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# On strong generation of $B(\mathcal{H})$ by two commutative $C^*$-algebras


by

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**Abstract.** The algebra $B(\mathcal{H})$ of all bounded operators on a Hilbert space $\mathcal{H}$ is generated in the strong operator topology by a single one-dimensional projection and a family of commuting unitary operators with cardinality not exceeding $\dim\mathcal{H}$. This answers Problem 8 posed by W. Żelazko in [6].


Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{S}$ be a subset of the algebra $B(\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$. We say that the algebra $B(\mathcal{H})$ is *strongly generated* by $\mathcal{S}$ if the smallest subalgebra of $B(\mathcal{H})$ closed in the strong operator topology and containing $\mathcal{S}$ coincides with $B(\mathcal{H})$. The first result on the strong generation of $B(\mathcal{H})$ was given by C. Davis in [2]. He proved, in the case when the Hilbert space $\mathcal{H}$ is separable, that the algebra $B(\mathcal{H})$ is strongly generated by two unitary operators. Later, E. Nordgren, M. Radjabalipour, H. Radjavi, and P. Rosenthal have shown ([3]) that two Hermitian operators strongly generate $B(\mathcal{H})$, which implies that $B(\mathcal{H})$ is singly generated as a von Neumann algebra. See also [5], pp. 160–163, for other results concerning generation of $B(\mathcal{H})$ when $\mathcal{H}$ is separable. These results show that for a separable Hilbert space $\mathcal{H}$ the algebra $B(\mathcal{H})$ is strongly generated by two commutative $C^*$-algebras. In [6] W. Żelazko raised the following

PROBLEM. *Is the algebra $B(\mathcal{H})$ of all operators on a complex Hilbert space $\mathcal{H}$ always generated in the strong operator topology by two commutative $C^*$-algebras?*

We show that the answer to this question is positive. Namely we prove the following

THEOREM. *Let $\mathcal{H}$ be a complex Hilbert space. The algebra $B(\mathcal{H})$ is strongly generated by a single one-dimensional (orthogonal) projection and a commuting family of unitary operators with cardinality not exceeding $\dim\mathcal{H}$.*


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We need the following simple

LEMMA. *Every non-empty set $I$ can be given a structure of an Abelian group.*

Proof. If the set $I$ is either finite or countable, then the result is clear. Assume that $I$ is uncountable. Take the family $\mathcal{M}$ of all subsets $J$ of $I$ having an Abelian group structure and order it by the following relation:

$$J_1 \le J_2 \quad \text{if and only if} \quad J_1 \text{ is a subgroup of } J_2.$$

The relation $\le$ is a partial order in $\mathcal{M}$. Notice that $\mathcal{M}$ is non-empty because every singleton $\{i_0\} \subset I$ can be given a group structure. If $(J_\alpha)$ is a chain in $\mathcal{M}$, then $J = \bigcup_\alpha J_\alpha$ is an Abelian group containing every $J_\alpha$ as a subgroup. By the Kuratowski–Zorn lemma $\mathcal{M}$ contains a maximal element $J_0$. If the cardinality of $J_0$ is smaller than that of $I$, then the sets $I \setminus J_0$ and $I$ have the same cardinality. Hence we can find a copy $J_1$ of $J_0$ in $I \setminus J_0$, i.e. there exists a one-to-one mapping $\varphi : J_0 \to I \setminus J_0$ with $J_1 = \varphi(J_0)$. Identifying $J_0 \cup J_1$ with $\mathbb{Z}_2 \times J_0$ via the map

$$(\varepsilon, j) \mapsto \begin{cases} j & \text{if } \varepsilon = 0, \\ \varphi(j) & \text{if } \varepsilon = 1, \end{cases}$$

we get an Abelian group structure on $J_0 \cup J_1$ with $J_0$ as a proper subgroup, contradicting the maximality of $J_0$. Therefore, $J_0$ must have the same cardinality as $I$ and via a bijection from $J_0$ onto $I$ we can define an Abelian group structure on $I$.

