteq \mathcal{R}(A) \text{ and } TM \subseteq M\}.$$

The latter notation is suggested by the use of $\mathscr{A}(A)$ in [7] to denote the set of those operators on H which leave $\mathcal{R}(A)$ invariant. Of course, $\mathscr{A}(A;M) \subseteq \mathscr{A}(A)$, and if $T \in \mathscr{A}(A)$, then it follows from the range inclusion theorem of R. G. Douglas (see [1]) that $TA = AW$, for some operator $W \in \mathcal{B}(H)$. Since A is injective this operator W is uniquely determined. The set of operators $\mathscr{A}(A;M)$ is a unital algebra and we denote by $\mathscr{A}(A;M)|_M$ the unital algebra of operators on M obtained by restricting each element of $\mathscr{A}(A;M)$ to M. The following proposition simplifies our constructions.

PROPOSITION 1. *Let $A \in \mathcal{B}(H)$ be positive, injective and non-invertible and let M be a non-zero finite-dimensional subspace of H satisfying $M \cap \mathcal{R}(A) = (0)$. Let $\mathcal{P}(A;M)$ be the pentagon subspace lattice on $H \oplus H$ given by $\mathcal{P}(A;M) = \{(0), G(-A), G(A), G(A) + (0) \oplus M, H \oplus H\}$. Then*

$$\operatorname{Lat}\operatorname{Alg}\mathcal{P}(A;M) = \mathcal{P}(A;M) \cup \{G(A) + (0) \oplus N : N \in \operatorname{Lat}\mathscr{A}(A;M)|_M\}.$$

Moreover, the mapping $\psi : N \mapsto G(A) + (0) \oplus N$ is a lattice-isomorphism of $\operatorname{Lat}\mathscr{A}(A;M)|_M$ onto the interval $[G(A), G(A) + (0) \oplus M]_{\operatorname{Lat}\operatorname{Alg}\mathcal{P}(A;M)}$ of $\operatorname{Lat}\operatorname{Alg}\mathcal{P}(A;M)$.

Proof. As remarked earlier,

$$\operatorname{Lat}\operatorname{Alg}\mathcal{P}(A;M) = \mathcal{P}(A;M) \cup [G(A), G(A) + (0) \oplus M]_{\operatorname{Lat}\operatorname{Alg}\mathcal{P}(A;M)}.$$

Using this, $\operatorname{Lat}\operatorname{Alg}\mathcal{P}(A;M)$ will have the required form if

(a) $G(A) + (0) \oplus N \in \operatorname{Lat}\operatorname{Alg}\mathcal{P}(A;M)$, for every $N \in \operatorname{Lat}\mathscr{A}(A;M)|_M$, and

(b) if $L \in \operatorname{Lat}\operatorname{Alg}\mathcal{P}(A;M)$ and $G(A) \subseteq L \subseteq G(A) + (0) \oplus M$, then $L = G(A) + (0) \oplus N$, for some $N \in \operatorname{Lat}\mathscr{A}(A;M)|_M$.

Let $N \in \operatorname{Lat}\mathscr{A}(A;M)|_M$. Then $N \subseteq M$ and $N \in \operatorname{Lat}\mathscr{A}(A;M)$. For every $u \in H$ and $v \in N$ and every operator $\left(\begin{smallmatrix} X+ZA & Z \\ AZA & Y+AZ \end{smallmatrix}\right)$ of $\operatorname{Alg}\mathcal{P}(A;M)$ it can be readily verified that

$$\begin{pmatrix} X+ZA & Z \\ AZA & Y+AZ \end{pmatrix} \begin{pmatrix} u \\ Au+v \end{pmatrix} = \begin{pmatrix} w \\ Aw+Yv \end{pmatrix},$$

for some $w \in H$. Since $YN \subseteq N$, it follows that $G(A) + (0) \oplus N \in \operatorname{Lat}\operatorname{Alg}\mathcal{P}(A;M)$.

Next, let L be a subspace of $H \oplus H$ satisfying $G(A) \subseteq L \subseteq G(A) + (0) \oplus M$. Put $N = \{x \in H : (0,x) \in L\}$. Then N is a subspace of H, $N \subseteq M$ and $L = G(A) + (0) \oplus N$. If additionally $L \in \operatorname{Lat}\operatorname{Alg}\mathcal{P}(A;M)$ then $N \in \operatorname{Lat}\mathscr{A}(A;M)|_M$. For, let $x \in N$ and $Y \in \mathscr{A}(A;M)$. Since $L \in \operatorname{Lat}\operatorname{Alg}\mathcal{P}(A;M)$,

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \begin{pmatrix} 0 \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ Yx \end{pmatrix} \in L,$$

where $YA = AX$, and so $Yx \in N$.

It is now easy to verify that the mapping ψ, as described in the statement of the proposition, has the required property. ∎

The preceding proposition shows that the extent of the non-triviality of the set of invariant subspaces of the finite-dimensional algebra $\mathscr{A}(A;M)|_M$ is a measure of the non-reflexivity of $\mathcal{P}(A;M)$.

