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Pointwise multipliers on weighted BMO spaces

by

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Abstract. Let E and F be spaces of real- or complex-valued functions defined on a set X. A real- or complex-valued function g defined on X is called a pointwise multiplier from E to F if the pointwise product fg belongs to F for each $f \in E$. We denote by $\mathrm{PWM}(E,F)$ the set of all pointwise multipliers from E to F. Let X be a space of homogeneous type in the sense of Coifman–Weiss. For $1 \le p < \infty$ and for $\phi : X \times \mathbb{R}_+ \to \mathbb{R}_+$, we denote by $\mathrm{bmo}_{\phi,p}(X)$ the set of all functions $f \in L^p_{\mathrm{loc}}(X)$ such that

$$\sup_{a\in X,r>0} \frac{1}{\phi(a,r)} \left(\frac{1}{\mu(B(a,r))} \int_{B(a,r)} |f(x) - f_{B(a,r)}|^p \, d\mu \right)^{1/p} < \infty,$$

where $B(a,r)$ is the ball centered at a and of radius r, and $f_{B(a,r)}$ is the integral mean of f on $B(a,r)$. Let $\mathrm{bmo}_\phi(X) = \mathrm{bmo}_{\phi,1}(X)$ and $\mathrm{bmo}(X) = \mathrm{bmo}_{1,1}(X)$. In this paper, we characterize $\mathrm{PWM}(\mathrm{bmo}_{\phi_1,p_1}(X), \mathrm{bmo}_{\phi_2,p_2}(X))$. The following are examples of our results.

$$\mathrm{PWM}(\mathrm{bmo}_{(\log(1/r))^{-\alpha}}(\mathbb{T}^n), \mathrm{bmo}_{(\log(1/r))^{-\beta}}(\mathbb{T}^n)) = \mathrm{bmo}_{(\log(1/r))^{\alpha-\beta-1}}(\mathbb{T}^n), \quad 0 \le \beta < \alpha < 1,$$

$$\mathrm{PWM}(\mathrm{bmo}_{(\log(1/r))^{-1}}(\mathbb{T}^n), \mathrm{bmo}(\mathbb{T}^n)) = \mathrm{bmo}_{(\log\log(1/r))^{-1}}(\mathbb{T}^n),$$

$$\mathrm{PWM}(\mathrm{bmo}(\mathbb{R}^n), \mathrm{bmo}_{\log(|a|+r+1/r),p}(\mathbb{R}^n)) = \mathrm{bmo}(\mathbb{R}^n), \quad 1 < p < \infty, \text{ etc.}$$

1. Introduction. Let E and F be spaces of real- or complex-valued functions defined on a set X. A real- or complex-valued function g defined on X is called a *pointwise multiplier* from E to F if the pointwise product fg belongs to F for each $f \in E$. We denote by $\mathrm{PWM}(E,F)$ the set of all pointwise multipliers from E to F.

For L^p-spaces on a σ-finite measure space X, it is known that

$$\mathrm{PWM}(L^{p_1}(X), L^{p_2}(X)) = L^{p_3}(X), \quad 1/p_1 + 1/p_3 = 1/p_2.$$

The purpose of this paper is to characterize

$$\mathrm{PWM}(\mathrm{bmo}_{\phi_1,p_1}(X), \mathrm{bmo}_{\phi_2,p_2}(X)),$$

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where $\mathrm{bmo}_{\phi_i,p_i}(X)$ $(i=1,2)$ are function spaces defined using the mean oscillation and weight functions $\phi_i : X\times\mathbb{R}_+\to\mathbb{R}_+$ $(i=1,2)$.

Janson [6] characterized $\mathrm{PWM}(\mathrm{bmo}_\phi(\mathbb{T}^n),\mathrm{bmo}_\phi(\mathbb{T}^n))$ on the n-dimensional torus $\mathbb{T}^n$ for a weight function $\phi:\mathbb{R}_+\to\mathbb{R}_+$. His result was extended in [13–15] to the cases of the n-dimensional Euclidean space $\mathbb{R}^n$ and spaces of homogeneous type. Bloom [2], Gotoh [5] and Yabuta [18] have also characterized pointwise multipliers from a weighted BMO space to itself. In this paper, we consider pointwise multipliers from $\mathrm{bmo}_{\phi_1,p_1}(X)$ to $\mathrm{bmo}_{\phi_2,p_2}(X)$.

Let $X=(X,d,\mu)$ be a space of homogeneous type in the sense of Coifman–Weiss [3, 4], i.e., X is a topological space endowed with a Borel measure μ and a quasi-distance d such that $d(x,y)\ge 0$, $d(x,y)=0$ if and only if $x=y$, $d(x,y)=d(y,x)$,

$$d(x,y)\le K_1(d(x,z)+d(z,y)),\quad x,y,z\in X, \tag{1.1}$$

the balls $B(x,r)=\{y\in X: d(x,y)<r\}$ centered at x and of radius $r>0$ form a basis of open neighborhoods of the point x, and

$$0<\mu(B(x,2r))\le K_2\,\mu(B(x,r))<\infty,\quad x\in X,\ r>0. \tag{1.2}$$

We assume that there are constants α_0 $(0<\alpha_0\le 1)$ and $K_3\ge 1$ such that

$$|d(x,z)-d(y,z)|\le K_3(d(x,z)+d(y,z))^{1-\alpha_0}d(x,y)^{\alpha_0},\quad x,y,z\in X. \tag{1.3}$$

If d is a distance, then (1.1) and (1.3) hold for $K_1=K_3=\alpha_0=1$.

For a function $f\in L^1_{\mathrm{loc}}(X)$ and for a ball B, let

$$f_B=\frac{1}{\mu(B)}\int_B f(x)\,d\mu,\quad \mathrm{MO}(f,B)=\frac{1}{\mu(B)}\int_B |f(x)-f_B|\,d\mu.$$

For $1\le p<\infty$ and for $\phi:X\times\mathbb{R}_+\to\mathbb{R}_+$, let

$$\mathrm{MO}_p(f,B(a,r))=\left(\frac{1}{\mu(B(a,r))}\int_{B(a,r)}|f(x)-f_{B(a,r)}|^p\,d\mu\right)^{1/p},$$

$$\mathrm{MO}_{\phi,p}(f,B(a,r))=\frac{1}{\phi(a,r)}\mathrm{MO}_p(f,B(a,r)).$$

We define

$$\mathrm{bmo}_{\phi,p}(X)=\{f\in L^p_{\mathrm{loc}}(X):\sup_B \mathrm{MO}_{\phi,p}(f,B)<\infty\},$$

$$\|f\|_{\mathrm{BMO}_{\phi,p}}=\sup_B \mathrm{MO}_{\phi,p}(f,B),$$

$$\|f\|_{\mathrm{bmo}_{\phi,p}}=\|f\|_{\mathrm{BMO}_{\phi,p}}+|f_{B(x_0,1)}|\quad\text{for fixed } x_0\in X.$$

Let $\mathrm{bmo}_\phi(X)=\mathrm{bmo}_{\phi,p}(X)$ for $p=1$ and $\mathrm{bmo}(X)=\mathrm{bmo}_\phi(X)$ for $\phi\equiv 1$. Then $\mathrm{bmo}_{\phi,p}(X)$ is a Banach space under the norm $\|f\|_{\mathrm{bmo}_{\phi,p}}$. The closed graph theorem shows that every pointwise multiplier from $\mathrm{bmo}_{\phi_1,p_1}(X)$ to $\mathrm{bmo}_{\phi_2,p_2}(X)$ is a bounded operator. For each ball $B(x_1,r_1)$, $\|f\|_{\mathrm{BMO}_{\phi,p}}+|f_{B(x_1,r_1)}|$ is comparable to $\|f\|_{\mathrm{bmo}_{\phi,p}}$ (see (3.2)). Moreover, if $\mu(X)<\infty$, then $\|f\|_{\mathrm{BMO}_{\phi,p}}+\|f\|_{L^p}$ is comparable to $\|f\|_{\mathrm{bmo}_{\phi,p}}$ (see (1.16)).

Usually, $\mathrm{bmo}_{\phi,p}$ is denoted by $\mathrm{BMO}_{\phi,p}$ and equipped with the seminorm $\|f\|_{\mathrm{BMO}_{\phi,p}}$. Then $\mathrm{BMO}_{\phi,p}$ modulo constants is a Banach space. But the pointwise multipliers are defined on function spaces or on such spaces modulo null-functions. To consider pointwise multipliers, the space $\mathrm{bmo}_{\phi,p}$ is therefore more suitable than $\mathrm{BMO}_{\phi,p}$.

For $1\le p<\infty$ and $\phi:X\times\mathbb{R}_+\to\mathbb{R}_+$, let

$$L_{\phi,p}(X)=\left\{f\in L^p_{\mathrm{loc}}(X):\sup_{B(a,r)}\frac{1}{\phi(a,r)}((|f|^p)_{B(a,r)})^{1/p}<\infty\right\},$$

$$\|f\|_{L_{\phi,p}}=\sup_{B(a,r)}\frac{1}{\phi(a,r)}((|f|^p)_{B(a,r)})^{1/p},$$

and $L_\phi(X)=L_{\phi,p}(X)$ for $p=1$. If $\phi\equiv 1$ then $L_{\phi,p}(X)=L^\infty(X)$. Let $X_0=\{x\in X:\phi(x,r)\to 0 \text{ as } r\to 0\}$. If $f\in L_{\phi,p}(X)$, then $f(x)=0$ a.e. $x\in X_0$.

We shall consider the following conditions on ϕ:

$$\frac{1}{A_1}\le\frac{\phi(a,s)}{\phi(a,r)}\le A_1,\quad \frac12\le\frac{s}{r}\le 2, \tag{1.4}$$

$$\frac{\phi(a,r)}{r^{\alpha_0}}\le A_2\frac{\phi(a,s)}{s^{\alpha_0}},\quad 0<s<r, \tag{1.5}$$

$$\int_0^r \mu(B(a,t))^{1/p}\frac{\phi(a,t)}{t}\,dt\le A_3\,\mu(B(a,r))^{1/p}\phi(a,r),\quad r>0, \tag{1.6}$$

$$\frac{1}{A_4}\le\frac{\phi(a,r)}{\phi(b,r)}\le A_4,\quad d(a,b)\le r, \tag{1.7}$$

where $A_i>0$ $(i=1,2,3,4)$ are independent of $r,s>0$, $a,b\in X$.

If there is a constant $A_5>0$ such that

$$\phi(a,r)\le A_5\phi(b,s)\quad\text{for } B(a,r)\subset B(b,s), \tag{1.8}$$

then, for $1<p<\infty$, we have

$$\mathrm{bmo}_{\phi,p}(X)=\mathrm{bmo}_\phi(X)\quad\text{and}\quad \|f\|_{\mathrm{bmo}_\phi}\le\|f\|_{\mathrm{bmo}_{\phi,p}}\le C_p\|f\|_{\mathrm{bmo}_\phi},$$

by Hölder's inequality and John–Nirenberg's inequality.

