STUDIA MATHEMATICA 125 (1) (1997)

On the spectral bound of the generator of a C_0-semigroup

by

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Abstract. We give several conditions implying that the spectral bound of the generator of a C_0-semigroup is negative. Applications to stability theory are considered.

1. Let X be a complex Banach space, $(T(t))_{t\ge0}$ a C_0-semigroup in X, and A its generator with the domain $D(A)$, the spectrum $\sigma(A)$ and the resolvent set $\varrho(A)$. Denote by $R(\lambda,A):=(\lambda-A)^{-1}$, $\lambda\in\varrho(A)$, the *resolvent* of the operator A, by $s(A):=\sup\{\operatorname{Re}\lambda:\lambda\in\sigma(A)\}$ the *spectral bound* of A, and by $\omega_0:=\lim_{t\to\infty}\ln\|T(t)\|/t$ the *type* of the semigroup $(T(t))_{t\ge0}$. Finally, for a linear operator A, denote by $C^\infty(A)$ the set $\bigcap_{n=1}^{\infty}D(A^n)$. Among other results, G. Weiss has proved in [14] the following

THEOREM 1. *Suppose that for every $x\in X$ and every $x^*\in X^*$,*

$$\int_0^\infty |(T(t)x,x^*)|^p\,dt<\infty,\qquad p\in[1,\infty).$$

Then $s(A)<0$.

(For $p=1$ this result appeared also in [3, Ch. 7].)

The main purpose of the present paper is to show that for the case $p=1$ the conditions of Theorem 1 can be essentially weakened. Namely, we require

$$\int_0^\infty |(T(t)x,x^*)|\,dt<\infty$$

only for $x\in C^\infty(A)$ and $x^*\in C^\infty(A^*)$. It is also established that if $\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda>0\}\subset\varrho(A)$ and

$$\sup_{s>0}\int_{-\infty}^{\infty}|(R(s+it,A)x,x^*)|\,dt<\infty$$

for $x\in C^\infty(A)$ and $x^*\in C^\infty(A)$, then a similar assertion holds. Combining our results with the results of [1] we derive a criterion for some kind of stability of $(T(t))_{t\ge0}$.

1991 *Mathematics Subject Classification*: 47D03, 47A05.

Recent developments related to the subject of this paper can be found in [10]. The role of $C^\infty(A)$ for stability theory was studied in [17].

2. For our considerations we shall need some background from the theory of operator semigroups. First, we recall some properties of the sets $C^\infty(A)$ and $C^\infty(A^*)$. It is well known that $\overline{C^\infty(A)} = X$. But, in general, the closure of $C^\infty(A^*)$ is not the whole X^*. This is due to the fact that, in general, the semigroup $(T^*(t))_{t\ge 0}$ is not strongly continuous on X^*. At the same time, the set

$$X^\odot = \{x^* \in X^* : T^*(t)x^* \text{ is continuous}, \ t \ge 0\}$$

is equal to $\overline{D(A^*)}$, and it is a $T^*(t)$-invariant subspace of X^*, $t \ge 0$; $T^*(t)|_{X^\odot}$ is a C_0-semigroup with the generator $A^\odot = A^*|_{X^\odot}$, called the *sun-dual* of $(T(t))_{t\ge 0}$. Thus

$$\overline{C^\infty(A^\odot)} = X^\odot, \quad C^\infty(A^\odot) \subset C^\infty(A^*). \tag{1}$$

(In fact, $C^\infty(A^\odot) = C^\infty(A^*)$. See [8] for the proof.) Furthermore, the subspace $X^\odot \subset X^*$ induces on X the prime-norm

$$\|x\|' := \sup\{|(x, x^\odot)| : x^\odot \in X^\odot, \ \|x^\odot\| \le 1\}, \quad x \in X,$$

which is equivalent to the original one. Therefore, there is $C > 0$ such that

$$\|x\| \le C\|x\|'. \tag{2}$$

For all these facts, we refer the reader to [3].