The remainder of this note will be devoted to showing, with $n = \dim M$, that $\operatorname{Lat}\mathscr{A}(A;M)|_M$ can be either (i) a nest with $n+1$ elements, or (ii) an atomic Boolean algebra with n atoms, or (iii) the lattice of all subspaces of M. In the third case, the last part of the proof of the preceding proposition then shows that $[G(A), G(A) + (0) \oplus M]_{\operatorname{Lat}\operatorname{Alg}\mathcal{P}(A;M)}$ consists of all those subspaces of $H \oplus H$ lying between $G(A)$ and $G(A) + (0) \oplus M$. For obvious reasons we take $n \geq 2$.

3. The examples.

In what follows we take $H = l^2$ for simplicity and identify each operator on H with its matrix relative to the usual orthonormal basis.

(I) *Nest case.* Let $n \in \mathbb{Z}^+$, $n \geq 2$, and let $0 < a < 1$. Let $A \in \mathcal{B}(H)$ be the diagonal matrix $A = \operatorname{diag}(1, a, a^2, \ldots)$. Then $\|A\| = 1$ and A is positive, injective and non-invertible (even compact). Clearly,

$$\mathcal{R}(A) \subseteq \mathcal{R}(A^{(n-1)/n}) \subseteq \ldots \subseteq \mathcal{R}(A^{2/n}) \subseteq \mathcal{R}(A^{1/n}).$$

In fact, the inclusions are strict. This follows from [1, Theorem 2.1] or more directly as follows. Define vectors e_j, $1 \leq j \leq n$, by $e_j = (a^{(n-j+1)(k-1)/n})_{k=1}^\infty$. Then $e_n \notin \mathcal{R}(A^{1/n})$ and $e_j = A^{(n-j)/n}e_n$, $1 \leq j \leq n-1$, from which it follows that $e_j \in \mathcal{R}(A^{(n-j)/n}) \setminus \mathcal{R}(A^{(n-j+1)/n})$, $1 \leq j \leq n-1$. Put $M = \langle e_1, \ldots, e_n \rangle$. A proof by induction shows that, if $1 \leq j \leq n$ and $\sum_{i=n-j+1}^n \alpha_i e_i \in \mathcal{R}(A^{j/n})$ where the α_i are scalars, then $\alpha_i = 0$, $n - j + 1 \leq i \leq n$. (Begin the proof of the inductive step by observing that $\mathcal{R}(A^{(j+1)/n}) \subseteq \mathcal{R}(A^{j/n})$ and $e_{n-j} \in \mathcal{R}(A^{j/n})$.) It now follows that $M \cap \mathcal{R}(A) = (0)$, that $\{e_1, \ldots, e_{n-j}\}$ is linearly independent and that $M \cap \mathcal{R}(A^{j/n}) = \langle e_1, \ldots, e_{n-j} \rangle$, $1 \leq j \leq n-1$. Thus we have a pentagon $\mathcal{P}(A;M)$ on $H \oplus H$ with gap-dimension n. We show that $\operatorname{Lat}\mathscr{A}(A;M)|_M$ is the nest $\{N_j : 0 \leq j \leq n\}$ where $N_0 = (0)$ and $N_j = \langle e_1, \ldots, e_j \rangle$, $1 \leq j \leq n$.

If $T \in \mathscr{A}(A; M)$ then, by Theorem 1, T leaves $N_j = M \cap \mathcal{R}(A^{(n-j)/n})$ invariant, for every $1 \le j \le n-1$. Thus $\{N_j : 0 \le j \le n\} \subseteq \operatorname{Lat} \mathscr{A}(A; M)|_M$. To prove the reverse inclusion it is enough to show that there is an element J of $\mathscr{A}(A; M)$ such that the matrix of $J|_M$ relative to the basis $\{e_1, \ldots, e_n\}$ of M is the elementary Jordan matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & 0 & \ldots & 0 \end{pmatrix},$$

since it is well-known (and easily proved) that $J|_M$ will then have only the obvious invariant subspaces, namely $\{N_j : 0 \le j \le n\}$. Let S^* be the backward shift operator on H. Then $S^*A = aAS^*$, so $S^*\mathcal{R}(A) \subseteq \mathcal{R}(A)$. It follows that, for any scalars γ_i, $0 \le i \le n-1$, the operator $J = A^{1/n} \sum_{i=0}^{n-1} \gamma_i (S^*)^i$ also leaves $\mathcal{R}(A)$ invariant. Note that, for $1 \le j \le n$, $S^* e_j = a^{(n-j+1)/n} e_j$, so $Je_j = \sum_{i=0}^{n-1} \gamma_i (a^{(n-j+1)/n})^i A^{1/n} e_j$. Since $A^{1/n} e_j = e_{j-1}$, $2 \le j \le n$, let us choose scalars γ_i, $0 \le i \le n-1$, such that $\sum_{i=0}^{n-1} \gamma_i a^i = 0$ and $\sum_{i=0}^{n-1} \gamma_i (a^{(n-j+1)/n})^i = 1$, $2 \le j \le n$. This choice is possible since the $n \times n$ Vandermonde determinant

