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# Constructions of cocycles over irrational rotations

by

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**Abstract.** We construct a coboundary cocycle which is of bounded variation, is homotopic to the identity and is Hölder continuous with an arbitrary Hölder exponent smaller than 1.

**Introduction.** This paper is a continuation of investigations from [5] and is devoted to constructions of Hölder continuous cocycles with nonzero topological degree which are coboundaries over some irrational rotations. We recall that Furstenberg [2] proved that no Lipschitz continuous cocycle with nonzero degree is a coboundary. The Lipschitz condition can be weakened to the absolute continuity (see [3], [4], [8]). However, in [5] a construction of a bounded variation continuous coboundary cocycle with nonzero degree has been presented showing that further weakenings are not possible. Here, by a refinement of the construction from [5] we give an example of a degree 1 bounded variation coboundary cocycle which is Hölder continuous for an arbitrary Hölder exponent smaller than 1. In fact, constructions of counterexamples of this type are equivalent to constructing special Cantor sets related to continued fraction expansion of an irrational number.

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**1. Notation.** Let $\alpha$ be an irrational number from $[0,1)$ and

$$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots}} = [0 : a_1, a_2, \ldots]$$

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be its continued fraction expansion. The positive integers $a_n$ are called the *partial quotients* of $\alpha$. Put

$$(1)\qquad \begin{cases} P_0=0, & P_1=1, \quad P_{n+1}=a_{n+1}P_n+P_{n-1},\\ Q_0=1, & Q_1=a_1, \quad Q_{n+1}=a_{n+1}Q_n+Q_{n-1}.\end{cases}$$

We have

$$(2)\qquad \frac{1}{Q_n(Q_{n+1}+Q_n)}<\left|\alpha-\frac{P_n}{Q_n}\right|<\frac{1}{Q_nQ_{n+1}},$$
$$Q_{n+1}\|Q_n\alpha\|+Q_n\|Q_{n+1}\alpha\|=1,$$

where $\|t\|$ denotes the distance of a real number $t$ from the set of integers. By $\{t\}$ we denote the fractional part of $t$.

Let $T$ denote the irrational translation mod 1 by $\alpha$ on $[0,1)$. From the continued fraction expansion of $\alpha$ we obtain, for each $n$, two Rokhlin towers $\xi_n,\overline{\xi}_n$ for $T$ whose union is the whole interval $[0,1)$. For $n$ even,

$$\xi_n=\{[0,\{Q_n\alpha\}),T[0,\{Q_n\alpha\}),\ldots,T^{(a_{n+1}Q_n+Q_{n-1})-1}[0,\{Q_n\alpha\})\},$$
$$\overline{\xi}_n=\{[1-\{Q_{n+1}\alpha\},1),T[1-\{Q_{n+1}\alpha\},1),\ldots,T^{Q_n-1}[1-\{Q_{n+1}\alpha\},1)\}.$$

Given a subsequence $\{n_k\}$ of natural numbers we define

$$I_k=[0,\{a_{2n_k+1}Q_{2n_k}\alpha\}),\qquad J_k^s=T^{(s-1)Q_{2n_k}}[0,\{Q_{2n_k}\alpha\}),$$

$s=1,\ldots,a_{2n_k+1}$. Then

$$I_k=\bigcup_{s=1}^{a_{2n_k+1}-1}J_k^s$$

and by (1), $\eta_k=\{I_k,TI_k,\ldots,T^{Q_{2n_k}-1}I_k\}$ is a Rokhlin tower.

Each measurable map $\varphi:\mathbb{S}^1\to\mathbb{S}^1$ will be called a *cocycle*. By a standard method we will identify $\mathbb{S}^1=[0,1)$ (with addition mod 1). Lebesgue measure on $\mathbb{S}^1$ will be denoted by $\mu$. After our identification, $T$ becomes a rotation

$$T(e^{2\pi ix})=e^{2\pi i(x+\alpha)},\qquad x\in[0,1).$$

We say that a cocycle $\varphi$ is a *coboundary* if $\varphi(x)=\xi(Tx)/\xi(x)$ for a measurable function $\xi:\mathbb{S}^1\to\mathbb{S}^1$. Notice that if $\varphi$ is a coboundary then the corresponding extension

$$T_\varphi:(\mathbb{S}^1\times\mathbb{S}^1,\widetilde{\mathcal{B}},\widetilde{\mu})\to(\mathbb{S}^1\times\mathbb{S}^1,\widetilde{\mathcal{B}},\widetilde{\mu}),\qquad T_\varphi(x,z)=(Tx,\varphi(x)z),$$

