On strong generation of $B(\mathcal{H})$
by two commutative $C^*$-algebras

by

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Abstract. The algebra $B(\mathcal{H})$ of all bounded operators on a Hilbert space $\mathcal{H}$ is generated in the strong operator topology by a single one-dimensional projection and a family of commuting unitary operators with cardinality not exceeding $\dim \mathcal{H}$. This answers Problem 3 posed by W. Żelazko in [6].

Let $\mathcal{H}$ be a complex Hilbert space and let $S$ be a subset of the algebra $B(\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$. We say that the algebra $B(\mathcal{H})$ is strongly generated by $S$ if the smallest $*$-subalgebra of $B(\mathcal{H})$ closed in the strong operator topology and containing $S$ coincides with $B(\mathcal{H})$. The first result on the strong generation of $B(\mathcal{H})$ was given by C. Davis in [2]. He proved, in the case when the Hilbert space $\mathcal{H}$ is separable, that the algebra $B(\mathcal{H})$ is strongly generated by two unitary operators. Later, E. Nordgren, M. Radjabaliportal, H. Radjavi, and P. Rosenthal have shown ([3]) that two Hermitian operators strongly generate $B(\mathcal{H})$, which implies that $B(\mathcal{H})$ is singly generated as a von Neumann algebra. See also [5], pp. 160–163, for other results concerning generation of $B(\mathcal{H})$ when $\mathcal{H}$ is separable. These results show that for a separable Hilbert space $\mathcal{H}$ the algebra $B(\mathcal{H})$ is strongly generated by two commutative $C^*$-algebras. In [6] W. Żelazko raised the following

**Problem.** Is the algebra $B(\mathcal{H})$ of all operators on a complex Hilbert space $\mathcal{H}$ always generated in the strong operator topology by two commutative $C^*$-algebras?

We show that the answer to this question is positive. Namely we prove the following

**Theorem.** Let $\mathcal{H}$ be a complex Hilbert space. The algebra $B(\mathcal{H})$ is strongly generated by a single one-dimensional (orthogonal) projection and a commuting family of unitary operators with cardinality not exceeding $\dim \mathcal{H}$.

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We need the following simple

**Lemma.** Every non-empty set \( I \) can be given a structure of an Abelian group.

**Proof.** If the set \( I \) is either finite or countable, then the result is clear. Assume that \( I \) is uncountable. Take the family \( \mathcal{M} \) of all subsets \( J \) of \( I \) having an Abelian group structure and order it by the following relation:

\[
J_1 \leq J_2 \quad \text{if and only if} \quad J_1 \text{ is a subgroup of } J_2.
\]

The relation \( \leq \) is a partial order in \( \mathcal{M} \). Notice that \( \mathcal{M} \) is non-empty because every singleton \( \{x_0\} \subset I \) can be given a group structure. If \( \{x_n\} \) is a chain in \( \mathcal{M} \), then \( J = \bigcup_{n=0}^{\infty} J_n \) is an Abelian group containing every \( J_n \) as a subgroup. By the Kuratowski–Zorn lemma \( \mathcal{M} \) contains a maximal element \( J_0 \). If the cardinality of \( J_0 \) is smaller than that of \( I \), then the sets \( I \setminus J_0 \) and \( I \) have the same cardinality. Hence we can find a copy \( J_1 \) of \( J_0 \) in \( I \setminus J_0 \), i.e., there exists a one-to-one mapping \( \varphi : J_0 \to I \setminus J_0 \) with \( J_1 = \varphi(J_0) \). Identifying \( J_0 \cup J_1 \) with \( \mathbb{Z}_2 \times J_0 \) via the map

\[
(e, j) \mapsto \begin{cases} 
    j & \text{if } e = 0, \\
    \varphi(j) & \text{if } e = 1,
\end{cases}
\]

we get an Abelian group structure on \( J_0 \cup J_1 \) with \( J_0 \) as a proper subgroup, contradicting the maximality of \( J_0 \). Therefore, \( J_0 \) must have the same cardinality as \( I \) and via a bijection from \( J_0 \) onto \( I \) we can define an Abelian group structure on \( I \).