For $\mu(X)=\infty$, we shall consider the following condition. There are constants $r_0\ge 0$ and $A_6>0$ such that

$$\int_{r_0}^r\left(\frac{\phi_2(x_0,t)}{\phi_1(x_0,t)}\right)^p\frac{\mu(B(x_0,t))}{t}\,dt \le A_6\left(\frac{\phi_2(x_0,r)}{\phi_1(x_0,r)}\right)^p\mu(B(x_0,r)),\quad r>r_0. \tag{1.9}$$

The following are equivalent (see Lemma 5.2 of [15]):

$$B(x_0, K_4 r) \setminus B(x_0, r) \neq \emptyset, \quad r > r_0, \text{ for some } K_4 > 1, \tag{1.10}$$

$$\mu(B(x_0, r)) \le \tfrac{1}{2}\mu(B(x_0, K_5 r)), \quad r > r_0, \text{ for some } K_5 > 1, \tag{1.11}$$

$$\int_{r_0}^{r} \frac{\mu(B(x_0,t))}{t}\,dt \le K_6 \mu(B(x_0,r)), \quad r > r_0, \text{ for some } K_6 > 0, \tag{1.12}$$

where K_4, K_5 and K_6 are independent of r. On the assumption that (1.10) holds, if there are constants $A_7 > 0$ and $\varepsilon > 0$ such that

$$\left(\frac{\phi_2(x_0,r)}{\phi_1(x_0,r)}\right)^{p+\varepsilon} \mu(B(x_0,r)) \le A_7 \left(\frac{\phi_2(x_0,s)}{\phi_1(x_0,s)}\right)^{p+\varepsilon} \mu(B(x_0,s)), \quad r_0 < r < s, \tag{1.13}$$

then (1.9) holds.

For ϕ, we define

$$\Phi^*(a,r) = \Phi^*_{x_0}(a,r) = \int_1^{\max(2,d(x_0,a),r)} \frac{\phi(x_0,t)}{t}\,dt, \tag{1.14}$$

$$\Phi^{**}(a,r) = \Phi^{**}_{x_0}(a,r) = \int_r^{\max(2,d(x_0,a),r)} \frac{\phi(a,t)}{t}\,dt. \tag{1.15}$$

If ϕ satisfies (1.4) and (1.7), then, for each $x_1 \in X$, $\Phi^*_{x_1} + \Phi^{**}_{x_1}$ is comparable to $\Phi^*_{x_0} + \Phi^{**}_{x_0}$. For ϕ_i $(i = 1,2,3)$, we define Φ^*_i and Φ^{**}_i by (1.14) and (1.15), respectively.

The letter C will always denote a constant, not necessarily the same one.

Our results are the following.

THEOREM 1.1. *Suppose that ϕ_1 satisfies (1.4)–(1.8) for some p_1 $(1 \le p_1 < \infty)$, ϕ_2 satisfies (1.4), (1.7) and (1.8), and $(\Phi^*_2 + \Phi^{**}_2)/\phi_2 \le C(\Phi^*_1 + \Phi^{**}_1)/\phi_1$. If $\mu(X) = \infty$, then assume that (1.9) holds with $p = 1+\varepsilon$ for some $\varepsilon > 0$. Let $\phi_3 = \phi_2/(\Phi^*_1 + \Phi^{**}_1)$. Then*

$$\mathrm{PWM}(\mathrm{bmo}_{\phi_1}(X), \mathrm{bmo}_{\phi_2}(X)) = \mathrm{bmo}_{\phi_3}(X) \cap L_{\phi_2/\phi_1}(X).$$

Moreover, the operator norm of $g \in \mathrm{PWM}(\mathrm{bmo}_{\phi_1}(X), \mathrm{bmo}_{\phi_2}(X))$ is comparable to $\|g\|_{\mathrm{BMO}_{\phi_3}} + \|g\|_{L_{\phi_2/\phi_1}}$.

COROLLARY 1.2. *Under the assumptions of Theorem 1.1, if $\Phi^*_3 + \Phi^{**}_3 \le C\phi_2/\phi_1$, then*

$$\mathrm{PWM}(\mathrm{bmo}_{\phi_1}(X), \mathrm{bmo}_{\phi_2}(X)) = \mathrm{bmo}_{\phi_3}(X).$$

Moreover, the operator norm of $g \in \mathrm{PWM}(\mathrm{bmo}_{\phi_1}(X), \mathrm{bmo}_{\phi_2}(X))$ is comparable to $\|g\|_{\mathrm{bmo}_{\phi_3}}$.

If $\mu(X) < \infty$, then there is a constant $R_0 > 0$ such that

$$X = B(x, R_0) \quad \text{for all } x \in X. \tag{1.16}$$

For ϕ, we define

$$\Phi(a,r) = \int_r^{2R_0} \frac{\phi(a,t)}{t}\,dt, \quad 0 < r \le R_0. \tag{1.17}$$

If $\inf_{a\in X} \phi(a, R_0) = 0$, then $\mathrm{bmo}_{\phi,p}(X) = \{\text{const.}\}$. So we may assume $\Phi \ge C > 0$. If ϕ satisfies (1.4), then Φ is comparable to $\Phi^* + \Phi^{**}$. For ϕ_i $(i = 1,2,3)$, we define Φ_i by (1.17).

THEOREM 1.3. *Let $\mu(X) < \infty$. Suppose that ϕ_1 satisfies (1.4)–(1.8) for some p_1 $(1 \le p_1 < \infty)$, ϕ_2 satisfies (1.4), (1.7) and (1.8), and ϕ_1/ϕ_2 satisfies (1.8). Let $\phi_3 = \phi_2/\Phi_1$. Then*

$$\mathrm{PWM}(\mathrm{bmo}_{\phi_1}(X), \mathrm{bmo}_{\phi_2}(X)) = \mathrm{bmo}_{\phi_3}(X) \cap L_{\phi_2/\phi_1}(X).$$

Moreover, the operator norm of $g \in \mathrm{PWM}(\mathrm{bmo}_{\phi_1}(X), \mathrm{bmo}_{\phi_2}(X))$ is comparable to $\|g\|_{\mathrm{BMO}_{\phi_3}} + \|g\|_{L_{\phi_2/\phi_1}}$.

COROLLARY 1.4. *Under the assumptions of Theorem 1.3, if $\Phi_3 \le C\phi_2/\phi_1$, then*

$$\mathrm{PWM}(\mathrm{bmo}_{\phi_1}(X), \mathrm{bmo}_{\phi_2}(X)) = \mathrm{bmo}_{\phi_3}(X).$$

Moreover, the operator norm of $g \in \mathrm{PWM}(\mathrm{bmo}_{\phi_1}(X), \mathrm{bmo}_{\phi_2}(X))$ is comparable to $\|g\|_{\mathrm{bmo}_{\phi_3}}$.

COROLLARY 1.5. *Under the assumptions of Theorem 1.3, if $\Phi_1 \le C$, then*

$$\mathrm{PWM}(\mathrm{bmo}_{\phi_1}(X), \mathrm{bmo}_{\phi_2}(X)) = \mathrm{bmo}_{\phi_2}(X).$$

Moreover, the operator norm of $g \in \mathrm{PWM}(\mathrm{bmo}_{\phi_1}(X), \mathrm{bmo}_{\phi_2}(X))$ is comparable to $\|g\|_{\mathrm{bmo}_{\phi_2}}$.

THEOREM 1.6. *Let $1 < p_2 < p_1 < \infty$ and $p_1 p_2 \ge p_1 + p_2$. Suppose that ϕ_1 and p_1 satisfy (1.4)–(1.7), ϕ_2 satisfies (1.4) and (1.7), and $(\Phi^*_2 + \Phi^{**}_2)/\phi_2 \le C(\Phi^*_1 + \Phi^{**}_1)/\phi_1$. If $\mu(X) = \infty$, then assume that (1.9) holds with $p = p_2$. Let $\phi_3 = \phi_2/(\Phi^*_1 + \Phi^{**}_1)$ satisfy (1.8). Then*

$$\mathrm{PWM}(\mathrm{bmo}_{\phi_1,p_1}(X), \mathrm{bmo}_{\phi_2,p_2}(X)) = \mathrm{bmo}_{\phi_3}(X) \cap L_{\phi_2/\phi_1}(X).$$

Moreover, the operator norm of $g \in \mathrm{PWM}(\mathrm{bmo}_{\phi_1,p_1}(X), \mathrm{bmo}_{\phi_2,p_2}(X))$ is comparable to $\|g\|_{\mathrm{BMO}_{\phi_3}} + \|g\|_{L_{\phi_2/\phi_1}}$.

COROLLARY 1.7. *Under the assumptions of Theorem 1.6, if $\Phi^*_3 + \Phi^{**}_3 \le C\phi_2/\phi_1$, then*

$$\mathrm{PWM}(\mathrm{bmo}_{\phi_1,p_1}(X), \mathrm{bmo}_{\phi_2,p_2}(X)) = \mathrm{bmo}_{\phi_3}(X).$$

Moreover, the operator norm of $g \in \mathrm{PWM}(\mathrm{bmo}_{\phi_1,p_1}(X), \mathrm{bmo}_{\phi_2,p_2}(X))$ *is comparable to* $\|g\|_{\mathrm{bmo}_{\phi_3}}$.

THEOREM 1.8. *Let* $1 \le p_2 \le p_1 < \infty$. *Suppose that* ϕ *and* p_1 *satisfy* (1.4)–(1.7). *If* $\mu(X) = \infty$, *then assume that* (1.10) *holds. Let* $\psi = \phi/(\Phi^* + \Phi^{**})$. *Then*

$$\mathrm{PWM}(\mathrm{bmo}_{\phi,p_1}(X), \mathrm{bmo}_{\phi,p_2}(X)) = \mathrm{bmo}_{\psi,p_2}(X) \cap L^\infty(X).$$

Moreover, the operator norm of $g \in \mathrm{PWM}(\mathrm{bmo}_{\phi,p_1}(X), \mathrm{bmo}_{\phi,p_2}(X))$ *is comparable to* $\|g\|_{\mathrm{BMO}_{\psi,p_2}} + \|g\|_{L^\infty}$.

