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On the L_1-convergence of Fourier series

by

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Abstract. Since the trigonometric Fourier series of an integrable function does not necessarily converge to the function in the mean, several additional conditions have been devised to guarantee the convergence. For instance, sufficient conditions can be constructed by using the Fourier coefficients or the integral modulus of the corresponding function. In this paper we give a Hardy–Karamata type Tauberian condition on the Fourier coefficients and prove that it implies the convergence of the Fourier series in integral norm, almost everywhere, and if the function itself is in the real Hardy space, then also in the Hardy norm. We also compare it to the previously known conditions.

Main result. Let $L_1 = L_1[-\pi,\pi]$ denote the space of complex-valued 2π-periodic functions that are integrable on $[-\pi,\pi]$ with the usual norm denoted by $\|\cdot\|_1$. We use the conventional notations for the Fourier coefficients and for the Fourier partial sums with respect to the complex and the real trigonometric systems, i.e. for any $f \in L_1$,

$$S_n f(t) = \sum_{|k|=0}^{n} \widehat{f}(k)\exp(ikt)$$

$$= a_0 + \sum_{k=1}^{n}(a_k\cos kt + b_k \sin kt) \qquad (-\pi \le t \le \pi,\ n \in \mathbb{N}).$$

$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{C}$ stand for the set of natural numbers, integers and complex numbers respectively.

Let $[x]$ denote the integer part of the real number x and for any sequence $(c_k, k \in \mathbb{Z})$ of complex numbers let $\Delta_k c = c_k - c_{k+1}$ if k is non-negative and $\Delta_k c = c_k - c_{k-1}$ if k is negative.

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Then our main result reads as follows.

THEOREM 1. *Let $f \in L_1$ be such that*

$$\lim_{\lambda \to 1+} \overline{\lim_{n\to\infty}} \sum_{|k|=n+1}^{[\lambda n]} |\Delta \widehat{f}(k)| \log^+ |k\Delta \widehat{f}(k)| = 0. \tag{1}$$

Then

(i) $\lim_{n\to\infty} \|f - S_n f\|_1 = 0$ *if and only if* $\lim_{|n|\to\infty} \widehat{f}(n) \log |n| = 0$,
(ii) $f(x) = g(x)/x$ $(0 < |x| \le \pi)$ *where* $g \in L_2$.

Remark 1. It is easy to see that for the real trigonometric system condition (1) has the following equivalent form:

$$\lim_{\lambda \to 1+} \overline{\lim_{n\to\infty}} \sum_{k=n+1}^{[\lambda n]} (|\Delta a_k| \log^+(k|\Delta a_k|) + |\Delta b_k| \log^+(k|\Delta b_k|)) = 0.$$

We note that the proof of Theorem 1 will be presented for this version, i.e. for the real trigonometric system, in order that we can point out the difference between the behavior of the even and the odd parts of the series.

Remark 2. The decomposition in (ii) of Theorem 1 provides information about the structure of integrable functions that satisfy condition (1). We note that by Carleson's Theorem and by the proof of (ii) one can deduce the a.e. convergence of $S_n f$. However, this also follows from a condition that is weaker than (1). Namely, Chen [4] proved that if f is an integrable function for which

$$\lim_{\lambda \to 1+} \overline{\lim_{n\to\infty}} \sum_{|k|=n+1}^{[\lambda n]} |\Delta \widehat{f}(k)| = 0$$

then the Fourier series of f converges almost everywhere.

Let H denote the real Hardy space on $[-\pi,\pi]$. Recall that H is the collection of those real-valued f in L_1 for which also the trigonometric conjugate $\widetilde{f}$ is integrable, and the norm is defined as $\|f\|_H = \|f\|_1 + \|\widetilde{f}\|_1$. The following corollary is an immediate consequence of Theorem 1 and of the well known relation between the Fourier coefficients of the function and of its trigonometric conjugate.

COROLLARY 1. *Let $f \in H$ with $Sf(t) = a_0 + \sum_{k=1}^{\infty}(a_k \cos kt + b_k \sin kt)$. If*

$$\lim_{\lambda \to 1+} \overline{\lim_{n\to\infty}} \sum_{k=n+1}^{[\lambda n]} (|\Delta a_k| \log^+(k|\Delta a_k|) + |\Delta b_k| \log^+(k|\Delta b_k|)) = 0$$

then

$$\lim_{n\to\infty} \|f - S_n f\|_H = 0 \quad \textit{if and only if} \quad \lim_{|n|\to\infty} (|a_n| + |b_n|) \log |n| = 0.$$

Throughout this paper C will always denote a positive constant not necessarily the same in different occurrences.

