Pointwise multipliers on weighted BMO spaces

by

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Abstract. Let $E$ and $F$ be spaces of real- or complex-valued functions defined on a set $X$. A real- or complex-valued function $g$ defined on $X$ is called a pointwise multiplier from $E$ to $F$ if the pointwise product $fg$ belongs to $F$ for each $f \in E$. We denote by $\text{PWM}(E, F)$ the set of all pointwise multipliers from $E$ to $F$. Let $X$ be a space of homogeneous type in the sense of Coifman–Weiss. For $1 \leq p < \infty$ and for $\phi : X \times \mathbb{R}_+ \to \mathbb{R}_+$, we denote by $\text{bmo}_\phi(X)$ the set of all functions $f \in L^p_{\text{loc}}(X)$ such that

$$\sup_{a \in X, r > 0} \frac{1}{\phi(a, r)} \left( \frac{1}{\mu(B(a, r))} \int_{B(a, r)} |f(x) - f_{B(a, r)}|^p \, d\mu \right)^{1/p} < \infty,$$

where $B(a, r)$ is the ball centered at $a$ and of radius $r$, and $f_{B(a, r)}$ is the integral mean of $f$ on $B(a, r)$. Let $\text{bmo}_2(X) = \text{bmo}_{\phi_1}(X)$ and $\text{bmo}_1(X) = \text{bmo}_{\phi_1}(X)$. In this paper, we characterize $\text{PWM}(\text{bmo}_{\phi_1}, \text{p}_1(X), \text{bmo}_{\phi_2}, \text{p}_2(X))$. The following are examples of our results.

\begin{align*}
\text{PWM}(\text{bmo}_{\phi(1/r)}^{-\alpha}((T^n), \text{bmo}_{\phi(1/r)}^{-\beta}((T^n))) & = \text{bmo}_{\phi(1/r)}^{-\alpha - \beta - 1}(T^n), \quad 0 \leq \beta < \alpha < 1, \\
\text{PWM}(\text{bmo}_{\phi(1/r)}^{-\alpha - 1}(T^n), \text{bmo}(T^n)) & = \text{bmo}_{\phi(1/r) + 1}(T^n), \\
\text{PWM}(\text{bmo}(T^n), \text{bmo}_{\phi(1/r)}^{-\alpha - 1}(T^n)) & = \text{bmo}(T^n), \quad 1 < p < \infty.
\end{align*}

1. Introduction. Let $E$ and $F$ be spaces of real- or complex-valued functions defined on a set $X$. A real- or complex-valued function $g$ defined on $X$ is called a pointwise multiplier from $E$ to $F$ if the pointwise product $fg$ belongs to $F$ for each $f \in E$. We denote by $\text{PWM}(E, F)$ the set of all pointwise multipliers from $E$ to $F$.

For $L^p$-spaces on a $\sigma$-finite measure space $X$, it is known that

$$\text{PWM}(L^{p_1}(X), L^{p_2}(X)) = L^{p_3}(X), \quad 1/p_1 + 1/p_2 = 1/p_3.$$

The purpose of this paper is to characterize

$$\text{PWM}(\text{bmo}_{\phi_1}, p_1(X), \text{bmo}_{\phi_2}, p_2(X)).$$


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where \( \text{bmo}_{\phi_i, p_i}(X) \) \((i = 1, 2)\) are function spaces defined using the mean oscillation and weight functions \( \phi_i : X \times \mathbb{R}_+ \to \mathbb{R}_+ \) \((i = 1, 2)\).

Janson [6] characterized PWM \((\text{bmo}_o(T^n), \text{bmo}_o(T^n))\) on the \(n\)-dimensional torus \(T^n\) for a weight function \( \phi : \mathbb{R}_+ \to \mathbb{R}_+ \). His result was extended in [13–15] to the cases of the \(n\)-dimensional Euclidean space \(\mathbb{R}^n\) and spaces of homogeneous type. Bloom [2], Gotoh [5] and Yabuta [18] have also characterized pointwise multipliers from a weighted BMO space to itself. In this paper, we consider pointwise multipliers from \( \text{bmo}_{\phi_1, p_1}(X) \) to \( \text{bmo}_{\phi_2, p_2}(X) \).

Let \( X = (X, d, \mu) \) be a space of homogeneous type in the sense of Coifman–Weiss [3, 4], i.e., \( X \) is a topological space endowed with a Borel measure \( \mu \) and a quasi-distance \( d \) such that \( d(x, y) \geq 0, d(x, x) = 0 \) if and only if \( x = y \), \( d(x, y) = d(y, x) \), \( d(x, y) \leq K_1(d(x, z) + d(z, y)) \), \( x, y, z \in X \),

\[
d(x, y) \leq K_1(d(x, z) + d(z, y)), \quad x, y, z \in X,
\]

the balls \( B(x, r) = \{y \in X : d(x, y) < r\} \) centered at \( x \) and of radius \( r > 0 \) form a basis of open neighborhoods of the point \( x \), and

\[
0 < \mu(B(x, 2r)) \leq K_2 \mu(B(x, r)) < \infty, \quad x \in X, \ r > 0.
\]

We assume that there are constants \( a_0 \) \((0 < a_0 \leq 1)\) and \( K_3 \geq 1 \) such that

\[
d(x, z) - d(y, z) \leq K_3(d(x, z) + d(y, z))^1 - a_0 d(y, z)^a_0, \quad x, y, z \in X.
\]

If \( d \) is a distance, then (1.1) and (1.3) hold for \( K_1 = K_3 = a_0 = 1 \).

For a function \( f \in L^1_{\text{loc}}(X) \) and for a ball \( B \), let

\[
f_B = \frac{1}{\mu(B)} \int_B f(x) \, d\mu, \quad \text{MO}_p(f, B) = \frac{1}{\mu(B)} \int_B |f(x) - f_B| \, d\mu.
\]

For \( 1 \leq p < \infty \) and for \( \phi : X \times \mathbb{R}_+ \to \mathbb{R}_+ \), let

\[
\text{MO}_p(f, B(a, r)) = \left( \frac{1}{\mu(B(a, r))} \right)^{1/p} \int_{B(a, r)} |f(x) - f_{B(a, r)}|^p \, d\mu,
\]

\[
\text{MO}_{\phi, p}(f, B(a, r)) = \frac{1}{\phi(a, r)} \text{MO}_p(f, B(a, r)).
\]

We define

\[
\text{BMO}_{\phi, p}(X) = \{ f \in L^p_{\text{loc}}(X) : \sup_B \text{MO}_{\phi, p}(f, B) < \infty \},
\]

\[
\|f\|_{\text{BMO}_{\phi, p}} = \sup_B \text{MO}_{\phi, p}(f, B),
\]

\[
\|f\|_{\text{bmo}_{\phi, p}} = \|f\|_{\text{BMO}_{\phi, p}} + \|f_{B(x_0, 1)}\|_{\text{BMO}_{\phi, p}} \quad \text{for fixed } x_0 \in X.
\]

Let \( \text{bmo}_o(X) = \text{bmo}_{\phi, p}(X) \) for \( p = 1 \) and \( \text{bmo}_o(X) = \text{bmo}_{\phi}(X) \) for \( \phi \equiv 1 \). Then \( \text{bmo}_{\phi, p}(X) \) is a Banach space under the norm \( \|f\|_{\text{bmo}_{\phi, p}} \). The closed graph theorem shows that every pointwise multiplier from \( \text{bmo}_{\phi_1, p_1}(X) \) to \( \text{bmo}_{\phi_2, p_2}(X) \) is a bounded operator. For each ball \( B(x_1, r_1), \|f\|_{\text{BMO}_{\phi, p}} + \|f_{B(x_1, r_1)}\|_{\text{BMO}_{\phi, p}} \) is comparable to \( \|f\|_{\text{bmo}_{\phi, p}} \) (see (3.2)). Moreover, if \( \mu(X) < \infty \), then \( \|f\|_{\text{BMO}_{\phi, p}} + \|f_{B(x_0, 1)}\|_{\text{BMO}_{\phi, p}} \) is comparable to \( \|f\|_{\text{bmo}_{\phi, p}} \) (see (1.16)).

Usually, \( \text{bmo}_{\phi, p} \) is denoted by \( \text{BMO}_{\phi, p} \) and equipped with the seminorm \( \|f\|_{\text{BMO}_{\phi, p}} \). Then \( \text{BMO}_{\phi, p} \) modulo constants is a Banach space. But the pointwise multipliers are defined on function spaces or on such spaces modulo null-functions. To consider pointwise multipliers, the space \( \text{bmo}_{\phi, p} \) is therefore more suitable than \( \text{BMO}_{\phi, p} \).

For \( 1 \leq p < \infty \) and \( \phi : X \times \mathbb{R}_+ \to \mathbb{R}_+ \), let

\[
L_{\phi, p}(X) = \left\{ f \in L^p_{\text{loc}}(X) : \sup_{B(a, r)} \frac{1}{\phi(a, r)} \max_{a, r} \left( \frac{1}{\phi(a, r)} \right)^{1/p} \right\},
\]

\[
\|f\|_{L_{\phi, p}} = \sup_{B(a, r)} \frac{1}{\phi(a, r)} \max_{a, r} \left( \frac{1}{\phi(a, r)} \right)^{1/p},
\]

and \( L_{\phi, p}(X) = L_{\phi, p}(X) \) for \( p = 1 \). If \( \phi \equiv 1 \) then \( L_{\phi, p}(X) = L^p(X) \). Let \( X_0 = \{ x \in X : \phi(x, r) \to 0 \text{ as } r \to 0 \} \). If \( f \in L_{\phi, p}(X) \), then \( f(x) = 0 \) a.e. \( x \in X_0 \).