Now, let $\mathcal{L}(X)$ be the Banach algebra of bounded linear operators on X and let $\mathcal{A}$ be a maximal abelian subalgebra of $\mathcal{L}(X)$ containing $(T(t))_{t\ge 0}$. It is known that $\mathcal{A}$ is closed under limits in the strong operator topology. So the set $\{R(\lambda, A) : \lambda \in \varrho(A)\}$ is contained in $\mathcal{A}$. Moreover, for every $S \in \mathcal{A}$ we have

$$\sigma_{\mathcal{A}}(S) = \sigma(S).$$

These basic facts can be found in [4]. For more details, see [5].

We start with the following statement.

THEOREM 2. *Suppose that*

(i) *for all $x \in C^\infty(A)$ and $x^* \in C^\infty(A^*)$,* $\displaystyle\int_0^\infty |(T(t)x, x^*)|\,dt < \infty.$

Then $s(A) < 0$.

Proof. Fix $x^* \in C^\infty(A^*)$. Consider the linear operator $M_{x^*} : C^\infty(A) \to L_1(\mathbb{R}^+)$ defined by

$$M_{x^*}x := (T(t)x, x^*), \quad t \ge 0. \tag{3}$$

The operator M_{x^*} is closed by standard arguments. Since $C^\infty(A)$ can be regarded as a Fréchet space with the system of norms

$$\|x\|_n = \|(\lambda_0 - A)^n x\|, \quad n \in \mathbb{N}, \ \lambda_0 \in \varrho(A) \text{ fixed},$$

the closed graph theorem implies the continuity of M_{x^*} on $C^\infty(A)$. Therefore, there is some $m \in \mathbb{N}$ such that

$$\|M_{x^*}x\|_1 \le C(x^*)\|(\lambda_0 - A)^m x\|, \quad C(x^*) > 0,$$

(here $\|\cdot\|_1$ stands for the L_1-norm) or

$$\|M_{x^*}R^m(\lambda_0, A)x\|_1 \le C(x^*)\|x\|, \quad x \in C^\infty(A).$$

The last inequality implies that the linear operator $M_{x^*}R^m(\lambda_0, A)$ is continuous on $C^\infty(A)$ with the topology induced from X. Therefore, from $\overline{C^\infty(A)} = X$ it follows that

$$\|M_{x^*}R^m(\lambda_0, A)x\|_1 \le C(x^*)\|x\|, \quad x \in X.$$

Now, fix $x \in C^\infty(A)$. Then, in the same way, the linear operator $M_x : C^\infty(A^\odot) \to L^1(\mathbb{R}^+)$ defined by the right side of (3) is continuous. So, there exists $l \in \mathbb{N}$ such that

$$\|M_x R^l(\lambda_0, A^\odot)x^\odot\|_1 \le C(x)\|x^\odot\|, \quad x^\odot \in C^\infty(A^\odot).$$

Using (1) one can obtain

$$\|M_x R^l(\lambda_0, A^\odot)x^\odot\|_1 \le C(x)\|x^\odot\|, \quad x^\odot \in X^\odot.$$

Thus the bilinear operator

$$\begin{aligned} B(x, x^\odot) &:= (T(t)R^m(\lambda_0, A)x, R^l(\lambda_0, A^\odot)x^\odot) \\ &= (T(t)R^{m+l}(\lambda_0, A)x, x^\odot), \quad (x, x^\odot) \in X \times X^\odot, \end{aligned}$$

is separately continuous on the Banach space $X \times X^\odot$. Consequently, there exists $C > 0$ such that

$$\|B(x, x^\odot)\|_1 \le C\|x\| \cdot \|x^\odot\|, \quad x \in X, \ x^\odot \in X^\odot.$$

In other words,

$$\int_0^\infty |(T(t)R^{m+l}(\lambda_0, A)x, x^\odot)|\,dt \le C\|x\| \cdot \|x^\odot\|;$$

hence

$$\Big|\Big(\int_0^a e^{its}T(t)R^{m+l}(\lambda_0, A)x\,dt, x^\odot\Big)\Big| \le C\|x\| \cdot \|x^\odot\|, \quad x \in X, \ x^\odot \in X^\odot,$$

for every $a \ge 0$ and every $s \in \mathbb{R}$. Using inequality (2) and the continuity of the function $T(t)R^{m+l}(\lambda_0, A)$ in the uniform operator topology we obtain

$$\sup\Big\{\Big\|\int_0^a e^{its}T(t)R^{m+l}(\lambda_0,A)\,dt\Big\| : a\ge 0,\ s\in\mathbb{R}\Big\}\le C_1. \tag{4}$$