Comparison with previous conditions. We note that the classical convergence conditions on the Fourier coefficients given by Young [19], Kolmogorov [9], Sidon [13] and the more recent ones by Telyakovskiĭ [18], Fomin [5], Č. V. Stanojević and V. B. Stanojević [16] all generate the equivalence in (i) of Theorem 1. This led to the concept of L_1-convergence classes. Let $\mathcal{F}$ denote the set of sequences of Fourier coefficients of integrable functions. Then a subset $\mathcal{C}$ of $\mathcal{F}$ is called an L_1-*convergence class* if for any $f \in L_1$ with $(\widehat{f}(k), k \in \mathbb{Z}) \in \mathcal{C}$ we have

$$\lim_{n\to\infty} \|S_n f - f\|_1 = 0 \quad \text{if and only if} \quad \lim_{|n|\to\infty} \widehat{f}(n) \log |n| = 0.$$

Using this terminology we may say by Theorem 1 that condition (1) induces an L_1-convergence class. Let us denote this class by $\mathcal{S}$.

Thus $(c_k, k \in \mathbb{Z}) \in \mathcal{S}$ if and only if there exists $f \in L_1$ such that

$$\widehat{f}(k) = c_k \quad (k \in \mathbb{Z}) \quad \text{and} \quad \lim_{\lambda \to 1+} \overline{\lim_{n\to\infty}} \sum_{|k|=n}^{[\lambda n]} |\Delta c_k| \log^+ |k\Delta c_k| = 0.$$

First we show that our condition is the best possible for lacunary Fourier series and for the case when the Fourier coefficients of a real function form a decreasing sequence. By the latter we mean (in terms of the real trigonometric system) that $(a_n, n \in \mathbb{N})$ and $(b_{n+1}, n \in \mathbb{N})$ are decreasing sequences. It is known (see e.g. [20]) that in both of these cases $\lim_{|n|\to\infty} \widehat{f}(n) \log |n| = 0$ itself is sufficient to conclude the L_1-convergence of the corresponding Fourier series. Thus the best we can expect is that (1) does not add an unnecessary restriction to that. Indeed, as is easy to see, if the coefficient sequence is lacunary then (1) reduces to

$$\lim_{|n|\to\infty} |\Delta \widehat{f}(n)| \log^+ |n\Delta \widehat{f}(n)| = 0,$$

which is even weaker than $\lim_{n\to\infty} \widehat{f}(n) \log |n| = 0$.

Similarly, if $(a_n, n \in \mathbb{N})$ and $(b_{n+1}, n \in \mathbb{N})$ are decreasing then

$$\sum_{|k|=n+1}^{[\lambda n]} (|\Delta a_k| \log^+ |k\Delta a_k| + |\Delta b_k| \log^+ |k\Delta b_k|)$$
$$\le C((a_{n+1} - a_{[\lambda n]+1}) \log n + (b_{n+1} - b_{[\lambda n]+1}) \log n)$$

where C does not depend on n. Consequently, (1) follows from $\lim_{n\to\infty}(|a_n| + |b_n|)\log|n| = 0$.

At the beginning of this section we listed papers of several authors that contain L_1-convergence classes. In the last decade especially Stanojević and his coauthors have done much research in this field (see [2], [3], [7], [14], [15]). They have constructed L_1-convergence classes that contain those in [19], [9], [13], [18], [5], [16]. (For a summary of previous conditions we refer to [4] and [17].) The final version of their results (see [7]) reads as follows. It is shown in [7] that $\mathcal{L}_p$ is an L_1-convergence class for any $1 \le p < \infty$ where $\mathcal{L}_p$ denotes the set of sequences $(\widehat{f}_k, k \in \mathbb{Z})$ of Fourier coefficients of integrable functions for which

$$\lim_{\lambda\to 1^+}\overline{\lim_{n\to\infty}}\sum_{|k|=n+1}^{[\lambda n]} k^{p-1}|\Delta\widehat{f}(k)|^p < \infty \quad (p > 1) \tag{2}$$

and

$$\lim_{\lambda\to 1^+}\overline{\lim_{n\to\infty}}\sum_{|k|=n+1}^{[\lambda n]} |\Delta\widehat{f}(k)|\log|k| = 0 \quad (p = 1). \tag{3}$$

We show that $\mathcal{S}$ contains all of these classes:

THEOREM 2.

$$\mathcal{S} \supsetneq \bigcup_{p\ge 1}\mathcal{L}_p.$$

Remark 3. The following condition was used in earlier papers of Stanojević and his coauthors (see e.g. [14], [3]):

$$\lim_{\lambda\to 1^+}\overline{\lim_{n\to\infty}}\sum_{|k|=n+1}^{[\lambda n]} k^{p-1}|\Delta\widehat{f}(k)|^p = 0. \tag{4}$$

We note that it was observed by Aubertin and Fournier [1] that although (2) looks weaker than (4) they are essentially equivalent. Namely (see [1]), if (2) holds for a sequence with some p then the sequence satisfies (4) with p replaced by any q smaller than p. Furthermore, it is easy to show (see e.g. [4]) that (2) is equivalent to

$$\overline{\lim_{n\to\infty}}\sum_{|k|=n+1}^{2n} k^{p-1}|\Delta\widehat{f}(k)|^p < \infty.$$

Finally, we note that several other conditions have been constructed implying the L_1-convergence of Fourier series. The most recent ones can be found in [7] and [1]. Here we only note that their structure is substantially

different from, and more complicated than, that of (1). Therefore they are quite difficult to check, and it would also be difficult to compare them.

