STUDIA MATHEMATICA 125 (2) (1997)

The density condition in quotients of quasinormable Fréchet spaces

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Abstract. It is proved that a separable Fréchet space is quasinormable if, and only if, every quotient space satisfies the density condition of Heinrich. This answers positively a conjecture of Bonet and Díaz in the class of separable Fréchet spaces.

The class of quasinormable locally convex spaces was introduced and studied by Grothendieck in [9]. Recently Bonet and Díaz [6, 7] and Díaz and Fernández [8] gave a characterization of the quasinormability of Fréchet–Köthe sequence spaces of order p, $1 \le p \le \infty$ or $p = 0$, in terms of the density condition of their quotient spaces. They proved that a Fréchet–Köthe sequence space of order p, $1 \le p \le \infty$ or $p = 0$, is quasinormable if, and only if, every quotient space satisfies the density condition of Heinrich [10]. Also, Bonet and Díaz conjectured (see [8]) that a Fréchet space is quasinormable if, and only if, every quotient space meets the density condition. This question was also recalled in the problem list of [1, Problem 15]. Here we show that, within the class of separable Fréchet spaces, quasinormable spaces are the only ones whose quotient spaces have the density condition (Theorem 4), thereby giving a positive answer to the above question. Moreover, we show that a separable non-quasinormable Fréchet space has a quotient space with a normalized basis (Theorem 7). These results extend the previous ones of Bellenot [2] on Schwartz and non-Schwartz Fréchet–Montel spaces.

1. Notation and preliminaries. In the sequel, given a Fréchet space E we denote by $(\| \ \|_k)_k$ a fundamental system of increasing seminorms defining the topology of E such that the sets $U_k := \{x \in E : \|x\|_k \le 1\}$ form a basis of 0-neighbourhoods in E. The dual seminorms are defined by $\|f\|'_k := \sup\{|f(x)| : x \in U_k\}$ for $f \in E'$; hence $\| \ \|'_k$ is the gauge of U_k° in E'. We denote by $E'_k := \{f \in E' : \|f\|'_k < \infty\}$ the linear span of U_k°

1991 *Mathematics Subject Classification*: Primary 46A04.
Research supported by the Italian MURST.

endowed with the norm topology defined by $\| \|'_k$. Clearly, $(E'_k, \| \|'_k)$ is a Banach space and $E'_k = (E/\ker \| \|_k, \| \|_k)'$.

Also, given a closed subspace F of E and denoting by T the canonical quotient map from E onto E/F, we define the seminorm $| \ |_k$ induced by $\| \|_k$ on E/F by $|Tx| := \inf\{\|y\|_k : Ty = Tx\}$.

If E is a Fréchet space with a continuous norm, we assume that each $\| \|_k$ is a norm on E.

A Fréchet space E is called *quasinormable* if there exists a bounded subset B of E such that

$$\forall n\ \exists m > n\ \forall \varepsilon > 0\ \exists \lambda > 0: \quad U_m \subset \lambda B + \varepsilon U_n.$$

By [5, Theorem] and [13, Theorem 7] a Fréchet space E is quasinormable if, and only if,

$$(1) \qquad \forall n\ \exists m > n\ \forall k > m\ \forall \varepsilon > 0\ \exists \lambda > 0: \quad U_m \subset \lambda U_k + \varepsilon U_n.$$

By polarization it follows that a Fréchet space E is quasinormable if, and only if,

$$(2) \qquad \forall n\ \exists m > n\ \forall k > m\ \forall \varepsilon > 0\ \exists \lambda > 0: \quad \lambda U_k^\circ \cap U_n^\circ \subset \varepsilon U_m^\circ.$$

The density condition was introduced by Heinrich in his study of ultrapowers of locally convex spaces [10]. A Fréchet space E is said to satisfy the *density condition* (see [4, Proposition 2]) if for any sequence $(\lambda_n)_n$ of strictly positive numbers there exists a bounded subset B of E such that

$$(3) \qquad \forall n\ \exists m > n\ \exists \lambda > 0: \quad \bigcap_{j=1}^{m} \lambda_j U_j \subset \lambda B + U_n.$$

The density condition was thoroughly studied for Fréchet and Köthe spaces by Bierstedt and Bonet [4]. It was proved there that a Fréchet space E satisfies the density condition if, and only if, the bounded subsets of its strong dual are metrizable. Every quasinormable and every Montel Fréchet space meets the density condition.