With a reasoning similar to [4, Th. 2.16], we conclude that for $\lambda\in\sigma(A)$ there exists a character χ_λ of the Banach algebra $\mathcal{A}$ such that

$$\chi_\lambda(T(t)R^{m+l}(\lambda_0,A)) = e^{\lambda t}(\lambda_0-\lambda)^{-(m+l)}.$$

Hence, again taking into account the continuity of $T(t)R^{m+l}(\lambda_0,A)$ in $\mathcal{L}(X)$,

$$\begin{aligned} C_1 &\ge \sup\Big\{\Big\|\int_0^a e^{ist}T(t)R^{m+l}(\lambda_0,A)\,dt\Big\| : a\ge 0,\ s\in\mathbb{R}\Big\}\\ &\ge \sup\Big\{\Big|\int_0^a e^{ist}\chi_\lambda(T(t)R^{m+l}(\lambda_0,A))\,dt\Big| : a\ge 0,\ s\in\mathbb{R}\Big\}\\ &\ge \sup\Big\{\Big|\int_0^a e^{ist}e^{\lambda t}(\lambda_0-\lambda)^{-(m+l)}\,dt\Big| : a\ge 0,\ s\in\mathbb{R}\Big\}. \end{aligned}$$

Setting $s=-\operatorname{Im}\lambda$, we obtain

$$C_1 \ge \sup_{a\ge 0}\Big|\int_0^a e^{t\operatorname{Re}\lambda}(\lambda_0-\lambda)^{-(m+l)}\,dt\Big|$$

$$= \sup_{a\ge 0}\begin{cases} |\lambda_0-\lambda|^{-(m+l)}a, & \operatorname{Re}\lambda = 0, \quad (5)\\ |\lambda_0-\lambda|^{-(m+l)}\Big|\dfrac{e^{a\operatorname{Re}\lambda}-1}{\operatorname{Re}\lambda}\Big|, & \operatorname{Re}\lambda\ne 0. \quad (6)\end{cases}$$

Next, assume $s(A)\ge 0$ and consider two cases according to (5), (6). If there is $\lambda\in\sigma(A)$ with $\operatorname{Re}\lambda = 0$, then by (5) we immediately obtain a contradiction. Otherwise, consider two possibilities.

a) $\sigma(A)\cap\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda>0\}\ne\emptyset$. Then from (6) a contradiction follows.

b) $\sigma(A)\subset\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda<0\}$, but $s(A)=0$. Then there is $\{\lambda_n : n\ge 1\}\subset\sigma(A)$ such that $\operatorname{Re}\lambda_n\to 0$ as $n\to\infty$. From (6) we obtain

$$\begin{aligned} C_1 &\ge \sup_{n\ge 1}\Big\{|\lambda_0-\lambda_n|^{-(m+l)}\Big|\frac{e^{a_n\operatorname{Re}\lambda_n}-1}{\operatorname{Re}\lambda_n}\Big| : a_n = \frac{1}{|\operatorname{Re}\lambda_n|}\Big\}\\ &= \sup_{n\ge 1}\Big\{|\lambda_0-\lambda_n|^{-(m+l)}\Big|\frac{e^{-1}-1}{\operatorname{Re}\lambda_n}\Big|\Big\} = \infty. \end{aligned}$$

Thus we get a contradiction again. ■

The next theorem is in some sense dual to Theorem 2.

THEOREM 3. *Suppose that*

(ii) $\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda>0\}\subset\varrho(A)$, *and for all* $x\in C^\infty(A)$ *and* $x^*\in C^\infty(A^*)$,

$$\sup_{s>0}\int_{-\infty}^{\infty}|(R(s+it,A)x,x^*)|\,dt<\infty.$$

Then $s(A)<0$.