Proofs. Let D_k and $\widetilde{D}_k$ denote the kth $(k \in \mathbb{N})$ trigonometric and conjugate trigonometric Dirichlet kernel resp. The following Sidon type inequality holds true for the conjugate kernels.

LEMMA 1. *There exists $C > 0$ such that*

$$\Big\|\sum_{k=K}^{N} c_k\widetilde{D}_k\Big\|_1 \le \sum_{k=K}^{N}|c_k|\Big(1 + \log^+\frac{|c_k|}{(N-K+1)^{-1}\sum_{j=K}^{N}|c_j|}\Big) \tag{5}$$

for any $c_k \in \mathbb{C}$ with $\sum_{k=K}^{N} c_k = 0$ $(K, N \in \mathbb{N})$.

(Here and later $0/0$ will be considered to be equal to 1.)

Proof. Only an outline will be given because the proof is similar to the one of the corresponding result for the D_k's (see [6]). Let us start with the Sidon type inequality proved by Schipp [11] for complex trigonometric kernels. Set $\mathcal{D}_k(t) = \sum_{j=0}^{n}\exp ijt$ $(k \in \mathbb{N}, i = \sqrt{-1})$. Then (see [11])

$$\frac{1}{K-N+1}\Big\|\sum_{k=K}^{N} c_k\mathcal{D}_k\Big\|_1 \le C\max_{K\le k\le N}|c_k| \quad (c_k \in \mathbb{C},\ K, N \in \mathbb{N}) \tag{6}$$

provided $\sum_{k=K}^{N} c_k = 0$. Schipp used this result to prove a Sidon type inequality for the D_k's employing the concept of atomic decomposition of the real non-periodic Hardy space. For this last concept we refer to [8]. (Concerning the method of proof see also [12].) Based on Schipp's result the author proved the following shifted Sidon type inequality in [6] by using the notion of atomic decomposition, certain norm inequalities and the uniform boundedness of the L_1-norms of the Fejér kernels:

$$\Big\|\sum_{k=K}^{N} c_k D_k\Big\|_1 \le C\Big(\log\frac{N}{N-K+1}\Big|\sum_{k=K}^{N} c_k\Big| + \sum_{k=K}^{N}|c_k|\Big(1 + \log^+\frac{|c_k|}{(N-K+1)^{-1}\sum_{j=K}^{N}|c_j|}\Big)\Big) \tag{7}$$

$(c_k \in \mathbb{C},\ K, N \in \mathbb{N})$. Then taking the imaginary part in (6) one can derive (5) by repeating the considerations used in [11] and [6] to prove (7). The only difference is the necessity of the additional condition $\sum_{k=K}^{N} c_k = 0$. The reason behind it is that the L_1-norms of the arithmetic means of the conjugate kernels are not uniformly bounded. ■

LEMMA 2. *There exist absolute positive constants C_1, C_2 such that for any $K \le N$ $(K, N \in \mathbb{N})$ we have*

$$C_1 \log(N-K+1) \le \frac{1}{N-K+1}\Big\| \sum_{k=K}^{N} (\widetilde{D}_k - \widetilde{D}_K)\Big\|_1 \le C_2 \log(N-K+1).$$

Proof. We may suppose that $N-K$ is large enough. An easy calculation shows that

$$\frac{1}{N-K+1}\sum_{k=K}^{N}(\widetilde{D}_k(x) - \widetilde{D}_K(x)) = \cos Kx\, \widetilde{F}_{N-K+1}(x) + \big(\sin Kx\, F_{N-K+1}(x) - \tfrac{1}{2}\cos Kx\big) \quad (K, N \in \mathbb{N}),$$

where F_n and $\widetilde{F}_n$ denote the nth $(n \in \mathbb{N})$ Fejér and conjugate Fejér kernel respectively.

Then the upper estimate follows from $\|\widetilde{D}_n\|_1 \le C \log(n+1)$ and from the uniform boundedness of $\|F_n\|_1$ $(n \in \mathbb{N})$.