**Proof of the Theorem.** Let \( \{e_i : i \in I\} \) be an orthonormal basis for the Hilbert space \( \mathcal{H} \). By the Lemma we can introduce an Abelian group structure on \( I \). Denote the group operation by + and let \( i_0 \) be the zero element. Let \( P_0 \) be the (orthogonal) projection on the one-dimensional subspace spanned by \( e_{i_0} \). For every \( j \in I \) we define a unitary operator \( S_j \) by

\[
S_j(e_i) = e_{i+1} \quad (i \in I).
\]

Commutativity of the group \( I \) implies that the operators \( S_j \) mutually commute. Let \( f \) be an arbitrary vector in \( \mathcal{H} \) of norm one and let \( P_f \) be the one-dimensional projection defined by

\[
P_f(x) = \langle x, f \rangle f.
\]

We will show that \( P_f \) belongs to the von Neumann algebra \( A \) generated by \( P_0 \) and \( \{S_j : j \in I\} \). Since we can approximate \( f \) by elements of the form \( g = \sum_{k=1}^{n} \lambda_k e_{i_k} \), we can approximate \( P_f \) uniformly by operators \( P_g \) given by

\[
P_g(x) = \sum_{k=1}^{n} \sum_{l=1}^{n} \lambda_k \lambda_l \langle x, e_{i_l} \rangle e_{i_l} = \sum_{k=1}^{n} \sum_{l=1}^{n} \overline{\lambda_k} \lambda_l S_i P_0 S_{-i_k} (x),
\]

which implies that \( P_g \in A \) and hence \( P_f \in A \) as well. Consequently, every finite-rank projection is in the algebra \( A \). Since every projection \( P \) is a limit in the strong operator topology of the net of all finite-rank projections with ranges included in the range of \( P \) (cf. [4], p. 106, Lemma 3.3.2) we find that \( A \) contains all projections. Since \( B(\mathcal{H}) \) is the norm closed linear span of all projections we conclude that \( A = B(\mathcal{H}) \) (see [1], p. 280, Prop. 4.8).

**Corollary 1.** \( B(\mathcal{H}) \) is generated in the strong operator topology by two commutative \( C^* \)-algebras, one of them being one-dimensional.

**Remark.** Using a similar method to the one in the proof above we can give another proof of the fact that in the separable case \( B(\mathcal{H}) \) is singly generated as a von Neumann algebra (see [3]).

Choose an orthonormal basis \( \{e_n\}_{n=0}^{\infty} \) and let \( S \) be the unilateral shift, i.e., \( S^*e_n = e_{n+1} \) for \( n = 0, 1, \ldots \). Then \( SS^* = S^*S \) is the projection \( P_0 \) onto \( \langle e_0 \rangle \). Every operator of the form \( P_g(x) = \langle x, g \rangle g \) with \( g = \sum_{k=0}^{n} \lambda_k e_k \) can be written as

\[
P_g = \sum_{k=0}^{n} \sum_{l=0}^{n} \overline{\lambda_k} \lambda_l S^l P_0 S^{*l*}.
\]

Hence \( P_g \) is in the \( * \)-algebra generated by \( S \). As above, this implies that every one-dimensional projection and therefore every projection is in the von Neumann algebra \( A \) generated by \( S \). Hence \( A = B(\mathcal{H}) \).

Now we show that in the non-separable case the Theorem is the best possible result with respect to the number of generators.

**Proposition 1.** Let \( \mathcal{H} \) be a non-separable Hilbert space. If \( \{T_j : j \in J\} \) is a family of operators of cardinality smaller than \( \dim \mathcal{H} \), then it has a non-trivial common invariant subspace. In particular, the von Neumann algebra generated by \( \{T_j : j \in J\} \) is a proper subalgebra of \( B(\mathcal{H}) \).

**Proof.** We may suppose that the identity belongs to the family \( \{T_j : j \in J\} \). Fix an arbitrary non-zero vector \( x \in \mathcal{H} \) and define

\[
M = \text{span}_Q \{T_j(1), \ldots, T_j(n)(x) : j(1), \ldots, j(n) \in J; n \in \mathbb{N}\},
\]

where \( \text{span}_Q \) denotes the linear span over the field \( Q \) of rational numbers. Then the closure \( \overline{M} \) of \( M \) is a closed subspace of \( \mathcal{H} \) containing \( x \) with \( T_j(\overline{M}) \subseteq \overline{M} \) for every \( j \in J \), hence \( \{0\} \neq \overline{M} \) is a common invariant subspace of \( \{T_j : j \in J\} \). Let \( \{e_i : i \in I\} \) be an orthonormal basis for \( \overline{M} \). Then, for each \( i \in I \), we can find an element \( x_i \in \mathcal{H} \) such that \( \|e_i - x_i\| < 1/2 \). This implies that the mapping \( i \mapsto x_i \) from \( I \) into \( \overline{M} \) is one-to-one, and hence, \( \text{card} \overline{M} \leq \text{card} \mathcal{H} \). Since the cardinality of \( \mathcal{H} \) is at most \( \max \{N_0, \text{card} J\} < \dim \mathcal{H} \), we see that \( \overline{M} \) is a proper closed subspace of \( \mathcal{H} \).