where $\widetilde{\mathcal{B}}$ is the product $\sigma$-algebra and $\widetilde{\mu}$ is the corresponding product measure, is not ergodic (the function $F(x,z)=\xi(x)z^{-1}$ is $T_\varphi$-invariant). Actually, $T_\varphi$ is ergodic iff for each $k\in\mathbb{Z}\setminus\{0\}$ the cocycle $\varphi^k$ is not a coboundary ([1]). Two cocycles will be called *cohomologous* if their quotient is a coboundary. Each cocycle is cohomologous to a continuous one ([7], see also [6], [9]); moreover, in the cohomology class of each cocycle $\varphi$ there is a continuous one with a given degree $d\in\mathbb{Z}$ (recall that for a continuous function $\psi:\mathbb{S}^1\to\mathbb{S}^1$ its degree $d(\psi)$ is defined as $\widetilde{\psi}(1)-\widetilde{\psi}(0)$, where $\widetilde{\psi}:[0,1]\to\mathbb{R}$ is continuous and $e^{2\pi i\widetilde{\psi}(x)}=\psi(e^{2\pi ix})$). Define

$$(3)\qquad \varphi^{(n)}(z)=\begin{cases}\varphi(z)\varphi(Tz)\ldots\varphi(T^{n-1}z), & n\ge1,\\ 1, & n=0,\\ (\varphi(T^nz)\ldots\varphi(T^{-1}z))^{-1}, & n\le-1.\end{cases}$$

Following [5] if there exists a set $Y\subset\mathbb{S}^1$ of positive measure with the property that $\varphi^{(n)}(z)=1$ whenever $z,T^nz\in Y$ (such a set is called a *fixing set* for $\varphi$) then $\varphi$ is a coboundary.

**2. Construction of $\alpha$ and a coboundary cocycle $\varphi:\mathbb{S}^1\to\mathbb{S}^1$ which is Hölder continuous, has bounded variation and is homotopic to the identity.** We start with the following simple observation.

LEMMA. *Suppose that $L\in\mathbb{N}$ is odd. Then for every odd $K\ge1$,*

$$\sum_{r=0}^{L-1}\sum_{i=1}^{K}\left(\frac{r}{L}+\frac{i}{KL}\right)\in\mathbb{N}.\ \blacksquare$$

Below, by a modification of the construction from [5] we will define a class of continuous bounded variation coboundary cocycles which are of degree 1. Then we will show that under certain additional assumptions, these cocycles are even Hölder continuous.

Let $(K_n)$ be an arbitrary sequence of odd numbers, $K_n\ge3$. Put

$$L_0=1,\quad L_n=K_nL_{n-1},\quad n\ge1.$$

Assume that $\alpha=[0:a_1,a_2,\ldots]$ has unbounded partial quotients and moreover that $\{a_{2k+1}:k\ge1\}$ is unbounded. Let $\varepsilon_j>0$ with

$$\sum_{j=1}^{\infty}\varepsilon_j<1.$$

We will define $f_j:[0,1]\to[0,1]$ continuous increasing with $f_j(0)=0$, $f_j(1)=1$ and $\sum_{j=1}^{\infty}\|f_{j+1}-f_j\|_\infty<\infty$. A sequence $(k_j)$ will be selected so that if we define

$$\Delta_j=J_j^1\cup\ldots\cup J_j^{K_j},\quad B_j=\bigcup_{s=0}^{Q_{2k_j}-1}\Delta_{j,s},\quad\text{where}\quad \Delta_{j,s}=T^s\Delta_j,$$

then $\mu(B_j)<\varepsilon_j$. The function $f_j$ will be constant on the gaps between $\Delta_{j,s}$ and linear (possibly constant) on $\Delta_{j,s}$. The set of those values of $f_j$ which are assumed on the intervals of constancy will be exactly $\{1/L_{j-1},2/L_{j-1},\ldots,1\}$. Moreover, on such intervals the limit function $f$ will coincide with $f_j$. If by

$$C_j=\Delta_{j,s_1}\cup\ldots\cup\Delta_{j,s_{t_j}}$$

we denote the union of those $\Delta_{j,s}$ on which $f_j$ is strictly increasing then $C_{j+1} \subset C_j$ and $f_{j+1} = f_j$ off $C_j$. Finally, we will show that all such cocycles are coboundaries by exhibiting fixing sets.