COROLLARY 1.9 ([15]). *Let* $1 \le p < \infty$. *Suppose* ϕ *and* p *satisfy* (1.4)–(1.7). *If* $\mu(X) = \infty$, *then assume that* (1.10) *holds. Let* $\psi = \phi/(\Phi^* + \Phi^{**})$. *Then*

$$\mathrm{PWM}(\mathrm{bmo}_{\phi,p}(X), \mathrm{bmo}_{\phi,p}(X)) = \mathrm{bmo}_{\psi,p}(X) \cap L^\infty(X).$$

Moreover, the operator norm of $g \in \mathrm{PWM}(\mathrm{bmo}_{\phi,p}(X), \mathrm{bmo}_{\phi,p}(X))$ *is comparable to* $\|g\|_{\mathrm{BMO}_{\psi,p}} + \|g\|_{L^\infty}$.

THEOREM 1.10. *Let* $1 \le p_1, p_2 < \infty$. *Suppose that* ϕ_1 *and* p_1 *satisfy* (1.4)–(1.7) *and* ϕ_2 *satisfies* (1.4). *Let* X^* *be the set of all points* $x \in X$ *such that there are constants* $K_x > 1$ *and* $r_x > 0$ *such that*

$$\mu(B(x,r)) \le \tfrac{1}{2}\mu(B(x, K_x r)), \quad 0 < r \le r_x,$$

and X_0 *the set of all points* $x \in X$ *such that*

$$\phi_2(x,r)/\phi_1(x,r) \to 0 \quad \text{as } r \to 0.$$

If $g \in \mathrm{PWM}(\mathrm{bmo}_{\phi_1,p_1}(X), \mathrm{bmo}_{\phi_2,p_2}(X))$, *then* $g(x) = 0$ *a.e.* $x \in X^* \cap X_0$.

In the next section we give some examples obtained from our results. We state some lemmas in Section 3 and propositions in Section 4 to prove the results in Section 5.

2. Examples. In this section, we assume that (X, d, μ) has the following property: there are constants $C > 0$ and $\delta > 0$ such that

$$\frac{\mu(B(x,t))}{\mu(B(x,r))} \le C\left(\frac{t}{r}\right)^\delta, \quad x \in X,\ 0 < t < r. \tag{2.1}$$

Then (1.11) follows from (2.1). If $\phi(a,r)r^\varepsilon$ $(0 \le \varepsilon < \delta/p)$ satisfies (1.8), then ϕ and p satisfy (1.6). For example, the Muckenhoupt A_p-weights on $\mathbb{R}^n$ satisfy (2.1).

2.1. *The case* $\mu(X) < \infty$

EXAMPLE 2.1. For $0 \le \beta < \alpha < 1$,

$$\mathrm{PWM}(\mathrm{bmo}_{(\log(1/r))^{-\alpha}}(X), \mathrm{bmo}_{(\log(1/r))^{-\beta}}(X)) = \mathrm{bmo}_{(\log(1/r))^{\alpha-\beta-1}}(X).$$

For $\alpha = 1/2$ and $\beta = 0$ in particular,

$$\mathrm{PWM}(\mathrm{bmo}_{(\log(1/r))^{-1/2}}(X), \mathrm{bmo}(X)) = \mathrm{bmo}_{(\log(1/r))^{-1/2}}(X).$$

EXAMPLE 2.2.

$$\mathrm{PWM}(\mathrm{bmo}_{(\log(1/r))^{-1}}(X), \mathrm{bmo}(X)) = \mathrm{bmo}_{(\log\log(1/r))^{-1}}(X).$$

EXAMPLE 2.3.

$$\mathrm{PWM}(\mathrm{bmo}_{(\log\log(1/r))^{-1}}(X), \mathrm{bmo}(X)) = \mathrm{bmo}_{(\mathrm{li}(\log(1/r)))^{-1}}(X) \cap L_{(\log\log(1/r))}(X),$$

where $\mathrm{li}(R) = \int_e^R (1/\log t)\,dt$.

EXAMPLE 2.4.

$$\mathrm{PWM}(\mathrm{bmo}(X), \mathrm{bmo}(X)) = \mathrm{bmo}_{(\log(1/r))^{-1}}(X) \cap L^\infty(X).$$

If $X = \mathbb{T}^n$, $d(x,y) = |x-y|$ and μ is Lebesgue measure, then the example above is known (Janson [6] and Stegenga [17]).

EXAMPLE 2.5. For $\alpha > 1$,

$$\mathrm{PWM}(\mathrm{bmo}_{(\log(1/r))^{-\alpha}}(X), \mathrm{bmo}(X)) = \mathrm{bmo}(X).$$

EXAMPLE 2.6. For $0 < \beta \le \alpha \le \alpha_0$,

$$\mathrm{PWM}(\mathrm{bmo}_{r^\alpha}(X), \mathrm{bmo}_{r^\beta}(X)) = \mathrm{bmo}_{r^\beta}(X).$$

If $X = \mathbb{T}^n$, $d(x,y) = |x-y|$ and μ is Lebesgue measure, then $\mathrm{bmo}_{r^\alpha}(\mathbb{T}^n) = \mathrm{Lip}_\alpha(\mathbb{T}^n)$. Therefore, for $0 < \beta \le \alpha \le 1$,

$$\mathrm{PWM}(\mathrm{Lip}_\alpha(\mathbb{T}^n), \mathrm{Lip}_\beta(\mathbb{T}^n)) = \mathrm{Lip}_\beta(\mathbb{T}^n).$$

EXAMPLE 2.7. For $-1 < \alpha < \beta \le \alpha + 1$, $1 < p_2 < p_1 < \infty$, $p_1 p_2 \ge p_1 + p_2$,

$$\mathrm{PWM}(\mathrm{bmo}_{(\log(1/r))^\alpha, p_1}(X), \mathrm{bmo}_{(\log(1/r))^\beta, p_2}(X)) = \mathrm{bmo}_{(\log(1/r))^{\beta-\alpha-1}}(X).$$

2.2. *The case* $\mu(X) = \infty$

EXAMPLE 2.8.

$$\mathrm{PWM}(\mathrm{bmo}(X), \mathrm{bmo}(X)) = \mathrm{bmo}_{(\log(d(x_0,a)+r+1/r))^{-1}}(X) \cap L^\infty(X).$$

If $X = \mathbb{R}^n$, $d(x,y) = |x-y|$ and μ is Lebesgue measure, then the example above is known ([14]).

EXAMPLE 2.9. For $0 < \beta \le \alpha \le \alpha_0$,

$$\mathrm{PWM}(\mathrm{bmo}_{r^\alpha}(X), \mathrm{bmo}_{r^\beta}(X)) = \mathrm{bmo}_{\frac{r^\beta}{(2+d(x_0,a)+r)^\alpha}}(X) \cap L_{r^{\beta-\alpha}}(X).$$

EXAMPLE 2.10. For $0 < \alpha \le \alpha_0$, $\beta \ge 0$, $\beta - \alpha + \delta > 0$,

$$\mathrm{PWM}(\mathrm{bmo}_{(2+d(x_0,a)+r)^\alpha}(X), \mathrm{bmo}_{(2+d(x_0,a)+r)^\beta}(X)) = \mathrm{bmo}_{\frac{(2+d(x_0,a)+r)^{\beta-\alpha}}{\log(d(x_0,a)+r+1/r)}}(X) \cap L_{(2+d(x_0,a)+r)^{\beta-\alpha}}(X).$$

EXAMPLE 2.11. For $1 < p < \infty$,

$$\mathrm{PWM}(\mathrm{bmo}(X), \mathrm{bmo}_{\log(d(x_0,a)+r+1/r),p}(X)) = \mathrm{bmo}(X).$$

EXAMPLE 2.12. Let w be an $A_{p'}$-weight on $\mathbb{R}^n$. Then

$$\phi(a,r) = \Big(\int\limits_{B(a,r)} w(x)\,dx \Big)^{\alpha}$$

satisfies (1.4)–(1.7) for $-1/(pp') < \alpha \le 1/(np')$, and (1.8) for $\alpha \ge 0$. Let

$$\phi_i(a,r) = \Big(\int\limits_{B(a,r)} w(x)\,dx \Big)^{\alpha_i}, \quad i = 1,2,\ 0 < \alpha_2 \le \alpha_1.$$

Then $(\Phi_2^* + \Phi_2^{**})/\phi_2 \le C(\Phi_1^* + \Phi_1^{**})/\phi_1$.

3. Lemmas. First, we state some simple inequalities and four lemmas of [15]. Let $1 \le p < \infty$. Then

$$(3.1) \quad |F(z_1) - F(z_2)| \le C|z_1 - z_2| \Rightarrow \mathrm{MO}_{\phi,p}(F(f), B) \le 2C\,\mathrm{MO}_{\phi,p}(f, B),$$

$$(3.2) \quad |f_{B_1} - f_{B_2}| \le \frac{\mu(B_2)}{\mu(B_1)} \mathrm{MO}(f, B_2) \quad \text{for } B_1 \subset B_2,$$

$$(3.3) \quad \mathrm{MO}_p(f, B_1) \le 2\Big(\frac{\mu(B_2)}{\mu(B_1)}\Big)^{1/p} \mathrm{MO}_p(f, B_2) \quad \text{for } B_1 \subset B_2,$$

$$(3.4) \quad |f_{B(a,r)} - f_{B(a,s)}| \le 2K_2^2(\log 2)^{-1} \int\limits_r^{2s} \frac{\mathrm{MO}(f, B(a,t))}{t}\,dt \quad \text{for } 0 < r < s.$$

If ϕ satisfies (1.4), then

$$(3.5) \quad \int\limits_r^{2s} \frac{\phi(a,t)}{t}\,dt \le (1 + A_1) \int\limits_r^{s} \frac{\phi(a,t)}{t}\,dt \quad \text{for } 0 < 2r \le s.$$

LEMMA 3.1. *Let $1 \le p < \infty$. Suppose that ϕ satisfies* (1.4)–(1.7). *Let*

$$f_a(x) = \int\limits_{d(a,x)}^{1} \frac{\phi(a,t)}{t}\,dt.$$

Then f_a is in $\mathrm{bmo}_{\phi,p}(X)$ for all $a \in X$, and there is a constant $C > 0$, independent of a, such that $\|f_a\|_{\mathrm{BMO}_{\phi,p}} \le C$.

LEMMA 3.2. *Let $1 \le p < \infty$. Suppose that ϕ satisfies* (1.4). *Then there is a constant $C > 0$ such that*

$$|f_{B(a,r)}| \le C\|f\|_{\mathrm{bmo}_{\phi,p}} (\Phi^*(a,r) + \Phi^{**}(a,r)),$$

where C is independent of $f \in \mathrm{bmo}_{\phi,p}(X)$, $a \in X$ and $r > 0$.