We shall consider the following conditions on \( \phi \):

\[
1 \leq \frac{1}{A_1} \leq \frac{\phi(a, s)}{\phi(a, r)} \leq A_1, \quad \frac{1}{2} \leq \frac{s}{r} \leq 2,
\]

\[
\frac{\phi(a, r)}{\phi(a, s)} \leq A_2 \frac{s}{s_0}, \quad 0 < s < r,
\]

\[
\frac{\mu(B(a, t))^{1/p} \frac{\phi(a, t)}{t}}{t} \leq A_3 \mu(B(a, r))^{1/p} \phi(a, r), \quad r > 0,
\]

\[
\frac{1}{A_4} \leq \frac{\phi(a, r)}{\phi(b, r)} \leq A_4, \quad d(a, b) \leq r,
\]

where \( A_1 > 0 \) \((i = 1, 2, 3, 4)\) are independent of \( r, s > 0, a, b \in X \).

If there is a constant \( A_5 > 0 \) such that

\[
\frac{\phi(a, r)}{\phi(b, r)} \leq A_5 \frac{b}{a} \quad \text{for } B(a, r) \subset B(b, s),
\]

then, for \( 1 < p < \infty \), we have

\[
\text{bmo}_{\phi, p}(X) = \text{bmo}_o(X) \quad \text{and} \quad \|f\|_{\text{bmo}_o} \leq \|f\|_{\text{bmo}_{\phi, p}} \leq C_p \|f\|_{\text{bmo}_o},
\]

by Hölder’s inequality and John–Nirenberg’s inequality.

For \( \mu(X) = \infty \), we shall consider the following condition. There are constants \( r_0 \geq 0 \) and \( A_6 > 0 \) such that

\[
\left\{ \int_{r_0}^{r} \frac{\phi_2(x_0, t)}{\phi_1(x_0, t)} \mu(B(x_0, t)) \frac{d(B(x_0, t))}{t} \right\}^{1/p} \leq A_6 \left( \int_{r_0}^{r} \frac{\phi_2(x_0, t)}{\phi_1(x_0, t)} \mu(B(x_0, r)) \right)^{1/p}, \quad r > r_0.
\]
The following are equivalent (see Lemma 5.2 of [15]):

\[ B(x_0, K_4 r) \setminus B(x_0, r) \neq \emptyset, \quad r > r_0, \text{ for some } K_4 > 1, \]

\[ \mu(B(x_0, r)) \leq \frac{1}{2} \mu(B(x_0, K_5 r)), \quad r > r_0, \text{ for some } K_5 > 1, \]

\[ \int_{r_0} B(x_0, t) \mu(B(x_0, r)) \leq K_6 \mu(B(x_0, r)), \quad r > r_0, \text{ for some } K_6 > 0, \]

where \( K_4, K_5 \) and \( K_6 \) are independent of \( r \). On the assumption that (1.10) holds, if there are constants \( A_7 > 0 \) and \( \varepsilon > 0 \) such that

\[ \left( \frac{\phi_2(x_0, r)}{\phi_1(x_0, r)} \right)^{p+\varepsilon} \mu(B(x_0, r)) \leq A_7 \left( \frac{\phi_2(x_0, s)}{\phi_1(x_0, s)} \right)^{p+\varepsilon} \mu(B(x_0, s)), \quad r_0 < r < s, \]

then (1.9) holds.

For \( \Phi \), we define

\[ \Phi^*(a, r) = \Phi^*_a(a, r) = \int_1^{\max(2, d(x_0, a), r)} \frac{\phi(x_0, t)}{t} \, dt, \]

\[ \Phi^{**}(a, r) = \Phi^{**}_a(a, r) = \int_r^{\max(2, d(x_0, a), r)} \frac{\phi(a, t)}{t} \, dt. \]

If \( \phi \) satisfies (1.4) and (1.7), then, for each \( x_1 \in X \), \( \Phi^*_a + \Phi^{**}_a \) is comparable to \( \Phi^*_a + \Phi^{**}_a \). For \( \Phi_i (i = 1, 2, 3) \), we define \( \Phi_i^* \) and \( \Phi_i^{**} \) by (1.14) and (1.15), respectively.

The letter \( C \) will always denote a constant, not necessarily the same one.

Our results are the following.

**Theorem 1.1.** Suppose that \( \Phi_1 \) satisfies (1.4)–(1.8) for some \( p_1 (1 \leq p_1 < \infty), \phi_2 \) satisfies (1.4), (1.7) and (1.8), and \( \phi^*_a + \phi^{**}_a \) \( \phi^{**}_a \). If \( \mu(X) = 0 \), then assume that (1.9) holds with \( p = 1 + \varepsilon \) for some \( \varepsilon > 0 \). Let \( \Phi_2 \) and \( \Phi_3 \) by (1.4) and (1.7), respectively.

The letter \( C \) will always denote a constant, not necessarily the same one.

Our results are the following.

**Theorem 1.1.** Suppose that \( \phi_1 \) satisfies (1.4)–(1.8) for some \( p_1 (1 \leq p_1 < \infty), \phi_2 \) satisfies (1.4), (1.7) and (1.8), and \( \Phi^*_a + \Phi^{**}_a \) \( \phi^{**}_a \). If \( \mu(X) = 0 \), then assume that (1.9) holds with \( p = 1 + \varepsilon \) for some \( \varepsilon > 0 \). Let \( \Phi_2 \) and \( \Phi_3 \) by (1.4) and (1.7), respectively.

**Corollary 1.2.** Under the assumptions of Theorem 1.1, if \( \Phi^*_a + \Phi^{**}_a \leq C \phi_2 / \phi_1 \), then

\[ \text{PWM}(\text{bmo}_{\phi_1}(X), \text{bmo}_{\phi_2}(X)) = \text{bmo}_{\phi_3}(X). \]

Moreover, the operator norm of \( g \in \text{PWM}(\text{bmo}_{\phi_1}(X), \text{bmo}_{\phi_2}(X)) \) is comparable to \( \|g\|_{\text{bmo}_{\phi_3}} + \|g\|_{L^{\phi_2} / \phi_1}. \)

**Corollary 1.3.** Under the assumptions of Theorem 1.1, if \( \Phi_2 \leq C \phi_2 / \phi_1 \), then

\[ \text{PWM}(\text{bmo}_{\phi_1}(X), \text{bmo}_{\phi_2}(X)) = \text{bmo}_{\phi_3}(X). \]

Moreover, the operator norm of \( g \in \text{PWM}(\text{bmo}_{\phi_1}(X), \text{bmo}_{\phi_2}(X)) \) is comparable to \( \|g\|_{\text{bmo}_{\phi_3}} + \|g\|_{L^{\phi_2} / \phi_1}. \)

**Theorem 1.6.** Let \( 1 < p_2 < p_1 < \infty \) and \( p_2, p_3 \geq p_1 + p_2 \). Suppose that \( \phi_1 \) and \( p_1 \) satisfy (1.4)–(1.7), \( \phi_2 \) satisfies (1.4) and (1.7), and \( \Phi^*_a + \Phi^{**}_a \) \( \phi^{**}_a \). If \( \mu(X) = 0 \), then assume that (1.9) holds with \( p = p_2 \). Let \( \phi_3 \) and \( \phi_4 \) by (1.4) and (1.7), respectively.

**Corollary 1.7.** Under the assumptions of Theorem 1.6, if \( \Phi^*_a + \Phi^{**}_a \leq C \phi_2 / \phi_1 \), then

\[ \text{PWM}(\text{bmo}_{\phi_2}(X), \text{bmo}_{\phi_4}(X)) = \text{bmo}_{\phi_3}(X). \]

Moreover, the operator norm of \( g \in \text{PWM}(\text{bmo}_{\phi_2}(X), \text{bmo}_{\phi_4}(X)) \) is comparable to \( \|g\|_{\text{bmo}_{\phi_3}} + \|g\|_{L^{\phi_2} / \phi_1}. \)
Moreover, the operator norm of \( g \in \operatorname{PWM}(\bmo_{\phi_1, p_1}(X), \bmo_{\phi_2, p_2}(X)) \) is comparable to \( \|g\|_{\bmo_{\phi_2}} \).

**Theorem 1.8.** Let \( 1 \leq p_2 \leq p_1 < \infty \). Suppose that \( \phi \) and \( p_1 \) satisfy (1.4)–(1.7). If \( \mu(X) = \infty \), then assume that (1.10) holds. Let \( \psi = \phi/(\delta^* + \delta^{**}) \). Then

\[
\operatorname{PWM}(\bmo_{\phi, p_1}(X), \bmo_{\phi, p_2}(X)) = \bmo_{\phi, p_2}(X) \cap L^\infty(X).
\]

Moreover, the operator norm of \( g \in \operatorname{PWM}(\bmo_{\phi, p_1}(X), \bmo_{\phi, p_2}(X)) \) is comparable to \( \|g\|_{\bmo_{\phi, p_2}} + \|g\|_{L^\infty} \).