DEFINITION OF $f_1$. Select $k_1$ so that

$$\frac{K_1}{a_{2k_1+1}} \le \varepsilon_1.$$

We define $f_1$ to be equal to 1 on $[0,1] \setminus \Delta_1$, $f_1(0) = 0$ and then complete $f_1$ to obtain a linear continuous function. Note that if $x \in J_1^1$ then

$$\sum_{i=0}^{K_1 Q_{2k_1}-1} f_1(T^i x) = f_1(x) + f_1(T^{Q_{2k_1}} x) + \ldots + f_1(T^{(K_1-1)Q_{2k_1}} x) + M_1,$$

hence

$$\sum_{i=0}^{K_1 Q_{2k_1}-1} f(T^i x) = f(x) + f(T^{Q_{2k_1}} x) + \ldots + f(T^{(K_1-1)Q_{2k_1}} x) + M_1,$$

where $M_1 \in \mathbb{N}$.

DEFINITION OF $f_2$. Select $k_2 > k_1$ so that

$$\frac{K_2}{a_{2k_2+1}} \le \varepsilon_2.$$

We have $\Delta_2 \subset I_2 \subset J_1^1$; consequently, $T^{rQ_{2k_1}} \Delta_2 \subset T^{rQ_{2k_1}} J_1^1 = J_1^{r+1}$, $r = 0, \ldots, K_1 - 1$. If we take $\Delta_1$ and consider its partition into $\Delta_{2,rQ_{2k_1}}$, $r = 0, \ldots, K_1 - 1$, and the corresponding gaps then we put consecutively the values $1/L_1, 2/L_1, \ldots, 1$ on the gaps and then complete $f_2$ linearly on the remaining intervals. Note that if $x \in J_2^1$ then

$$\sum_{i=0}^{K_2 Q_{2k_2}-1} f_2(T^i x) = \sum_{s=0}^{K_1-1} \sum_{j=0}^{K_2-1} f_2(T^{sQ_{2k_1}+jQ_{2k_2}} x) + M_2,$$

where $M_2 \in \mathbb{N}$, since if $T^r x \in J_1^1 \setminus \Delta_2$ then $\sum_{i=0}^{p-1} f(T^{r+i}x) \in \mathbb{N}$ by the Lemma and the definition of $f_1$, where $p \ge 1$ is the smallest natural number such that $T^p x \in J_1^1$. Hence

$$\sum_{i=0}^{K_2 Q_{2k_2}-1} f(T^i x) = \sum_{s=0}^{K_1-1} \sum_{j=0}^{K_2-1} f(T^{sQ_{2k_1}+jQ_{2k_2}} x) + M_2.$$

Moreover, observe that if $x \in J_1^1 \setminus \Delta_2$ then by definition of $f_2$ and the Lemma



we have

$$\begin{aligned}\sum_{i=0}^{K_1 Q_{2k_1}-1} f(T^i x) &= f(x) + f(T^{Q_{2k_1}} x) + \ldots + f(T^{(K_1-1)Q_{2k_1}} x) + M_1\\ &= f_2(x) + f_2(T^{Q_{2k_1}} x) + \ldots + f_2(T^{(K_1-1)Q_{2k_1}} x) + M_1\\ &= M_1',\end{aligned}$$

where $M_1' \in \mathbb{N}$. Finally, note that $\|f_2 - f_1\| \le 1/L_1$.

DEFINITION OF $f_3$. Select $k_3 > k_2$ so that

$$\frac{K_3}{a_{2k_3+1}} \le \varepsilon_3.$$

We have $\Delta_3 \subset I_3 \subset J_2^1$; so $\Delta_{3,p}$ is a left-hand subinterval of $T^p J_2^1$. We know that

$$\Delta_{2,0},\ \Delta_{2,Q_{2k_1}},\ \ldots,\ \Delta_{2,(K_1-1)Q_{2k_1}}$$

are all the intervals where $f_2$ is strictly increasing. The appropriate translations of $\Delta_3$ will partition $\Delta_{2,rQ_{2k_1}}$ into $K_2$ translations of $\Delta_3$ and $K_2$ gaps. We put

$$\frac{r}{L_1} + \frac{1}{L_2},\ \frac{r}{L_1} + \frac{2}{L_2},\ \ldots,\ \frac{r+1}{L_1}$$