Proof. Fix $\lambda_0\in\varrho(A)$. Considering the bilinear operator

$$B_0(x,x^\odot) = (R(\lambda,A)x,x^\odot),\quad x\in C^\infty(A),\ x^\odot\in C^\infty(A^\odot),$$

with values in the Hardy space $H^1(\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda>0\})$, similarly to the proof of Theorem 2, one can show that

$$\sup_{s>0}\int_{s-i\infty}^{s+i\infty}|(R(\lambda,A)R^m(\lambda_0,A)x,x^\odot)|\,d\lambda \le C\|x\|\cdot\|x^\odot\|,\quad x\in X,\ x^\odot\in X^\odot, \tag{7}$$

for some $m\in\mathbb{N}$ and $C>0$. Then, for any $s>0$ and $(a,b)\subset\mathbb{R}$, we obtain

$$\Big|\Big(\int_{s+ia}^{s+ib}R(\lambda,A)R^m(\lambda_0,A)x\,d\lambda,x^\odot\Big)\Big|\le\int_{s+ia}^{s+ib}|(R(\lambda,A)R^m(\lambda_0,A)x,x^\odot)|\,d\lambda \le C\|x\|\cdot\|x^\odot\|.$$

Hence by (2),

$$\sup\Big\{\Big\|\int_{s+ia}^{s+ib}R(\lambda,A)R^m(\lambda_0,A)\,d\lambda\Big\| : s>0,\ a<b,\ a,b\in\mathbb{R}\Big\}\le C_1$$

for some $C_1>0$.

Suppose that $\sigma(A)\cap i\mathbb{R}\ne\emptyset$ and $\alpha\in\sigma(A)\cap i\mathbb{R}$. By the spectral mapping theorem for the resolvent, we have $1/(\lambda_0-\alpha)\in\sigma(R(\lambda_0,A))$. Let χ_α be the character of the algebra $\mathcal{A}$ such that $\chi_\alpha(R(\lambda_0,A)) = 1/(\lambda_0-\alpha)$. Now, the first resolvent identity implies $\chi_\alpha(R(\lambda,A)) = 1/(\lambda-\alpha)$, $\lambda\in\varrho(A)$. Hence

$$\begin{aligned} C_1 &\ge \sup\Big\{\Big\|\int_{s+ia}^{s+ib}R(\lambda,A)R^m(\lambda_0,A)\,d\lambda\Big\| : s>0,\ a<b,\ a,b\in\mathbb{R}\Big\}\\ &\ge \sup\Big\{\Big|\int_{s+ia}^{s+ib}\chi_\alpha(R(\lambda,A))\chi_\alpha(R^m(\lambda_0,A))\,d\lambda\Big| : s>0,\ a<b,\ a,b\in\mathbb{R}\Big\}\\ &= |\lambda_0-\alpha|^{-m}\sup\Big\{\Big|\int_{s+ia}^{s+ib}\frac{1}{\lambda-\alpha}\,d\lambda\Big| : s>0,\ a<b,\ a,b\in\mathbb{R}\Big\} \end{aligned}$$

$$= |\lambda_0 - \alpha|^{-m} \sup\left\{\left|\ln\frac{s+ib-\alpha}{s+ia-\alpha}\right| : s>0,\ a<b,\ a,b\in\mathbb{R}\right\}.$$

Setting $a = \operatorname{Im}\alpha$, $b = \operatorname{Im}\alpha + 1$, we obtain

$$C_1 \geq |\lambda_0 - \alpha|^{-m} \sup_{s>0}\left|\ln\frac{s+i(\operatorname{Im}\alpha+1)-\alpha}{s+i\operatorname{Im}\alpha-\alpha}\right|$$
$$= |\lambda_0 - \alpha|^{-m} \sup_{s>0}\left|\ln\frac{s+i}{s}\right| = \infty,$$

a contradiction.