LEMMA 3.3. *Let $1 \le p < \infty$. Suppose that ϕ satisfies* (1.4)–(1.7). *If $\mu(X) = \infty$, then assume that* (1.10) *holds. For any ball $B(a,r)$, there is a function $f \in \mathrm{bmo}_{\phi,p}(X)$ such that*

$$\|f\|_{\mathrm{bmo}_{\phi,p}} \le C_1, \quad f_{B(a,r)} \ge C_2(\Phi^*(a,r) + \Phi^{**}(a,r)),$$

where C_1, C_2 are independent of $B(a,r)$ and $f \in \mathrm{bmo}_{\phi,p}(X)$.

LEMMA 3.4. *Let $1 \le p < \infty$. Suppose $f \in \mathrm{bmo}_{\phi,p}(X)$ and $g \in L^\infty(X)$. Then fg belongs to $\mathrm{bmo}_{\phi,p}(X)$ if and only if*

$$\sup_B |f_B| \mathrm{MO}_{\phi,p}(g, B) < \infty.$$

In this case,

$$\big|\|fg\|_{\mathrm{BMO}_{\phi,p}} - \sup_B |f_B| \mathrm{MO}_{\phi,p}(g,B)\big| \le 2\|f\|_{\mathrm{BMO}_{\phi,p}} \|g\|_{L^\infty}.$$

We need more precise lemmas.

LEMMA 3.5. *Let $1 \le p < \infty$. Suppose ϕ satisfies* (1.4). *Then*

$$\mathrm{bmo}_{\phi,p}(X) \subset L_{\Phi^*+\Phi^{**},p}(X) \quad \text{and} \quad \|f\|_{L_{\Phi^*+\Phi^{**},p}} \le C\|f\|_{\mathrm{bmo}_{\phi,p}},$$

where C is independent of $f \in \mathrm{bmo}_{\phi,p}(X)$.

Proof. If $2r \le \max(d(x_0,a), 2)$, then

$$\mathrm{MO}_p(f, B(a,r)) \le \phi(a,r)\|f\|_{\mathrm{BMO}_{\phi,p}} \le C'\Phi^{**}(a,r)\|f\|_{\mathrm{BMO}_{\phi,p}}.$$

If $2r > \max(d(x_0,a), 2)$, then $B(a,r) \subset B(x_0, 3K_1r)$. From (3.3) and (3.5), it follows that

$$\begin{aligned}\mathrm{MO}_p(f, B(a,r)) &\le 2\Big(\frac{\mu(B(x_0,3K_1r))}{\mu(B(a,r))}\Big)^{1/p} \mathrm{MO}_p(f, B(x_0,3K_1r)) \\ &\le C''\phi(x_0,3K_1r)\|f\|_{\mathrm{BMO}_{\phi,p}} \\ &\le C'''\Phi^*(a,r)\|f\|_{\mathrm{BMO}_{\phi,p}}.\end{aligned}$$

By Lemma 3.2, we have

$$\begin{aligned}\Big(\frac{1}{\mu(B(a,r))} \int\limits_{B(a,r)} |f(x)|^p\,d\mu\Big)^{1/p} &\le \mathrm{MO}_p(f, B(a,r)) + |f_{B(a,r)}| \\ &\le C(\Phi^*(a,r) + \Phi^{**}(a,r))\|f\|_{\mathrm{bmo}_{\phi,p}}. \ \blacksquare\end{aligned}$$

COROLLARY 3.6. *Let $\mu(X) < \infty$ and $1 \le p < \infty$. Suppose ϕ satisfies* (1.4). *Then*

$$\mathrm{bmo}_{\phi,p}(X) \subset L_{\Phi,p}(X) \quad \text{and} \quad \|f\|_{L_{\Phi,p}} \le C\|f\|_{\mathrm{bmo}_{\phi,p}},$$

where C is independent of $f \in \mathrm{bmo}_{\phi,p}(X)$.

LEMMA 3.7. *Let $\mu(X)=\infty$ and $1\le p<\infty$. Suppose ϕ satisfies (1.4)–(1.7). Let $r_1\ge 2$ and*

$$f(x)=\int\limits_1^{\max(2,d(x_0,x))}\frac{\phi(x_0,t)}{t}\,dt. \tag{3.6}$$

Then f is in $\mathrm{bmo}_{\phi,p}(X)$ and there are constants $C_i>0$ $(i=1,2,3)$, independent of $B(a,r)$, such that:

(i) *if $r<2K_1d(x_0,a)$, then*

$$f(x)\ge C_1\Phi^*(a,r)\quad \text{for } x\in B(a,r/(2K_1)^2);$$

(ii) *if $2K_1d(x_0,a)\le r<2r_1$, then*

$$f(x)\ge C_2\Phi^*(a,r)\quad \text{for } x\in B(a,r);$$

(iii) *if $2K_1d(x_0,a)\le r$ and $2^kr_1\le r<2^{k+1}r_1$ for some positive integer k, then*

$$f(x)\ge C_3\Phi^*(x_0,2^{-j}r)\quad \text{for } x\in E_j,\ j=0,1,\ldots,k-1,$$

where

$$E_j=B(x_0,2^{-j}r)\setminus B(x_0,2^{-j-1}r),\qquad j=0,1,\ldots,k-1, \tag{3.7}$$

$$B(a,r/(2K_1))\subset\Big(\bigcup_{j=0}^{k-1}E_j\Big)\cup B(x_0,2^{-k}r). \tag{3.8}$$

Proof. Since $f(x)=\max(-f_{x_0}(x),\int_1^2\phi(x_0,t)t^{-1}\,dt)$, f is in $\mathrm{bmo}_{\phi,p}(X)$ by Lemma 3.1 and (3.1). Next we show (i)–(iii) by using (3.5).

(i) Since $B(a,r/(2K_1)^2)\cap B(x_0,d(x_0,a)/(2K_1))=\emptyset$,

$$\begin{aligned}\Phi^*(a,r)&\le\int\limits_1^{\max(2(2K_1)^2,d(x_0,a),r)}\frac{\phi(x_0,t)}{t}\,dt\\ &\le C\int\limits_1^{\max(2,d(x_0,a)/(2K_1)^2,r/(2K_1)^2)}\frac{\phi(x_0,t)}{t}\,dt\\ &\le C\int\limits_1^{\max(2,d(x_0,a)/(2K_1))}\frac{\phi(x_0,t)}{t}\,dt\\ &\le Cf(x).\end{aligned}$$

(ii) Since $d(x_0,a),r\le 2r_1$,

$$\Phi^*(a,r)\le\int\limits_1^{2r_1}\frac{\phi(x_0,t)}{t}\,dt\le C'\int\limits_1^{2}\frac{\phi(x_0,t)}{t}\,dt\le C'f(x).$$

(iii) For $x\in E_j$, $j=0,1,\ldots,k-1$,

$$\Phi^*(x_0,2^{-j}r)=\int\limits_1^{2^{-j}r}\frac{\phi(x_0,t)}{t}\,dt\le C''\int\limits_1^{2^{-j-1}r}\frac{\phi(x_0,t)}{t}\,dt\le C''f(x). \ \blacksquare$$

The next two lemmas have been proved in the proof of Lemma 3.3 of [15].

LEMMA 3.8. *Let $\mu(X)<\infty$ and $1\le p<\infty$. Suppose ϕ satisfies (1.4)–(1.7). Let f be defined by (3.6). Then f is in $\mathrm{bmo}_{\phi,p}(X)$ and there is a constant $C>0$ such that, for all $B(a,r)$,*

$$f(x)\ge C\Phi^*(a,r)\quad \text{for } x\in B(a,r/(2K_1)^2).$$

LEMMA 3.9. *Let $1\le p<\infty$. Suppose ϕ satisfies (1.4)–(1.7). For any ball $B(a,r)$, let*

$$f(x)=\max\Big(0,\int\limits_{d(a,x)}^{\max(1/K_1,d(x_0,a)/(2K_1))}\frac{\phi(a,t)}{t}\,dt\Big). \tag{3.9}$$

Then f is in $\mathrm{bmo}_{\phi,p}(X)$ and there are constants $C_1,C_2>0$ such that

$$\|f\|_{\mathrm{bmo}_{\phi,p}}\le C_1,$$
$$f(x)\ge C_2\Phi^{**}(a,r)\quad \text{for } x\in B(a,r/(2K_1)),$$

where C_1,C_2 are independent of $B(a,r)$ and $f\in\mathrm{bmo}_{\phi,p}(X)$.

LEMMA 3.10. *Let $1\le p_1,p_2,p_3<\infty$ and $1/p_1+1/p_3=1/p_2$. Suppose $f\in\mathrm{bmo}_{\phi_1,p_1}(X)$ and $g\in L_{\phi_2/\phi_1,p_3}$. Then $fg\in\mathrm{bmo}_{\phi_2,p_2}(X)$ if and only if*

$$\sup_B|f_B|\mathrm{MO}_{\phi_2,p_2}(g,B)<\infty.$$

In this case,

$$\Big|\|fg\|_{\mathrm{BMO}_{\phi_2,p_2}}-\sup_B|f_B|\mathrm{MO}_{\phi_2,p_2}(g,B)\Big| \le 2\|f\|_{\mathrm{BMO}_{\phi_1,p_1}}\|g\|_{L_{\phi_2/\phi_1,p_3}}. \tag{3.10}$$

Proof. As in the proof of Lemma 3.4, for any ball $B=B(a,r)$, we have

$$\begin{aligned}&\big|\|(fg)(\cdot)-(fg)_B\|_{L^{p_2}(B)}-|f_B|\|g(\cdot)-g_B\|_{L^{p_2}(B)}\big|\\ &\le 2\Big(\int\limits_B|(f(x)-f_B)g(x)|^{p_2}\,d\mu\Big)^{1/p_2}\\ &\le 2\Big(\int\limits_B|f(x)-f_B|^{p_1}\,d\mu\Big)^{1/p_1}\Big(\int\limits_B|g(x)|^{p_3}\,d\mu\Big)^{1/p_3}\\ &\le 2\mu(B)^{1/p_1}\phi_1(a,r)\|f\|_{\mathrm{BMO}_{\phi_1,p_1}}\times\mu(B)^{1/p_3}\frac{\phi_2(a,r)}{\phi_1(a,r)}\|g\|_{L_{\phi_2/\phi_1,p_3}}\\ &\le 2\mu(B)^{1/p_2}\phi_2(a,r)\|f\|_{\mathrm{BMO}_{\phi_1,p_1}}\|g\|_{L_{\phi_2/\phi_1,p_3}}.\end{aligned}$$