as the constant values on the consecutive gaps and then complete $f_3$ linearly on the remaining intervals. If now $x \in J_3^1$ then

$$\sum_{i=0}^{K_3 Q_{2k_3}-1} f_3(T^i x) = \sum_{q=0}^{K_1-1} \sum_{s=0}^{K_2-1} \sum_{j=0}^{K_3-1} f_3(T^{qQ_{2k_1}+sQ_{2k_2}+jQ_{2k_3}} x) + M_3,$$

where $M_3 \in \mathbb{N}$, since if $T^r x \in J_2^1 \setminus \Delta_3$ then $\sum_{i=0}^{p-1} f(T^{r+i}x) \in \mathbb{N}$ by the Lemma and the definition of $f_2$, where $p \ge 1$ is the smallest natural number such that $T^p x \in J_1^2$. Moreover, observe that if $x \in J_2^1 \setminus \Delta_3$ then

$$\begin{aligned}\sum_{i=0}^{K_2 Q_{2k_2}-1} f(T^i x) &= \sum_{s=0}^{K_1-1} \sum_{j=0}^{K_2-1} f(T^{sQ_{2k_1}+jQ_{2k_2}} x) + M_2\\ &= \sum_{s=0}^{K_1-1} \sum_{j=0}^{K_2-1} f_3(T^{sQ_{2k_1}+jQ_{2k_2}} x) + M_2 = M_2',\end{aligned}$$

where $M_2' \in \mathbb{N}$ by the definition of $f_3$ and the Lemma. Finally, $\|f_3 - f_2\| \le 1/L_2$.

Continuing, we define $f_n$ in such a way that

$$\frac{K_n}{a_{2k_n+1}} \le \varepsilon_n$$

and if

$$\Delta_{n-1,s_1},\ \Delta_{n-1,s_2},\ \ldots,\ \Delta_{n-1,s_{t_{n-1}}}\qquad (s_1=0)$$

are all the intervals where $f_{n-1}$ is strictly increasing then the appropriate $K_{n-1}$ translations of $\Delta_n$ will partition each $\Delta_{n-1,s_j}$ into $K_{n-1}$ subintervals and $K_{n-1}$ gaps and if we fix $s_j$ then

$$\frac{r}{L_{n-2}}+\frac{1}{L_{n-1}},\ \frac{r}{L_{n-2}}+\frac{2}{L_{n-1}},\ \ldots,\ \frac{r+1}{L_{n-2}}$$

are the constant values of $f_n$ on the consecutive gaps, where $r/L_{n-1}$ is the biggest value of constancy of $f_{n-1}$ not exceeding the values of $f_{n-1}$ on $\Delta_{n-1,s_j}$. Then $f_n$ is completed linearly. Also, if $x\in J_n^1$ then by the Lemma,

$$\sum_{i=0}^{K_nQ_{2k_n}-1} f_n(T^ix)=\sum_{j_1=0}^{K_1-1}\sum_{j_2=0}^{K_2-1}\ldots\sum_{j_n=0}^{K_n-1} f_n(T^{j_1Q_{2k_1}+j_2Q_{2k_2}+\ldots+j_nQ_{2k_n}}x)+M_n,$$

where $M_n\in\mathbb{N}$. As before, for $x\in J_{n-1}^1\setminus\Delta_n$ by the definition of $f_n$ and the Lemma we get

$$\begin{aligned}\sum_{i=0}^{K_{n-1}Q_{2k_{n-1}}-1} & f(T^ix)\\ &=\sum_{j_1=0}^{K_1-1}\sum_{j_2=0}^{K_2-1}\ldots\sum_{j_{n-1}=0}^{K_{n-1}-1} f(T^{j_1Q_{2k_1}+j_2Q_{2k_2}+\ldots+j_{n-1}Q_{2k_{n-1}}}x)+M_{n-1}\\ &=\sum_{j_1=0}^{K_1-1}\sum_{j_2=0}^{K_2-1}\ldots\sum_{j_{n-1}=0}^{K_{n-1}-1} f_n(T^{j_1Q_{2k_1}+j_2Q_{2k_2}+\ldots+j_{n-1}Q_{2k_{n-1}}}x)+M_{n-1}=M'_{n-1},\end{aligned}$$

where $M'_{n-1}\in\mathbb{N}$. Finally, $\|f_n-f_{n-1}\|\le 1/L_{n-1}$. We have

$$\sum_{n\ge1}\|f_{n+1}-f_n\|\le 1/L_n<\infty$$

and therefore $f=\lim_{n\to\infty}f_n$ is well-defined, increasing continuous and $f(0)=0$, $f(1)=1$.

