On the spectral bound of the generator of a $C_0$-semigroup

by

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Abstract. We give several conditions implying that the spectral bound of the generator of a $C_0$-semigroup is negative. Applications to stability theory are considered.

1. Let $X$ be a complex Banach space, $(T(t))_{t \geq 0}$ a $C_0$-semigroup in $X$, and $A$ its generator with the domain $D(A)$, the spectrum $\sigma(A)$ and the resolvent set $\rho(A)$. Denote by $R(\lambda, A) := (\lambda - A)^{-1}$, $\lambda \in \rho(A)$, the resolvent of the operator $A$, by $s(A) := \sup \{ \Re \lambda : \lambda \in \sigma(A) \}$ the spectral bound of $A$, and by $\omega_0 := \lim_{t \to -\infty} \ln \| T(t) \| / t$ the type of the semigroup $(T(t))_{t \geq 0}$. Finally, for a linear operator $A$, denote by $C^\infty(A)$ the set $\bigcap_{n=1}^\infty D(A^n)$. Among other results, G. Weiss has proved in [14] the following

**Theorem 1.** Suppose that for every $x \in X$ and every $x^* \in X^*$,

$$\int_0^\infty \|(T(t)x, x^*)\|^p \, dt < \infty, \quad p \in [1, \infty).$$

Then $s(A) < 0$.

(For $p = 1$ this result appeared also in [3, Ch. 7].)

The main purpose of the present paper is to show that for the case $p = 1$ the conditions of Theorem 1 can be essentially weakened. Namely, we require

$$\int_0^\infty \|(T(t)x, x^*)\| \, dt < \infty$$

only for $x \in C^\infty(A)$ and $x^* \in C^\infty(A^*)$. It is also established that if $\{ \lambda \in \mathbb{C} : \Re \lambda > 0 \} \subset \rho(A)$ and

$$\sup_{s>0} \int_{-\infty}^\infty \|(R(s + it, A)x, x^*)\| \, dt < \infty$$

for $x \in C^\infty(A)$ and $x^* \in C^\infty(A)$, then a similar assertion holds. Combining our results with the results of [1] we derive a criterion for some kind of stability of $(T(t))_{t \geq 0}$.

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Recent developments related to the subject of this paper can be found in [10]. The role of $C^\infty(A)$ for stability theory was studied in [17].

2. For our considerations we shall need some background from the theory of operator semigroups. First, we recall some properties of the sets $C^\infty(A)$ and $C^\infty(A^*)$. It is well known that $C^\infty(\bar{A}) = X$. But, in general, the closure of $C^\infty(A^*)$ is not the whole $X^*$. This is due to the fact that, in general, the semigroup $(T^*(t))_{t \geq 0}$ is not strongly continuous on $X^*$. At the same time, the set

$$X^0 = \{ x^* \in X^* : T^*(t)x^* \text{ is continuous, } t \geq 0 \}$$

is equal to $D(A^*)$, and it is a $T^*(t)$-invariant subspace of $X^*$, $t \geq 0$; $T^*(t)|_{X^0}$ is a $C_0$-semigroup with the generator $A^0 = A^*|_{X^0}$, called the sun-dual of $(T(t))_{t \geq 0}$. Thus

$$\overline{C^\infty(A^0)} = X^0, \quad C^\infty(A^0) \subset C^\infty(A^*).$$

(In fact, $C^\infty(A^0) = C^\infty(A^*)$. See [8] for the proof.) Furthermore, the subspace $X^0 \subset X^*$ induces on $X$ the prime-norm

$$\|x\| := \sup \{|(x, x^0) : x^0 \in X^0, \|x^0\| \leq 1\}, \quad x \in X,$

which is equivalent to the original one. Therefore, there is $C > 0$ such that

$$\|x\| \leq C\|x\|^\prime.$$  

For all these facts, we refer the reader to [3].