$$\begin{vmatrix} 1 & 1 & 1 & \ldots & 1 \\ a^{1/n} & a^{2/n} & a^{3/n} & \ldots & a \\ (a^{1/n})^2 & (a^{2/n})^2 & (a^{3/n})^2 & \ldots & a^2 \\ \vdots & \vdots & \vdots & & \vdots \\ (a^{1/n})^{n-1} & (a^{2/n})^{n-1} & (a^{3/n})^{n-1} & \ldots & a^{n-1} \end{vmatrix} = \prod_{\substack{i=1 \\ j>i}}^{n} (a^{j/n} - a^{i/n})$$

is non-zero. With this choice $Je_1 = 0$ and $Je_j = e_{j-1}$, $2 \le j \le n$. Thus $J \in \mathscr{A}(A; M)$ and the matrix of $J|_M$ relative to $\{e_1, \ldots, e_n\}$ is elementary Jordan.

(II) *Boolean algebra case*. For the remaining examples we need the following lemma.

LEMMA 1. *Let $s \in \mathbb{Z}^+$ and, for every $1 \le m \le s$, let $(b_j^{(m)})_{j=1}^\infty$ be a strictly decreasing sequence of positive real numbers converging to zero. Then there exists a strictly decreasing sequence $(t_j)_{j=1}^\infty$ of positive real numbers converging to zero, with $t_1 = 1$, such that each of the piecewise linear functions $\varphi_m : [0,1] \to \mathbb{R}$, $1 \le m \le s$, given by $\varphi_m(0) = 0$ and, for every $j \ge 1$,*

$$\varphi_m(x) = b_j^{(m)} + \left(\frac{b_j^{(m)} - b_{j+1}^{(m)}}{t_j - t_{j+1}} \right)(x - t_j), \quad t_{j+1} \le x \le t_j,$$

is non-negative, strictly increasing, continuous and concave.

Proof. For every $j \ge 2$, $(b_j^{(m)} - b_{j+1}^{(m)})/(b_{j-1}^{(m)} - b_j^{(m)}) > 0$, for every $1 \le m \le s$. For each $j \ge 2$ choose $\delta_j > 0$ such that

(1) $\delta_j \le \min_{1\le m\le s} (b_j^{(m)} - b_{j+1}^{(m)})/(b_{j-1}^{(m)} - b_j^{(m)})$, and

(2) $\sum_{k=2}^\infty (\prod_{j=2}^k \delta_j)$ converges.

Define $(\Delta_j)_{j=1}^\infty$ by $\Delta_1 = 1/(1 + \sum_{k=2}^\infty (\prod_{j=2}^k \delta_j))$, $\Delta_j = (\prod_{k=2}^j \delta_k)\Delta_1$, $j \ge 2$. Define $(t_j)_{j=1}^\infty$ by $t_1 = 1$, $t_j = 1 - \sum_{k=1}^{j-1} \Delta_k$, $j \ge 2$. Note that $\sum_{j=1}^\infty \Delta_j = 1$, so $t_j = \sum_{k=j}^\infty \Delta_k$, for every $j \ge 1$. Clearly, $(t_j)_{j=1}^\infty$ is a strictly decreasing sequence of positive real numbers converging to zero. For each $1 \le m \le s$ define the piecewise linear function $\varphi_m : [0,1] \to \mathbb{R}$ as in the statement of the lemma. Then each φ_m is non-negative, strictly increasing and continuous on $[0,1]$. Note that (as far as gradients of line segments are concerned), for $1 \le m \le s$,

$$\frac{b_j^{(m)} - b_{j+1}^{(m)}}{t_j - t_{j+1}} \ge \frac{b_{j-1}^{(m)} - b_j^{(m)}}{t_{j-1} - t_j}, \quad j \ge 2.$$

For, if $1 \le m \le s$ and $j \ge 2$ we have

$$\delta_j = \frac{\Delta_j}{\Delta_{j-1}} = \frac{t_j - t_{j+1}}{t_{j-1} - t_j} \le \frac{b_j^{(m)} - b_{j+1}^{(m)}}{b_{j-1}^{(m)} - b_j^{(m)}}.$$

That each φ_m is concave now follows from [2, Theorem 2.3, p. 351] (applied to $-\varphi_m$). ∎