Proof of the Theorem. Let $\{e_i : i \in I\}$ be an orthonormal basis for the Hilbert space $\mathcal{H}$. By the Lemma we can introduce an Abelian group structure on $I$. Denote the group operation by $+$ and let $i_0$ be the zero element. Let $P_0$ be the (orthogonal) projection on the one-dimensional subspace spanned by $e_{i_0}$. For every $j \in I$ we define a unitary operator $S_j$ by $S_j(e_i) = e_{i+j}$ $(i \in I)$. Commutativity of the group $I$ implies that the operators $S_j$ mutually commute. Let $f$ be an arbitrary vector in $\mathcal{H}$ of norm one and let $P_f$ be the one-dimensional projection defined by

$$P_f(x) = \langle x, f\rangle f.$$

We will show that $P_f$ belongs to the von Neumann algebra $\mathcal{A}$ generated by $P_0$ and $\{S_j : j \in J\}$. Since we can approximate $f$ by elements of the form $g = \sum_{k=1}^n \lambda_k e_{i_k}$ we can approximate $P_f$ uniformly by operators $P_g$ given by $P_g(x) = \langle x, g\rangle g$ with $g$ as above. But we have

$$P_g(x) = \sum_{k=1}^n \sum_{l=1}^n \overline{\lambda}_k \lambda_l \langle x, e_{i_k}\rangle e_{i_l} = \sum_{k=1}^n \sum_{l=1}^n \overline{\lambda}_k \lambda_l S_{i_l} P_0 S_{-i_k}(x),$$

which implies that $P_g \in \mathcal{A}$ and hence $P_f \in \mathcal{A}$ as well. Consequently, every finite-rank projection is in the algebra $\mathcal{A}$. Since every projection $P$ is a limit in the strong operator topology of the net of all finite-rank projections with ranges included in the range of $P$ (cf. [4], p. 106, Lemma 3.3.2) we find that $\mathcal{A}$ contains all projections. Since $B(\mathcal{H})$ is the norm closed linear span of all projections we conclude that $\mathcal{A} = B(\mathcal{H})$ (see [1], p. 280, Prop. 4.8).



COROLLARY 1. *$B(\mathcal{H})$ is generated in the strong operator topology by two commutative $C^*$-algebras, one of them being one-dimensional.*

Remark. Using a similar method to the one in the proof above we can give another proof of the fact that in the separable case $B(\mathcal{H})$ is singly generated as a von Neumann algebra (see [3]).

Choose an orthonormal basis $(e_n)_{n=0}^\infty$ and let $S$ be the unilateral shift, i.e. $S(e_n) = e_{n+1}$ for $n = 0, 1, \ldots$ Then $S^*S - SS^*$ is the projection $P_0$ onto $\operatorname{span}\{e_0\}$. Every operator of the form $P_g(x) = \langle x, g\rangle g$ with $g = \sum_{k=0}^n \lambda_k e_k$ can be written as

$$P_g = \sum_{k=0}^n \sum_{l=0}^n \overline{\lambda}_k \lambda_l S^l P_0 S^{*k}.$$

Hence $P_g$ is in the $*$-algebra generated by $S$. As above, this implies that every one-dimensional projection and therefore every projection is in the von Neumann algebra $\mathcal{A}$ generated by $S$. Hence $\mathcal{A} = B(\mathcal{H})$.

Now we show that in the non-separable case the Theorem is the best possible result with respect to the number of generators.

PROPOSITION 1. *Let $\mathcal{H}$ be a non-separable Hilbert space. If $\{T_j : j \in J\}$ is a family of operators of cardinality smaller than $\dim \mathcal{H}$, then it has a non-trivial common invariant subspace. In particular, the von Neumann algebra generated by $\{T_j : j \in J\}$ is a proper subalgebra of $B(\mathcal{H})$.*

Proof. We may suppose that the identity belongs to the family $\{T_j : j \in J\}$. Fix an arbitrary non-zero vector $x \in \mathcal{H}$ and define

$$M = \operatorname{span}_{\mathbb{Q}}\{T_{j(1)} \ldots T_{j(n)}(x) : j(1), \ldots, j(n) \in J;\ n \in \mathbb{N}\},$$

where $\operatorname{span}_{\mathbb{Q}}$ denotes the linear span over the field $\mathbb{Q}$ of rational numbers. Then the closure $\overline{M}$ of $M$ is a closed subspace of $\mathcal{H}$ containing $x$ with $T_j(\overline{M}) \subset \overline{M}$ for every $j \in J$, hence $(0) \ne \overline{M}$ is a common invariant subspace of $\{T_j : j \in J\}$. Let $\{e_i : i \in I\}$ be an orthonormal basis for $\overline{M}$. Then, for each $i \in I$, we can find an element $x_i \in M$ such that $\|e_i - x_i\| < 1/2$. This implies that the mapping $i \mapsto x_i$ from $I$ into $M$ is one-to-one, and hence, $\operatorname{card} I \le \operatorname{card} M$. Since the cardinality of $M$ is at most $\max\{\aleph_0, \operatorname{card} J\} < \dim \mathcal{H}$, we see that $\overline{M}$ is a proper closed subspace of $\mathcal{H}$.