Let E be a Fréchet space. A sequence $(x_n)_n$ in E is a *basis* of E if for every $x \in E$ there is a unique sequence of scalars $(\alpha_n)_n$ so that $x = \sum_{n=1}^\infty \alpha_n x_n$. In this case, the sequence $(f_n)_n \subset E'$ defined by $f_n(x_m) = \delta_{nm}$ is called the *dual basis* of $(x_n)_n$.

A sequence $(x_n)_n$ in E which is a basis of its closed linear span $[x_n : n \in \mathbb{N}]$ is called a *basic sequence*. If $\| \|$ is a seminorm on $[x_n : n \in \mathbb{N}]$, then $(x_n)_n$ is a *K-basic sequence with respect to* $\| \|$ if for all scalars $(\alpha_n)_n$ and integers p and q,

$$\Big\|\sum_{n=1}^{p} \alpha_n x_n\Big\| \le K \Big\|\sum_{n=1}^{p+q} \alpha_n x_n\Big\|.$$

Moreover, a basic sequence $(x_n)_n$ in E is said to be *normalized* if it is bounded and there exists a 0-neighbourhood U with $x_n \notin U$ for each $n \in \mathbb{N}$.

For all undefined notation we refer to [11, 12].

2. The results. We start with a preliminary result which is the basic step towards Theorems 4 and 7 and which seems to be interesting in itself.

THEOREM 1. *Let E be a separable Fréchet space with a continuous norm. Let $(\| \|_n)_n$ be a fundamental sequence of norms for E with dual norms $(\| \|'_n)_n$. If E is non-quasinormable, then E has a quotient space F which has a continuous norm and a basis $(z_{jk})_{j,k\in\mathbb{N}}$ with biorthogonal functionals $(g_{jk})_{j,k\in\mathbb{N}}$ such that:*

(a) *$(z_{jk})_{j,k\in\mathbb{N}}$ is basic with respect to each norm $| \ |_n$,*
(b) *$\sup\{|z_{jk}|_n : j \in \mathbb{N}\} = \alpha_{nk} < \infty$ for all $k \in \mathbb{N}$ and $n \le k$,*
(c) *$|g_{jk}|'_1 \le 1$ for all $j, k \in \mathbb{N}$,*
(d) *$\lim_{j\to\infty} |g_{jk}|'_{k+1} = 0$ for all $k \in \mathbb{N}$,*

where $(| \ |_n)_n$ is the system of norms induced by $(\| \|_n)_n$ on F and $(| \ |'_n)_n$ is the system of dual norms on F'.

Proof. Let E be a non-quasinormable separable Fréchet space. Then, by (2),

$$\exists n\ \forall m > n\ \exists k > m\ \exists \alpha > 0\ \forall \lambda > 0: \quad \lambda U_k^\circ \cap U_n^\circ \not\subset \alpha U_m^\circ.$$

Without loss of generality, we may then assume that there is a decreasing sequence $(\alpha_k)_{k\in\mathbb{N}}$ of numbers with $0 < \alpha_k < 1$ so that

$$\forall k \in \mathbb{N}\ \forall \lambda > 0: \quad \lambda U_{k+1}^\circ \cap U_1^\circ \not\subset \alpha_k U_k^\circ,$$

or equivalently

$$(4) \qquad \forall k \in \mathbb{N}: \quad \inf\{\|f\|'_{k+1} : f \in E',\ \|f\|'_1 \le 1,\ \|f\|'_k > \alpha_k\} = 0.$$

Actually, we have more. If $G = \{f \in E' : f(x) = 0 \text{ for all } x \in L\}$ with L some finite subset of E, then from (4) it follows that, for each $k \in \mathbb{N}$ and $0 < \varepsilon < \alpha_k$,

$$(5) \qquad \inf\{\|f\|'_{k+1} : f \in G,\ \|f\|'_1 \le 1 + \varepsilon,\ \|f\|'_k > \alpha_k - \varepsilon\} = 0.$$

To prove (5) assume that $L = (z_i)_{i=1}^m$ is linearly independent. Since $\| \|_1$ is a norm on E, E'_1 is $\sigma(E', E)$-dense in E' so we can find $(g_i)_{i=1}^m \subset E'_1$ such that $(z_i, g_i)_{i=1}^m$ is biorthogonal. Then the map $P : E'_1 \to E'_1$ defined by $Pf = \sum_{i=1}^m f(z_i) g_i$ is a projection with $\ker P = G$ and P is continuous with respect to each norm $\| \|'_k$.