Next suppose that $\sigma(A)\cap i\mathbb{R} = \emptyset$, but there exists a sequence $\{\alpha_n : n\geq 1\} \subset \sigma(A)$ such that $\operatorname{Re}\alpha_n \to 0$ as $n\to\infty$. Then, for the sequences χ_{α_n} and $a_n = \operatorname{Im}\alpha_n$, $b_n = \operatorname{Im}\alpha_n + 1$, $n\geq 1$, we have

$$C_1 \geq \sup_{s>0}\left|\int\limits_{s+ia_n}^{s+ib_n} \chi_{\alpha_n}(R(\lambda,A))\chi_{\alpha_n}(R(\lambda_0,A))\,d\lambda\right|$$
$$= |\lambda_0 - \alpha_n|^{-m}\sup_{s>0}\ln\left|\frac{s+i-\operatorname{Re}\alpha_n}{s-\operatorname{Re}\alpha_n}\right|, \qquad n\geq 1.$$

Letting $n\to\infty$, we obtain a contradiction. So $s(A)<0$. ∎

Remark 1. If $(T(t))_{t\geq 0}$ is continuous in the uniform operator topology for $t\geq t_0$ (for example, if it is differentiable or compact), then $s(A) = \omega_0$ (see [3]). Thus, each of the conditions (i), (ii) ensures the uniform stability of such semigroups.

Remark 2. For the discrete counterparts of Theorems 2 and 3, see [7, 9, 10, 15], and especially [11].

Remark 3. Conditions similar to (ii) were used in [10, 18] for the study of the exponential stability of C_0-semigroups.

Now we shall show that each of the conditions (i), (ii) implies the stability of the "sufficiently smooth" orbits of $(T(t))_{t\geq 0}$. The next statement, which is a special case of a result given in [1, p. 803], will be essentially used.

Theorem 4. *Assume that* $\sigma(A)\subset\{\lambda\in\mathbb{C} : \operatorname{Re}\lambda < 0\}$ *and* $\sup_{t\geq 0}\|T(t)S\| < \infty$ *for some* $S\in\mathcal{L}(X)$. *Then*

$$\|T(t)A^{-1}S\| \to 0 \quad \text{as } t\to\infty.$$

(For a survey of this type results, see also [2].)

Corollary 1. *Suppose that one of the conditions* (i), (ii) *is satisfied. Then there is* $m_0\in\mathbb{N}$ *such that*

$$\|T(t)R^{m_0}(\lambda_0,A)\| \to 0 \quad \text{as } t\to\infty,$$

for some $\lambda_0\in\varrho(A)$.

(Observe that, in view of the first resolvent identity, the conclusion of Corollary 1 does not depend on the choice of $\lambda_0\in\varrho(A)$.)

Proof. If (i) or (ii) is true then, according to Theorems 2 and 3, we have $s(A)<0$. Next we prove that each of the conditions (i), (ii) implies the existence of $n\in\mathbb{N}$ such that $\sup_{t\geq 0}\|T(t)A^{-n}\| < \infty$. Then from Theorem 4 we shall obtain the required conclusion for $\lambda_0 = 0$.

(a) Assume (i) holds. Then (4) holds for some $m\in\mathbb{N}$ and $C_1>0$. In particular, we have

$$\sup_{t\geq 0}\left\|\int\limits_0^t T(s)A^{-m}\,ds\right\| \leq C_1.$$

From the identity

$$T(t)A^{-(m+1)} - A^{-(m+1)} = \int\limits_0^t T(s)A^{-m}\,ds \qquad (m\in\mathbb{N})$$

we obtain

$$\sup_{t\geq 0}\|T(t)A^{-(m+1)}\| \leq \|A^{-(m+1)}\| + \sup_{t\geq 0}\left\|\int\limits_0^t T(s)A^{-m}\,ds\right\|$$
$$\leq \|A^{-(m+1)}\| + C_1.$$

The desired result follows for $m_0 = m+1$.