For the lower estimate write

$$\widetilde{F}_{N-K+1}(x) = \frac{1}{N-K+1}\Big(((N-K+1) - D_{N-K+1}(x))\cot\frac{x}{2} + \widetilde{D}_{N-K+1}(x)\Big).$$

Using $|D_n(x)| \le n/2$ $(x > \pi/n)$ and $\|\widetilde{D}_n\|_1 \le C\log(n+1)$ $(n \in \mathbb{N}, n > 0)$ again we have

$$\int_0^{\pi} \cos Kx\, \widetilde{F}_{N-K+1}(x)\,dx \ge \frac{1}{2}\int_{\pi/(N-K+1)}^{\pi} |\cos Kx| \cot\frac{x}{2}\,dx - C\frac{\log(N-K+1)}{N-K+1} \ge C\log(N-K+1).$$

Consequently,

$$\frac{1}{N-K+1}\Big\| \sum_{k=K}^{N} (\widetilde{D}_k - \widetilde{D}_K)\Big\|_1 \ge C_2 \log(N-K+1). \ \blacksquare$$

LEMMA 3. *Let $(c_k, k \in \mathbb{N})$ be a sequence of non-negative real numbers. Then*

$$\lim_{\lambda\to 1+}\overline{\lim_{n\to\infty}} \sum_{k=n+1}^{[\lambda n]} c_k \log^+(kc_k) = 0$$

implies

$$(8) \qquad \lim_{\lambda\to 1+}\overline{\lim_{n\to\infty}} \sum_{k=n+1}^{[\lambda n]} c_k\Big(1 + \log^+ \frac{([\lambda n]-n)c_k}{\sum_{j=n+1}^{[\lambda n]} c_j}\Big) = 0.$$

Proof. We use the fact that the sum in (8) can be associated with an Orlicz norm. Let L_M denote the Orlicz space generated by the Young function

$$M(x) = \begin{cases} (1/2)|x|^2 & \text{if } 0 \le |x| < 1, \\ 1/2 + |x|\log|x| & \text{if } |x| \ge 1. \end{cases}$$

For the definition and properties of Orlicz spaces we refer to [10]. It is known (see e.g. [6]) that

$$(9) \qquad \|h\|_{L_M} \approx \int_0^1 |h|\Big(1 + \log^+ \frac{|h|}{\|h\|_1}\Big) \quad (h \in L_M).$$

Let χ_A stand for the characteristic function of $A \subset [0,1)$. If for a complex vector $(c_k)_j^l = (c_j, \ldots, c_l)$ the step function $\mathbf{\Gamma}(c_k)_j^l$ is defined as follows:

$$\mathbf{\Gamma}(c_k)_j^l = \sum_{k=1}^{n} c_k \chi_{[(k-j)(l-j+1)^{-1}, (k-j+1)(l-j+1)^{-1})}$$

then (8) can be written in the form

$$\lim_{\lambda\to 1+}\overline{\lim_{n\to\infty}} ([\lambda n]-n)\|\mathbf{\Gamma}(c_k)_{n+1}^{[\lambda n]}\|_{L_M} = 0.$$

In order to prove this form of (8) we modify the c_k's $(k \in \mathbb{N})$ as follows:

$$d_k = \begin{cases} c_k & \text{if } c_k \ge e/k, \\ e/k & \text{if } c_k < e/k, \end{cases} \qquad (k \in \mathbb{N}, k > 0).$$

Since the Orlicz norms are monotonic, i.e. $\|h\|_{L_M} \le \|g\|_{L_M}$ whenever $|h| \le |g|$ $(h, g \in L_M)$, we have

$$\|\mathbf{\Gamma}(c_k)_{n+1}^{[\lambda n]}\|_{L_M} \le \|\mathbf{\Gamma}(d_k)_{n+1}^{[\lambda n]}\|_{L_M} \quad (n \in \mathbb{N}).$$

Recall that $d_k \ge e/k$ $(k \in \mathbb{N}, k > 0)$. Consequently, by (9) we have

$$\lim_{\lambda\to 1+}\overline{\lim_{n\to\infty}} ([\lambda n]-n)\|\mathbf{\Gamma}(d_k)_{n+1}^{[\lambda n]}\|_{L_M} \le \lim_{\lambda\to 1+}\overline{\lim_{n\to\infty}} \sum_{k=n+1}^{[\lambda n]} d_k(1 + \log^+(nd_k)) \le 2\lim_{\lambda\to 1+}\overline{\lim_{n\to\infty}} \sum_{k=n+1}^{[\lambda n]} d_k\log^+(kd_k) \le 2\lim_{\lambda\to 1+}\overline{\lim_{n\to\infty}} \sum_{k=n+1}^{[\lambda n]} (c_k\log^+(kc_k) + e/k) = 0. \ \blacksquare$$

Proof of Theorem 1. Suppose that $f \in L_1$ satisfies condition (1). In our proof of (i) we will need the concept of the generalized de la Vallée Poussin means, which are defined as

$$V_{\lambda,n}f = \frac{1}{[\lambda n]-n+1}\sum_{k=n}^{[\lambda n]} S_k f \quad (n \in \mathbb{N},\ \lambda > 1).$$

It is known that $\lim_{n\to\infty}\|V_{\lambda,n}f - f\|_1 = 0$ for any $f \in L_1$ and $\lambda > 1$. Therefore $\lim_{n\to\infty}\|S_n f - f\|_1 = 0$ is equivalent to

$$\lim_{\lambda\to 1+}\overline{\lim_{n\to\infty}}\|V_{\lambda,n}f - S_n f\|_1 = 0. \tag{10}$$