Now, let $L(X)$ be the Banach algebra of bounded linear operators on $X$ and let $A$ be a maximal abelian subalgebra of $L(X)$ containing $(T(t))_{t \geq 0}$. It is known that $A$ is closed under limits in the strong operator topology. So the set $\{ R(\lambda, A) : \lambda \in \rho(A) \}$ is contained in $A$. Moreover, for every $S \in A$ we have

$$\sigma_A(S) = \sigma(S).$$

These basic facts can be found in [4]. For more details, see [5].

We start with the following statement.

**Theorem 2.** Suppose that

(i) for all $x \in C^\infty(A)$ and $x^* \in C^\infty(A^*)$,

$$\int_0^\infty \|(T(t)x, x^*)\| \, dt < \infty.$$ 

Then $s(A) < 0$.

**Proof.** Fix $x^* \in C^\infty(A^*)$. Consider the linear operator $M_{x^*} : C^\infty(A) \to L_1(\mathbb{R}^+)$ defined by

$$M_{x^*}x := (T(t)x, x^*), \quad t \geq 0.$$ 

The operator $M_{x^*}$ is closed by standard arguments. Since $C^\infty(A)$ can be regarded as a Fréchet space with the system of norms

$$\|x\| = \| (\lambda_0 - A)^nx \|, \quad n \in \mathbb{N}, \quad \lambda_0 \in \rho(A) \text{ fixed},$$

the closed graph theorem implies the continuity of $M_{x^*}$ on $C^\infty(A)$. Therefore, there is some $m \in \mathbb{N}$ such that

$$\|M_{x^*}x\|_1 \leq C(x^*)\| (\lambda_0 - A)^m x \|, \quad C(x^*) > 0,$$

(here $\| \cdot \|_1$ stands for the $L_1$-norm) or

$$\|M_{x^*}R^m(\lambda_0, A)x\|_1 \leq C(x^*)\|x\|, \quad x \in C^\infty(A).$$

The last inequality implies that the linear operator $M_{x^*}R^m(\lambda_0, A)$ is continuous on $C^\infty(A)$ with the topology induced from $X$. Therefore, from $C^\infty(A) = X$ it follows that

$$\|M_{x^*}R^m(\lambda_0, A)x\|_1 \leq C(x^*)\|x\|, \quad x \in X.$$

Now, fix $x \in C^\infty(A)$. Then, in the same way, the linear operator $M_x : C^\infty(A^0) \to L_1(\mathbb{R}^+)$ defined by the right side of (3) is continuous. So, there exists $l \in \mathbb{N}$ such that

$$\|M_x R^l(\lambda_0, A^0)x^0\|_1 \leq C(x)\|x^0\|, \quad x^0 \in C^\infty(A^0).$$

Using (1) one can obtain

$$\|M_x R^l(\lambda_0, A^0)x^0\|_1 \leq C(x)\|x^0\|, \quad x^0 \in X^0.$$ 

Thus the bilinear operator

$$B(x, x^0) := (T(t)R^m(\lambda_0, A)x, R^l(\lambda_0, A^0)x^0)$$

$$= (T(t)R^{m+l}(\lambda_0, A)x, x^0), \quad (x, x^0) \in X \times X^0,$$

is separately continuous on the Banach space $X \times X^0$. Consequently, there exists $C > 0$ such that

$$\|B(x, x^0)\|_1 \leq C\|x\| \cdot \|x^0\|, \quad x \in X, \quad x^0 \in X^0.$$ 

In other words,

$$\int_0^\infty \|(T(t)R^{m+l}(\lambda_0, A)x, x^0)\| \, dt \leq C\|x\| \cdot \|x^0\|,$$

hence

$$\left| \left\{ e^{s(t)}(T(t)R^{m+l}(\lambda_0, A)x, x^0) \right\} \right| \leq C\|x\| \cdot \|x^0\|, \quad x \in X, \quad x^0 \in X^0,$$

for every $c \geq 0$ and every $s \in \mathbb{R}$. Using inequality (2) and the continuity of
the function $T(t)R^{m+1}(\lambda_0, A)$ in the uniform operator topology we obtain

\[
(4) \quad \sup \left\{ \left\| \int_0^a e^{int} T(t) R^{m+1}(\lambda_0, A) \, dt \right\| : a \geq 0, \ s \in \mathbb{R} \right\} \leq C_1.
\]