Below, several sequences will be defined in equally-sized "blocks" of terms and sometimes these blocks will consist of equally-sized "sub-blocks". In general, if $p \in \mathbb{Z}^+$ then the terms of a sequence $(\xi_j)_{j=1}^\infty$ in its pth block are taken to be the terms $(\xi_j)_{j=1+r(p-1)}^{rp}$, where $r \in \mathbb{Z}^+$ is the (common) size of each block. Additionally, if $s \in \mathbb{Z}^+$ divides r, then each block consists of $t = r/s$ sub-blocks and the terms in the qth sub-block of the pth block, where $1 \le q \le t$, are $(\xi_j)_{j=1+s(q-1)+r(p-1)}^{sq+r(p-1)}$. Also, if $i, n \in \mathbb{Z}^+$ we define

$$[i]_n = \begin{cases} i & \text{if } i \le n, \\ i - n & \text{if } i > n. \end{cases}$$

Let $n \in \mathbb{Z}^+$, $n \ge 2$, and let $0 < a < 1$. Let $(r_k)_{k=1}^\infty$ be the sequence $r_k = (n+1)k(k-1)$, $k \ge 1$. Consider the $n \times n$ array

$$\begin{array}{lllll}
r_k+k & r_k+2k & r_k+4k & \dots & r_k+(2n-2)k \\
r_k & r_k+3k & r_k+4k & \dots & r_k+(2n-2)k \\
r_k & r_k+2k & r_k+5k & \dots & r_k+(2n-2)k \\
\vdots & \vdots & \vdots & & \vdots \\
r_k & r_k+2k & r_k+4k & \dots & r_k+(2n-2)k \\
r_k & r_k+2k & r_k+4k & \dots & r_k+(2n-1)k.
\end{array}$$

For each $1 \le m \le n$, let $(s_j^{(m)})_{j=1}^{\infty}$ be the sequence of natural numbers which has the mth row of this array as the terms in its kth block (so each block has size n). Then $(s_j^{(m)})_{j=1}^{\infty}$ is strictly increasing. For each $1 \le m \le n$ put $(b_j^{(m)})_{j=1}^{\infty} = (a^{s_j^{(m)}})_{j=1}^{\infty}$. Then $(b_j^{(m)})_{j=1}^{\infty}$ is a strictly decreasing sequence of positive real numbers. Also, $(b_j^{(m)})_{j=1}^{\infty}$ is square-summable, so $b_j^{(m)} \to 0$ as $j \to \infty$. On $H = l^2$ define operators B_m, $1 \le m \le n$, by $B_m = \operatorname{diag}(b_1^{(m)}, b_2^{(m)}, \dots)$. Next, define sequences $f_m = (f_j^{(m)})_{j=1}^{\infty}$, $1 \le m \le n$, each of them in blocks of size $n(n-1)$ with each block consisting of $n-1$ sub-blocks of size n, by specifying that the terms of f_m in the qth sub-block of the pth block (where $p \ge 1$ and $1 \le q \le n-1$) be given by

$$0, \dots, 0, b^{([q+m]_n)}_{[q+m]_n + v_{p,q}}, 0, \dots, 0$$

where $[q+m]_n$ is as defined on p. 193, where $v_{p,q} = n(q-1) + n(n-1)(p-1)$ and where the subscript on the non-zero term is purely positional. Thus the pth block of $n(n-1)$ terms of f_m is the mth row of the array

$$\left|\begin{array}{cccccc|cccccc|}
0 & b_*^{(2)} & 0 & \dots & 0 & 0 & 0 & 0 & b_*^{(3)} & 0 & \dots & 0 \\
0 & 0 & b_*^{(3)} & \dots & 0 & 0 & 0 & 0 & 0 & b_*^{(4)} & \dots & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & 0 & \dots & 0 & b_*^{(n)} & b_*^{(1)} & 0 & 0 & 0 & \dots & 0 \\
b_*^{(1)} & 0 & 0 & \dots & 0 & 0 & 0 & b_*^{(2)} & 0 & 0 & \dots & 0
\end{array}\right.$$

$$\left.\begin{array}{cccccc|c|cccccc}
0 & 0 & 0 & b_*^{(4)} & \dots & 0 & \dots & 0 & 0 & \dots & 0 & 0 & b_*^{(n)} \\
0 & 0 & 0 & 0 & \dots & 0 & \dots & b_*^{(1)} & 0 & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots \\
0 & b_*^{(2)} & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & b_*^{(n-2)} & 0 & 0 \\
0 & 0 & b_*^{(3)} & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 & b_*^{(n-1)} & 0
\end{array}\right|$$

where the subscripts are purely positional and the sub-blocks have been separated for the sake of clarity. The sum, f say, of the f_m, $1 \le m \le n$,

occurs in blocks of size n, its terms in the kth block ($k \ge 1$) being

$$b^{(1)}_{1+n(k-1)}, b^{(2)}_{2+n(k-1)}, \dots, b^{(n)}_{nk},$$

or, more precisely,

$$a^{r_k+k}, a^{r_k+3k}, a^{r_k+5k}, \dots, a^{r_k+(2n-1)k}.$$

It follows that $f \in l^2$, so $f_m \in l^2$, for every $1 \le m \le n$. For $1 \le m \le n$, the non-zero terms of $(f_j^{(m)}/b_j^{(m)})_{j=1}^{\infty}$ are, in order, simply $a, a^2, a^3, \dots$, so $(f_j^{(m)}/b_j^{(m)})_{j=1}^{\infty} \in l^2$ and $f_m \in \mathcal{R}(B_m)$.