Since

$$V_{\lambda,n}f(t) - S_n f(t) = \frac{1}{[\lambda n]-n+1}\sum_{k=n+1}^{[\lambda n]}\sum_{j=n+1}^{k} a_j \cos jt + \frac{1}{[\lambda n]-n+1}\sum_{k=n+1}^{[\lambda n]}\sum_{j=n+1}^{k} b_j \sin jt = A_{\lambda,n}(t) + B_{\lambda,n}(t)$$

we see that (10) holds if and only if

$$\lim_{\lambda\to 1+}\overline{\lim_{n\to\infty}}\|A_{\lambda,n}\|_1 = 0 \quad \text{and} \quad \lim_{\lambda\to 1+}\overline{\lim_{n\to\infty}}\|B_{\lambda,n}\|_1 = 0.$$

Let us start with $A_{\lambda,n}$. By an Abel transfromation we have

$$A_{\lambda,n} = \frac{[\lambda n]-n}{[\lambda n]-n+1}(-a_{n+1})D_n + \frac{1}{[\lambda n]-n+1}\sum_{k=n+1}^{[\lambda n]-1}([\lambda n]-k)\Delta a_k D_k + \frac{1}{[\lambda n]-n+1}\sum_{k=n+1}^{[\lambda n]} a_k D_k = A^1_{\lambda,n} + A^2_{\lambda,n} + A^3_{\lambda,n}. \tag{11}$$

To estimate the norm of the third term we use the well known inequality

$$\frac{1}{n}\Big\|\sum_{k=1}^{n} c_k D_k\Big\|_1 \le C \max_{1\le k\le n}|c_k| \quad (c_k \in \mathbb{C},\ n \in \mathbb{N}) \tag{12}$$

that can be easily derived from (7). (For its original proof see [18].) Thus

$$\|A^3_{\lambda,n}\|_1 \le C\frac{\lambda}{\lambda-1}\max_{n\le k\le[\lambda n]}|a_k|.$$

Since $f \in L_1$ implies $a_k \to 0$ as $n \to \infty$,

$$\lim_{n\to\infty}\|A^3_{\lambda,n}\|_1 = 0 \quad (\lambda > 1).$$

To estimate $\|A^2_{\lambda,n}\|_1$ let us reformulate (7) by using (9) as follows:

$$\Big\|\sum_{k=K}^{N} c_k D_k\Big\|_1 \le C\Big(\log\frac{N}{N-K+1}\Big|\sum_{k=K}^{N} c_k\Big| + (N-K+1)\|\boldsymbol{\Gamma}(c_k)_K^N\|_{L_M}\Big)$$

($c_k \in \mathbb{C}$, $K, N \in \mathbb{N}$). Applying it to $A^2_{\lambda,n}$ we obtain

$$\|A^2_{\lambda,n}\|_1 \le C\frac{1}{[\lambda n]-n+1}\log\frac{[\lambda n]}{[\lambda n]-n+1}\Big|\sum_{k=n+1}^{[\lambda n]-1}([\lambda n]-k)\Delta a_k\Big| + C\|\boldsymbol{\Gamma}(([\lambda n]-k)\Delta a_k)_{n+1}^{[\lambda n]-1}\|_{L_M} = X_1 + X_2.$$

It is easy to see that X_1 converges to 0. Indeed,

$$X_1 = C\frac{1}{[\lambda n]-n+1}\log\frac{[\lambda n]}{[\lambda n]-n+1}\Big|\sum_{k=n+1}^{[\lambda n]}(a_{n+1}-a_k)\Big| \le C\log\frac{\lambda}{\lambda-1}\Big(|a_{n+1}| + \frac{1}{[\lambda n]-n+1}\sum_{k=n+1}^{[\lambda n]}|a_k|\Big) \le C\log\frac{\lambda}{\lambda-1}\max_{n+1\le k\le[\lambda n]}|a_k|.$$

Then similarly to $\|A^3_{\lambda,n}\|_1$ we can conclude that $X_1 \to 0$ as $n \to \infty$ ($\lambda > 1$).