With a reasoning similar to [4, Th. 2.16], we conclude that for $\lambda \in \sigma(A)$ there exists a character $\chi_\lambda$ of the Banach algebra $A$ such that

\[
\chi_\lambda(T(t)R^{m+1}(\lambda_0, A)) = e^{\lambda t}(\lambda - \lambda)^{-(m+1)}.
\]

Hence, again taking into account the continuity of $T(t)R^{m+1}(\lambda_0, A)$ in $L(X)$,

\[
C_1 \geq \sup \left\{ \left\| \int_0^a e^{int} \chi_\lambda(T(t)R^{m+1}(\lambda_0, A)) \, dt \right\| : a \geq 0, \ s \in \mathbb{R} \right\}
\]

\[
\geq \sup \left\{ \left\| \int_0^a \left( e^{int} \chi_\lambda(\lambda_0 - \lambda) \right)^{-(m+1)} \, dt \right\| : a \geq 0, \ s \in \mathbb{R} \right\}
\]

\[
\geq \sup \left\{ \left\| e^{int} e^{it\lambda} (\lambda_0 - \lambda)^{-(m+1)} \right\| : a \geq 0, \ s \in \mathbb{R} \right\}.
\]

Setting $s = -\text{Im} \lambda$, we obtain

\[
C_1 \geq \sup_{a \geq 0} \left\| \int_0^a e^{it \text{Re} \lambda} (\lambda_0 - \lambda)^{-(m+1)} \, dt \right\|
\]

\[
= \sup_{a \geq 0} \left\{ \left| \lambda_0 - \lambda \right|^{-(m+1)} a \right\} \text{Re} \lambda = 0,
\]

\[
\text{Re} \lambda \neq 0.
\]

Next, assume $s(A) \geq 0$ and consider two cases according to (5), (6). If there is $\lambda \in \sigma(A)$ with $\text{Re} \lambda = 0$, then by (5) we immediately obtain a contradiction. Otherwise, consider two possibilities.

a) $\sigma(A) \cap \{ \lambda \in \mathbb{C} : \text{Re} \lambda > 0 \} \neq \emptyset$. Then from (6) a contradiction follows.

b) $\sigma(A) \subset \{ \lambda \in \mathbb{C} : \text{Re} \lambda < 0 \}$, but $s(A) = 0$. Then there is $\lambda_n : n \geq 1 \subset \sigma(A)$ such that $\text{Re} \lambda_n \to 0$ as $n \to \infty$. From (6) we obtain

\[
C_1 \geq \sup_{n \geq 1} \left\{ \left| \lambda_0 - \lambda_n \right|^{-(m+1)} \left| \frac{e^{n \text{Re} \lambda_n} - 1}{\text{Re} \lambda_n} \right| a_n = \frac{1}{|\text{Re} \lambda_n|} \right\}
\]

\[
= \sup_{n \geq 1} \left\{ \left| \lambda_0 - \lambda_n \right|^{-(m+1)} \left| \frac{e^{-1} - 1}{\text{Re} \lambda_n} \right| \right\} = \infty.
\]

Thus we get a contradiction again. \n
The next theorem is in some sense dual to Theorem 2.

**Theorem 3.** Suppose that

\[
\{ \lambda \in \mathbb{C} : \text{Re} \lambda > 0 \} \subset \rho(A), \text{ and for all } x \in C^\infty(A) \text{ and } x^* \in C^\infty(A^*),
\]

\[
\sup_{s > 0} \int_{-\infty}^\infty \left\| (R(s + it, A)x, x^*) \right\| dt < \infty.
\]

Then $s(A) < 0$.