Hence

$$|\mathrm{MO}_{\phi_2,p_2}(fg,B)-|f_B|\mathrm{MO}_{\phi_2,p_2}(g,B)|\le 2\|f\|_{\mathrm{BMO}_{\phi_1,p_1}}\|g\|_{L_{\phi_2/\phi_1,p_3}},$$

which shows (3.10). ■

LEMMA 3.11. *Let* $1\le p_1,p_2<\infty$. *Suppose that* ϕ_1 *satisfies* (1.8) *and* $\phi_1\le C\phi_2$. *Then*

$$\mathrm{bmo}_{\phi_1,p_1}(X)\cap L_{\phi_2,p_2}(X)=\mathrm{bmo}_{\phi_1}(X)\cap L_{\phi_2}(X),$$
$$\|f\|_{\mathrm{bmo}_{\phi_1}}+\|f\|_{L_{\phi_2}}\le\|f\|_{\mathrm{bmo}_{\phi_1,p_1}}+\|f\|_{L_{\phi_2,p_2}}\le C_{p_1,p_2}(\|f\|_{\mathrm{bmo}_{\phi_1}}+\|f\|_{L_{\phi_2}}).$$

Proof. By Hölder's inequality, we have

$$\|f\|_{\mathrm{bmo}_{\phi_1}}+\|f\|_{L_{\phi_2}}\le\|f\|_{\mathrm{bmo}_{\phi_1,p_1}}+\|f\|_{L_{\phi_2,p_2}}.$$

By John–Nirenberg's inequality, we have

$$\|f\|_{\mathrm{BMO}_{\phi_1,p_i}}\le C_{p_i}\|f\|_{\mathrm{BMO}_{\phi_1}},\quad i=1,2.$$

For any ball $B=B(a,r)$,

$$\begin{aligned}\Big(\int_B|f(x)|^{p_2}\,d\mu\Big)^{1/p_2}&\le\Big(\int_B|f(x)-f_B|^{p_2}\,d\mu\Big)^{1/p_2}+\Big(\int_B|f_B|^{p_2}\,d\mu\Big)^{1/p_2}\\&\le\mu(B)^{1/p_2}(\phi_1(a,r)\|f\|_{\mathrm{BMO}_{\phi_1,p_2}}+|f_B|)\\&\le\mu(B)^{1/p_2}(C_{p_2}\phi_1(a,r)\|f\|_{\mathrm{BMO}_{\phi_1}}+\phi_2(a,r)\|f\|_{L_{\phi_2}})\\&\le C\mu(B)^{1/p_2}\phi_2(a,r)(C_{p_2}\|f\|_{\mathrm{BMO}_{\phi_1}}+\|f\|_{L_{\phi_2}}).\end{aligned}$$

Hence

$$\|f\|_{\mathrm{bmo}_{\phi_1,p_1}}+\|f\|_{L_{\phi_2,p_2}}\le C_{p_1,p_2}(\|f\|_{\mathrm{bmo}_{\phi_1}}+\|f\|_{L_{\phi_2}}).\ ■$$

LEMMA 3.12. *Let* $\mu(X)=\infty$ *and* $1\le p<\infty$. *Suppose that* ϕ_1 *and* ϕ_2 *satisfy* (1.4) *and* (1.7)–(1.9). *Let* $\phi_3=\phi_2/(\Phi_1^*+\Phi_1^{**})$. *Then*

$$\mathrm{bmo}_{\phi_3}(X)\cap L_{\phi_2/\phi_1}(X)=\mathrm{bmo}_{\phi_3}(X)\cap L_{\phi_2/\phi_1,p}(X),$$
$$\|f\|_{\mathrm{bmo}_{\phi_3}}+\|f\|_{L_{\phi_2/\phi_1}}\le\|f\|_{\mathrm{bmo}_{\phi_3}}+\|f\|_{L_{\phi_2/\phi_1,p}}\le C_p(\|f\|_{\mathrm{bmo}_{\phi_3}}+\|f\|_{L_{\phi_2/\phi_1}}).$$

Proof. We show that, for any ball $B(a,r)$,

$$(3.11)\quad\Big(\frac{1}{\mu(B(a,r))}\int_{B(a,r)}|f(x)|^p\,d\mu\Big)^{1/p}\le C_p\frac{\phi_2(a,r)}{\phi_1(a,r)}(\|f\|_{\mathrm{bmo}_{\phi_3}}+\|f\|_{L_{\phi_2/\phi_1}}).$$

First we note that

$$\Big(\frac{1}{\mu(B(a,r))}\int_{B(a,r)}|f(x)|^p\,d\mu\Big)^{1/p}\le\mathrm{MO}_p(f,B(a,r))+|f_{B(a,r)}|,$$

and

$$|f_{B(a,r)}|\le\frac{\phi_2(a,r)}{\phi_1(a,r)}\|f\|_{L_{\phi_2/\phi_1}}.$$

Case 1: $r\le d(x_0,a)/(5K_1^2)$. If $b\in B(a,r)$, then $d(x_0,b)>d(x_0,a)/K_1-r\ge 4K_1r$. If $B(b,s)\subset B(a,r)$, then we may assume $s\le 2K_1r$. It follows from (1.4) and (1.7) that

$$\Phi_1^{**}(b,s)\ge\int_{2K_1r}^{4K_1r}\frac{\phi_1(b,t)}{t}\,dt\ge A_1{}^{-1}\phi_1(b,2K_1r)\ge C\phi_1(a,r).$$

By John–Nirenberg's inequality, we have

$$\begin{aligned}\mathrm{MO}_p(f,B(a,r))&\le C_p\sup_{B(b,s)\subset B(a,r)}\mathrm{MO}(f,B(b,s))\\&\le C_p\Big(\sup_{B(b,s)\subset B(a,r)}\frac{\phi_2(b,s)}{\Phi_1^{**}(b,s)}\Big)\|f\|_{\mathrm{bmo}_{\phi_3}}\\&\le C_p'\frac{\phi_2(a,r)}{\phi_1(a,r)}\|f\|_{\mathrm{bmo}_{\phi_3}}.\end{aligned}$$

Case 2: $a=x_0$, $r\le r_0$. Then

$$\begin{aligned}\mathrm{MO}_p(f,B(x_0,r))&\le C_p\sup_{B(b,s)\subset B(x_0,r)}\mathrm{MO}(f,B(b,s))\\&\le C_p\Big(\sup_{B(b,s)\subset B(x_0,r)}\frac{\phi_2(b,s)}{\Phi_1^{*}(b,s)}\Big)\|f\|_{\mathrm{bmo}_{\phi_3}}\\&\le C_p'\frac{\phi_1(x_0,r_0)}{\int_1^2\frac{\phi_1(x_0,t)}{t}\,dt}\cdot\frac{\phi_2(x_0,r)}{\phi_1(x_0,r)}\|f\|_{\mathrm{bmo}_{\phi_3}}.\end{aligned}$$

Case 3: $a=x_0$, $r>r_0$. Let $2^{k-1}r_0<r\le 2^kr_0$ and $E_j=B(x_0,2^jr_0)\setminus B(x_0,2^{j-1}r_0)$, $j=1,\dots,k$. If $E_j=\emptyset$, then $\int_{E_j}|f(x)|^p\,d\mu=0$. If $E_j\ne\emptyset$, then there are balls $B_{j,m}$ $(m=1,\dots,m_j)$ such that

$$B_{j,m}=B(b_{j,m},s_j),\quad b_{j,m}\in E_j,\quad s_j=2^{j-1}r_0/(20K_1^3),$$
$$E_j\subset\bigcup_{m=1}^{m_j}B(b_{j,m},4K_1s_j),\quad B_{j,m}\cap B_{j,n}=\emptyset\quad(m\ne n)$$

(see [3], pp. 68–69). We note that $\phi_i(b_{j,m},4K_1s_j)$ $(i=1,2)$ are comparable to $\phi_i(x_0, 2^j r_0)$ $(i = 1, 2)$, respectively, and

$$\sum_{m=1}^{m_j} \mu(B(b_{j,m}, 4K_1 s_j)) \le C \sum_{m=1}^{m_j} \mu(B(b_{j,m}, s_j)) \le C' \mu(B(x_0, 2^j r_0)).$$

Since $B(b_{j,m}, 4K_1 s_j)$ is in Case 1,

$$\int_{E_j} |f(x)|^p \, d\mu \le \sum_{m=1}^{m_j} \int_{B(b_{j,m}, 4K_1 s_j)} |f(x)|^p \, d\mu$$

$$\le C_p \sum_{m=1}^{m_j} \left(\frac{\phi_2(b_{j,m}, 4K_1 s_j)}{\phi_1(b_{j,m}, 4K_1 s_j)} \right)^p \mu(B(b_{j,m}, 4K_1 s_j))(\|f\|_{\mathrm{bmo}_{\phi_3}} + \|f\|_{L_{\phi_2/\phi_1}})^p$$

$$\le C_p' \left(\frac{\phi_2(x_0, 2^j r_0)}{\phi_1(x_0, 2^j r_0)} \right)^p \mu(B(x_0, 2^j r_0))(\|f\|_{\mathrm{bmo}_{\phi_3}} + \|f\|_{L_{\phi_2/\phi_1}})^p,$$

for $j = 1, \ldots, k$. Since $B(x_0, r_0)$ is in Case 2,

$$\int_{B(x_0, r_0)} |f(x)|^p \, d\mu \le C_p \left(\frac{\phi_2(x_0, r_0)}{\phi_1(x_0, r_0)} \right)^p \mu(B(x_0, r_0))(\|f\|_{\mathrm{bmo}_{\phi_3}} + \|f\|_{L_{\phi_2/\phi_1}})^p.$$

It follows from (1.9) that

$$\sum_{j=0}^{k} \left(\frac{\phi_2(x_0, 2^j r_0)}{\phi_1(x_0, 2^j r_0)} \right)^p \mu(B(x_0, 2^j r_0)) \le C \sum_{j=0}^{k} \int_{2^j r_0}^{2^{j+1} r_0} \left(\frac{\phi_2(x_0, t)}{\phi_1(x_0, t)} \right)^p \frac{\mu(B(x_0, t))}{t} \, dt$$

$$\le A_6 C \left(\frac{\phi_2(x_0, 2^{k+1} r_0)}{\phi_1(x_0, 2^{k+1} r_0)} \right)^p \mu(B(x_0, 2^{k+1} r_0))$$

$$\le C' \left(\frac{\phi_2(x_0, r)}{\phi_1(x_0, r)} \right)^p \mu(B(x_0, r)).$$

Therefore we have (3.11).

Case 4: $r > d(x_0, a)/(5K_1^2)$. In this case, $B(a, r)$ is included in $B(x_0, 6K_1^3 r)$ which is in Case 2 or in Case 3. Since $\phi_i(x_0, 6K_1^3 r)$ $(i = 1, 2)$ and $\mu(B(x_0, 6K_1^3 r))$ are comparable to $\phi_i(a, r)$ $(i = 1, 2)$ and $\mu(B(a, r))$, respectively, we have (3.11). ■

4. Propositions.

We now show some propositions.