**Corollary 1.9 (15).** Let \( 1 \leq p < \infty \). Suppose \( \phi \) and \( p \) satisfy (1.4)–(1.7). If \( \mu(X) = \infty \), then assume that (1.10) holds. Let \( \psi = \phi/(\delta^* + \delta^{**}) \). Then

\[
\operatorname{PWM}(\bmo_{\phi, p}(X), \bmo_{\phi, p}(X)) = \bmo_{\phi, p}(X) \cap L^\infty(X).
\]

Moreover, the operator norm of \( g \in \operatorname{PWM}(\bmo_{\phi, p}(X), \bmo_{\phi, p}(X)) \) is comparable to \( \|g\|_{\bmo_{\phi, p}} + \|g\|_{L^\infty} \).

**Theorem 1.10.** Let \( 1 \leq p_1, p_2 < \infty \). Suppose that \( \phi_1 \) and \( p_1 \) satisfy (1.4)–(1.7) and \( \phi_2 \) satisfies (1.4). Let \( X^* \) be the set of all points \( x \in X \) such that there are constants \( K_x > 1 \) and \( r_x > 0 \) such that

\[
\mu(B(x, r)) \leq \frac{1}{2} \mu(B(x, K_x r)), \quad 0 < r \leq r_x,
\]

and \( X_0 \) the set of all points \( x \in X \) such that

\[
\phi_2(x, r)/\phi_1(x, r) \to 0 \quad \text{as} \quad r \to 0.
\]

If \( g \in \operatorname{PWM}(\bmo_{\phi_1, p_1}(X), \bmo_{\phi_2, p_2}(X)) \), then \( g(x) = 0 \) a.e. \( x \in X^* \cap X_0 \).

In the next section we give some examples obtained from our results. We state some lemmas in Section 3 and propositions in Section 4 to prove the results in Section 5.

**2. Examples.** In this section, we assume that \((X, d, \mu)\) has the following property: there are constants \( C > 0 \) and \( \delta > 0 \) such that

\[
(2.1) \quad \frac{\mu(B(x, t))}{\mu(B(x, r))} \leq C \left( \frac{t}{r} \right)^\delta, \quad x \in X, \quad 0 < t < r.
\]

Then (1.11) follows from (2.1). If \( \phi(a, r)^{\alpha}/(\delta \leq \delta/p) \) satisfies (1.8), then \( \phi \) and \( p \) satisfy (1.6). For example, the Muckenhoupt \( A_p \)-weights on \( \mathbb{R}^n \) satisfy (2.1).

**2.1. The case \( \mu(X) < \infty \)**

**Example 2.1.** For \( 0 \leq \beta < \alpha < 1 \),

\[
\operatorname{PWM}(\bmo_{(\log(1/r))^{\alpha}}(X), \bmo_{(\log(1/r))^{-\beta}}(X)) = \bmo_{(\log(1/r))^{\alpha - \beta - 1}}(X).
\]

For \( \alpha = 1/2 \) and \( \beta = 0 \) in particular,

\[
\operatorname{PWM}(\bmo_{(\log(1/r))^{\alpha}}(X), \bmo(X)) = \bmo_{(\log(1/r))^{\alpha - \beta}}(X).
\]

**Example 2.2.**

\[
\operatorname{PWM}(\bmo_{(\log(1/r))^{\alpha}}(X), \bmo(X)) = \bmo_{(\log(1/r))^{\alpha - \beta}}(X).
\]

**Example 2.3.**

\[
\operatorname{PWM}(\bmo_{(\log(1/r))^{\alpha}}(X), \bmo(X)) = \bmo_{(\log(1/r))^{\alpha - \beta}}(X) \cap L_{(\log(1/r))^{\alpha - \beta}}(X),
\]

where \( \mu(R) = \int_0^1 (1/\log t) \, dt \).

**Example 2.4.**

\[
\operatorname{PWM}(\bmo_{(\log(1/r))^{\alpha}}(X), \bmo(X)) = \bmo_{(\log(1/r))^{\alpha - \beta}}(X) \cap L_{(\log(1/r))^{\alpha - \beta}}(X).
\]

If \( X = \mathbb{T}, \, d(x, y) = |x - y| \) and \( \mu \) is Lebesgue measure, then the example above is known (Janson [6] and Stegenga [17]).

**Example 2.5.** For \( \alpha > 1 \),

\[
\operatorname{PWM}(\bmo_{(\log(1/r))^{\alpha}}(X), \bmo(X)) = \bmo_{(\log(1/r))^{\alpha - \beta}}(X).
\]

**Example 2.6.** For \( 0 < \beta \leq \alpha \leq \alpha_0 \),

\[
\operatorname{PWM}(\bmo_{(\log(1/r))^{\alpha}}(X), \bmo(X)) = \bmo_{(\log(1/r))^{\alpha - \beta}}(X).
\]

If \( X = \mathbb{T}, \, d(x, y) = |x - y| \) and \( \mu \) is Lebesgue measure, then \( \bmo_{(\log(1/r))^{\alpha}}(X) = \bmo_{(\log(1/r))^{\alpha - \beta}}(X) \). Therefore, for \( 0 < \beta \leq \alpha \leq 1 \),

\[
\operatorname{PWM}(\bmo_{(\log(1/r))^{\alpha}}(X), \bmo(X)) = \bmo_{(\log(1/r))^{\alpha - \beta}}(X).
\]

**Example 2.7.** For \(-1 < \alpha < \beta \leq \alpha + 1, \, 1 < p_2 < p_1 < \infty, \, p_1 p_2 \geq \beta_1 + p_2 \),

\[
\operatorname{PWM}(\bmo_{(\log(1/r))^{\alpha}}(X), \bmo_{(\log(1/r))^{\alpha - \beta}}(X)) = \bmo_{(\log(1/r))^{\alpha - \beta - 1}}(X).
\]

**2.2. The case \( \mu(X) = \infty \)**

**Example 2.8.**

\[
\operatorname{PWM}(\bmo_{(\log(1/r))^{\alpha}}(X), \bmo_{(\log(1/r))^{\alpha - \beta}}(X)) = \bmo_{(\log(1/r))^{\alpha - \beta - 1}}(X). \cap L_{(\log(1/r))^{\alpha - \beta}}(X).
\]

If \( X = \mathbb{T}, \, d(x, y) = |x - y| \) and \( \mu \) is Lebesgue measure, then the example above is known (14)).

**Example 2.9.** For \( 0 < \beta \leq \alpha \leq \alpha_0 \),

\[
\operatorname{PWM}(\bmo_{(\log(1/r))^{\alpha}}(X), \bmo_{(\log(1/r))^{\alpha - \beta}}(X)) = \bmo_{(\log(1/r))^{\alpha - \beta - 1}}(X) \cap L_{(\log(1/r))^{\alpha - \beta}}(X).
\]

**Example 2.10.** For \( 0 < \alpha \leq \alpha_0, \, 0 \leq \beta \leq \alpha + \delta > 0 \),

\[
\operatorname{PWM}(\bmo_{(\log(1/r))^{\alpha}}(X), \bmo_{(\log(1/r))^{\alpha - \beta}}(X)) = \bmo_{(\log(1/r))^{\alpha - \beta - 1}}(X) \cap L_{(\log(1/r))^{\alpha - \beta}}(X).
\]
Example 2.11. For $1 < p < \infty$, \[
\text{PWM}(\text{bmo}(X), \text{bmo}_{\log}(d(x_0, a) + r^{1/\alpha}), p(X)) = \text{bmo}(X).\]

Example 2.12. Let $w$ be an $A_p^\alpha$-weight on $\mathbb{R}^n$. Then
\[
\phi(a, r) = \left( \int_{B(a,r)} w(x) \, dx \right)^{\alpha} \quad \text{satisfies (1.4)–(1.7) for } -1/(pp') < \alpha \leq 1/(np'), \text{ and (1.8) for } \alpha \geq 0.
\]
Let \[
\phi_i(a, r) = \left( \int_{B(a,r)} w(x) \, dx \right)^{\alpha_i}, \quad i = 1, 2, \quad 0 < \alpha_2 \leq \alpha_1.
\]
Then \[
(\phi_2^{\alpha_2} + \phi_2^{\alpha_2})/\phi_2 \leq C(\phi_1^{\alpha_1} + \phi_1^{\alpha_1})/\phi_1.
\]

3. Lemmas. First, we state some simple inequalities and four lemmas of [15]. Let $1 \leq p < \infty$. Then
\[
|F(x_1) - F(x_2)| \leq C|x_1 - x_2| \Rightarrow \text{MO}_{\phi,p}(F(f), B) \leq 2C\text{MO}_{\phi,p}(f, B),
\]

(3.2) \[
|f_{B_1} - f_{B_2}| \leq \frac{\mu(B_2)}{\mu(B_1)}\text{MO}(f, B_2) \quad \text{for } B_1 \subset B_2,
\]

(3.3) \[
\text{MO}_p(f, B_1) \leq 2\left( \frac{\mu(B_2)}{\mu(B_1)} \right)^{1/p}\text{MO}(f, B_2) \quad \text{for } B_1 \subset B_2,
\]

(3.4) \[
|f_{B(a,r)} - f_{B(a,s)}| \leq 2K_2^2(\log 2)^{-1}\int_{r}^{s} \frac{\text{MO}(f, B(a,t))}{t} \, dt \quad \text{for } 0 < r < s.
\]

If $\phi$ satisfies (1.4), then
\[
\int_{r}^{s} \frac{\phi(a, t)}{t} \, dt \leq (1 + A_1)\int_{r}^{s} \frac{\phi(a, t)}{t} \, dt \quad \text{for } 0 < 2r \leq s.
\]

Lemma 3.1. Let $1 \leq p < \infty$. Suppose that $\phi$ satisfies (1.4)–(1.7). Let
\[
f_a(x) = \frac{1}{d(a, x)} \int_{d(a, x)} \frac{\phi(a, t)}{t} \, dt.
\]
Then $f_a$ is in $\text{bmo}_{\phi,p}(X)$ for all $a \in X$, and there is a constant $C > 0$, independent of $a$, such that $\|f_a\|_{\text{BMO}_{\phi,p}} \leq C$.