**Proof.** Fix $\lambda_0 \in \rho(A)$. Considering the bilinear operator

\[
B_0(x, x^0) = (R(\lambda, A)x, x^0), \quad x \in C^\infty(A), \ x^0 \in C^\infty(A^0),
\]

with values in the Hardy space $H^1(\{ \lambda \in \mathbb{C} : \text{Re} \lambda > 0 \})$, similarly to the proof of Theorem 2, one can show that

\[
(7) \quad \sup_{s > 0} \int_{s \to -\infty} \int_{s \to +\infty} \left\| (R(\lambda, A)^m(\lambda_0, A)x, x^0) \right\| d\lambda
\]

\[
\leq C\|x\| \cdot \|x^0\|, \quad \text{for some } m \in \mathbb{N} \text{ and } C > 0.
\]

Then, for any $s > 0$ and $(a, b) \in \mathbb{R}$, we obtain

\[
\left\| \left( \int_{a \to +\infty}^{s + b} \int_{a \to -\infty}^{s + a} R(\lambda, A)^m(\lambda_0, A)x \lambda d\lambda \right) \right\| \leq C\|x\| \cdot \|x^0\|
\]

Hence by (2),

\[
\sup_{s > 0} \int_{s \to a}^{s \to b} \int_{s \to a}^{s \to b} \left\| (R(\lambda, A)^m(\lambda_0, A)x, x^0) \right\| d\lambda
\]

\[
\leq C\|x\| \cdot \|x^0\|
\]

for some $C_1 > 0$.

Suppose that $\sigma(A) \cap \mathbb{R} \neq \emptyset$ and $\alpha \in \sigma(A) \cap \mathbb{R}$. By the spectral mapping theorem for the resolvent, we have $1/(\lambda_0 - \alpha) \in \sigma(R(\lambda_0, A))$. Let $\chi_\alpha$ be the character of the algebra $A$ such that $\chi_\alpha(R(\lambda_0, A)) = 1/(\lambda_0 - \alpha)$. Now, the first resolvent identity implies $\chi_\alpha(R(\lambda, A)) = 1/(\lambda - \alpha)$, $\lambda \in \rho(A)$. Hence

\[
C_1 \geq \sup_{s > 0} \int_{s \to a}^{s \to b} \int_{s \to a}^{s \to b} \left\| R(\lambda, A)^m(\lambda_0, A) \right\| d\lambda
\]

\[
\geq \sup_{s > 0} \int_{s \to a}^{s \to b} \chi_\alpha(R(\lambda, A)) \chi_\alpha(R(\lambda_0, A)) d\lambda
\]

\[
= |\lambda_0 - \alpha|^{-m} \sup_{s > 0} \int_{s \to a}^{s \to b} \frac{1}{\lambda - \alpha} d\lambda
\]

\[
= |\lambda_0 - \alpha|^{-m} \sup_{s > 0} \int_{s \to a}^{s \to b} \frac{1}{\lambda - \alpha} d\lambda
\]

\[
\leq C\|x\| \cdot \|x^0\|
\]
\[ = |\lambda_0 - \alpha|^{-m} \sup_{s > 0, \ a < b, \ a, b \in \mathbb{R}} \left\{ \ln \frac{s + ib - \alpha}{s + ia - \alpha} : s > 0, \ a < b, \ a, b \in \mathbb{R} \right\}. \]

Setting \( a = \text{Im} \alpha, \ b = \text{Im} \alpha + 1, \) we obtain
\[ C_1 \geq |\lambda_0 - \alpha|^{-m} \sup_{s > 0} \left| \ln \frac{s + i(\text{Im} \alpha + 1) - \alpha}{s + i\text{Im} \alpha - \alpha} \right| = |\lambda_0 - \alpha|^{-m} \sup_{s > 0} \left| \frac{s + i - \text{Re} \alpha_n}{s - \text{Re} \alpha_n} \right|, \ n \geq 1. \]

Letting \( n \to \infty, \) we obtain a contradiction. So \( s(A) < 0. \]

**Remark 1.** If \((T(t))_{t \geq 0}\) is continuous in the uniform operator topology for \( t \geq t_0 \) (for example, if it is differentiable or compact), then \( s(A) = \omega_0 \) (see [3]). Thus, each of the conditions (i), (ii) ensures the uniform stability of such semigroups.