Put $M = \langle f_1, \dots, f_n \rangle$. Then $M \cap \mathcal{R}(B_m) = \langle f_m \rangle$, $1 \le m \le n$. For, suppose that $(\xi_j)_{j=1}^{\infty} \in l^2$ and $B_m((\xi_j)_{j=1}^{\infty}) = (b_j^{(m)} \xi_j)_{j=1}^{\infty} = \sum_{i=1}^n \beta_i f_i$ with $\beta_i \in \mathbb{C}$, $1 \le i \le n$. For every $1 \le i \le n$, $i \ne m$, there is a unique $1 \le q_i \le n-1$ such that $[q_i + i]_n = m$. If $i \ne m$ and $p \ge 1$ put $j_{p,i} = m + v_{p,q_i}$. Then the $j_{p,i}$th term of $\sum_{u=1}^n \beta_u f_u$ is simply $\beta_i f^{(i)}_{j_{p,i}} = \beta_i b^{(m)}_{j_{p,i}}$, so $\beta_i b^{(m)}_{j_{p,i}} = b^{(m)}_{j_{p,i}} \xi_{j_{p,i}}$, which gives $\beta_i = \xi_{j_{p,i}}$. Since the latter is true for every $p \ge 1$ and $(\xi_{j_{p,i}})_{p=1}^{\infty} \in l^2$, we have $\beta_i = 0$.

Now let $(t_j)_{j=1}^{\infty}$ be the sequence arising from $\{(b_j^{(m)})_{j=1}^{\infty} : 1 \le m \le n\}$ as in Lemma 1 and let $\varphi_m : [0,1] \to \mathbb{R}$, $1 \le m \le n$, be the associated concave functions. Let $A \in \mathcal{B}(H)$ be the operator $A = \operatorname{diag}(t_1, t_2, \dots)$. Then $\|A\| = t_1 = 1$ and A is positive, injective and non-invertible (even compact). Also, $B_m = \varphi_m(A)$, $1 \le m \le n$. It is clear that $\{f_1, \dots, f_n\}$ is linearly independent. By Theorem 1, for every $1 \le m \le n$, $\mathcal{R}(B_m)$ is an invariant operator range of $\mathscr{A}(A)$, so, since every non-zero invariant operator range of $\mathscr{A}(A)$ contains $\mathcal{R}(A)$ (cf. [7]), we have $\mathcal{R}(A) \subsetneq \mathcal{R}(B_m)$. It follows that $M \cap \mathcal{R}(A) = (0)$, so we have a pentagon $\mathcal{P}(A; M)$ on $H \oplus H$ with gap-dimension n. We show that $\operatorname{Lat} \mathscr{A}(A; M)|_M$ is the atomic Boolean algebra with atoms $\{\langle f_j \rangle : 1 \le j \le n\}$. In fact we show that $\mathscr{A}(A; M)|_M$ consists precisely of those operators on M whose matrix relative to the basis $\{f_1, \dots, f_n\}$ is diagonal.

For every $1 \le m \le n$ let $A_m \in \mathcal{B}(H)$ be the operator whose matrix (relative to the usual orthonormal basis of l^2) is diagonal, the diagonal sequence having a one exactly where f_m has a non-zero term, with zeroes elsewhere. Then $A_m A = A A_m$, so $A_m \in \mathscr{A}(A)$, $1 \le m \le n$. Also, $A_m f_l = \delta_{lm} f_m$, $1 \le l, m \le n$. Hence, for each $1 \le m \le n$, $A_m \in \mathscr{A}(A; M)$ and the matrix of $A_m|_M$ relative to the basis $\{f_1, \dots, f_n\}$ is diagonal with mth diagonal entry 1 and 0 otherwise. It follows that $\mathscr{A}(A; M)|_M$ contains every operator whose matrix is diagonal relative to $\{f_1, \dots, f_n\}$. For the reverse inclusion, let $T \in \mathscr{A}(A; M)$ and let $1 \le m \le n$. Then, by Theorem 1, $T\mathcal{R}(B_m) \subseteq \mathcal{R}(B_m)$, so T leaves $M \cap \mathcal{R}(B_m) = \langle f_m \rangle$ invariant. Hence $T f_m = \lambda_m f_m$, for some $\lambda_m \in \mathbb{C}$, so the matrix of $T|_M$ relative to $\{f_1, \dots, f_n\}$ is diagonal.