Lemma 3.2. Let $1 \leq p < \infty$. Suppose that $\phi$ satisfies (1.4). Then there is a constant $C > 0$ such that
\[
|f_{B(a,r)}| \leq C\|f\|_{\text{bmo}_{\phi,p}}(\Phi^*(a, r) + \Phi^{**}(a, r)),
\]
where $C$ is independent of $f \in \text{bmo}_{\phi,p}(X)$, $a \in X$ and $r > 0$.

Lemma 3.3. Let $1 \leq p < \infty$. Suppose that $\phi$ satisfies (1.4)–(1.7). If $\mu(X) = \infty$, then assume that (1.10) holds. For any ball $B(a, r)$, there is a function $f \in \text{bmo}_{\phi,p}(X)$ such that
\[
\|f\|_{\text{bmo}_{\phi,p}} \leq C_1, \quad f_{B(a,r)} \geq C_2(\Phi^*(a, r) + \Phi^{**}(a, r)),
\]
where $C_1, C_2$ are independent of $B(a, r)$ and $f \in \text{bmo}_{\phi,p}(X)$.

Lemma 3.4. Let $1 \leq p < \infty$. Suppose $f \in \text{bmo}_{\phi,p}(X)$ and $g \in L^\infty(X)$. Then $fg$ belongs to $\text{bmo}_{\phi,p}(X)$ if and only if
\[
\sup_{B} |f_B|\text{MO}_{\phi,p}(g, B) < \infty.
\]

In this case,
\[
\|fg\|_{\text{BMO}_{\phi,p}} - \sup_{B} |f_B|\text{MO}_{\phi,p}(g, B) \leq 2\|f\|_{\text{BMO}_{\phi,p}}\|g\|_{L^\infty}.
\]

We need more precise lemmas.

Lemma 3.5. Let $1 \leq p < \infty$. Suppose $\phi$ satisfies (1.4). Then $\text{bmo}_{\phi,p}(X) \subset L_{\Phi^*, \Phi^{**}, p}(X)$ and $\|f\|_{L_{\Phi^*, \Phi^{**}, p}} \leq C\|f\|_{\text{bmo}_{\phi,p}}$, where $C$ is independent of $f \in \text{bmo}_{\phi,p}(X)$.

Proof. If $2r \leq \max(d(x_0, a), 2)$, then
\[
\text{MO}_p(f, B(a,r)) \leq \phi(a,r)\|f\|_{\text{MO}_{\phi,p}} \leq C\Phi^*(a,r)\|f\|_{\text{MO}_{\phi,p}}.
\]

If $2r > \max(d(x_0, a), 2)$, then $B(a,r) \subset B(x_0, 3K_1 r)$. From (3.3) and (3.5), it follows that
\[
\text{MO}_p(f, B(a,r)) \leq 2\left( \frac{\mu(B(x_0, 3K_1 r))}{\mu(B(a,r))} \right)^{1/p}\text{MO}_p(f, B(x_0, 3K_1 r)) \leq C'\Phi^*(a,r)\|f\|_{\text{BMO}_{\phi,p}}.
\]

By Lemma 3.2, we have
\[
\left( \frac{1}{\mu(B(a,r))} \int_{B(a,r)} |f(x)|^p \, d\mu \right)^{1/p} \leq \text{MO}_p(f, B(a,r)) \leq C(\Phi^*(a,r) + \Phi^{**}(a,r))\|f\|_{\text{bmo}_{\phi,p}}.
\]

Corollary 3.6. Let $\mu(X) < \infty$ and $1 \leq p < \infty$. Suppose $\phi$ satisfies (1.4). Then $\text{bmo}_{\phi,p}(X) \subset L_{\Phi^*, p}(X)$ and $\|f\|_{L_{\Phi^*, p}} \leq C\|f\|_{\text{bmo}_{\phi,p}}$ where $C$ is independent of $f \in \text{bmo}_{\phi,p}(X)$. 

Lemma 3.7. Let \( \mu(X) = \infty \) and \( 1 \leq p < \infty \). Suppose \( \phi \) satisfies (1.4)–(1.7). Let \( r_1 \geq 2 \) and
\[
(3.6) \quad f(x) = \max_{d(x_0, a) \leq 1} \left( \int_1^{\max(2, d(x_0, a))} \frac{\phi(x_0, t)}{t} \, dt \right).
\]
Then \( f \) is in \( \text{bmo}_{\phi,p}(X) \) and there are constants \( C_i > 0 \) (\( i = 1, 2, 3 \)), independent of \( B(a, r) \), such that:

(i) if \( r < 2K_1 d(x_0, a) \), then
\[
f(x) \geq C_1 \Phi^*(a, r) \quad \text{for} \quad x \in B(a, r/(2K_1)^2); \]
(ii) if \( 2K_1 d(x_0, a) \leq r < 2r_1 \), then
\[
f(x) \geq C_2 \Phi^*(a, r) \quad \text{for} \quad x \in B(a, r); \]
(iii) if \( 2K_1 d(x_0, a) \leq r \) and \( 2^k r_1 \leq r < 2^{k+1} r_1 \) for some positive integer \( k \), then
\[
f(x) \geq C_3 \Phi^*(x_0, 2^{-j} r) \quad \text{for} \quad x \in E_j, \; j = 0, 1, \ldots, k - 1,
\]
where
\[
E_j = B(x_0, 2^{-j} r) \setminus B(x_0, 2^{-j-1} r), \quad j = 0, 1, \ldots, k - 1,
\]
(3.7)
\[
B(a, r/(2K_1)^2) \subset \left( \bigcup_{j=0}^{k-1} E_j \right) \cup B(x_0, 2^{-k} r).
\]

Proof. Since \( f(x) = \max(-f_{x_0}(x), \int_1^{\max(2, d(x_0, a))} \frac{\phi(x_0, t)}{t} \, dt) \), \( f \) is in \( \text{bmo}_{\phi,p}(X) \) by Lemma 3.1 and (3.5). Next we show (i)–(iii) by using (3.5).

(i) Since \( B(a, r/(2K_1)^2) \cap B(x_0, d(x_0, a)/(2K_1)) = \emptyset \),
\[
\Phi^*(a, r) \leq \max_{(2, d(x_0, a)/(2K_1)^2)} \int_1^{\max(2, d(x_0, a), r)} \frac{\phi(x_0, t)}{t} \, dt \leq C \int_1^{\max(2, d(x_0, a)/(2K_1)^2)} \frac{\phi(x_0, t)}{t} \, dt \leq C \int_1^{d(x_0, a)/(2K_1)} \frac{\phi(x_0, t)}{t} \, dt \leq Cf(x).
\]

(ii) Since \( d(x_0, a), r \leq 2r_1 \),
\[
\Phi^*(a, r) \leq \int_1^{2r_1} \frac{\phi(x_0, t)}{t} \, dt \leq C' \int_1^2 \frac{\phi(x_0, t)}{t} \, dt \leq C' f(x).
\]

(iii) For \( x \in E_j, \; j = 0, 1, \ldots, k - 1 \),
\[
\Phi^*(x_0, 2^{-j} r) \leq \int_1^{2^{-j} r} \phi(x_0, t) \, dt \leq C' \int_1^{2^{-j} r} \phi(x_0, t) \, dt \leq C' f(x).
\]

The next two lemmas have been proved in the proof of Lemma 3.3 of [15].