**Remark 2.** For the discrete counterparts of Theorems 2 and 3, see [7, 9, 10, 15], and especially [11].

**Remark 3.** Conditions similar to (ii) were used in [10, 18] for the study of the exponential stability of \( C_0\)-semigroups.

Now we shall show that each of the conditions (i), (ii) implies the stability of the "sufficiently smooth" orbits of \((T(t))_{t \geq 0}\). The next statement, which is a special case of a result given in [1, p. 803], will be essentially used.

**Theorem 4.** Assume that \( \sigma(A) \subseteq \{ \lambda \in \mathbb{C} : \text{Re} \lambda < 0 \} \) and \( \sup_{t \geq 0} \| T(t)S \| < \infty \) for some \( S \in \mathcal{L}(X). \) Then
\[ \| T(t)A^{-m}S \| \to 0 \quad \text{as} \quad t \to \infty. \]

(For a survey of this type results, see also [2].)

**Corollary 1.** Suppose that one of the conditions (i), (ii) is satisfied. Then there is \( m_0 \in \mathbb{N} \) such that
\[ \| T(t)R^{m_0}(\lambda_0, A) \| \to 0 \quad \text{as} \quad t \to \infty, \]

for some \( \lambda_0 \in \sigma(A). \) (Observe that, in view of the first resolvent identity, the conclusion of Corollary 1 does not depend on the choice of \( \lambda_0 \in \sigma(A). \))

**Proof.** If (i) or (ii) is true then, according to Theorems 2 and 3, we have \( s(A) < 0. \) Next we prove that each of the conditions (i), (ii) implies the existence of \( \epsilon \in \mathbb{N} \) such that \( \sup_{t \geq 0} \| T(t)A^{-n} \| < \infty. \) Then from Theorem 4 we shall obtain the required conclusion for \( \lambda_0 = 0. \)

(a) Assume (i) holds. Then (4) holds for some \( m \in \mathbb{N} \) and \( C_1 > 0. \) In particular, we have
\[ \sup_{t \geq 0} \left\| \int_0^t T(s)A^{-m}ds \right\| \leq C_1. \]

From the identity
\[ T(t)A^{-(m+1)} - A^{-(m+1)} = \int_0^t T(s)A^{-m}ds \quad (m \in \mathbb{N}) \]

we obtain
\[ \sup_{t \geq 0} \| T(t)A^{-(m+1)} \| \leq \| A^{-(m+1)} \| + \sup_{t \geq 0} \left\| \int_0^t T(s)A^{-m}ds \right\| \leq \| A^{-(m+1)} \| + C_1. \]

The desired result follows for \( m_0 = m + 1. \)

(b) Assume (ii) is true. Then there are \( m \in \mathbb{N} \) and \( C > 0 \) such that (7) holds for \( A_0 = 0. \) Fix \( x \in X \) and \( x^0 \in X^0. \) According to (7), the function \((R(\lambda, A)A^{-m}x, x^0)\) belongs to the Hardy space \( H^1(\{ \lambda \in \mathbb{C} : \text{Re} \lambda > 0 \}). \) Therefore, it is bounded in every halfplane \( \{ \lambda \in \mathbb{C} : \text{Re} \lambda > \varepsilon \}, \ \varepsilon > 0 \) (see [6], for example). On the other hand, using the first part of the proof of Theorem 2.1 in [13] we obtain the representation
\[ T(t)A^{-m}x = -\sum_{k=0}^{m-1} \frac{A^{-(m-k)}}{k!} x \]
\[ = \frac{1}{2\pi i} \int_{s-i\infty}^{s+i\infty} e^{\lambda t} R(\lambda, A)A^{-m} x d\lambda, \quad s > \max(0, \omega_0), \ x \in X \]