(b) Assume (ii) is true. Then there are $m\in\mathbb{N}$ and $C>0$ such that (7) holds for $\lambda_0 = 0$. Fix $x\in X$ and $x^{\odot}\in X^{\odot}$. According to (7), the function $(R(\lambda,A)A^{-m}x, x^{\odot})$ belongs to the Hardy space $H^1(\{\lambda\in\mathbb{C} : \operatorname{Re}\lambda > 0\})$. Therefore, it is bounded in every halfplane $\{\lambda\in\mathbb{C} : \operatorname{Re}\lambda\geq\varepsilon\}$, $\varepsilon>0$ (see [6], for example). On the other hand, using the first part of the proof of Theorem 2.1 in [13] we obtain the representation

$$T(t)A^{-m}x - \sum_{k=0}^{m-1}\frac{A^{-(m-k)}t^k}{k!}x$$
$$= \frac{1}{2\pi i}\int\limits_{s-i\infty}^{s+i\infty}\frac{e^{\lambda t}R(\lambda,A)}{\lambda^m}x\,d\lambda, \qquad s>\max(0,\omega_0),\ x\in X$$

(the integral exists in the principal value sense). Then from the identity

$$R(\lambda,A)A^{-m} = \lambda^{-m}R(\lambda,A) + \sum_{k=0}^{m-1}\lambda^{-(k+1)}A^{-(m-k)}$$

and the Jordan lemma, shifting the contour of integration we have for every $\varepsilon>0$,

$$T(t)A^{-m}x = \frac{1}{2\pi i}\int_{s-i\infty}^{s+i\infty} e^{\lambda t}R(\lambda,A)A^{-m}x\,d\lambda, \qquad s>\varepsilon.$$

Hence we obtain

$$|(T(t)A^{-m}x, x^{\odot})| \le \frac{1}{2\pi}e^{2\varepsilon t}\int_{s-i\infty}^{s+i\infty} |(R(\lambda,A)A^{-m}x, x^{\odot})|\,d\lambda, \qquad s>\varepsilon.$$

Letting $\varepsilon \to 0+$, we get

$$|(T(t)A^{-m}x, x^{\odot})| \le \frac{1}{2\pi}C\|x\|\cdot\|x^{\odot}\|, \quad C>0.$$

From (2) the necessary statement follows for $m_0 = m$. ∎

Finally, we give one more result, which includes conditions different from those of Theorems 2 and 3, but has the same flavour. In [12] A. Pazy has proved the following

THEOREM 5. *Suppose that for every $x \in X$,*

(iii) $$\int_0^\infty \|T(t)x\|^p\,dt < \infty, \qquad p\in[1,\infty).$$

Then $\omega_0 < 0$.

Remark 4. If condition (iii) holds for a fixed $x\in X$, then $\|T(t)x\| \to 0$ as $t\to\infty$. This was observed by G. Weiss [16].

Our $C^\infty(A)$-version of Theorem 5 is as follows.

PROPOSITION 1. *Suppose that for every $x \in C^\infty(A)$,*

$$\int_0^\infty \|T(t)x\|^p\,dt < \infty, \qquad p\in[1,\infty).$$

Then $s(A) < 0$.

Proof. Fix $\lambda_0 \in \varrho(A)$. By a reasoning similar to the proof of Theorem 2 we can show that there are $m\in\mathbb{N}$ and $C>0$ such that

$$\int_0^\infty \|T(t)R^m(\lambda_0,A)x\|^p\,dt \le C\|x\|^p, \qquad x\in X.$$

In view of Remark 4, we have $\|T(t)R^m(\lambda_0,A)x\| \to 0$ as $t\to\infty$, for every $x\in X$. The uniform boundedness principle implies

$$\sup_{t\ge0}\|T(t)R^m(\lambda_0,A)\| \le C$$

for some $C>0$. Now, following the proof of Pazy's theorem from [3], we obtain

$$\begin{aligned}t\|T(t)R^{2m}(\lambda_0,A)x\|^p &= \int_0^t \|T(t)R^{2m}(\lambda_0,A)x\|^p\,ds\\ &\le \int_0^t \|T(t-s)R^m(\lambda_0,A)\|^p\|T(s)R^m(\lambda_0,A)x\|^p\,ds\\ &\le C^p\int_0^t \|T(s)R^m(\lambda_0,A)x\|^p\,ds.\end{aligned}$$