PROPOSITION 4.1. *Let $1 \le p_1, p_2 < \infty$. Suppose that ϕ_1 and p_1 satisfy (1.4)–(1.7), ϕ_2 satisfies (1.4) and (1.7), and $(\Phi_2^* + \Phi_2^{**})/\phi_2 \le C(\Phi_1^* + \Phi_1^{**})/\phi_1$. If $\mu(X) = \infty$, then assume that there are constants $r_0 \ge 0$ and $A_6' > 0$ such that*

$$(4.1)\quad \int_{r_0}^{r} \left(\frac{\Phi_2^*(x_0, t)}{\Phi_1^*(x_0, t)} \right)^{p_2} \frac{\mu(B(x_0, t))}{t} \, dt \le A_6' \left(\frac{\phi_2(x_0, r)}{\phi_1(x_0, r)} \right)^{p_2} \mu(B(x_0, r)), \quad r > r_0.$$

Then

$$\mathrm{PWM}(\mathrm{bmo}_{\phi_1, p_1}(X), \mathrm{bmo}_{\phi_2, p_2}(X)) \subset L_{\phi_2/\phi_1, p_2}(X),$$
$$\|g\|_{L_{\phi_2/\phi_1, p_2}} \le C\|g\|_{\mathrm{Op}},$$

where $\|g\|_{\mathrm{Op}}$ is the operator norm of $g \in \mathrm{PWM}(\mathrm{bmo}_{\phi_1, p_1}(X), \mathrm{bmo}_{\phi_2, p_2}(X))$.

COROLLARY 4.2. *Let $1 \le p_1, p_2 < \infty$. Suppose that ϕ_1 and p_1 satisfy (1.4)–(1.7) and ϕ_2 satisfies (1.4). Let*

$$X_1 = \left\{ x \in X : \int_r^1 \frac{\phi_2(x, t)}{t} \, dt \Big/ \int_r^1 \frac{\phi_1(x, t)}{t} \, dt \to 0 \text{ as } r \to 0 \right\}.$$

If $g \in \mathrm{PWM}(\mathrm{bmo}_{\phi_1, p_1}(X), \mathrm{bmo}_{\phi_2, p_2}(X))$, then $g(x) = 0$ a.e. $x \in X_1$.

PROPOSITION 4.3. *Let $1 < p_1, p_2 < \infty$ and $p_3 = p_1 p_2/(p_1 + p_2) \ge 1$. Suppose that ϕ_1 and p_1 satisfy (1.4)–(1.7), ϕ_2 satisfies (1.4) and (1.7), and $(\Phi_2^* + \Phi_2^{**})/\phi_2 \le C(\Phi_1^* + \Phi_1^{**})/\phi_1$. If $\mu(X) = \infty$, then assume that (4.1) holds. Let $\phi_3 = \phi_2/(\Phi_1^* + \Phi_1^{**})$. Then*

$$\mathrm{PWM}(\mathrm{bmo}_{\phi_1, p_1}(X), \mathrm{bmo}_{\phi_2, p_2}(X)) \subset \mathrm{bmo}_{\phi_3, p_3}(X) \cap L_{\phi_2/\phi_1, p_2}(X),$$
$$\|g\|_{\mathrm{BMO}_{\phi_3, p_3}} + \|g\|_{L_{\phi_2/\phi_1, p_2}} \le C\|g\|_{\mathrm{Op}},$$

where $\|g\|_{\mathrm{Op}}$ is the operator norm of $g \in \mathrm{PWM}(\mathrm{bmo}_{\phi_1, p_1}(X), \mathrm{bmo}_{\phi_2, p_2}(X))$.

PROPOSITION 4.4. *Suppose that ϕ_1 and ϕ_2 satisfy (1.4). Let $\phi_3 = \phi_2/(\Phi_1^* + \Phi_1^{**})$. If $1 \le p_2 < p_1 < \infty$ and $p_4 \ge p_1 p_2/(p_1 - p_2)$, then*

$$(4.2)\quad \mathrm{PWM}(\mathrm{bmo}_{\phi_1, p_1}(X), \mathrm{bmo}_{\phi_2, p_2}(X)) \supset \mathrm{bmo}_{\phi_3, p_2}(X) \cap L_{\phi_2/\phi_1, p_4}(X),$$

$$(4.3)\quad \|g\|_{\mathrm{Op}} \le C(\|g\|_{\mathrm{BMO}_{\phi_3, p_2}} + \|g\|_{L_{\phi_2/\phi_1, p_4}}),$$

where $\|g\|_{\mathrm{Op}}$ is the operator norm of $g \in \mathrm{PWM}(\mathrm{bmo}_{\phi_1, p_1}(X), \mathrm{bmo}_{\phi_2, p_2}(X))$.

PROPOSITION 4.5. *Suppose that ϕ satisfies (1.4). Let $\psi = \phi/(\Phi^* + \Phi^{**})$. If $1 \le p_2 \le p_1 < \infty$, then*

$$(4.4)\quad \mathrm{PWM}(\mathrm{bmo}_{\phi, p_1}(X), \mathrm{bmo}_{\phi, p_2}(X)) \supset \mathrm{bmo}_{\psi, p_2}(X) \cap L^\infty(X),$$

$$(4.5)\quad \|g\|_{\mathrm{Op}} \le C(\|g\|_{\mathrm{BMO}_{\psi, p_2}} + \|g\|_{L^\infty}),$$

where $\|g\|_{\mathrm{Op}}$ is the operator norm of $g \in \mathrm{PWM}(\mathrm{bmo}_{\phi, p_1}(X), \mathrm{bmo}_{\phi, p_2}(X))$.

Proof of Prop. 4.1. Let $g \in \mathrm{PWM}(\mathrm{bmo}_{\phi_1,p_1}(X), \mathrm{bmo}_{\phi_2,p_2}(X))$. Then g is a bounded operator. We show that, for any $a \in X$ and for any $r > 0$,

$$(4.6)\quad \left(\frac{1}{\mu(B(a,r/(2K_1)^2))}\int_{B(a,r/(2K_1)^2)} |g(x)|^{p_2}\,d\mu\right)^{1/p_2} \le C\|g\|_{\mathrm{Op}}\frac{\phi_2(a,r/(2K_1)^2)}{\phi_1(a,r/(2K_1)^2)}.$$

For any $f \in \mathrm{bmo}_{\phi_1,p_1}(X)$, fg is in $\mathrm{bmo}_{\phi_2,p_2}(X)$. From Lemma 3.5 it follows that, for any ball $B(a,r)$,

$$(4.7)\quad \left(\frac{1}{\mu(B(a,r))}\int_{B(a,r)} |f(x)g(x)|^{p_2}\,d\mu\right)^{1/p_2} \le C\|fg\|_{\mathrm{bmo}_{\phi_2,p_2}}(\Phi_2^*(a,r)+\Phi_2^{**}(a,r)) \le C\|f\|_{\mathrm{bmo}_{\phi_1,p_1}}\|g\|_{\mathrm{Op}}(\Phi_2^*(a,r)+\Phi_2^{**}(a,r)).$$

Applying (4.7) with f defined by (3.9) and using Lemma 3.9, we have

$$(4.8)\quad \Phi_1^{**}(a,r)\left(\frac{1}{\mu(B(a,r))}\int_{B(a,r/(2K_1))} |g(x)|^{p_2}\,d\mu\right)^{1/p_2} \le C\|g\|_{\mathrm{Op}}(\Phi_2^*(a,r)+\Phi_2^{**}(a,r)).$$

If $\mu(X) < \infty$, then, applying (4.7) with f defined by (3.6) and using Lemma 3.8, we have

$$(4.9)\quad \Phi_1^{*}(a,r)\left(\frac{1}{\mu(B(a,r))}\int_{B(a,r/(2K_1)^2)} |g(x)|^{p_2}\,d\mu\right)^{1/p_2} \le C\|g\|_{\mathrm{Op}}(\Phi_2^*(a,r)+\Phi_2^{**}(a,r)).$$

By (4.8), (4.9) and the inequality $(\Phi_2^*+\Phi_2^{**})/\phi_2 \le C(\Phi_1^*+\Phi_1^{**})/\phi_1$, we have

$$(4.10)\quad \left(\frac{1}{\mu(B(a,r))}\int_{B(a,r/(2K_1)^2)} |g(x)|^{p_2}\,d\mu\right)^{1/p_2} \le C\|g\|_{\mathrm{Op}}\frac{\phi_2(a,r)}{\phi_1(a,r)}.$$

Since $\phi_i(a,r)$ $(i=1,2)$ and $\mu(B(a,r))$ are comparable to $\phi_i(a,r/(2K_1)^2)$ $(i=1,2)$ and $\mu(B(a,r/(2K_1)^2))$, respectively, we have (4.6).

If $\mu(X) = \infty$, then, applying (4.7) with f defined by (3.6) and using Lemma 3.7(i), (ii) with $r_1 = \max(2,r_0)$, we have (4.9) and hence (4.6) for $r < 2K_1 d(x_0,a)$ and for $2K_1 d(x_0,a) \le r < 2\max(2,r_0)$. For $2K_1 d(x_0,a) \le r$ and $2^k\max(2,r_0) \le r < 2^{k+1}\max(2,r_0)$ $(k=1,2,\ldots)$, let E_j be defined by (3.7) and $s_j = 2^{-j}r$. Since $s_j \ge 2$, $\Phi_i^{**}(x_0,s_j) = 0$ $(i=1,2;\ j=0,1,\ldots,k)$. Using Lemma 3.7(iii), we have, for $j=0,1,\ldots,k-1$,

$$(4.11)\quad \Phi_1^*(x_0,s_j)\left(\frac{1}{\mu(B(x_0,s_j))}\int_{E_j} |g(x)|^{p_2}\,d\mu\right)^{1/p_2} \le C\|g\|_{\mathrm{Op}}\Phi_2^*(x_0,s_j).$$

Since $s_k < 2\max(2,r_0)$, $B(x_0,s_k)$ is in case (ii) of Lemma 3.7. Thus

$$(4.12)\quad \Phi_1^*(x_0,s_k)\left(\frac{1}{\mu(B(x_0,s_k))}\int_{B(x_0,s_k)} |g(x)|^{p_2}\,d\mu\right)^{1/p_2} \le C\|g\|_{\mathrm{Op}}\Phi_2^*(x_0,s_k).$$

By (3.8), (4.11) and (4.12), we have

$$\begin{aligned}\int_{B(a,r/(2K_1))} |g(x)|^{p_2}\,d\mu &\le C\|g\|_{\mathrm{Op}}^{p_2}\sum_{j=0}^{k}\left(\frac{\Phi_2^*(x_0,s_j)}{\Phi_1^*(x_0,s_j)}\right)^{p_2}\mu(B(x_0,s_j))\\ &\le C'\|g\|_{\mathrm{Op}}^{p_2}\sum_{j=0}^{k}\int_{s_j}^{s_{j-1}}\left(\frac{\Phi_2^*(x_0,t)}{\Phi_1^*(x_0,t)}\right)^{p_2}\frac{\mu(B(x_0,t))}{t}\,d\mu\\ &\le C'\|g\|_{\mathrm{Op}}^{p_2}\int_{r_0}^{2r}\left(\frac{\Phi_2^*(x_0,t)}{\Phi_1^*(x_0,t)}\right)^{p_2}\frac{\mu(B(x_0,t))}{t}\,d\mu\\ &\le C'A_6'\|g\|_{\mathrm{Op}}^{p_2}\left(\frac{\phi_2(x_0,2r)}{\phi_1(x_0,2r)}\right)^{p_2}\mu(B(x_0,2r)).\end{aligned}$$

Since $\phi_i(x_0,2r)$ $(i=1,2)$ and $\mu(B(x_0,2r))$ are comparable to $\phi_i(a,r/(2K_1)^2)$ $(i=1,2)$ and $\mu(B(a,r/(2K_1)^2))$, respectively, we have (4.6).