Lemma 3.8. Let \( \mu(X) < \infty \) and \( 1 \leq p < \infty \), Suppose \( \phi \) satisfies (1.4)–(1.7). Let \( f \) be defined by (3.6). Then \( f \) is in \( \text{bmo}_{\phi,p}(X) \) and there is a constant \( C > 0 \) such that, for all \( B(a, r) \),
\[
f(x) \geq C \Phi^*(a, r) \quad \text{for} \quad x \in B(a, r/(2K_1)^2). \]

Lemma 3.9. Let \( 1 \leq p < \infty \). Suppose \( \phi \) satisfies (1.4)–(1.7). For any ball \( B(a, r) \), let
\[
(3.9) \quad f(x) = \max \left( \frac{\max(1/K_1, d(x_0, a)/(2K_1))}{d(x_0, a)} \int_1^{\max(1/K_1, d(x_0, a)/(2K_1))} \frac{\phi(x_0, t)}{t} \, dt \right).
\]

Then \( f \) is in \( \text{bmo}_{\phi,p}(X) \) and there are constants \( C_1, C_2 > 0 \) such that
\[
\|f\|_{\text{bmo}_{\phi,p}} \leq C_1, \quad f(x) \geq C_2 \Phi^{**}(a, r) \quad \text{for} \quad x \in B(a, r/(2K_1)),
\]
where \( C_1, C_2 \) are independent of \( B(a, r) \) and \( f \) is in \( \text{bmo}_{\phi,p}(X) \).

Lemma 3.10. Let \( 1 \leq p_1, p_2, p_3 < \infty \) and \( 1/p_1 + 1/p_2 = 1/p_3 \). Suppose \( f \in \text{bmo}_{\phi_1, p_1}(X) \) and \( g \in L_{\phi_2, p_2}(X) \). Then \( f \in \text{bmo}_{\phi_3, p_3}(X) \) if and only if
\[
sup_B |f|_{B} |\text{MO}_{\phi_3, p_3}(g, B)| < \infty.
\]

In this case,
\[
(3.10) \quad \|fg\|_{\text{bmo}_{\phi_3, p_3}} = \sup_B |f|_{B} |\text{MO}_{\phi_3, p_3}(g, B)| \leq C \|\text{bmo}_{\phi_1, p_1}(g)\|_{L_{\phi_2, p_1} \cap L_{\phi_3, p_3}}.
\]

Proof. As in the proof of Lemma 3.4, for any ball \( B = B(a, r) \), we have
\[
\|fg\|_{L_{p_3}(B)} \leq \|fg\|_{L_{p_1}(B)} \|g\|_{L_{p_2}(B)} \leq 2 \left( \int_B |f(x) - f_B| \mu(x) \right)^{1/p_3} \mu(B)^{1/p_3} \|g\|_{L_{p_2}(B)} \mu(B)^{1/p_3}
\]
\[
\leq 2 \left( \int_B |f(x) - f_B|^p \mu(x) \right)^{1/p_3} \left( \int_B |g(x)|^p \mu(x) \right)^{1/p_3} \mu(B)^{1/p_3}
\]
\[
\leq 2 \mu(B)^{1/p_3} \phi_1(a, r) \|f\|_{\text{bmo}_{\phi_1, p_1}} \times \mu(B)^{1/p_3} \phi_2(a, r) \|g\|_{L_{\phi_2, p_2}} \mu(B)^{1/p_3}
\]
\[
\leq 2 \mu(B)^{1/p_3} \phi_3(a, r) \|f\|_{\text{bmo}_{\phi_1, p_1}} \|g\|_{L_{\phi_3, p_3}} \mu(B)^{1/p_3}
\]


Hence
\[ |MO_{\phi_2,p_2}(f_B,B) - |f_B||MO_{\phi_2,p_2}(g,B)| \leq 2\|f\|_{bmo_{\phi_1,p_1}} \|g\|_{L^{p_2/\theta_1}\theta_1}, \]

which shows (3.10). \( \blacksquare \)

**Lemma 3.11.** Let \( 1 \leq p_1, p_2 < \infty \). Suppose that \( \phi_1 \) satisfies (1.8) and \( \phi_1 \leq C\phi_2 \). Then
\[ bmo_{\phi_1,p_1}(X) \cap L^{p_2,p_2}(X) = bmo_{\phi_1}(X) \cap L^{p_2}(X), \]
\[ \|f\|_{bmo_{\phi_1}} + \|f\|_{L^{p_2}} \leq \|f\|_{bmo_{\phi_1,p_1}} + \|f\|_{L^{p_2,p_2}} \leq C_{p_1,p_2} (\|f\|_{bmo_{\phi_1}} + \|f\|_{L^{p_2}}). \]

**Proof.** By Hölder’s inequality, we have
\[ \|f\|_{bmo_{\phi_1}} + \|f\|_{L^{p_2}} \leq \|f\|_{bmo_{\phi_1,p_1}} + \|f\|_{L^{p_2,p_2}}. \]

By John–Nirenberg’s inequality, we have
\[ \|f\|_{bmo_{\phi_1,p_1}} \leq C_{p_1} \|f\|_{bmo_{\phi_1}} , \quad i = 1, 2. \]

For any ball \( B = B(a,r) \),
\[ \left( \frac{1}{\mu(B(a,r))} \int_{B(a,r)} |f(x)|^p \, d\mu \right)^{1/p} \leq \left( \frac{1}{\mu(B)} \int_{B} |f(x) - f_B|^p \, d\mu \right)^{1/p} + \left( \frac{1}{\mu(B)} \int_{B} |f_B|^p \, d\mu \right)^{1/p} \]
\[ \leq \mu(B)^{1/p_2} \|\phi_1(a,r)\| \|f\|_{bmo_{\phi_1,p_1}} + |f_B| \]
\[ \leq \mu(B)^{1/p_2} (C_{p_1} \phi_1(a,r) \|f\|_{bmo_{\phi_1}} + \phi_2(a,r) ||f||_{L^{p_2}}) \]
\[ \leq C_{p_2} \mu(B)^{1/p_2} \phi_2(a,r) (C_{p_2} \|f\|_{bmo_{\phi_1}} + ||f||_{L^{p_2}}). \]

Hence
\[ \|f\|_{bmo_{\phi_1,p_1}} + \|f\|_{L^{p_2,p_2}} \leq C_{p_1,p_2} (\|f\|_{bmo_{\phi_1}} + ||f||_{L^{p_2}}). \] \( \blacksquare \)

**Lemma 3.12.** Let \( \mu(X) = \infty \) and \( 1 \leq p < \infty \). Suppose that \( \phi_1 \) and \( \phi_2 \) satisfy (1.4) and (1.7)–(1.9). Let \( \phi_3 = \frac{\phi_2}{\phi_1^* + \phi_1^*} \). Then
\[ bmo_{\phi_3}(X) \cap L^{p_2/p_3}(X) = bmo_{\phi_3}(X) \cap L^{p_2}(X), \]
\[ \|f\|_{bmo_{\phi_3}} + \|f\|_{L^{p_2}} \leq \|f\|_{bmo_{\phi_3}} + \|f\|_{L^{p_2,p_3}} \leq C_p (\|f\|_{bmo_{\phi_3}} + ||f||_{L^{p_2}}). \]

**Proof.** We show that, for any ball \( B(a,r) \),
\[ \left( \frac{1}{\mu(B(a,r))} \int_{B(a,r)} |f(x)|^p \, d\mu \right)^{1/p} \leq C_p \frac{\phi_2(a,r)}{\phi_1(a,r)} (\|f\|_{bmo_{\phi_3}} + \|f\|_{L^{p_2}}). \]

First we note that
\[ \left( \frac{1}{\mu(B(a,r))} \int_{B(a,r)} |f(x)|^p \, d\mu \right)^{1/p} \leq MO_p(f, B(a,r)) + |f_{B(a,r)}|, \]

and
\[ |f_{B(a,r)}| \leq C \frac{\phi_2(a,r)}{\phi_1(a,r)} ||f||_{L^{p_2/p_3}}. \]

**Case 1:** \( r \leq d(x_0, a)/(5K_1^2) \). If \( b \in B(a, r) \), then \( d(x_0, b) > d(x_0, a)/K_1 \)
\[ r \geq 4K_1^2 (r \geq 4K_1^2). \]

If \( B(b, s) \subset B(a, r) \), then we may assume \( s \leq 2K_1^2 \). It follows from (1.4) and (1.7) that
\[ \phi_1^{**}(b, s) \geq \frac{4K_1^2}{2K_1^2} \int_{2K_1^2} \phi_1(b, t) \, dt \geq A_1^{-1} \phi_1(b, 2K_1^2) \geq C \phi_1(a, r). \]

By John–Nirenberg’s inequality, we have
\[ MO_p(f, B(a,r)) \leq C_p \sup_{B(b,s) \subset B(a,r)} MO(f, B(b,s)) \]
\[ \leq C_p \left( \sup_{B(b,s) \subset B(a,r)} \frac{\phi_2(b, s)}{\phi_1(b, s)} \right) ||f||_{bmo_{\phi_3}} \]
\[ \leq C_p \frac{\phi_2(a,r)}{\phi_1(a,r)} ||f||_{bmo_{\phi_3}}. \]

**Case 2:** \( a = x_0, r \geq r_0. \) Then
\[ MO_p(f, B(x_0, r)) \leq C_p \sup_{B(b,s) \subset B(x_0, r)} MO(f, B(b,s)) \]
\[ \leq C_p \left( \sup_{B(b,s) \subset B(x_0, r)} \frac{\phi_2(b, s)}{\phi_1(b, s)} \right) ||f||_{bmo_{\phi_3}} \]
\[ \leq C_p \frac{\phi_2(x_0, r_0)}{\phi_1(x_0, r_0)} ||f||_{bmo_{\phi_3}}. \]

**Case 3:** \( a = x_0, r > r_0. \) Let \( 2^{j-1} r_0 < r \leq 2^k r_0 \) and \( E_j = B(x_0, 2^j r_0) \cap B(x_0, 2^{j-1} r_0), j = 1, \ldots, k \). If \( E_j = \emptyset \), then \( \int_{E_j} |f(x)|^p \, d\mu = 0. \) If \( E_j \neq \emptyset \), then there are balls \( B_i, j = 1, \ldots, m_j \) such that
\[ B_j = B(b_j, m_j, \epsilon), \quad b_j, m_j \in E_j, \quad \epsilon = 2^j - 2^{j-1} r_0/(20K_1^2), \]
\[ E_j \subset \bigcup_{m_j} B(b_j, m_j, 4K_1^2 \epsilon), \quad B_j \cap B_j = \emptyset \quad (m_j \neq n). \]

(see [3], pp. 68–69). We note that \( \phi_i(b_j, m_j, 4K_1^2 \epsilon) \) \( (i = 1, 2) \) are comparable.
4. Propositions. We now show some propositions.