For X_2 we take advantage of the monotony of Orlicz norms. By (9) we have

$$X_2 \le C([\lambda n]-n-1)\|\boldsymbol{\Gamma}(\Delta a_k)_{n+1}^{[\lambda n]-1}\|_{L_M} \le C\sum_{k=n+1}^{[\lambda n]-1}|\Delta a_k|\Big(1 + \log^+\frac{([\lambda n]-n-1)|\Delta a_k|}{\sum_{j=n+1}^{[\lambda n]-1}|\Delta a_j|}\Big).$$

Thus by Lemma 3 it is clear that (1) implies

$$\lim_{\lambda\to 1+}\overline{\lim_{n\to\infty}}\|A^2_{\lambda,n}\|_1 = 0.$$

Finally, it is obvious that $\lim_{\lambda\to 1+}\overline{\lim}_{n\to\infty}\|A^1_{\lambda,n}\|_1 = 0$ if and only if $\lim_{n\to\infty}|a_n|\log n = 0$. In summary, if (1) holds for $f \in L_1$ then

$$\lim_{\lambda\to 1+}\overline{\lim_{n\to\infty}}\|A_{\lambda,n}\|_1 = 0 \quad \text{if and only if} \quad \lim_{n\to\infty}|a_n|\log n = 0.$$

To reach the same conclusion for $B_{\lambda,n}$ we need a slight modification of the above method. We begin with the decomposition that corresponds to (7), i.e.

$$B_{\lambda,n}=\frac{[\lambda n]-n}{[\lambda n]-n+1}(-b_{n+1})\widetilde{D}_n+\frac{1}{[\lambda n]-n+1}\sum_{k=n+1}^{[\lambda n]}([\lambda n]-k)\Delta b_k\widetilde{D}_k+\frac{1}{[\lambda n]-n+1}\sum_{k=n+1}^{[\lambda n]}b_k\widetilde{D}_k.$$

Observe that to estimate the middle term we can only use Lemma 1 if the sum of the coefficients is zero. To this end let us modify the above decomposition. Noticing that

$$\sum_{k=n+1}^{[\lambda n]}([\lambda n]-n)\Delta b_k=\sum_{k=n+1}^{[\lambda n]}(b_{n+1}-b_k)$$

we have

$$\begin{aligned}B_{\lambda,n}&=b_{n+1}\frac{1}{[\lambda n]-n+1}\sum_{k=n}^{[\lambda n]}(\widetilde{D}_k-\widetilde{D}_n)\\&\quad+\frac{1}{[\lambda n]-n+1}\\&\quad\times\sum_{k=n+1}^{[\lambda n]}\bigg(([\lambda n]-k)\Delta b_k-\frac{1}{[\lambda n]-n}\sum_{j=n+1}^{[\lambda n]}(b_{n+1}-b_j)\bigg)\widetilde{D}_k\\&\quad+\frac{1}{[\lambda n]-n+1}\sum_{k=n+1}^{[\lambda n]}\bigg(b_k-\frac{1}{[\lambda n]-n}\sum_{j=n+1}^{[\lambda n]}b_j\bigg)\widetilde{D}_k\\&=B^1_{\lambda,n}+B^2_{\lambda,n}+B^3_{\lambda,n}.\end{aligned}$$

We can basically use the same considerations as for the corresponding $A^j_{\lambda,n}$ ($j=1,2,3$) terms. Indeed, by Lemma 1, (9) and again using the monotony of the Orlicz norms we have

$$\|B^3_{\lambda,n}\|_1\le C\bigg\|\Gamma\bigg(b_k-\frac{1}{[\lambda n]-n}\sum_{j=n+1}^{[\lambda n]}b_j\bigg)_{n+1}^{[\lambda n]}\bigg\|_{L_M}\le C\max_{n+1\le k\le[\lambda n]}|b_k|.$$

Hence, similarly to $\|A^3_{\lambda,n}\|_1$, we infer

$$\lim_{n\to\infty}\|B^3_{\lambda,n}\|_1=0\quad(\lambda>1).$$

To estimate $\|B^2_{\lambda,n}\|_1$ first observe that the sum of the coefficients is zero and

$$\|h-\|h\|_1\chi_{[0,1)}\|_{L_M}\le 2\|h\|_{L_M}\quad(h\in L_M).$$

Then using Lemma 1 and following the steps applied for the estimation of X_2 in $\|A^2_{\lambda,n}\|_1$ we find that the hypothesis of Theorem 1 implies

$$\lim_{\lambda\to1+}\overline{\lim_{n\to\infty}}\|B^2_{\lambda,n}\|_1=0.$$

Finally, by Lemma 2 we have

$$C_1\,b_{n+1}(\log(\lambda-1)+\log n)\le\|B^1_{\lambda,n}\|_1\le C_2\,b_{n+1}(\log(\lambda-1)+\log n)$$

for n large enough. Consequently, $\lim_{\lambda\to1+}\overline{\lim}_{n\to\infty}\|B^1_{\lambda,n}\|_1=0$ is equivalent to $\lim_{n\to\infty}\|B^1_{\lambda,n}\|_1=0$ ($\lambda>1$). Clearly, for the latter it is necessary and sufficient that $\lim_{n\to\infty}b_n\log n=0$.

In summary, we have

$$\lim_{\lambda\to1+}\overline{\lim_{n\to\infty}}\|B_{\lambda,n}\|_1=0\quad\text{if and only if}\quad\lim_{n\to\infty}b_n\log n=0.$$

Part (i) of Theorem 1 is proved.