Proof of Corollary 4.2. Just as we proved (4.8) and (4.9), we get

$$\left(\frac{1}{\mu(B(a,r/(2K_1)^2))}\int_{B(a,r/(2K_1)^2)} |g(x)|^{p_2}\,d\mu\right)^{1/p_2} \le C\|g\|_{\mathrm{Op}}\frac{\Phi_2^*(a,r)+\Phi_2^{**}(a,r)}{\Phi_1^*(a,r)+\Phi_1^{**}(a,r)},$$

for small r. If $a \in X_1$ then $\int_r^1 \phi_1(a,t)t^{-1}\,dt \to \infty$ as $r \to 0$. Therefore

$$\lim_{r\to 0}\frac{\Phi_2^*(a,r)+\Phi_2^{**}(a,r)}{\Phi_1^*(a,r)+\Phi_1^{**}(a,r)} = 0.$$

Proof of Prop. 4.3. Let $g \in \mathrm{PWM}(\mathrm{bmo}_{\phi_1,p_1}(X), \mathrm{bmo}_{\phi_2,p_2}(X))$. By Proposition 4.1, g is in $L_{\phi_2/\phi_1,p_2}(X)$ and $\|g\|_{L_{\phi_2/\phi_1,p_2}} \le C\|g\|_{\mathrm{Op}}$. By

Lemma 3.3, for any $B(a,r)$, we have a function f such that

$$\|f\|_{\mathrm{bmo}_{\phi_1,p_1}} \le C_1, \tag{4.13}$$

$$f_{B(a,r)} \ge C_2(\Phi_1^*(a,r)+\Phi_1^{**}(a,r)). \tag{4.14}$$

Since $1/p_1+1/p_2=1/p_3$, it follows from Lemma 3.10 that

$$\begin{aligned}(4.15)\quad |f_{B(a,r)}|\mathrm{MO}_{\phi_2,p_3}(g,B(a,r)) &\le 2\|f\|_{\mathrm{BMO}_{\phi_1,p_1}}\|g\|_{L_{\phi_2/\phi_1,p_2}}+\|fg\|_{\mathrm{BMO}_{\phi_2,p_3}}\\ &\le 2\|f\|_{\mathrm{BMO}_{\phi_1,p_1}}\|g\|_{L_{\phi_2/\phi_1,p_2}}+\|fg\|_{\mathrm{BMO}_{\phi_2,p_2}}\\ &\le C\|f\|_{\mathrm{bmo}_{\phi_1,p_1}}\|g\|_{\mathrm{Op}}.\end{aligned}$$

By (4.13)–(4.15), g is in $\mathrm{bmo}_{\phi_3,p_3}(X)$ and $\|g\|_{\mathrm{BMO}_{\phi_3,p_3}}\le C\|g\|_{\mathrm{Op}}$.

Proof of Propositions 4.4 and 4.5. Let $g\in \mathrm{bmo}_{\phi_3,p_2}(X)\cap L_{\phi_2/\phi_1,p_4}(X)$. By Lemmas 3.10 and 3.2, for any $f\in \mathrm{bmo}_{\phi_1,p_1}(X)$, we have

$$\begin{aligned}&\|fg\|_{\mathrm{BMO}_{\phi_2,p_2}}\\ &\quad\le \sup_{B(a,r)}|f_{B(a,r)}|\mathrm{MO}_{\phi_2,p_2}(g,B(a,r))+2\|f\|_{\mathrm{BMO}_{\phi_1,p_1}}\|g\|_{L_{\phi_2/\phi_1,p_4}}\\ &\quad\le C\|f\|_{\mathrm{bmo}_{\phi_1,p_1}}\sup_{B(a,r)}(\Phi_1^*(a,r)+\Phi_1^{**}(a,r))\mathrm{MO}_{\phi_2,p_2}(g,B(a,r))\\ &\qquad+2\|f\|_{\mathrm{BMO}_{\phi_1,p_1}}\|g\|_{L_{\phi_2/\phi_1,p_4}}\\ &\quad\le C\|f\|_{\mathrm{bmo}_{\phi_1,p_1}}\|g\|_{\mathrm{BMO}_{\phi_3,p_2}}+2\|f\|_{\mathrm{BMO}_{\phi_1,p_1}}\|g\|_{L_{\phi_2/\phi_1,p_4}}\\ &\quad\le C'\|f\|_{\mathrm{bmo}_{\phi_1,p_1}}(\|g\|_{\mathrm{BMO}_{\phi_3,p_2}}+\|g\|_{L_{\phi_2/\phi_1,p_4}}).\end{aligned}$$

We also note that

$$\begin{aligned}|(fg)_{B(x_0,1)}| &\le \Big(\frac{1}{\mu(B(x_0,1))}\int_{B(x_0,1)}|f(x)|^{p_1}\,d\mu\Big)^{1/p_1}\\ &\quad\times\Big(\frac{1}{\mu(B(x_0,1))}\int_{B(x_0,1)}|g(x)|^{p_4}\,d\mu\Big)^{1/p_4}\\ &\le(\mathrm{MO}_{p_1}(f,B(x_0,1))+|f_{B(x_0,1)}|)\Big(\frac{\phi_2(x_0,1)}{\phi_1(x_0,1)}\|g\|_{L_{\phi_2/\phi_1,p_4}}\Big)\\ &\le C\|f\|_{\mathrm{bmo}_{\phi_1,p_1}}\|g\|_{L_{\phi_2/\phi_1,p_4}}.\end{aligned}$$

Therefore we have (4.2) and (4.3).

In the same way, by Lemmas 3.4 and 3.2, we have (4.4) and (4.5).

5. Proofs of the theorems and the corollaries

Proof of Theorem 1.1. First we note that ϕ_1 satisfies (1.6) for any p_1 $(1\le p_1<\infty)$ (see Lemma 5.3 of [15]). If $\mu(X)=\infty$, then we may assume $r_0\ge 1$ in (1.9). For $t\ge 1$, $\Phi_1^*(x_0,t)+\Phi_1^{**}(x_0,t)\le 2\Phi_1^*(x_0,t)$. By (1.9) with $p=1+\varepsilon$ and by the inequality $(\Phi_2^*+\Phi_2^{**})/\phi_2\le C(\Phi_1^*+\Phi_1^{**})/\phi_1$, we have (4.1) with $p_2=1+\varepsilon$. From Proposition 4.3 it follows that

$$\mathrm{PWM}(\mathrm{bmo}_{\phi_1,(1+\varepsilon)/\varepsilon}(X),\mathrm{bmo}_{\phi_2,1+\varepsilon}(X))\subset \mathrm{bmo}_{\phi_3}(X)\cap L_{\phi_2/\phi_1,1+\varepsilon}(X),$$
$$\|g\|_{\mathrm{BMO}_{\phi_3}}+\|g\|_{L_{\phi_2/\phi_1,1+\varepsilon}}\le C\|g\|_{\mathrm{Op}},$$

where $\|g\|_{\mathrm{Op}}$ is the operator norm of $g\in\mathrm{PWM}(\mathrm{bmo}_{\phi_1,(1+\varepsilon)/\varepsilon}(X),\mathrm{bmo}_{\phi_2,1+\varepsilon}(X))$. From Proposition 4.4 it follows that

$$\mathrm{PWM}(\mathrm{bmo}_{\phi_1,(1+\varepsilon)/\varepsilon}(X),\mathrm{bmo}_{\phi_2}(X))\supset \mathrm{bmo}_{\phi_3}(X)\cap L_{\phi_2/\phi_1,1+\varepsilon}(X),$$
$$\|g\|_{\mathrm{BMO}_{\phi_3}}+\|g\|_{L_{\phi_2/\phi_1,1+\varepsilon}}\ge C\|g\|_{\mathrm{Op}},$$

where $\|g\|_{\mathrm{Op}}$ is the operator norm of $g\in\mathrm{PWM}(\mathrm{bmo}_{\phi_1,(1+\varepsilon)/\varepsilon}(X),\mathrm{bmo}_{\phi_2}(X))$. By John–Nirenberg's inequality and Hölder's inequality, we have

$$\mathrm{bmo}_{\phi_1,(1+\varepsilon)/\varepsilon}(X)=\mathrm{bmo}_{\phi_1}(X),\qquad \mathrm{bmo}_{\phi_2,1+\varepsilon}(X)=\mathrm{bmo}_{\phi_2}(X).$$

Moreover, the operator norms of g from $\mathrm{bmo}_{\phi_1,(1+\varepsilon)/\varepsilon}(X)$ to $\mathrm{bmo}_{\phi_2,1+\varepsilon}(X)$, from $\mathrm{bmo}_{\phi_1,(1+\varepsilon)/\varepsilon}(X)$ to $\mathrm{bmo}_{\phi_2}(X)$, and from $\mathrm{bmo}_{\phi_1}(X)$ to $\mathrm{bmo}_{\phi_2}(X)$ are comparable. From Lemma 3.12 it follows that

$$\begin{aligned}\mathrm{bmo}_{\phi_3}(X)\cap L_{\phi_2/\phi_1,1+\varepsilon}(X)&=\mathrm{bmo}_{\phi_3}(X)\cap L_{\phi_2/\phi_1}(X),\\ \|g\|_{\mathrm{BMO}_{\phi_3}}+\|g\|_{L_{\phi_2/\phi_1}}&\le\|g\|_{\mathrm{BMO}_{\phi_3}}+\|g\|_{L_{\phi_2/\phi_1,1+\varepsilon}}\\ &\le C(\|g\|_{\mathrm{BMO}_{\phi_3}}+\|g\|_{L_{\phi_2/\phi_1}}).\end{aligned}$$

Therefore we have Theorem 1.1.