Now, fix $k \in \mathbb{N}$ and $0 < \varepsilon < \alpha_k$. By (4) we can find a sequence $(f_j)_j \subset E'_1$ with $\|f_j\|'_1 \le 1$, $\|f_j\|'_k > \alpha_k$ for each $j \in \mathbb{N}$ and $\lim_{j\to\infty} \|f_j\|'_{k+1} = 0$. Then $\lim_{j\to\infty} \|Pf_j\|'_{k+1} = 0$; hence, since $P(E'_1)$ is finite-dimensional, also $\lim_{j\to\infty} \|Pf_j\|'_1 = 0$. Therefore, there exists a $j_0 \in \mathbb{N}$ such that $\|Pf_j\|'_1 < \varepsilon$

for each $j \geq j_0$. It follows that the sequence $(h_j)_{j\geq j_0} = (f_j - Pf_j)_{j\geq j_0} \subset G$ satisfies

$$\|h_j\|_1' \leq \|f_j\|_1' + \|Pf_j\|_1' \leq 1+\varepsilon, \qquad \|h_j\|_k' \geq \|f_j\|_k' - \|Pf_j\|_k' > \alpha_k - \varepsilon,$$

for each $j \geq j_0$. Clearly, also $\lim_{j\to\infty} \|h_j\|_{k+1}' = 0$. Hence we conclude that condition (5) holds.

We can now construct the desired quotient. Our construction is by induction.

Let $(d_n)_n$ be a dense sequence in E and E_0 the vector subspace it generates. Let $\gamma : \mathbb{N} \to \mathbb{N}^2$ be a bijective map and put $\gamma(n) = (\gamma_1(n), \gamma_2(n))$.

For $n = 1$, by (4), we find $f_{\gamma(1)} \in E_1'$ with $\|f_{\gamma(1)}\|_1' \leq 1$, $\|f_{\gamma(1)}\|_{\gamma_2(1)}' > \alpha_{\gamma_2(1)}$ and $\|f_{\gamma(1)}\|_{\gamma_2(1)+1}' < 1$. Since E_0 is dense in E there is $x_{\gamma(1)} \in E_0$ such that $f_{\gamma(1)}(x_{\gamma(1)}) = 1$.

We now consider the space

$$G_{\gamma(2)} = \{f \in E' : f(x_{\gamma(1)}) = f(d_1) = 0\}.$$

Fix $k = \gamma_2(n)$ and $\varepsilon = \alpha_{\gamma_2(n)}/2$. Then, by (5), we can find $f_{\gamma(2)} \in G_{\gamma(2)}$ with $\|f_{\gamma(2)}\|_{\gamma_2(2)}' > \alpha_{\gamma_2(2)}/2$, $\|f_{\gamma(2)}\|_1' \leq 1 + \alpha_{\gamma_2(2)}/2 < 2$ and $\|f_{\gamma(2)}\|_{\gamma_2(2)+1}' < \gamma_2(2)^{-\gamma_1(2)}$. Since E_0 is dense in E, we can choose $x_{\gamma(2)} \in E_0$ such that $f_{\gamma(2)}(x_{\gamma(2)}) = 1$ and $f_{\gamma(1)}(x_{\gamma(2)}) = 0$.

Turning to the induction step we assume that we have determined a biorthogonal system $(x_{\gamma(i)}, f_{\gamma(i)})_{i=1}^{n-1}$ such that, for each $i = 1, \ldots, n-1$,

$$(x_{\gamma(i)})_{i=1}^{n-1} \subset E_0, \qquad f_{\gamma(i)}(d_j) = 0, \qquad \forall j \leq i-1,$$
$$\|f_{\gamma(i)}\|_1' \leq 1 + \alpha_{\gamma_2(i)}/2 < 2,$$
$$\|f_{\gamma(i)}\|_{\gamma_2(i)}' > \alpha_{\gamma_2(i)}/2,$$
$$\|f_{\gamma(i)}\|_{\gamma_2(i)+1}' < \gamma_2(i)^{-\gamma_1(i)}.$$

