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An irreducible semigroup of idempotents

by

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Abstract. We construct a semigroup of bounded idempotents with no nontrivial invariant closed subspace. This answers a question which was open for some time.

Let $\mathcal{B}(X)$ denote the algebra of all bounded linear operators on a (real or complex) Banach space X. A (multiplicative) semigroup $\mathcal{S}$ in $\mathcal{B}(X)$ is said to be *irreducible* if the only closed subspaces of X invariant under all members of $\mathcal{S}$ are $\{0\}$ and X. Otherwise, $\mathcal{S}$ is called *reducible*. An operator $P \in \mathcal{B}(X)$ is called *idempotent* if $P^2 = P$. In this note we answer the following question negatively:

Is every semigroup of idempotents reducible?

It seems that this problem was first considered by Heydar Radjavi [3]. He proved that a semigroup $\mathcal{S}$ of idempotents on a Hilbert space is reducible provided $\mathcal{S}$ contains a nonzero finite-rank operator. The above question was explicitly mentioned in the paper [1] by P. Fillmore, G. MacDonald, M. Radjabalipour and H. Radjavi, where it was shown that a finitely generated semigroup of idempotents on a Banach space is reducible. Recently, the problem has also been mentioned in the survey article [4] by H. Radjavi.

Our construction of an irreducible semigroup of idempotents was inspired by the construction of a weakly dense semigroup of nilpotent operators (see [2]).

THEOREM. *There exists a semigroup $\mathcal{S}$ of idempotents on the Hilbert space l^2, which is weakly dense in $\mathcal{B}(l^2)$. In particular, the semigroup $\mathcal{S}$ is irreducible.*

Proof. If A and B are $k \times k$ matrices, then let $P_{A,B}$ be the $3k \times 3k$ matrix

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$$P_{A,B} = \begin{bmatrix} A \\ A \\ I \end{bmatrix} [B \quad -B \quad I] = \begin{bmatrix} AB & -AB & A \\ AB & -AB & A \\ B & -B & I \end{bmatrix},$$

where I is the identity matrix of order k. Let $T_{A,B}$ be the infinite block-diagonal matrix

$$T_{A,B} = \operatorname{diag}\{D_0, D_1, D_2, D_3, \ldots\},$$

where the block D_i is equal to $P_{A,B}$ if the number i is representable in the ternary system by 0's and 1's only, and D_i equals the identity matrix of order $3k$ otherwise. We regard $T_{A,B}$ as an operator on the Hilbert space l^2. It is easy to see that $T_{A,B}T_{C,D} = T_{A,D}$ for all $k \times k$ matrices A, B, C and D. In particular, $T_{A,B}^2 = T_{A,B}$.

For each $n = 0, 1, 2, 3, \ldots$ let $\mathcal{S}_n$ denote the set of all operators $T_{A,B}$ as A and B range over all $3^n \times 3^n$ matrices, and let $\mathcal{S}$ be the union of all $\mathcal{S}_n$. It is clear that $\mathcal{S}_n$ is a semigroup of idempotents. If $T_{A,B} \in \mathcal{S}_n$ and $S \in \mathcal{S}_m$ with $m < n$, then there exists a $3^n \times 3^n$ matrix M such that

$$S = \operatorname{diag}\{C_0, C_1, C_2, C_3, \ldots\},$$

where the block C_i is equal to M if the number i is representable in the ternary system by 0's and 1's only, and C_i equals the identity matrix of order 3^n otherwise. Since

$$\begin{bmatrix} A \\ A \\ I \end{bmatrix} [B \quad -B \quad I] \begin{bmatrix} M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} A \\ A \\ I \end{bmatrix} [BM \quad -BM \quad I],$$

we have $T_{A,B}S = T_{A,BM}$. We conclude similarly that $ST_{A,B} = T_{MA,B}$, and so $\mathcal{S}_m\mathcal{S}_n \subseteq \mathcal{S}_n$ and $\mathcal{S}_n\mathcal{S}_m \subseteq \mathcal{S}_n$ for all m and n with $m < n$. Therefore, $\mathcal{S}$ is a semigroup of idempotents on l^2.

In order to prove that $\mathcal{S}$ is weakly dense in the algebra $\mathcal{B}(l^2)$ we first observe that

$$\|T_{A,B}\|^2 \le 4\|AB\|^2 + 2\|A\|^2 + 2\|B\|^2 + 1 \le (2\|A\|^2 + 1)(2\|B\|^2 + 1)$$

for all A and B. Choose $T \in \mathcal{B}(l^2)$ and $x, y \in l^2$. For each $n \in \mathbb{N}$ there exists an operator $T_n := T_{A,I} \in \mathcal{S}_n \subseteq \mathcal{S}$ such that the matrix of $T - T_n$ is of the form

$$\begin{bmatrix} 0 & U \\ V & W \end{bmatrix},$$

where the 0 is $3^n \times 3^n$ zero matrix. Writing $x = (x_1, x_2)$ and $y = (y_1, y_2)$ with respect to the same decomposition of the space l^2, we have

$$|\langle (T - T_n)x, y\rangle| \le \|U\| \cdot \|x_2\| \cdot \|y_1\| + \|V\| \cdot \|x_1\| \cdot \|y_2\| + \|W\| \cdot \|x_2\| \cdot \|y_2\|.$$

Since $\|x_1\| \le \|x\|$, $\|y_1\| \le \|y\|$ and

$$\begin{aligned} \max\{\|U\|, \|V\|, \|W\|\} &\le \|T - T_n\| \le \|T\| + \|T_n\| \\ &\le \|T\| + \sqrt{3(2\|A\|^2 + 1)} \\ &\le \|T\| + \sqrt{3(2\|T\|^2 + 1)} =: K, \end{aligned}$$

we obtain

$$|\langle (T - T_n)x, y\rangle| \le K(\|y\| \cdot \|x_2\| + \|x\| \cdot \|y_2\| + \|x_2\| \cdot \|y_2\|).$$

If n is sufficiently large, then $\max\{\|x_2\|, \|y_2\|\}$ is arbitrarily small, which implies that the semigroup $\mathcal{S}$ is weakly dense in $\mathcal{B}(l^2)$. From this it follows easily that $\mathcal{S}$ is an irreducible semigroup. ∎

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