Proof of Theorem 1.3. Since $\int_s^{2s}\phi_i(a,t)t^{-1}\,dt$ $(i=1,2)$ are comparable to $\phi_i(a,s)$ $(i=1,2)$, respectively,

$$\int_s^{2s}\frac{\phi_2(a,t)}{t}\,dt\Big/\int_s^{2s}\frac{\phi_1(a,t)}{t}\,dt\le C\frac{\phi_2(a,s)}{\phi_1(a,s)}\le C'\frac{\phi_2(a,r)}{\phi_1(a,r)},\qquad r\le s.$$

Thus, for $R_0\le 2^kr<2R_0$,

$$\begin{aligned}(5.1)\quad \Phi_2(a,r)/\Phi_1(a,r)&\\ &\le C''\Big(\sum_{j=0}^k\int_{2^jr}^{2^{j+1}r}\frac{\phi_2(a,t)}{t}\,dt\Big)\Big/\Big(\sum_{j=0}^k\int_{2^jr}^{2^{j+1}r}\frac{\phi_1(a,t)}{t}\,dt\Big)\\ &\le C'C''\frac{\phi_2(a,r)}{\phi_1(a,r)}.\end{aligned}$$

We also note that ϕ_3 satisfies (1.8) and $\phi_3 \le C\phi_2/\phi_1$. Therefore, using Propositions 4.3, 4.4 and Lemma 3.11, we have Theorem 1.3.

Proof of Theorem 1.6. By Propositions 4.3 and 4.4, we have

$$\begin{aligned}\mathrm{bmo}_{\phi_3,p_2}(X) \cap L_{\phi_2/\phi_1,p_1p_2/(p_1-p_2)}(X) &\subset \mathrm{PWM}(\mathrm{bmo}_{\phi_1,p_1}(X), \mathrm{bmo}_{\phi_2,p_2}(X))\\ &\subset \mathrm{bmo}_{\phi_3,p_1p_2/(p_1+p_2)}(X) \cap L_{\phi_2/\phi_1,p_2}(X),\end{aligned}$$

$$\begin{aligned}C_1(\|g\|_{\mathrm{BMO}_{\phi_3,p_1p_2/(p_1+p_2)}} + \|g\|_{L_{\phi_2/\phi_1,p_2}}) \\ \le \|g\|_{\mathrm{Op}} \le C_2(\|g\|_{\mathrm{BMO}_{\phi_3,p_2}} + \|g\|_{L_{\phi_2/\phi_1,p_1p_2/(p_1-p_2)}}).\end{aligned}$$

We note that $\phi_3 \le C\phi_2/\phi_1$. By Lemma 3.11, we have Theorem 1.6.

Proof of Theorem 1.8. Let $g \in \mathrm{PWM}(\mathrm{bmo}_{\phi,p_1}(X), \mathrm{bmo}_{\phi,p_2}(X))$. By Proposition 4.1, $g \in L_{\phi/\phi,p_2}(X) = L^\infty(X)$ and $\|g\|_{L^\infty} \le C\|g\|_{\mathrm{Op}}$. From Lemma 3.3 it follows that, for any ball $B(a,r)$, there is a function $f \in \mathrm{bmo}_{\phi,p_1}(X)$ such that

$$\|f\|_{\mathrm{bmo}_{\phi,p_1}} \le C_1, \tag{5.2}$$

$$f_{B(a,r)} \ge C_2(\Phi^*(a,r) + \Phi^{**}(a,r)). \tag{5.3}$$

Since f is in $\mathrm{bmo}_{\phi,p_2}(X)$, from Lemma 3.4 it follows that

$$\begin{aligned}|f_{B(a,r)}|\mathrm{MO}_{\phi,p_2}(g, B(a,r)) &\le 2\|f\|_{\mathrm{BMO}_{\phi,p_2}}\|g\|_{L^\infty} + \|fg\|_{\mathrm{BMO}_{\phi,p_2}}\\ &\le C\|f\|_{\mathrm{bmo}_{\phi,p_1}}\|g\|_{\mathrm{Op}}.\end{aligned} \tag{5.4}$$

By (5.2)–(5.4), g is in $\mathrm{bmo}_{\psi,p_2}(X)$ and $\|g\|_{\mathrm{BMO}_{\psi,p_2}} \le C\|g\|_{\mathrm{Op}}$. Conversely, by Proposition 4.5, we have

$$\mathrm{PWM}(\mathrm{bmo}_{\phi,p_1}(X), \mathrm{bmo}_{\phi,p_2}(X)) \supset \mathrm{bmo}_{\psi,p_2}(X) \cap L^\infty(X),$$
$$\|g\|_{\mathrm{Op}} \le C(\|g\|_{\mathrm{bmo}_{\psi,p_2}} + \|g\|_{L^\infty}).$$

Proof of Theorem 1.10. Let $g \in \mathrm{PWM}(\mathrm{bmo}_{\phi_1,p_1}(X), \mathrm{bmo}_{\phi_2,p_2}(X))$ and $a \in X^* \cap X_0$. We show

$$\lim_{r\to 0} \frac{1}{\mu(B(a,r))} \int_{B(a,r)} |g(x)|\,d\mu = 0. \tag{5.5}$$

If $\int_r^1 \phi_1(a,t)t^{-1}\,dt \to \infty$ as $r \to 0$, then

$$\lim_{r\to 0} \int_r^1 \phi_2(a,t)t^{-1}\,dt \Big/ \int_r^1 \phi_1(a,t)t^{-1}\,dt = \lim_{r\to 0} \frac{\phi_2(a,r)}{\phi_1(a,r)} = 0.$$

By Corollary 4.2, we have (5.5).

If $\int_0^1 \phi_1(a,t)t^{-1}\,dt < \infty$, then $\int_0^1 \phi_2(a,t)t^{-1}\,dt < \infty$. Let

$$f(x) = \int_0^{d(a,x)} \frac{\phi_1(a,t)}{t}\,dt.$$

Then f is in $\mathrm{bmo}_{\phi_1,p_1}(X)$. From (3.4) it follows that, for $0 < s < r$,

$$\big||fg|_{B(a,s)} - |fg|_{B(a,r)}\big| \le C\|fg\|_{\mathrm{bmo}_{\phi_2,p_2}} \int_s^{2r} \frac{\phi_2(a,t)}{t}\,dt.$$

Letting $s \to 0$, we have $|fg|_{B(a,s)} \to |f(a)g(a)| = 0$ and

$$|fg|_{B(a,r)} \le C' \int_0^{2r} \frac{\phi_2(a,t)}{t}\,dt \le C'A_1 \int_0^r \frac{\phi_2(a,t)}{t}\,dt.$$

Since

$$f(x) \ge \int_0^{r/2} \frac{\phi_1(a,t)}{t}\,dt \ge A_1{}^{-1} \int_0^r \frac{\phi_1(a,t)}{t}\,dt \quad \text{for } x \in B(a,r) \setminus B(a,r/2),$$

we have

$$\int_{B(a,r)\setminus B(a,r/2)} |g(x)|\,d\mu \le C''\mu(B(a,r)) \int_0^r \frac{\phi_2(a,t)}{t}\,dt \Big/ \int_0^r \frac{\phi_1(a,t)}{t}\,dt,$$

and

$$\begin{aligned}\int_{B(a,r)} |g(x)|\,d\mu &= \sum_{j=0}^\infty \int_{B(a,2^{-j}r)\setminus B(a,2^{-j-1}r)} |g(x)|\,d\mu\\ &\le C'' \sum_{j=0}^\infty \mu(B(a,2^{-j}r)) \int_0^{2^{-j}r} \frac{\phi_2(a,t)}{t}\,dt \Big/ \int_0^{2^{-j}r} \frac{\phi_1(a,t)}{t}\,dt.\end{aligned}$$

We note that $a \in X^*$ if and only if there is a constant $C_a > 0$ such that

$$\sum_{j=0}^\infty \mu(B(a,2^{-j}r)) \le C_a\mu(B(a,r)), \quad 0 < r < r_a$$

(see Lemma 5.3 of [15]). By the equality

$$\lim_{r\to 0} \int_0^r \frac{\phi_2(a,t)}{t}\,dt \Big/ \int_0^r \frac{\phi_1(a,t)}{t}\,dt = \lim_{r\to 0} \frac{\phi_2(a,r)}{\phi_1(a,r)} = 0,$$

we have (5.5).

Proofs of Corollaries. If $\Phi_3^* + \Phi_3^{**} \le C\phi_2/\phi_1$ or if $\Phi_3 \le C\phi_2/\phi_1$, then, by Lemma 3.5 or Corollary 3.6, it follows that $\mathrm{bmo}_{\phi_3}(X) \subset L_{\phi_2/\phi_1}(X)$ and $\|g\|_{L_{\phi_2/\phi_1}} \le C\|g\|_{\mathrm{bmo}_{\phi_3}}$. Therefore we have Corollaries 1.2, 1.4 and 1.7.

Under the assumptions of Theorem 1.3, if $\Phi_1 \le C$, then ϕ_3 is comparable to ϕ_2. By (5.1), we have $\Phi_3 \le C\phi_2/\phi_1$. Therefore, Corollary 1.5 follows from Corollary 1.4.

Finally, Corollary 1.9 follows from Theorem 1.8.

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Spreading sequences in *JT*

by

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Abstract. We prove that a normalized non-weakly null basic sequence in the James tree space JT admits a subsequence which is equivalent to the summing basis for the James space J. Consequently, every normalized basic sequence admits a spreading subsequence which is either equivalent to the unit vector basis of l_2 or to the summing basis for J.

1. Introduction. We study subsequences of normalized basic sequences $\{x_i\}_{i=1}^{\infty}$ in the James tree space JT. Amemiya and Ito [1] proved that if $\{x_i\}_{i=1}^{\infty} \subset JT$ is weakly null then it has a subsequence which is equivalent to the unit vector basis of l_2.

We prove, following an idea of Hagler [7], that if $\{x_i\}_{i=1}^{\infty}$ is not weakly null then there is a subsequence equivalent to the summing basis for the James space J. In particular, this yields a classification of all the spreading models of JT, extending the work of Andrew [2] for the space J.

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We first introduce some necessary notation and recall the definitions of J and JT constructed by James in [8] and [9] respectively. Most of the material referring to these spaces used here can be found in [5].

DEFINITION 1. The *James space* J is the Banach space of real sequences $b = (b_l)_{l=1}^{\infty}$ with the norm

$$\|b\| = \sup\Big(\sum_{\nu=1}^{M}\Big(\sum_{l=n(\nu)}^{\kappa(\nu)} b_l\Big)^2\Big)^{1/2},$$

where the sup is taken over all finite collections $S_1, \ldots, S_M$ of disjoint intervals of natural numbers with $S_\nu = \{n(\nu), n(\nu)+1, \ldots, \kappa(\nu)\}$.

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