Then we consider the space

$$G_{\gamma(n)} = \{f \in E' : f(x_{\gamma(i)}) = f(d_i) = 0 \text{ for } i = 1, \ldots, n-1\}$$

and we apply condition (5) with $k = \gamma_2(n)$ and $\varepsilon = \alpha_{\gamma_2(n)}/2$. So, we find $f_{\gamma(n)} \in G_{\gamma(n)}$ with

$$\|f_{\gamma(n)}\|_1' \leq 1 + \alpha_{\gamma_2(n)}/2 < 2,$$
$$\|f_{\gamma(n)}\|_{\gamma_2(n)}' > \alpha_{\gamma_2(n)}/2,$$
$$\|f_{\gamma(n)}\|_{\gamma_2(n)+1}' < \gamma_2(n)^{-\gamma_1(n)}.$$

Since E_0 is dense in E, we can choose $x_{\gamma(n)} \in E_0$ such that $f_{\gamma(n)}(x_{\gamma(n)}) = 1$ and $f_{\gamma(i)}(x_{\gamma(n)}) = 0$ for $i = 1, \ldots, n-1$. Proceeding inductively, we construct a biorthogonal system $(x_{\gamma(n)}, f_{\gamma(n)})_n$ satisfying

$$(x_{\gamma(n)})_n \subset E_0, \tag{6}$$

$$(f_{\gamma(n)}) \subset E_1' \quad \text{and} \quad \|f_{\gamma(n)}\|_1' < 2, \quad \forall n \in \mathbb{N}, \tag{7}$$
$$f_{\gamma(n)}(d_m) = 0, \quad \forall n \in \mathbb{N}, \ \forall m < n, \tag{8}$$
$$\|f_{\gamma(n)}\|_{\gamma_2(n)}' > \alpha_{\gamma_2(n)}/2, \quad \forall n \in \mathbb{N}, \tag{9}$$
$$\|f_{\gamma(n)}\|_{\gamma_2(n)+1}' < \gamma_2(n)^{-\gamma_1(n)}, \quad \forall n \in \mathbb{N}. \tag{10}$$

Conditions (6)–(8) mean that the biorthogonal system $(x_{\gamma(n)}, f_{\gamma(n)})_n$ satisfies assumptions (a) and (b) of Theorem 1 of [3] (see also Proposition 5 of [3]). Therefore, there exists a subsequence $(x_{\gamma(n(i))}, f_{\gamma(n(i))})_i$ of $(x_{\gamma(n)}, f_{\gamma(n)})_n$ such that the quotient space $F = E/\bigcap_{i\in\mathbb{N}} \ker f_{\gamma(n(i))}$ has a continuous norm and the image of $(x_{\gamma(n(i))})_i$ under the canonical quotient map T from E onto F is a basis with respect to each norm $|\ |_n$ induced by $\|\ \|_n$ on F ($|\ |_n$ is a norm on F because $\bigcap_{i\in\mathbb{N}} \ker f_{\gamma(n(i))}$ is closed with respect to $\|\ \|_n$ for each n by (7)). Moreover, by recalling the proofs of Proposition 1 and Theorem 1 of [3] it is easy to see that we can choose the subsequence $(\gamma(n(i)))_i$ of couples of integers in such a way that, for each $k \in \mathbb{N}$, $\gamma_2(n(i)) = k$ for infinitely many i. (Indeed, it suffices to repeat the proof of Theorem 1 of [3] with minor changes as follows. By (6)–(8) we can apply Proposition 1 of [3] to $(E_0, \|\ \|_1)$, $(d_n)_n$, $(x_{\gamma(n)}, f_{\gamma(n)})_n$ to obtain a subsequence $(n_1(i))_i$ so that if $M_1 = \bigcap_{i\in\mathbb{N}} \ker f_{\gamma(n_1(i))}$ and $T_1 : (E_0, \|\ \|_1) \to (E_0, \|\ \|_1)/M_1$ is the quotient map, then $(T_1(x_{\gamma(n_1(i))}))_i$ is 4-basic in $(E_0, \|\ \|_1)/M_1$ and spans it. Also, by the proof of Proposition 1 of [3], we can select $(n_1(i))_i$ in such a way that, for each $i \in \mathbb{N}$, $\gamma_2(n_1(i)) = \gamma_2(i)$.