(III) *Extreme case.* Let $n \in \mathbb{Z}^+$, $n \geq 2$, and let $0 < a < 1$. Let $(r_k)_{k=1}^\infty$, $(b_j^{(m)})_{j=1}^\infty$ and $\{B_m : 1 \leq m \leq n\}$ be as defined in our discussion of the preceding case. Define sequences $g_m = (g_j^{(m)})_{j=1}^\infty$, $1 \leq m \leq n$, each of them in blocks of size n^2 with each block consisting of n sub-blocks of size n, by specifying that the terms of g_m in the qth sub-block of the pth block where $p \geq 1$ and $1 \leq q \leq n-1$ be given by

$$0, \ldots, 0, b_{[q+m]_n + w_{p,q}}^{([q+m]_n)}, 0, \ldots, 0$$

where $[q+m]_n$ is as before, where $w_{p,q} = n(q-1) + n^2(p-1)$ and where the subscript on the non-zero term is purely positional. Also, we specify that the terms of g_m, $1 \leq m \leq n$, in the nth sub-block of the pth block be given by

$$b_{1+w_{p,n}}^{(1)}, b_{2+w_{p,n}}^{(2)}, \ldots, b_{m-1+w_{p,n}}^{(m-1)}, 0, b_{m+1+w_{p,n}}^{(m+1)}, \ldots, b_{n+w_{p,n}}^{(n)}$$

where $w_{p,n} = n(n-1) + n^2(p-1)$. Thus the pth block of n^2 terms of g_m is the mth row of the array

$$\left|\begin{array}{ccccc|cccccc|}
0 & b_*^{(2)} & 0 & \ldots & 0 & 0 & 0 & b_*^{(3)} & 0 & \ldots & 0 \\
0 & 0 & b_*^{(3)} & \ldots & 0 & 0 & 0 & 0 & b_*^{(4)} & \ldots & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & 0 & \ldots & b_*^{(n)} & b_*^{(1)} & 0 & 0 & 0 & \ldots & 0 \\
b_*^{(1)} & 0 & 0 & \ldots & 0 & 0 & b_*^{(2)} & 0 & 0 & \ldots & 0
\end{array}\right.$$

$$\begin{array}{c|ccccc|cccccc|}
\ldots & 0 & \ldots & 0 & 0 & b_*^{(n)} & 0 & b_*^{(2)} & b_*^{(3)} & \ldots & b_*^{(n-1)} & b_*^{(n)} \\
\ldots & b_*^{(1)} & \ldots & 0 & 0 & 0 & b_*^{(1)} & 0 & b_*^{(3)} & \ldots & b_*^{(n-1)} & b_*^{(n)} \\
 & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
\ldots & 0 & \ldots & b_*^{(n-2)} & 0 & 0 & b_*^{(1)} & b_*^{(2)} & b_*^{(3)} & \ldots & 0 & b_*^{(n)} \\
\ldots & 0 & \ldots & 0 & b_*^{(n-1)} & 0 & b_*^{(1)} & b_*^{(2)} & b_*^{(3)} & \ldots & b_*^{(n-1)} & 0
\end{array}$$

where the subscripts are purely positional and the sub-blocks have, once again, been separated for clarity. A similar proof to that given in the discussion of the preceding case (begun by adding the g_m's) shows that $g_m \in l^2$, $1 \leq m \leq n$. Also, for every $1 \leq m \leq n$, the non-zero terms of $(g_j^{(m)}/b_j^{(m)})_{j=1}^\infty$ are, in order, simply

$$a, a^2, \ldots, a^{n-1}, \underbrace{a^n, \ldots, a^n}_{n-1 \text{ terms}}, a^{n+1}, a^{n+2}, \ldots, a^{2n-1},$$
$$\underbrace{a^{2n}, \ldots, a^{2n}}_{n-1 \text{ terms}}, a^{2n+1}, a^{2n+2}, \ldots$$

so, since the latter sequence is square-summable, $g_m \in \mathcal{R}(B_m)$.

Put $M = \langle g_1, \ldots, g_n \rangle$. Again with proof as before, $M \cap \mathcal{R}(B_m) = \langle g_m \rangle$, $1 \leq m \leq n$ (take $j_{p,i} = m + w_{p,q_i}$ this time).

Assume that $n \geq 3$ (the case $n = 2$ will be considered separately). For $n+1 \leq m \leq 2n-1$ define the sequence $(b_j^{(m)})_{j=1}^\infty$, in blocks of size n^2 with each block consisting of n sub-blocks of size n, by specifying that the terms of $(b_j^{(m)})_{j=1}^\infty$ in the pth block (where $p \geq 1$) be given by the $(m-n)$th row of the array