Proposition 4.1. Let $1 \leq p_1, p_2 < \infty$. Suppose that $\phi_1$ and $p_1$ satisfy (1.4)–(1.7), $\phi_2$ satisfies (1.4) and (1.7), and $(\Phi_2 + \Phi_2^*)/\phi_2 \leq C(\Phi_2^* + \Phi_2^{**})/\phi_1$. If $\mu(X) = \infty$, then assume that there are constants $r_0 > 0$ and $A_8 > 0$ such that

$\int_0^{r} \left( \frac{\Phi_2(x_0, t)}{\Phi_1(x_0, t)} \right)^{p_2} \frac{\mu(B(x_0, t))}{t} \, dt \leq A_8 \left( \frac{\phi_2(x_0, r)}{\phi_1(x_0, r)} \right)^{p_3} \mu(B(x_0, r)), \quad r > r_0.

Then $PWM(\text{bmo}_{\phi_1, p_1}(X), \text{bmo}_{\phi_2, p_2}(X)) \subset L_{\phi_2/\phi_1, p_2}(X)$,

$\|g\|_{L_{\phi_2/\phi_1, p_2}} \leq C\|g\|_{OP}$,

where $\|g\|_{OP}$ is the operator norm of $g$ in $PWM(\text{bmo}_{\phi_2, p_1}(X), \text{bmo}_{\phi_2, p_2}(X))$.

Corollary 4.2. Let $1 \leq p_1, p_2 < \infty$. Suppose that $\phi_1$ and $p_1$ satisfy (1.4)–(1.7) and $\phi_2$ satisfies (1.4). Let

$X_1 = \left\{ x \in X : \frac{\phi_2(x_0, r)}{t} \frac{dt}{\phi_1(x_0, t)} \to 0 \quad \text{as} \quad r \to 0 \right\}$.

If $g \in PWM(\text{bmo}_{\phi_1, p_1}(X), \text{bmo}_{\phi_2, p_2}(X))$, then $g(x) = 0$ a.e. $x \in X_1$.

Proposition 4.3. Let $1 < p_1, p_2 < \infty$ and $p_8 = p_1/p_2 + p_2 \geq 1$. Suppose that $\phi_1$ and $p_1$ satisfy (1.4)–(1.7), $\phi_2$ satisfies (1.4) and (1.7), and $(\Phi_2 + \Phi_2^*)/\phi_2 \leq C(\Phi_2^* + \Phi_2^{**})/\phi_1$. If $\mu(X) = \infty$, then assume that (4.1) holds. Let $\phi_3 = \phi_2/(\Phi_2^* + \Phi_2^{**})$. Then

$PWM(\text{bmo}_{\phi_1, p_1}(X), \text{bmo}_{\phi_2, p_2}(X)) \subset \text{bmo}_{\phi_3, p_8}(X) \cap L_{\phi_2/\phi_1, p_2}(X)$,

$\|g\|_{\text{bmo}_{\phi_3, p_8}} + \|g\|_{L_{\phi_2/\phi_1, p_2}} \leq C\|g\|_{OP}$,

where $\|g\|_{OP}$ is the operator norm of $g$ in $PWM(\text{bmo}_{\phi_1, p_1}(X), \text{bmo}_{\phi_2, p_2}(X))$.

Proposition 4.4. Suppose that $\phi_1$ and $\phi_2$ satisfy (1.4). Let $\phi_3 = \phi_2/(\Phi_2^* + \Phi_2^{**})$. If $1 \leq p_2 < p_1 < \infty$ and $p_4 = p_1/p_2(p_1 - p_2)$, then

$PWM(\text{bmo}_{\phi_1, p_1}(X), \text{bmo}_{\phi_2, p_2}(X)) \subset \text{bmo}_{\phi_3, p_8}(X) \cap L_{\phi_2/\phi_1, p_4}(X)$,

$\|g\|_{OP} \leq C(\|g\|_{\text{bmo}_{\phi_3, p_8}} + \|g\|_{L_{\phi_2/\phi_1, p_4}})$,

where $\|g\|_{OP}$ is the operator norm of $g$ in $PWM(\text{bmo}_{\phi_1, p_1}(X), \text{bmo}_{\phi_2, p_2}(X))$.

Proposition 4.5. Suppose that $\phi$ satisfies (1.4). Let $\psi = \phi/(\Phi_1^* + \Phi_1^{**})$. If $1 \leq p_2 \leq p_1 < \infty$, then

$PWM(\text{bmo}_{\phi_1, p_1}(X), \text{bmo}_{\phi_2, p_2}(X)) \subset \text{bmo}_{\phi_3, p_8}(X) \cap L_{\psi, p_2}(X)$,

$\|g\|_{OP} \leq C(\|g\|_{\text{bmo}_{\phi_3, p_8}} + \|g\|_{L^\infty})$,

where $\|g\|_{OP}$ is the operator norm of $g$ in $PWM(\text{bmo}_{\phi_1, p_1}(X), \text{bmo}_{\phi_2, p_2}(X))$. 

Proof of Prop. 4.1. Let \( g \in \text{PWM}(b\text{mo}_{\phi_1,p_1}(X), b\text{mo}_{\phi_2,p_2}(X)) \). Then \( g \) is a bounded operator. We show that, for any \( a \in X \) and for any \( r > 0 \),

\[
(4.6) \quad \frac{1}{\mu (B(a,r/(2K_1)^2))} \int_{B(a,r/(2K_1)^2)} |g(x)|^p \, d\mu \leq C \|g\|_{op} \phi_2^*(a,r/(2K_1)^2) \phi_1^*(a,r/(2K_1)^2).
\]

For any \( f \in \text{bmo}_{\phi_1,p_1}(X) \), \( fg \) is in \( \text{bmo}_{\phi_2,p_2}(X) \). From Lemma 3.5 it follows that, for any ball \( B(a,r) \),

\[
(4.7) \quad \frac{1}{\mu(B(a,r))} \int_{B(a,r)} |f(x)g(x)|^p \, d\mu \leq C \|fg\|_{b\text{mo}_{\phi_2,p_2}} (\phi_2^*(a,r) + \phi_2^{**}(a,r)) \leq C \|f\|_{b\text{mo}_{\phi_1,p_1}} \|g\|_{op} (\phi_2^*(a,r) + \phi_2^{**}(a,r)).
\]

Applying (4.7) with \( f \) defined by (3.9) and using Lemma 3.9, we have

\[
(4.8) \quad \phi_1^*(a,r) \left( \frac{1}{\mu(B(a,r))} \int_{B(a,r/(2K_1)^2)} |g(x)|^p \, d\mu \right)^{1/p} \leq C \|g\|_{op} (\phi_2^*(a,r) + \phi_2^{**}(a,r)).
\]

If \( \mu(X) < \infty \), then, applying (4.7) with \( f \) defined by (3.6) and using Lemma 3.8, we have

\[
(4.9) \quad \phi_1^*(a,r) \left( \frac{1}{\mu(B(a,r))} \int_{B(a,r/(2K_1)^2)} |g(x)|^p \, d\mu \right)^{1/p} \leq C \|g\|_{op} (\phi_2^*(a,r) + \phi_2^{**}(a,r)).
\]

By (4.8), (4.9) and the inequality \( (\phi_2^* + \phi_2^{**})/\phi_2 \leq C (\phi_2^* + \phi_2^{**})/\phi_1 \), we have

\[
(4.10) \quad \left( \frac{1}{\mu(B(a,r))} \int_{B(a,r/(2K_1)^2)} |g(x)|^p \, d\mu \right)^{1/p} \leq C \|g\|_{op} \frac{\phi_2^*(a,r)}{\phi_1^*(a,r)}.
\]

Since \( \phi_i(a,r) (i = 1, 2) \) and \( \mu(B(a,r)) \) are comparable to \( \phi_i(a,r/(2K_1)^2) \) \((i = 1, 2)\) and \( \mu(B(a,r/(2K_1)^2)) \), respectively, we have (4.6).

If \( \mu(X) = \infty \), then, applying (4.7) with \( f \) defined by (3.6) and using Lemma 3.7(iii), we have \( \phi_1^*(a,r) \) and \( \mu(B(a,r/(2K_1)^2)) \) \((i = 1, 2)\) and \( \mu(B(a,r/(2K_1)^2)) \), respectively, we have (4.6).