Proceeding inductively, we apply Proposition 1 of [3] to $(E_0, \|\ \|_h)$, $(d_n)_n$, $(x_{\gamma(n_{h-1}(i))}, f_{\gamma(n_{h-1}(i))})_i$ to obtain a subsequence $(n_h(i))_i$ of $(n_{h-1}(i))_i$ so that if $M_h = \bigcap_{i\in\mathbb{N}} \ker f_{\gamma(n_h(i))}$ and $T_h : (E_0, \|\ \|_h) \to (E_0, \|\ \|_h)/M_h$, then $(T_h(x_{\gamma(n_h(i))}))_i$ is 4-basic in $(E_0, \|\ \|_h)/M_h$ and spans it and also, for each $i \in \mathbb{N}$, $\gamma_2(n_h(i)) = \gamma_2(i)$.

Now we consider the diagonal sequence $(n_i(i))_i$. Then, for each $i \in \mathbb{N}$, $\gamma_2(n_i(i)) = \gamma_2(i)$ and so, for each $k \in \mathbb{N}$, $\gamma_2(n_i(i)) = k$ for infinitely many i. From this point on, one proceeds exactly as in the proof of Theorem 1 of [3].)

Thus, for simplicity, we can also denote by $(x_{\gamma(n)}, f_{\gamma(n)})_n$ such a subsequence.

Now put, for each $(j,k) \in \mathbb{N}^2$,

$$z_{jk} = 2Tx_{jk}.$$

Then the vectors z_{jk} form a basis of F which is basic with respect to each norm $|\ |_n$; hence (a) follows.

Also, define $g_{jk} \in F'$ by

$$g_{jk}(Tx) = \tfrac{1}{2} f_{jk}(x).$$

Then g_{jk} is well defined, linear and continuous since $F = E/\bigcap_{j,k\in\mathbb{N}} \ker f_{jk}$. It is immediate to verify that $(z_{jk}, g_{jk})_{j,k\in\mathbb{N}}$ is a biorthogonal system and so $(g_{jk})_{j,k\in\mathbb{N}}$ is the dual basis of $(z_{jk})_{j,k\in\mathbb{N}}$. Moreover, if $|\ |'_n$ is the dual norm of $|\ |_n$, we have

$$|g_{jk}|'_n = \sup_{|Tx|_n<1} |g_{jk}(Tx)| = \tfrac{1}{2} \sup_{\|x\|_n<1} |f_{jk}(x)| = \tfrac{1}{2}\|f_{jk}\|'_n.$$

It follows, by (7), (9) and (10), that

$$(11)\qquad |g_{jk}|'_1 \le 1, \quad |g_{jk}|'_k > \alpha_k/4, \quad |g_{jk}|'_{k+1} < \tfrac{1}{2}k^{-j},$$

and so we have (c) and (d). Finally, since $(z_{jk})_{j,k\in\mathbb{N}}$ is basic with respect to each norm $|\ |_n$ and $(g_{jk})_{j,k\in\mathbb{N}}$ is its dual basis, for each $k \in \mathbb{N}$ there exists $C_k > 0$ such that, for each $j \in \mathbb{N}$,

$$1/|g_{jk}|'_k \le |z_{jk}|_k \le C_k/|g_{jk}|'_k$$

and so, by using (11), for each $k, j \in \mathbb{N}$,

$$|z_{jk}|_k \le 4C_k/\alpha_k,$$

which implies that (b) holds. Thus the proof is complete.

In order to state and prove our first main result we also need the following two technical lemmas. The first one, due to Bonet and Díaz [7] and stated in a dual version in [8, Lemma 1], gives sufficient conditions to ensure that a Fréchet space E does not satisfy the density condition.

LEMMA 2. *Let E be a Fréchet space. Assume we can find $(z_{ij})_{i,j\in\mathbb{N}}$ in E and $(g_{ij})_{i,j\in\mathbb{N}}$ in E' with the following properties:*

(a) $g_{ij}(z_{ij}) = 1$ *for all* $i, j \in \mathbb{N}$,
(b) $\|g_{ij}\|'_1 \le 1$ *for all* $i, j \in \mathbb{N}$,
(c) $\sup\{\|z_{in}\|_j : i \in \mathbb{N},\ n \ge j\} < \infty$ *for all* $j \in \mathbb{N}$,
(d) $\lim_{i\to\infty} \|g_{ij}\|'_{j+1} = 0$ *for all* $j \in \mathbb{N}$.

Then E does not satisfy the density condition.