$$\left|\begin{array}{cccc|cccc|}
b_*^{(2)} & b_*^{(1)} & \ldots & b_*^{(1)} & b_*^{(2)} & b_*^{(1)} & \ldots & b_*^{(1)} \\
b_*^{(2)} & b_*^{(1)} & \ldots & b_*^{(1)} & b_*^{(2)} & b_*^{(1)} & \ldots & b_*^{(1)} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
b_*^{(2)} & b_*^{(1)} & \ldots & b_*^{(1)} & b_*^{(2)} & b_*^{(1)} & \ldots & b_*^{(1)} \\
b_*^{(2)} & b_*^{(1)} & \ldots & b_*^{(1)} & b_*^{(2)} & b_*^{(1)} & \ldots & b_*^{(1)}
\end{array}\right.$$

$$\begin{array}{c|cccc|cccc|}
\ldots & b_*^{(2)} & b_*^{(1)} & \ldots & b_*^{(1)} & b_*^{(3)} & b_*^{(3)} & \ldots & b_*^{(3)} \\
\ldots & b_*^{(2)} & b_*^{(1)} & \ldots & b_*^{(1)} & b_*^{(4)} & b_*^{(4)} & \ldots & b_*^{(4)} \\
 & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
\ldots & b_*^{(2)} & b_*^{(1)} & \ldots & b_*^{(1)} & b_*^{(n)} & b_*^{(n)} & \ldots & b_*^{(n)} \\
\ldots & b_*^{(2)} & b_*^{(1)} & \ldots & b_*^{(1)} & b_*^{(1)} & b_*^{(1)} & \ldots & b_*^{(1)}
\end{array}$$

where the subscripts are purely positional and the sub-blocks have been clearly separated. Then $(b_j^{(m)})_{j=1}^\infty$, $n+1 \leq m \leq 2n-1$, is a strictly decreasing sequence of positive real numbers converging to zero. Define operators B_m, $n+1 \leq m \leq 2n-1$, on H by $B_m = \operatorname{diag}(b_1^{(m)}, b_2^{(m)}, \ldots)$. Then $g_m \notin \mathcal{R}(B_{m+n})$, for every $1 \leq m \leq n-1$, since the jth term of g_m equals $b_j^{(m+n)}$ for infinitely many values of j (consider the nth sub-block (size n) in each block (size n^2)). Also, $g_m - g_{m+1} \in \mathcal{R}(B_{m+n})$, for every $1 \leq m \leq n-1$. For, if $1 \leq m \leq n-1$, the pth block ($p \geq 1$) of $g_m - g_{m+1}$ is

$$|\, 0 \ldots 0\ b_*^{(m+1)}\ -b_*^{(m+2)}\ 0 \ldots 0 \,|\, 0 \ldots 0\ b_*^{(m+2)}\ -b_*^{(m+3)}\ 0 \ldots 0 \,|$$
$$\ldots |\, 0 \ldots 0\ b_*^{(n-1)}\ -b_*^{(n)} \,|\, -b_*^{(1)}\ 0 \ldots 0\ b_*^{(n)} \,|\, b_*^{(1)}\ -b_*^{(2)}\ 0 \ldots 0 \,|$$
$$\ldots |\, 0 \ldots 0\ b_*^{(m-1)}\ -b_*^{(m)}\ 0 \ldots 0 \,|\, 0 \ldots 0\ -b_*^{(m)}\ b_*^{(m+1)} 0 \ldots 0 \,|$$

where the subscripts are purely positional and the superscripts give the positions in each sub-block. Dividing by the corresponding terms of $(b_j^{(m+n)})_{j=1}^\infty$ we find that the absolute values of non-zero terms arising in the pth block are, respectively,

$$a^{k_p}, a^{k_p}, a^{k_p+1}, a^{k_p+1}, a^{k_p+2}, a^{k_p+2}, \ldots, a^{k_p+n}, a^{k_p+n}$$

where $k_p = (p-1)n + 1$. Since $(a^k)_{k=1}^{\infty} \in l^2$, $g_m - g_{m+1} \in \mathcal{R}(B_{m+n})$.

This time let $(t_j)_{j=1}^{\infty}$ be the sequence arising from $\{(b_j^{(m)})_{j=1}^{\infty} : 1 \le m \le 2n-1\}$ as in Lemma 1 and let $\varphi_m : [0,1] \to \mathbb{R}$, $1 \le m \le 2n-1$, be the associated concave functions. Let $A \in \mathcal{B}(H)$ be the operator $A = \operatorname{diag}(t_1, t_2, \ldots)$. Then $\|A\| = t_1 = 1$ and A is positive, injective and non-invertible (even compact). Also, $B_m = \varphi_m(A)$, $1 \le m \le 2n-1$. Clearly, $\{g_1, \ldots, g_n\}$ is linearly independent and by Theorem 1, $\mathcal{R}(B_m)$ is an invariant operator range of $\mathscr{A}(A)$, so $\mathcal{R}(A) \subseteq \mathcal{R}(B_m)$, $1 \le m \le 2n-1$. Since $M \cap \mathcal{R}(B_m) = \langle g_m \rangle$, $1 \le m \le n$, it follows that $M \cap \mathcal{R}(A) = (0)$. Once again we have a pentagon $\mathcal{P}(A; M)$ on $H \oplus H$ with gap-dimension n. We show that $\mathscr{A}(A; M)|_M = \mathbb{C}I$ (then, of course, $\operatorname{Lat} \mathscr{A}(A; M)|_M$ is the set of all subspaces of M).