(the integral exists in the principal value sense). Then from the identity
\[ R(\lambda, A)A^{-m} = \lambda^{-m} R(\lambda, A) + \sum_{k=0}^{m-1} \lambda^{-(k+1)} A^{-(m-k)} \]

and the Jordan lemma, shifting the contour of integration we have for every \( \varepsilon > 0, \)
\[ T(t)A^{-m}x = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{s't} R(\lambda, A)A^{-m}x \, d\lambda, \quad s > \varepsilon. \]

Hence we obtain
\[ |(T(t)A^{-m}x, x^0)| \leq \frac{1}{2\pi} e^{2\varepsilon t} \int_{-\infty}^{\infty} |(R(\lambda, A)A^{-m}x, x^0)| \, d\lambda, \quad s > \varepsilon. \]

Letting \( \varepsilon \to 0^+ \), we get
\[ |(T(t)A^{-m}x, x^0)| \leq \frac{1}{2\pi} C\|x\| \cdot \|x^0\|, \quad C > 0. \]

From (2) the necessary statement follows for \( m_0 = m \).

Finally, we give one more result, which includes conditions different from those of Theorems 2 and 3, but has the same flavour. In [12] A. Pazy has proved the following

**Theorem 5.** Suppose that for every \( x \in X \),

(iii) \[ \int_0^\infty \|T(t)x\|^p \, dt < \infty, \quad p \in [1, \infty). \]

Then \( \omega_0 < 0 \).

**Remark 4.** If condition (iii) holds for a fixed \( x \in X \), then \( \|T(t)x\| \to 0 \) as \( t \to \infty \). This was observed by G. Weiss [16].

Our \( C^\infty(A) \)-version of Theorem 5 is as follows.

**Proposition 1.** Suppose that for every \( x \in C^\infty(A) \),

\[ \int_0^\infty \|T(t)x\|^p \, dt < \infty, \quad p \in [1, \infty). \]

Then \( s(A) < 0 \).

**Proof.** Fix \( \lambda_0 \in \sigma(A) \). By a reasoning similar to the proof of Theorem 2 we can show that there are \( m \in \mathbb{N} \) and \( C > 0 \) such that

\[ \int_0^\infty \|T(t)R^m(\lambda_0, A)x\|^p \, dt \leq C\|x\|^p, \quad x \in X. \]

In view of Remark 4, we have \( \|T(t)R^m(\lambda_0, A)x\| \to 0 \) as \( t \to \infty \), for every \( x \in X \). The uniform boundedness principle implies

\[ \sup_{t \geq 0} \|T(t)R^m(\lambda_0, A)\| \leq C \]

for some \( C > 0 \). Now, following the proof of Pazy’s theorem from [3], we obtain

\[ t\|T(t)R^{2m}(\lambda_0, A)x\|^p = \int_0^t \|T(t)R^{2m}(\lambda_0, A)x\|^p \, ds \]

\[ \leq \int_0^t \|T(t-s)R^m(\lambda_0, A)\|^p \|T(s)R^m(\lambda_0, A)x\|^p \, ds \]

\[ \leq C^p \int_0^t \|T(s)R^m(\lambda_0, A)x\|^p \, ds. \]

Again by the uniform boundedness principle

\[ \|T(t)R^{2m}(\lambda_0, A)\| \to 0 \quad \text{as} \quad t \to \infty. \]