LEMMA 3. *If E is a non-quasinormable Fréchet space, then it has a quotient space which is non-quasinormable and has a continuous norm.*

Proof. Suppose E is a non-quasinormable Fréchet space. Then, by (1),

$$\exists n \in \mathbb{N}\ \forall m > n\ \exists k > m\ \exists \varepsilon > 0\ \forall \lambda > 0: \quad U_m \not\subset \lambda U_k + \varepsilon U_n.$$

Put $F = E/\ker\|\ \|_n$. Then F has a continuous norm and $(T(U_m))_m$ is a basis of 0-neighbourhoods in F such that

$$\forall m > n\ \exists k > m\ \exists \varepsilon > 0\ \forall \lambda > 0: \quad T(U_m) \not\subset \lambda T(U_k) + \varepsilon T(U_n).$$

This means, by (1), that F is non-quasinormable. The proof is complete.

We now have our main result which gives a positive answer to a conjecture of Bonet and Díaz in the setting of separable Fréchet spaces.

THEOREM 4. *A separable Fréchet space E is quasinormable if, and only if, every quotient space satisfies the density condition.*

Proof. Every quotient space of a quasinormable space is again quasinormable and, hence, satisfies the density condition. Thus the necessity of the condition follows.

We now suppose that E is a separable non-quasinormable Fréchet space. By Lemma 3 we may assume that E has a continuous norm. Let $(\|\ \|_n)_n$ be a fundamental sequence of norms for E. Then, by Theorem 1, E has a quotient space F which has a continuous norm and a basis $(z_{jk})_{j,k\in\mathbb{N}}$ with biorthogonal functionals $(g_{jk})_{j,k\in\mathbb{N}}$ satisfying conditions (a)–(d) of Theorem 1.

We will show that F has a quotient space without the density condition, hence so does E. To see this we write the first $\mathbb{N}$ in $\mathbb{N}\times\mathbb{N}$ as a countable union of disjoint infinite subsets, hence we may write $(z_{ijk}, g_{ijk})_{i,j,k\in\mathbb{N}}$ instead of $(z_{jk}, g_{jk})_{j,k\in\mathbb{N}}$. Thus,

$$(12)\qquad \sup\{|z_{ijk}|_n : i, j \in \mathbb{N}\} = \alpha_{nk} < \infty, \quad \forall k \in \mathbb{N},\ \forall n \le k,$$

$$(13)\qquad |g_{ijk}|'_1 \le 1, \quad \forall i, j, k \in \mathbb{N},$$

$$(14)\qquad \lim_{i\to\infty} |g_{ijk}|'_{k+1} = 0, \quad \forall j, k \in \mathbb{N},$$

where $|\ |_n$ denotes the norm induced by $\|\ \|_n$ on F and $|\ |'_n$ the dual norm of $|\ |_n$.

Now, proceeding in a similar way to [8], we put

$$c(i,j,k) = \begin{cases} \max\{|g_{ijs}|'_{j+1} : 1 \le s \le j\} & \text{if } j < k, \\ 1 & \text{if } 1 \le k \le j. \end{cases}$$

We also define

$$\overline{g}_{ij}: F \to \mathbb{R}, \quad x \to \sum_{k=1}^{\infty} g_{ijk}(x) \frac{c(i,j,k)}{2^k}.$$

Then, by (13), for each $i, j \in \mathbb{N}$ we have

$$|\overline{g}_{ij}(x)| \le |x|_1 \sum_{k=1}^{\infty} |g_{ijk}|'_1 2^{-k} \le |x|_1.$$

Therefore, for each $i, j \in \mathbb{N}$,

$$(15)\qquad |\overline{g}_{ij}|'_1 \le 1.$$

Moreover, by (13) again,

$$|\overline{g}_{ij}(x)| \le |x|_{j+1}\Big(\sum_{k=1}^{j} |g_{ijk}|'_{j+1} 2^{-k} + \sum_{k>j}^{\infty} |g_{ijk}|'_{j+1} \frac{c(i,j,k)}{2^k}\Big)$$
$$\le \max\{|g_{ijk}|'_{j+1} : 1 \le k \le j\}|x|_{j+1};$$

then $|\overline{g}_{ij}|'_{j+1} \le \max\{|g_{ijk}|'_{j+1} : 1 \le k \le j\}$, where $\max\{|g_{ijk}|'_{j+1} : 1 \le k \le j\} \le \max\{|g_{ijk}|'_{k+1} : 1 \le k \le j\}$ and so, by (14),

$$\lim_{i\to\infty} |\overline{g}_{ij}|'_{j+1} = 0. \tag{16}$$

Now we are ready to construct the desired quotient space.