Let $T \in \mathscr{A}(A; M)$. As in the discussion of the preceding case, for every $1 \le m \le n$, T leaves $M \cap \mathcal{R}(B_m) = \langle g_m \rangle$ invariant, so $Tg_m = \lambda_m g_m$ for some $\lambda_m \in \mathbb{C}$. Let $1 \le m \le n-1$. By Theorem 1, T also leaves $\mathcal{R}(B_{m+n})$ invariant, so since $g_m - g_{m+1} \in \mathcal{R}(B_{m+n})$, we have $T(g_m - g_{m+1}) = \lambda_m g_m - \lambda_{m+1} g_{m+1} \in \mathcal{R}(B_{m+n})$. Hence

$$(\lambda_{m+1} - \lambda_m) g_m = \lambda_{m+1}(g_m - g_{m+1}) - (\lambda_m g_m - \lambda_{m+1} g_{m+1}) \in \mathcal{R}(B_{m+n}).$$

But $g_m \notin \mathcal{R}(B_{m+n})$, so $\lambda_m = \lambda_{m+1}$. Hence $\lambda_1 = \lambda_2 = \ldots = \lambda_n$, and $T|_M \in \mathbb{C}I$.

Finally, consider the case where $n = 2$. The required example can be found by considering the case where $n = 3$. Our discussion of the latter case provides $A \in \mathcal{B}(H)$ positive, injective and non-invertible with $\|A\| = 1$ and operator ranges $\mathcal{R}(B_m)$, $1 \le m \le 5$, each invariant under every member of $\mathscr{A}(A)$ together with linearly independent vectors g_1, g_2, g_3 of H satisfying

$$\langle g_1, g_2, g_3 \rangle \cap \mathcal{R}(A) = (0),$$
$$\langle g_1, g_2, g_3 \rangle \cap \mathcal{R}(B_m) = \langle g_m \rangle, \quad m = 1, 2, 3,$$
$$g_1 - g_2 \in \mathcal{R}(B_4), \quad g_1 \notin \mathcal{R}(B_4),$$
$$g_2 - g_3 \in \mathcal{R}(B_5), \quad g_2 \notin \mathcal{R}(B_5).$$

Hence, with $M = \langle g_1, g_2 \rangle$, we have $\dim M = 2$, $M \cap \mathcal{R}(A) = (0)$, $M \cap \mathcal{R}(B_m) = \langle g_m \rangle$ for $m = 1, 2$, and $g_1 - g_2 \in \mathcal{R}(B_4)$, $g_1 \notin \mathcal{R}(B_4)$. With this M, the pentagon $\mathcal{P}(A; M)$ is the required example for the case where $n = 2$.

4. Conclusion. Since $\operatorname{Lat} \mathcal{F}$ is reflexive for any collection $\mathcal{F}$ of operators on H (clearly, $\operatorname{Lat} \operatorname{Alg} \operatorname{Lat} \mathcal{F} \subseteq \operatorname{Lat} \mathcal{F}$), our examples when $n = 2$ show that there exist subspace lattices isomorphic to those given in Figures 2(a) and 2(b) which are reflexive and have $\dim M \ominus L = 2$. Non-reflexive subspace lattices of these two lattice types can also be found. Indeed, let $\mathcal{P} = \{(0), K, L, M, H\}$ be a pentagon with $\dim M \ominus L = 2$, with $\operatorname{Lat} \operatorname{Alg} \mathcal{P} = \mathcal{P} \cup \{N : N \text{ is a subspace of } H \text{ and } L \subseteq N \subseteq M\}$. (The latter is schematically represented in Figure 2(c).) Let N_1 and N_2 be distinct subspaces satisfying $L \subset N_i \subset M$, $i = 1, 2$. If $\mathcal{L}_1 = \mathcal{P} \cup \{N_1\}$ and $\mathcal{L}_2 = \mathcal{P} \cup \{N_1, N_2\}$ then neither $\mathcal{L}_1$ nor $\mathcal{L}_2$ is reflexive since $\operatorname{Lat} \operatorname{Alg} \mathcal{P}$ is infinite and $\operatorname{Lat} \operatorname{Alg} \mathcal{P} \subseteq \operatorname{Lat} \operatorname{Alg} \mathcal{L}_i$, $i = 1, 2$. An analogous remark can also be made for every $n > 2$.

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Department of Mathematics
University of Crete
Iraklion, Crete
Greece
E-mail: lambrou@talos.cc.uch.gr

Department of Mathematics
The University of Western Australia
Nedlands, Western Australia 6907
Australia
E-mail: longstaf@maths.uwa.edu.au

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