Next, as in the final part of the proof of Theorem 2, for every \( \lambda \in \sigma(A) \) there is a character \( \chi_\lambda \) of the Banach algebra \( A \) such that

\[ \chi_\lambda(\|T(t)R^{2m}(\lambda_0, A)\|) = e^{\lambda t}(\lambda_0 - \lambda)^{-2m}. \]

Since

\[ \|T(t)R^m(\lambda_0, A)\| \geq \sup_{\lambda \in \sigma(A)} |\chi_\lambda(T(t)R^{2m}(\lambda_0, A))| \]

\[ = \sup_{\lambda \in \sigma(A)} |e^{\lambda t}(\lambda_0 - \lambda)^{-2m}| = \sup_{\lambda \in \sigma(A)} e^{\Re \lambda |\lambda_0 - \lambda|^{-2m}}, \]

by (8) we obtain \( s(A) < 0 \).

3. We shall show that the conclusions of Theorems 2 and 3 are not reversible.

**Example 1.** We use the construction of the classical example due to Zabczyk [19]. Consider the Hilbert space \( X = \bigoplus_{n=1}^\infty C^0 \) and the \( C_0 \)-semigroup \( (T(t))_{t \geq 0} \) on \( X \) given by

\[ T(t) = \bigoplus_{n=1}^\infty e^{(in-1/3)t}A_n t, \]

where \( A_n = (a_{ij})_{i,j=1}^n \) is the \( n \times n \)-matrix with \( a_{i,i+1} = 1, i = 1, \ldots, n-1 \), and \( a_{ij} = 0 \) otherwise.

From the reasoning in [19] it follows that the generator of \( (T(t))_{t \geq 0} \) is defined by

\[ A = \bigoplus_{n=1}^\infty (A_n + (in-1/3)) \]

and has the domain \( \{(x_n)_{n=1}^\infty \in X : (nx_n)_{n=1}^\infty \in X \} \) and the spectrum \( \{in-1/3 : n \in \mathbb{N} \} \). So \( s(A) = -1/3 \).
Next, it is easy to see that \( x_0 = (e^{-n/2})_{n=1}^{\infty} \) with \( x_0^n = (e^{-n/2}, \ldots, e^{-n/2}) \), \( n \geq 1 \), belongs to \( C^\infty(A) \cap C^\infty(A^*) \). Moreover, we have
\[
(T(t)x, x) \geq e^{-t/3} \sum_{n=0}^{\infty} e^{-n} \sum_{k=0}^{n} \frac{t^k}{k!}, \quad t \geq 0.
\]
Observe that the series on the right side is uniformly convergent on compact sets in \( \mathbb{R} \). So for \( f(t) := e^{-t/3} \sum_{n=0}^{\infty} e^{-n} \sum_{k=0}^{n} \frac{t^k}{k!} \), we have
\[
f'(t) = \frac{1}{3} f(t) + e^{-t/3} \sum_{n=0}^{\infty} e^{-(n-1)} \sum_{k=0}^{n} \frac{t^k}{k!} = (e^{-1} - 1/3)f(t),
\]
\[
f(0) = \sum_{n=0}^{\infty} e^{-n} = \frac{e}{e - 1},
\]
and hence
\[
f(t) = \frac{e}{e - 1} e^{(e-1-1/3)t}.
\]
Thus, for \( x = x^* = x_0 \) condition (i) is violated.

Remark 4. Similarly one can show that Proposition 1 is not reversible.

Example 2. Consider a diagonal operator \( A \) on \( \mathbb{C}^n \) with the spectrum \( \{\lambda_1, \ldots, \lambda_n\} \in \mathbb{C} \subseteq \mathbb{R} \lambda < 0 \). Obviously, such an operator has a negative spectral bound but its resolvent \( R(\lambda, A) \) defined by
\[
R(\lambda, A) x = \left( \frac{1}{\lambda_i - \lambda} x_i \right)_{i=1}^{n}, \quad x = (x_i)_{i=1}^{n} \in \mathbb{C}^n,
\]
is not integrable in the sense of condition (ii).

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References