Using Lemma 3.7(iii), we have, for \( j = 0, 1, \ldots, k - 1 \),

\[
(4.11) \quad \phi_1^*(a,s_j) \left( \frac{1}{\mu(B(a,s_j))} \int_{B(a,s_j)} |g(x)|^p \, d\mu \right)^{1/p} \leq C \|g\|_{op} \phi_2^*(a,s_j).
\]

Since \( s_k < 2 \max(2,r_0) \), \( B(x_0,s_k) \) is in case (ii) of Lemma 3.7. Thus

\[
(4.12) \quad \phi_1^*(a,s_k) \left( \frac{1}{\mu(B(a,s_k))} \int_{B(a,s_k)} |g(x)|^p \, d\mu \right)^{1/p} \leq C \|g\|_{op} \phi_2^*(a,s_k).
\]

By (3.8), (4.11) and (4.12), we have

\[
\frac{1}{\mu(B(a,r/(2K_1)^2))} \int_{B(a,r/(2K_1)^2)} |g(x)|^p \, d\mu \leq C \|g\|_{op} \phi_2^*(a,r/(2K_1)^2) \phi_1^*(a,r/(2K_1)^2).
\]

Since \( \phi_1(x_0,2r) (i = 1, 2) \) and \( \mu(B(x_0,2r)) \) are comparable to \( \phi_1(a,r/(2K_1)^2) \) \((i = 1, 2)\) and \( \mu(B(a,r/(2K_1)^2)) \), respectively, we have (4.6).

Proof of Corollary 4.2. Just as we proved (4.8) and (4.9), we get

\[
\left( \frac{1}{\mu(B(a,r/(2K_1)^2))} \int_{B(a,r/(2K_1)^2)} |g(x)|^p \, d\mu \right)^{1/p} \leq C \|g\|_{op} \frac{\phi_2^*(a,r) + \phi_2^{**}(a,r)}{\phi_1^*(a,r)}.
\]

for small \( r \). If \( a \in X_1 \) then \( \int_0^r \phi_1(a,t)^{-1} \, dt \to \infty \) as \( r \to 0 \). Therefore

\[
\lim_{r \to 0} \phi_1^*(a,r) + \phi_2^*(a,r) + \phi_2^{**}(a,r) = 0.
\]

Proof of Prop. 4.3. Let \( g \in \text{PWM}(b\text{mo}_{\phi_1,p_1}(X), b\text{mo}_{\phi_2,p_2}(X)) \). By Proposition 4.1, \( g \) is in \( L_{\phi_2/\phi_1,p_1}(X) \) and \( \|g\|_{L_{\phi_2/\phi_1,p_1}} \leq C \|g\|_{op} \). By
Lemma 3.3, for any \( B(a, r) \), we have a function \( f \) such that

\[
\|f\|_{\text{bmo}_{\phi_1, p_1}} \leq C_1,
\]
\[
\left|f B(a, r) \right| \geq C_2 (\Phi_1^*(a, r) + \Phi_1^*(a, r)).
\]

Since \( 1/p_1 + 1/p_2 = 1/p_3 \), it follows from Lemma 3.10 that

\[
\|f B(a, r) \|_{\text{MO}_{\phi_3, p_3}(g, B(a, r))} \leq 2\|f\|_{\text{bmo}_{\phi_1, p_1}} \|g\|_{L^{p_2}/p_{2, 1}, p_2} + 2\|f\|_{\text{bmo}_{\phi_2, p_2}} \|g\|_{L^{p_2}/p_{2, 1}, p_2} \leq C\|f\|_{\text{bmo}_{\phi_1, p_1}} \|g\|_{\text{bmo}_{\phi_3, p_3}} \leq C\|g\|_{\text{bmo}_{\phi_3, p_3}}.
\]

5. Proofs of the theorems and the corollaries

Proof of Theorem 1.1. First we note that \( \Phi_1 \) satisfies (1.6) for any \( p_1 (1 \leq p_1 < \infty) \) (see Lemma 5.3 of [15]). If \( \Phi_1 \) is finite, then we may assume \( r_0 \geq 1 \) in (1.9). For \( t \geq 1, \Phi_1^*(x_0, t) + \Phi_1^*(x_0, t) \leq 2\Phi_1^*(x_0, t) \). By (1.9) with \( p = 1 + \varepsilon \) and by the inequality (\( \Phi_1^* + \Phi_1^* \))/\( \Phi_1^* \leq C(\Phi_1^* + \Phi_1^*)/\Phi_1 \), we have (4.1) with \( p_2 = 1 + \varepsilon \). From Proposition 4.3 it follows that

\[
\text{PVM}(\operatorname{bmo}\phi_1, (1 + \varepsilon)/(\varepsilon), \text{bmo}_{\phi_2, 1 + \varepsilon}(X)) \subset \operatorname{bmo}\phi_1(X) \cap L_{\phi_2/\phi_1, 1 + \varepsilon}(X),
\]
\[
\|g\|_{\text{bmo}_{\phi_3}} + \|g\|_{L_{\phi_2/\phi_1, 1 + \varepsilon}} \leq C\|g\|_{\text{bmo}_{\phi_2}}.
\]

where \( \|g\|_{\text{bmo}_{\phi_3}} \) is the operator norm of \( g \in \text{PVM}(\operatorname{bmo}\phi_1, (1 + \varepsilon)/(\varepsilon), \text{bmo}_{\phi_2, 1 + \varepsilon}(X)) \). From Proposition 4.4 it follows that

\[
\text{PVM}(\operatorname{bmo}\phi_1, (1 + \varepsilon)/(\varepsilon), \text{bmo}_{\phi_2, 1 + \varepsilon}(X)) \subset \operatorname{bmo}\phi_1(X) \cap L_{\phi_2/\phi_1, 1 + \varepsilon}(X),
\]
\[
\|g\|_{\text{bmo}_{\phi_3}} + \|g\|_{L_{\phi_2/\phi_1, 1 + \varepsilon}} \leq C\|g\|_{\text{bmo}_{\phi_2}}.
\]

where \( \|g\|_{\text{bmo}_{\phi_3}} \) is the operator norm of \( g \in \text{PVM}(\operatorname{bmo}\phi_1, (1 + \varepsilon)/(\varepsilon), \text{bmo}_{\phi_2, 1 + \varepsilon}(X)) \). By John–Nirenberg’s inequality and Hölder’s inequality, we have

\[
\text{bmo}_{\phi_1, (1 + \varepsilon)/(\varepsilon)}(X) = \text{bmo}_{\phi_1}(X), \quad \text{bmo}_{\phi_2, 1 + \varepsilon}(X) = \text{bmo}_{\phi_2}(X).
\]

Moreover, the operator norms of \( g \) from \( \text{bmo}_{\phi_1, (1 + \varepsilon)/(\varepsilon)}(X) \) to \( \text{bmo}_{\phi_2, 1 + \varepsilon}(X) \), from \( \text{bmo}_{\phi_1, 1 + \varepsilon}(X) \) to \( \text{bmo}_{\phi_2}(X) \), and from \( \text{bmo}_{\phi_1}(X) \) to \( \text{bmo}_{\phi_2}(X) \) are comparable. From Lemma 3.12 it follows that

\[
\text{bmo}_{\phi_2}(X) \cap L_{\phi_2/\phi_1, 1 + \varepsilon}(X) = \text{bmo}_{\phi_2}(X) \cap L_{\phi_2/\phi_1, 1 + \varepsilon}(X),
\]
\[
\|g\|_{\text{bmo}_{\phi_3}} + \|g\|_{L_{\phi_2/\phi_1, 1 + \varepsilon}} \leq \|g\|_{\text{bmo}_{\phi_3}} + \|g\|_{L_{\phi_2/\phi_1, 1 + \varepsilon}} \leq C\|g\|_{\text{bmo}_{\phi_2}} + \|g\|_{L_{\phi_2/\phi_1, 1 + \varepsilon}}.
\]

Therefore we have Theorem 1.1.

Proof of Theorem 1.3. Since \( \int_0^2 \phi_1(a, t) \frac{dt}{t} \) (\( i = 1, 2 \)) are comparable to \( \phi_1(a, s) \) (i = 1, 2), respectively,

\[
\int_0^2 \frac{\phi_2(a, t)}{t} \frac{dt}{t} \leq \int_0^2 \frac{\phi_1(a, t)}{t} \frac{dt}{t} \leq C \frac{\phi_2(a, s)}{\phi_1(a, s)} \leq C \frac{\phi_2(a, r)}{\phi_1(a, r)}.
\]

Thus, for \( R_0 \leq 2^{k}r \leq 2R_0 \),

\[
\left( \sum_{j=0}^{k-1} \frac{\phi_2(a, t)}{t} \frac{dt}{t} \right) \left( \sum_{j=0}^{k-1} \frac{\phi_1(a, t)}{t} \frac{dt}{t} \right) \leq C \frac{\phi_2(a, r)}{\phi_1(a, r)}.
\]
We also note that \( \phi_3 \) satisfies (1.8) and \( \phi_3 \leq C \phi_2 / \phi_1 \). Therefore, using Propositions 4.3, 4.4 and Lemma 3.11, we have Theorem 1.3.