Finally, we show that in the Theorem one cannot replace the strong operator topology by the uniform topology.
PROPOSITION 2. Let \( \{ T_\lambda : \lambda \in \Lambda \} \) be a family of operators on an infinite-dimensional Hilbert space \( \mathcal{H} \) such that the algebra generated by it is uniformly dense in \( B(\mathcal{H}) \). Then the cardinality of \( \Lambda \) is at least \( 2^{\dim \mathcal{H}} \).

Proof. Since the algebra generated by \( \{ T_\lambda : \lambda \in \Lambda \} \) is uniformly dense in \( B(\mathcal{H}) \), the same is true for
\[
\mathcal{A}_0 = \text{span} \{ T_{\lambda(1)} \ldots T_{\lambda(n)} : \lambda(1), \ldots, \lambda(n) \in \Lambda; n \in \mathbb{N} \},
\]
which has cardinality at most \( \max \{ n_0, \text{card} \Lambda \} \).

Now choose an orthonormal basis \( \{ e_i : i \in I \} \) of \( \mathcal{H} \). For every subset \( J \) of \( I \) let \( P_J \) denote the projection onto \( \text{span} \{ e_j : j \in J \} \). For every \( J \in I \) we can find \( T_J \in \mathcal{A}_0 \) with \( \| P_J - T_J \| < 1/2 \). Since \( \| P_J - P_{J'} \| = 1 \) for \( J \neq J' \), we deduce that the mapping \( J \mapsto T_J \) from the family of all subsets of \( I \) into \( \mathcal{A}_0 \) is one-to-one, hence card \( \mathcal{A}_0 \geq 2^{\dim \mathcal{H}} \). This implies card \( \Lambda \geq 2^{\dim \mathcal{H}} \).

COROLLARY 2. The \( C^* \)-algebra generated by a countable family of operators on a separable Hilbert space \( \mathcal{H} \) is always a proper subalgebra of \( B(\mathcal{H}) \).

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References


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Diffeomorphisms between spheres and hyperplanes in infinite-dimensional Banach spaces

by

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Abstract. We prove that for every infinite-dimensional Banach space \( X \) with a Fréchet differentiable norm, the sphere \( S_X \) is diffeomorphic to each closed hyperplane in \( X \). We also prove that every infinite-dimensional Banach space \( Y \) having a (not necessarily equivalent) \( C^p \) norm (with \( p \in \mathbb{N} \cup \{ \infty \} \)) is \( C^p \) diffeomorphic to \( Y \setminus \{ 0 \} \).

In 1966 C. Bessaga [1] proved that every infinite-dimensional Hilbert space \( H \) is \( C^\infty \) diffeomorphic to its unit sphere. The key to proving this astonishing result was the construction of a diffeomorphism between \( H \) and \( H \setminus \{ 0 \} \) which is the identity outside a ball, and this construction was possible thanks to the existence of a \( C^\infty \) non-complete norm in \( H \). In [5], T. Dobrowolski developed Bessaga's non-complete norm technique and proved that every infinite-dimensional Banach space \( X \) which is linearly injectable into some \( c_0(\Gamma) \) is \( C^\infty \) diffeomorphic to \( X \setminus \{ 0 \} \). More generally, he proved that every infinite-dimensional Banach space \( X \) having a \( C^p \) non-complete norm is \( C^p \) diffeomorphic to \( X \setminus \{ 0 \} \). If in addition \( X \) has an equivalent \( C^p \) smooth norm \( \| \cdot \| \) then one can deduce that the sphere \( S = \{ x \in X : \| x \| = 1 \} \) is \( C^p \) diffeomorphic to any of the hyperplanes in \( X \). So, regarding the generalization of Bessaga and Dobrowolski's results to every infinite-dimensional Banach space having a differentiable norm (resp. \( C^p \) smooth norm, with \( p \in \mathbb{N} \cup \{ \infty \} \)), the following problem naturally arises: does every infinite-dimensional Banach space with a \( C^p \) smooth equivalent norm have a \( C^p \) smooth non-complete norm? Surprisingly enough, this seems to be a difficult question which still remains unsolved. Without proving the existence of smooth non-complete norms we show that every infinite-dimensional Banach space \( X \) with a Fréchet differentiable (resp. \( C^p \) smooth) norm \( \| \cdot \| \) is diffeomorphic (resp. \( C^p \) diffeomorphic) to \( X \setminus \{ 0 \} \), and we deduce...