In order to prove (ii) write

$$S_nf(x)=\sum_{k=0}^{n}\Delta a_k\,D_k(x)+\sum_{k=1}^{n}\Delta b_k\,\widetilde{D}_k(x)+(a_{n+1}D_n(x)+b_{n+1}\widetilde{D}_n(x)).$$

Then

$$\begin{aligned}2\sin\frac{x}{2}\,S_nf(x)&=\bigg(\sum_{k=0}^{n}\Delta a_k\sin\Big(\frac{2k+1}{2}x\Big)+\sum_{k=1}^{n}\Delta b_k\cos\Big(\frac{2k+1}{2}x\Big)\bigg)\\&\quad+\bigg(a_{n+1}\sin\Big(\frac{2n+1}{2}x\Big)-b_{n+1}\cos\Big(\frac{2n+1}{2}x\Big)+b_1\cos\frac{x}{2}\bigg)\\&=A_n(x)+B_n(x).\end{aligned}$$

By (i) we know that the left side converges to an integrable function h as $n\to\infty$, where $h(x)=2\sin(x/2)f(x)$ ($0<|x|\le\pi$). We will show that the right side is convergent in the norm of L_2. Hence $h\in L_2$, which was to be proved.

Obviously, $\lim_{n\to\infty}B_n(x)=0$ uniformly in x.

On the other hand, A_n converges in the norm of L_2 if and only $\sum_{k=1}^{\infty}(|\Delta a_k|^2+|\Delta b_k|^2)<\infty$. It is easy to see that if (1) holds then

$$\overline{\lim_{n\to\infty}}\sum_{k=n+1}^{2n}|\Delta a_k|\log^+(k|\Delta a_k|)<\infty.$$

Moreover, we may suppose that

(13) $$\sum_{k=n+1}^{2n}|\Delta a_k|\log^+(k|\Delta a_k|)<\frac12\quad(n\in\mathbb{N}).$$

For $j,n\in\mathbb{N}$ define

$$\Omega_{j,n}=\{k\in\mathbb{N}:2^{-j-1}<|\Delta a_k|\le 2^{-j},\ 2^n<k\le 2^{n+1}\}.$$

Then by (13) we have

$$|\Omega_{j,n}|\frac{n-j-1}{2^{j+1}}\le\sum_{k\in\Omega_{j,n}}|\Delta a_k|\log^+(k|\Delta a_k|)\le\frac12\quad(j\in\mathbb{N}).$$

($|\Omega_{j,n}|$ denotes the cardinality of $\Omega_{j,n}$.) Hence we infer $|\Omega_{j,n}|\le 2^j/(n-j-1)$ $(j<n-1)$.

As a consequence of (13) we see that $|\Delta a_k|$ cannot be too large. Indeed, $\Omega_{j,n}=\emptyset$ for any $j<[\log n]$ $(n\ge 2)$. Thus

$$\begin{aligned}\sum_{k=2^n+1}^{2^{n+1}}|\Delta a_k|^2&=\sum_{j=[\log n]}^{\infty}\sum_{k\in\Omega_{j,n}}|\Delta a_k|^2\\&\le\sum_{j=[\log n]}^{[2n/3]}|\Omega_{j,n}|\Big(\frac{1}{2^j}\Big)^2+2^n\Big(\frac{1}{2^{[2n/3]}}\Big)^2\\&\le\sum_{j=[\log n]}^{[2n/3]}\frac{1}{2^j(n-j-1)}+\frac{4}{2^{n/3}}\\&\le C\Big(\frac1n\sum_{j=[\log n]}^{[2n/3]}\frac{1}{2^j}+\frac{1}{2^{n/3}}\Big)\le C\frac{1}{n^2}.\end{aligned}$$

In a similar way we can show that $\sum_{k=1}^{\infty}|\Delta b_k|^2<\infty$. The proof of Theorem 1 is complete. ■

Proof of Theorem 2. First we note that $\mathcal{S}\supset\mathcal{L}_1$ is trivial. Suppose now that (2) holds for $(\widehat{f}(k),k\in\mathbb{Z})$ where $f\in L_1$. Then, as mentioned in Remark 3, there exist $p>1$ and $C>0$ such that

$$(14)\qquad \sum_{|k|=n+1}^{[\lambda n]}|k|^{p-1}|\Delta\widehat{f}(k)|^p<C\quad(n\in\mathbb{N},\ 1<\lambda\le 2).$$

Let $1<q<p$, and let $x_q>1$ be such that $x\log x\le x^q$ whenever $x\ge x_q$. Then

$$\begin{aligned}\sum_{|k|=n+1}^{[\lambda n]}|\Delta\widehat{f}(k)|\log^+|k\Delta\widehat{f}(k)|&\le C(\lambda-1)\sum_{|k|=n+1}^{[\lambda n]-n}\frac{(|k\Delta\widehat{f}(k)|+x_q)^q}{[\lambda n]-n}\\&\le 2^pC(\lambda-1)\sum_{|k|=n+1}^{[\lambda n]-n}\frac{|k\Delta\widehat{f}(k)|^q}{[\lambda n]-n}+2^p(\lambda-1)x_q^q.\end{aligned}$$