Again by the uniform boundedness principle

(8) $$\|T(t)R^{2m}(\lambda_0,A)\| \to 0 \quad \text{as } t\to\infty.$$

Next, as in the final part of the proof of Theorem 2, for every $\lambda\in\sigma(A)$ there is a character χ_λ of the Banach algebra $\mathcal{A}$ such that

$$\chi_\lambda(T(t)R^{2m}(\lambda_0,A)) = e^{\lambda t}(\lambda_0-\lambda)^{-2m}.$$

Since

$$\begin{aligned}\|T(t)R^{2m}(\lambda_0,A)\| &\ge \sup_{\lambda\in\sigma(A)}|\chi_\lambda(T(t)R^{2m}(\lambda_0,A))|\\ &= \sup_{\lambda\in\sigma(A)}|e^{\lambda t}(\lambda_0-\lambda)^{-2m}| = \sup_{\lambda\in\sigma(A)} e^{t\,\mathrm{Re}\,\lambda}|\lambda_0-\lambda|^{-2m},\end{aligned}$$

by (8) we obtain $s(A)<0$. ∎

3. We shall show that the conclusions of Theorems 2 and 3 are not reversible.

EXAMPLE 1. We use the construction of the classical example due to Zabczyk [19]. Consider the Hilbert space $X = \bigoplus_{n=1}^\infty \mathbb{C}^n$ and the C_0-semigroup $(T(t))_{t\ge0}$ on X given by

$$T(t) = \bigoplus_{n=1}^\infty e^{(in-1/3)t}e^{A_n t},$$

where $A_n = (a_{ij})_{i,j=1}^n$ is the $n\times n$-matrix with $a_{i,i+1}=1$, $i=1,\dots,n-1$, and $a_{ij}=0$ otherwise.

From the reasoning in [19] it follows that the generator of $(T(t))_{t\ge0}$ is defined by

$$A = \bigoplus_{n=1}^\infty (A_n + (in-1/3))$$

and has the domain $\{(x^n)_{n=1}^\infty \in X : (nx^n)_{n=1}^\infty \in X\}$ and the spectrum $\{in-1/3 : n\in\mathbb{N}\}$. So $s(A) = -1/3$.

Next, it is easy to see that $x_0 = (x_0^n)_{n=1}^{\infty}$ with $x_0^n = (e^{-n/2}, \ldots, e^{-n/2})$, $n \ge 1$, belongs to $C^\infty(A) \cap C^\infty(A^*)$. Moreover, we have

$$(T(t)x, x) \ge e^{-t/3} \sum_{n=0}^{\infty} e^{-n} \sum_{k=0}^{n} \frac{t^k}{k!}, \quad t \ge 0.$$

Observe that the series on the right side is uniformly convergent on compact sets in $\mathbb{R}$. So for $f(t) := e^{-t/3} \sum_{n=0}^{\infty} e^{-n} \sum_{k=0}^{n} t^k/k!$, we have

$$f'(t) = -\frac{1}{3} f(t) + e^{-t/3} \sum_{n=0}^{\infty} e^{-(n+1)} \sum_{k=0}^{n} \frac{t^k}{k!} = (e^{-1} - 1/3) f(t),$$

$$f(0) = \sum_{n=0}^{\infty} e^{-n} = \frac{e}{e-1},$$

hence

$$f(t) = \frac{e}{e-1} e^{(e^{-1}-1/3)t}.$$

Thus, for $x = x^* = x_0$ condition (i) is violated.

Remark 4. Similarly one can show that Proposition 1 is not reversible.

EXAMPLE 2. Consider a diagonal operator A on $\mathbb{C}^n$ with the spectrum $\{\lambda_1, \ldots, \lambda_n\}$ in $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda < 0\}$. Obviously, such an operator has a negative spectral bound but its resolvent $R(\lambda, A)$ defined by

$$R(\lambda, A)x = \left(\frac{1}{\lambda_i - \lambda} x_i \right)_{i=1}^{n}, \quad x = (x_i)_{i=1}^{n} \in \mathbb{C}^n,$$

is not integrable in the sense of condition (ii).

Acknowledgments. I would like to thank the referee for his helpful suggestions and remarks, and Prof. J. Zemánek for providing me with the references [10–12].

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Received November 10, 1995 (3570)
Revised version January 6, 1997