Proof of Theorem 1.6. By Propositions 4.3 and 4.4, we have
\[
\text{bmo}_{\phi_3, \phi_2}(X) \cap L_{\phi_1 / \phi_2}(X, \mu_{\phi_3, \phi_2}(X)) \\
\subseteq \text{PWM}(\text{bmo}_{\phi_3, \phi_2}(X), \text{bmo}_{\phi_3, \phi_2}(X)) \\
\subseteq \text{bmo}_{\phi_3, \phi_2}(X) \cap L_{\phi_1 / \phi_2}(X, \mu_{\phi_3, \phi_2}(X)),
\]
\[
C_1(\|g\|_{\text{bmo}_{\phi_3, \phi_2}(X)} + \|g\|_{L^{p}(X)}) \\
\leq C_2(\|g\|_{\text{bmo}_{\phi_3, \phi_2}(X)} + \|g\|_{L^{p}(X)}).
\]

We note that \( \phi_3 \leq C \phi_2 / \phi_1 \). By Lemma 3.11, we have Theorem 1.6.

Proof of Theorem 1.8. Let \( g \in \text{PWM}(\text{bmo}_{\phi_3}(X), \text{bmo}_{\phi_3}(X)) \).

By Proposition 4.1, \( g \in L_{\phi_1}^{\omega}(X) = L^{\infty}(X) \) and \( \|g\|_{L^{\infty}} \leq C \|g\|_{\text{op}} \). From Lemma 3.3 it follows that, for any ball \( B(a, r) \), there is a function \( f \in \text{bmo}_{\phi_3, \phi_2}(X) \) such that
\[
\|f\|_{\text{bmo}_{\phi_3, \phi_2}(X)} \leq C_1 \phi_2(a, r) / \phi_1(a, r),
\]
\[
f_{B(a, r)} \geq C_2 (\phi_1^*(a, r) + \phi_2^*(a, r)).
\]

Since \( f \) is in \( \text{bmo}_{\phi_3, \phi_2}(X) \), from Lemma 3.4 it follows that
\[
\|f\|_{\text{bmo}_{\phi_3, \phi_2}(X)} \leq C_3 \|f\|_{L^{\infty}(X)} + \|f\|_{\text{bmo}_{\phi_3, \phi_2}(X)}
\]
\[
\leq C_4(\|f\|_{\text{bmo}_{\phi_3, \phi_2}(X)} + \|g\|_{\text{op}}).
\]

By (5.2)-(5.4), \( g \) is in \( \text{bmo}_{\phi_3, \phi_2}(X) \) and \( \|g\|_{\text{bmo}_{\phi_3, \phi_2}(X)} \leq C \|g\|_{\text{op}} \). Conversely, by Proposition 4.5, we have
\[
\text{PWM}(\text{bmo}_{\phi_3, \phi_2}(X), \text{bmo}_{\phi_3, \phi_2}(X)) \supseteq \text{bmo}_{\phi_3, \phi_2}(X) \cap L_{\phi_1}^{\omega}(X),
\]
\[
\|g\|_{\text{op}} \leq C(\|g\|_{\text{bmo}_{\phi_3, \phi_2}(X)} + \|g\|_{L^{\infty}}).
\]

Proof of Theorem 1.10. Let \( g \in \text{PWM}(\text{bmo}_{\phi_3, \phi_2}(X), \text{bmo}_{\phi_3, \phi_2}(X)) \) and \( a \in X^* \cap X_0 \). We show
\[
\lim_{r \to 0} \frac{1}{\mu(B(a, r))} \int_{B(a, r)} |g(x)| \, d\mu = 0.
\]

If \( \int_{0}^{1} \phi_1(a, t) t^{-1} \, dt \to \infty \) as \( r \to 0 \), then
\[
\lim_{r \to 0} \int_{r}^{1} \phi_2(a, t) t^{-1} \, dt = \lim_{r \to 0} \int_{r}^{1} \phi_2(a, t) t^{-1} \, dt = 0.
\]

By Corollary 4.2, we have (5.5).

If \( \int_{0}^{1} \phi_1(a, t) t^{-1} \, dt < \infty \), then \( \int_{0}^{1} \phi_0(a, t) t^{-1} \, dt < \infty \). Let
\[
f(x) = \int_{0}^{1} \phi_1(a, t) t^{-1} \, dt.
\]

Then \( f \) is in \( \text{bmo}_{\phi_1, \phi_3}(X) \). From (3.4) it follows that, for \( 0 < s < r \),
\[
\|f\|_{\text{bmo}_{\phi_1, \phi_3}(X)} \leq C \|f\|_{\text{bmo}_{\phi_1, \phi_3}(X)} + \|f\|_{\text{bmo}_{\phi_1, \phi_3}(X)} \\
\leq C \phi_1(a, t) \int_{s}^{r} \phi_1(a, t) t^{-1} \, dt.
\]

Letting \( s \to 0 \), we have \( \int_{B(a, r)} |f(x)\, g(x)| = 0 \) and
\[
\int_{B(a, r)} |f(x)| \, dx \leq C(\int_{0}^{r} \phi_1(a, t) t^{-1} \, dt).
\]

Since
\[
f(x) = \int_{0}^{r} \frac{\phi_1(a, t)}{t} \, dt \geq \int_{0}^{1} \frac{\phi_1(a, t)}{t} \, dt \quad \text{for } x \in B(a, r) \setminus B(a, r/2),
\]
we have
\[
\int_{B(a, r)} |g(x)| \, dx \leq C^\prime \mu(B(a, r)) \int_{0}^{r} \phi_1(a, t) t^{-1} \, dt.
\]

And
\[
\int_{B(a, r)} \mu(B(a, 2r)) \int_{0}^{r} \phi_1(a, t) t^{-1} \, dt \leq C^\prime \mu(B(a, 2r)) \int_{0}^{r} \phi_1(a, t) t^{-1} \, dt.
\]

We note that \( a \in X^* \) if and only if there is a constant \( C_a > 0 \) such that
\[
\int_{0}^{r} \mu(B(a, 2r)) \mu(B(a, r)) \int_{0}^{r} \phi_1(a, t) t^{-1} \, dt \leq C_a \mu(B(a, r)) \quad 0 < r < r_a
\]
(see Lemma 5.3 of [15]). By the equality
\[
\int_{0}^{r} \frac{\phi_2(a, t)}{t} \, dt = \lim_{r \to 0} \int_{0}^{\phi_2(a, r)} \frac{\phi_2(a, r)}{t} \, dt = \lim_{r \to 0} \frac{\phi_2(a, r)}{r} = 0,
\]
we have (5.5).

Proofs of Corollaries. If \( \phi_3^* + \phi_3^* \leq C \phi_2 / \phi_1 \) or if \( \phi_3 \leq C \phi_2 / \phi_1 \),
then, by Lemma 3.5 or Corollary 3.6, it follows that \( \text{bmo}_{\phi_3}(X) \subseteq L_{\phi_1 / \phi_2}(X) \)
and \( \|g\|_{L_{\phi_1}^{\omega}(X)} \leq C \|g\|_{\text{bmo}_{\phi_3}(X)} \). Therefore we have Corollaries 1.2, 1.4 and 1.7.

Under the assumptions of Theorem 1.3, if \( \phi_3 \leq C \), then \( \phi_3 \) is comparable to \( \phi_2 \). By (5.1), we have \( \phi_3 \leq C \phi_2 / \phi_1 \). Therefore, Corollary 1.5 follows from Corollary 1.4.

Finally, Corollary 1.9 follows from Theorem 1.8.
Spreading sequences in $JT$

by

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Abstract. We prove that a normalized non-weakly null basic sequence in the James tree space $JT$ admits a subsequence which is equivalent to the summing basis for the James space $J$. Consequently, every normalized basic sequence admits a spreading subsequence which is either equivalent to the unit vector basis of $l_2$ or to the summing basis for $J$.

1. Introduction. We study subsequences of normalized basic sequences $(x_i)_{i=1}^\infty$ in the James tree space $JT$. Amemiya and Ito [1] proved that if $(x_i)_{i=1}^\infty \subset JT$ is weakly null then it has a subsequence which is equivalent to the unit vector basis of $l_2$.

We prove, following an idea of Hagler [7], that if $(x_i)_{i=1}^\infty$ is not weakly null then there is a subsequence equivalent to the summing basis for the James space $J$. In particular, this yields a classification of all the spreading models of $JT$, extending the work of Andrew [2] for the space $J$.

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We first introduce some necessary notation and recall the definitions of $J$ and $JT$ constructed by James in [8] and [9] respectively. Most of the material referring to these spaces used here can be found in [5].

DEFINITION 1. The James space $J$ is the Banach space of real sequences $b = (b_i)_{i=1}^\infty$ with the norm

$$||b|| = \sup \left( \sum_{i=1}^M \left( \sum_{j=1}^{n(i)} b_j \right)^2 \right)^{1/2},$$

where the sup is taken over all finite collections $S_1, \ldots, S_M$ of disjoint intervals of natural numbers with $S_i = \{n(\nu), n(\nu) + 1, \ldots, n(\nu) + \kappa(\nu)\}$. 

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