Clearly, the second term tends to 0 as $\lambda\to 1^+$. For the first term, by (14) we have

$$\begin{aligned}(\lambda-1)\sum_{|k|=n+1}^{[\lambda n]-n}\frac{|k\Delta\widehat{f}(k)|^q}{[\lambda n]-n}&\le(\lambda-1)\Big(\sum_{|k|=n+1}^{[\lambda n]-n}\frac{|k\Delta\widehat{f}(k)|^p}{[\lambda n]-n}\Big)^{q/p}\\&\le C(\lambda-1)^{1-q/p}\Big(\sum_{|k|=n+1}^{[\lambda n]-n}|k|^{p-1}|\Delta\widehat{f}(k)|^p\Big)^{q/p}\\&\le C(\lambda-1)^{1-q/p}.\end{aligned}$$

Consequently, $\mathcal{S}\supseteq\bigcup_{p\ge1}\mathcal{L}_p$. In order to show that the inclusion is proper we use the sequences constructed by Grow and Stanojević in [7] when discussing the relation between $\mathcal{L}_1$ and $\bigcup_{p>1}\mathcal{L}_p$. Namely, let

$$c_k^{(1)}=\begin{cases}1/\log^2|k| & \text{if } |k|=2^n,\ n\in\mathbb{N},\\ 0 & \text{otherwise,}\end{cases}\qquad c_k^{(2)}=\begin{cases}1/|k| & \text{if } |k| \text{ even } k\ne 0,\\ 0 & \text{otherwise.}\end{cases}$$

It is shown in [7] that

$$c^{(1)}\in\mathcal{L}_1\setminus\bigcup_{p>1}\mathcal{L}_p\quad\text{and}\quad c^{(2)}\in\bigcup_{p>1}\mathcal{L}_p\setminus\mathcal{L}_1.$$

Let $c=c^{(1)}+c^{(2)}$. We conclude from $\mathcal{S}\supseteq\bigcup_{p\ge1}\mathcal{L}_p$ that both $c^{(1)}$ and $c^{(2)}$ belong to $\mathcal{S}$. Hence, $c\in\mathcal{S}$. On the other hand, since

$$|\Delta c_k|\ge\tfrac12\max\{|\Delta c_k^{(1)}|,|\Delta c_k^{(2)}|\}$$

for k large enough, we have $c\notin\bigcup_{p\ge1}\mathcal{L}_p$. Consequently,

$$c\in\mathcal{S}\setminus\bigcup_{p\ge1}\mathcal{L}_p.\ ■$$

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On strong generation of $B(\mathcal{H})$ by two commutative C^*-algebras

by

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Abstract. The algebra $B(\mathcal{H})$ of all bounded operators on a Hilbert space $\mathcal{H}$ is generated in the strong operator topology by a single one-dimensional projection and a family of commuting unitary operators with cardinality not exceeding $\dim\mathcal{H}$. This answers Problem 8 posed by W. Żelazko in [6].

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{S}$ be a subset of the algebra $B(\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$. We say that the algebra $B(\mathcal{H})$ is *strongly generated* by $\mathcal{S}$ if the smallest subalgebra of $B(\mathcal{H})$ closed in the strong operator topology and containing $\mathcal{S}$ coincides with $B(\mathcal{H})$. The first result on the strong generation of $B(\mathcal{H})$ was given by C. Davis in [2]. He proved, in the case when the Hilbert space $\mathcal{H}$ is separable, that the algebra $B(\mathcal{H})$ is strongly generated by two unitary operators. Later, E. Nordgren, M. Radjabalipour, H. Radjavi, and P. Rosenthal have shown ([3]) that two Hermitian operators strongly generate $B(\mathcal{H})$, which implies that $B(\mathcal{H})$ is singly generated as a von Neumann algebra. See also [5], pp. 160–163, for other results concerning generation of $B(\mathcal{H})$ when $\mathcal{H}$ is separable. These results show that for a separable Hilbert space $\mathcal{H}$ the algebra $B(\mathcal{H})$ is strongly generated by two commutative C^*-algebras. In [6] W. Żelazko raised the following

PROBLEM. *Is the algebra $B(\mathcal{H})$ of all operators on a complex Hilbert space $\mathcal{H}$ always generated in the strong operator topology by two commutative C^*-algebras?*

We show that the answer to this question is positive. Namely we prove the following

THEOREM. *Let $\mathcal{H}$ be a complex Hilbert space. The algebra $B(\mathcal{H})$ is strongly generated by a single one-dimensional (orthogonal) projection and a commuting family of unitary operators with cardinality not exceeding* $\dim\mathcal{H}$.

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