Let $G = \bigcap_{i,j\in\mathbb{N}} \ker \overline{g}_{ij}$. We consider the quotient space F/G, with quotient map $S : F \to F/G$, and we check that it does not satisfy the density condition.

We denote by $(\||\ \||_n)_n$ the quotient norms induced by $(|\ |_n)_n$ on F/G (each $\||\ \||_n$ is a norm because G is closed with respect to each $|\ |_n$) and put, for each $i, j \in \mathbb{N}$,

$$\overline{\overline{g}}_{ij}(Sx) = \overline{g}_{ij}(x).$$

Clearly $\overline{\overline{g}}_{ij}$ is well defined, linear and continuous. Further, since $\||\overline{\overline{g}}_{ij}\||'_n = |\overline{g}_{ij}|'_n$ for each $i, j, n \in \mathbb{N}$, it follows from (15) and (16) that

$$\||\overline{\overline{g}}_{ij}\||'_1 \le 1 \quad \text{and} \quad \lim_{i\to\infty} \||\overline{\overline{g}}_{ij}\||'_{j+1} = 0;$$

hence conditions (b) and (d) of Lemma 2 hold.

Next, put $\overline{\overline{z}}_{ij} = S(2z_{ij1})$ for each $i, j \in \mathbb{N}$. Then $\overline{\overline{g}}_{ls}(\overline{\overline{z}}_{ij}) = \delta_{li}\delta_{sj}$, implying that condition (a) of Lemma 2 is also satisfied. Finally, fix $j \in \mathbb{N}$. Given $n \ge j$, we observe that $2z_{in1} - 2^j z_{inj} \in G$ and so, by (12),

$$\||\overline{\overline{z}}_{in}\||_j \le |2z_{in1} - (2z_{in1} - 2^j z_{inj})|_j \le 2^j |z_{inj}|_j \le 2^j \alpha_{jj};$$

it follows that $\sup\{\||\overline{\overline{z}}_{in}\||_j : i \in \mathbb{N},\ n \ge j\} < \infty$ for each $j \in \mathbb{N}$ and hence condition (c) of Lemma 2 is also satisfied. Thus, Lemma 2 can be applied to conclude that F/G does not satisfy the density condition and the result follows.

Remark. The quotient space F/G in the above proof can be constructed in such a way that it also has a basis. To see this, we proceed as follows.

We use the notation of Theorems 1 and 4. Let e_n be equal to $T(d_n)$ for each $n \in \mathbb{N}$. Then $(e_n)_n$ is a dense sequence in F; let F_0 be the vector subspace it generates. Now, we observe that from the proof of Theorem 1 it follows that the biorthogonal system $(z_{jk}, g_{jk})_{j,k\in\mathbb{N}}$, besides satisfying conditions (a)–(d) of Theorem 1, has the following properties:

$$(z_{jk})_{j,k\in\mathbb{N}} \subset F_0, \qquad g_{jk}(e_n) = 0 \quad \text{for each } j, k \in \mathbb{N} \text{ with } \gamma^{-1}(j,k) > n.$$

Next, for each $k \in \mathbb{N}$ we write $(z_{ijk}, g_{ijk})_{i,j\in\mathbb{N}}$ instead of $(z_{jk}, g_{jk})_{j\in\mathbb{N}}$ so as to have

$$g_{ijk}(e_n) = 0$$

for each $j \in \mathbb{N}$ and i sufficiently large (depending only on n).

Now, defining $\overline{g}_{ij} \in F'$ as in the proof of Theorem 4, we see that $(\overline{g}_{ij})_{i,j} \subset F'_1$, the sequence $(2z_{ij1})_{i,j} \subset F_0$ is biorthogonal to $(\overline{g}_{ij})_{i,j}$, and $\overline{g}_{ij}(e_n) = 0$ for each $j \in \mathbb{N}$ and i sufficiently large (depending only on n). Then we can proceed exactly as in the proof of Theorem 1 to conclude that there is a quotient space F/G, with quotient map S, such that $(S(2z_{ij1}))_{i,j}$ is a basis of G whose biorthogonal functionals are defined by

$$\overline{\overline{g}}_{ij}(Sx) = \overline{g}_{ij}(x).$$

Also, G has a continuous norm and does not satisfy the density condition as can be shown by repeating the same argument of Theorem 4.