Finally, we show that in the Theorem one cannot replace the strong operator topology by the uniform topology.

PROPOSITION 2. *Let $\{T_\lambda : \lambda \in \Lambda\}$ be a family of operators on an infinite-dimensional Hilbert space $\mathcal{H}$ such that the algebra generated by it is uniformly dense in $B(\mathcal{H})$. Then the cardinality of $\Lambda$ is at least $2^{\dim \mathcal{H}}$.*

Proof. Since the algebra generated by $\{T_\lambda : \lambda \in \Lambda\}$ is uniformly dense in $B(\mathcal{H})$, the same is true for

$$\mathcal{A}_0 = \operatorname{span}_{\mathbb{Q}}\{T_{\lambda(1)} \ldots T_{\lambda(n)} : \lambda(1), \ldots, \lambda(n) \in \Lambda;\ n \in \mathbb{N}\},$$

which has cardinality at most $\max\{\aleph_0, \operatorname{card} \Lambda\}$.

Now choose an orthonormal basis $\{e_i : i \in I\}$ of $\mathcal{H}$. For every subset $J$ of $I$ let $P_J$ denote the projection onto $\overline{\operatorname{span}}\{e_j : j \in J\}$. For every $J \subset I$ we can find $T_J \in \mathcal{A}_0$ with $\|P_J - T_J\| < 1/2$. Since $\|P_J - P_{J'}\| = 1$ for $J \neq J'$, we deduce that the mapping $J \mapsto T_J$ from the family of all subsets of $I$ into $\mathcal{A}_0$ is one-to-one, hence $\operatorname{card} \mathcal{A}_0 \geq 2^{\dim \mathcal{H}}$. This implies $\operatorname{card} \Lambda \geq 2^{\dim \mathcal{H}}$.

COROLLARY 2. *The $C^*$-algebra generated by a countable family of operators on a separable Hilbert space $\mathcal{H}$ is always a proper subalgebra of $B(\mathcal{H})$.*

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# Diffeomorphisms between spheres and hyperplanes in infinite-dimensional Banach spaces

by

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**Abstract.** We prove that for every infinite-dimensional Banach space $X$ with a Fréchet differentiable norm, the sphere $S_X$ is diffeomorphic to each closed hyperplane in $X$. We also prove that every infinite-dimensional Banach space $Y$ having a (not necessarily equivalent) $C^p$ norm (with $p \in \mathbb{N} \cup \{\infty\}$) is $C^p$ diffeomorphic to $Y \setminus \{0\}$.

In 1966 C. Bessaga [1] proved that every infinite-dimensional Hilbert space $H$ is $C^\infty$ diffeomorphic to its unit sphere. The key to proving this astonishing result was the construction of a diffeomorphism between $H$ and $H \setminus \{0\}$ which is the identity outside a ball, and this construction was possible thanks to the existence of a $C^\infty$ non-complete norm in $H$. In [5], T. Dobrowolski developed Bessaga's non-complete norm technique and proved that every infinite-dimensional Banach space $X$ which is linearly injectable into some $c_0(\Gamma)$ is $C^\infty$ diffeomorphic to $X \setminus \{0\}$. More generally, he proved that every infinite-dimensional Banach space $X$ having a $C^p$ non-complete norm is $C^p$ diffeomorphic to $X \setminus \{0\}$. If in addition $X$ has an equivalent $C^p$ smooth norm $\|\cdot\|$ then one can deduce that the sphere $S = \{x \in X : \|x\| = 1\}$ is $C^p$ diffeomorphic to any of the hyperplanes in $X$. So, regarding the generalization of Bessaga and Dobrowolski's results to every infinite-dimensional Banach space having a differentiable norm (resp. $C^p$ smooth norm, with $p \in \mathbb{N} \cup \{\infty\}$), the following problem naturally arises: does every infinite-dimensional Banach space with a $C^p$ smooth equivalent norm have a $C^p$ smooth non-complete norm? Surprisingly enough, this seems to be a difficult question which still remains unsolved. Without proving the existence of smooth non-complete norms we show that every infinite-dimensional Banach space $X$ with a Fréchet differentiable (resp. $C^p$ smooth) norm $\|\cdot\|$ is diffeomorphic (resp. $C^p$ diffeomorphic) to $X \setminus \{0\}$, and we deduce

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