As an immediate consequence we obtain a well known result of Bellenot [2, Corollary 5.3]:

COROLLARY 6. *A Fréchet space E is Schwartz if, and only if, every quotient space is Montel.*

Finally, from Theorem 1 we also obtain the following:

THEOREM 7. *Let E be a separable Fréchet non-quasinormable space. Then E has an infinite-dimensional quotient space with a continuous norm and with a normalized basis.*

Proof. Let E be a separable Fréchet non-quasinormable space. By Lemma 3, we may assume that E has a continuous norm. Let $(\|\ \|_n)_n$ be a fundamental sequence of norms for E. Then, by Theorem 1, E has a quotient space F which has a continuous norm and a basis $(z_{jk})_{j,k\in\mathbb{N}}$ with biorthogonal functionals $(g_{jk})_{j,k\in\mathbb{N}}$ satisfying conditions (a)–(d) of Theorem 1.

We will show that F has a quotient space with a continuous norm and with a normalized basis, and hence so does E.

We define, for each $j \in \mathbb{N}$,

$$g_j : F \to \mathbb{R}, \quad x \to \sum_{k=1}^{\infty} \frac{g_{jk}(x)}{2^k}.$$

By (b) and (c) of Theorem 1, it follows that, for each $j, k \in \mathbb{N}$,

$$2^{-k}\alpha_{kk}^{-1} \le |g_j|'_k \le |g_j|'_1 \le 1, \tag{17}$$

where $|\ |_n$ denotes the norm induced by $\|\ \|_n$ on F and $|\ |'_n$ the dual norm of $|\ |_n$.

Now, using the notation of Theorem 1, let e_n be equal to $T(d_n)$ for each $n \in \mathbb{N}$. Then $(e_n)_n$ is a dense sequence in F; let F_0 be the vector subspace it generates. Moreover, $g_j(e_n) = 0$ for j sufficiently large (depending on n). Put $z_j = 2z_{j1}$ for each $j \in \mathbb{N}$, $(z_j)_j \subset F_0$ and is biorthogonal to $(g_j)_j$, as is easy to verify. We can then apply Theorem 1 of [3] to conclude that there exists a subsequence $(z_{j(i)}, g_{j(i)})_i$ of $(z_j, g_j)_j$ such that the quotient space $G = F/\bigcap_{i\in\mathbb{N}} \ker g_{j(i)}$ has a continuous norm and the image of $(z_{j(i)})_i$ under the quotient map S from F onto G is a basis with respect to each norm $|||\ |||_n$ induced by $|\ |_n$ on G.

Next, we observe that the functionals defined by

$$\overline{g}_{j(i)}(Sx) = g_{j(i)}(x)$$

form the dual basis of $(Sz_{j(i)})_i$ and, for each i and n in $\mathbb{N}$, $|||\overline{g}_{j(i)}|||'_n = |g_{j(i)}|'_n$. Then, by (17),

$$2^{-k}\alpha_{kk}^{-1} \le |||\overline{g}_{j(i)}|||'_k \le |||\overline{g}_{j(i)}|||'_1 \le 1 \tag{18}$$

for each i, $k \in \mathbb{N}$. Since $(Sz_{j(i)})_i$ is basic with respect to each $|||\ |||_n$, for each $k \in \mathbb{N}$ there exists a constant $C_k > 0$ so that

$$1/|||\overline{g}_{j(i)}|||'_k \le |||Sz_{j(i)}|||_k \le C_k/|||\overline{g}_{j(i)}|||'_k$$

for each $i \in \mathbb{N}$. This together with (18) implies that, for each $i, k \in \mathbb{N}$,

$$1 \le |||Sz_{j(i)}|||_1 \le |||Sz_{j(i)}|||_k \le 2^k \alpha_{kk} C_k.$$

Thus, $(Sz_{j(i)})_i$ is a normalized basis of G and the proof is complete.

Theorem 7 implies a well known result of Bellenot [2, Theorem 5.1].

COROLLARY 8. *Let E be a Montel Fréchet non-Schwartz space. Then E has an infinite-dimensional quotient space with a normalized basis.*

Thus, Theorems 4 and 7 are proper extensions of Theorem 5.1 and Corollary 5.2 of [2] to the case of separable quasinormable and non-quasinormable Fréchet spaces.

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Received July 15, 1996
Revised version February 3, 1997 (3713)