THEOREM. *If $f$ is defined as above then the cocycle $e^{2\pi if}$ is a coboundary.*

Proof. We have $\mu(B_j)<\varepsilon_j$, $j\ge1$, so $0<\mu(Y)<1$, where $Y=[0,1)\setminus\bigcup_{j=1}^{\infty}B_j$. It remains to prove that $Y$ is a fixing set for $e^{2\pi if}$.

All we need to show is that if $x,T^Nx\in Y$ then

$$f(x)+f(Tx)+\ldots+f(T^{N-1}x)\in\mathbb{Z}. \tag{4}$$

First, note that if also $Tx,T^2x,\ldots,T^{N-1}x\in Y$ then $f(T^ix)=1$ for $i=0,\ldots,N-1$ and so we are done. Therefore, assume that $T^nx\notin Y$ for some $0<n<N$ and let $n$ be minimal with this property. Now $f(x),f(Tx),\ldots,f(T^{n-1}x)=1$ and there exists $j\ge1$ such that $T^nx\in B_j$. Since $T^{n-1}x\notin B_j$ we have $T^nx\in\Delta_j$, but $B_j$ can be considered as a Rokhlin tower with base $J_j^1$ and height $Q_{2k_j}K_j-1$, so we must have $T^nx\in J_j^1$. Assume that $j$ is the biggest such that $x\in\Delta_j$. We will prove that

$$f(T^nx)+\ldots+f(T^{n+Q_{2k_j}K_j-1}x)\in\mathbb{N}. \tag{5}$$

Indeed, first notice that $T^nx\notin B_{j+1}$ because $T^nx\notin\Delta_{j+1}$ by our choice of $j$ and if $T^nx\in B_{j+1}\setminus\Delta_{j+1}$ then $T^{n-1}x\in B_{j+1}$, so $T^{n-1}x$ would belong to $Y$, a contradiction. Therefore, by the definition of $f_j$ we get

$$\begin{aligned}\sum_{i=0}^{K_jQ_{2k_j}-1} & f_j(T^{n+i}x)\\ &=\sum_{m_1=0}^{K_1-1}\sum_{m_2=0}^{K_2-1}\ldots\sum_{m_j=0}^{K_j-1} f_j(T^{m_1Q_{2k_1}+m_2Q_{2k_2}+\ldots+m_jQ_{2k_j}+n}x)+M_j,\end{aligned}$$

where $M_j\in\mathbb{N}$. But since $T^nx\notin B_{j+1}$, we see that $f_j(T^{n+i}x)$ can be different from $f(T^{n+i}x)$ only for those $i$ which are of the form $i=m_1Q_{2k_1}+m_2Q_{2k_2}+\ldots+m_jQ_{2k_j}$. Consequently

$$\begin{aligned}\sum_{i=0}^{K_jQ_{2k_j}-1} & f(T^{n+i}x)\\ &=\sum_{m_1=0}^{K_1-1}\sum_{m_2=0}^{K_2-1}\ldots\sum_{m_j=0}^{K_j-1} f(T^{m_1Q_{2k_1}+m_2Q_{2k_2}+\ldots+m_jQ_{2k_j}+n}x)+M_j.\end{aligned}$$

However, $T^nx\notin\Delta_{j+1}$, so that

$$f(T^{m_1Q_{2k_1}+m_2Q_{2k_2}+\ldots+m_jQ_{2k_j}+n}x)=f_{j+1}(T^{m_1Q_{2k_1}+m_2Q_{2k_2}+\ldots+m_jQ_{2k_j}+n}x).$$

By the definition of $f_{j+1}$ we obtain

$$\sum_{m_1=0}^{K_1-1}\sum_{m_2=0}^{K_2-1}\ldots\sum_{m_j=0}^{K_j-1} f_{j+1}(T^{m_1Q_{2k_1}+m_2Q_{2k_2}+\ldots+m_jQ_{2k_j}+n}x)\in\mathbb{N},$$

so (5) has been proved.

Now, note that $x_1=T^{n+Q_{2k_j}K_j}x\in J_{j-1}^1\setminus\Delta_j$ (since in fact $x_1\in J_j^{K_j+1}$), and therefore we can repeat the same arguments for $x_1,\ldots,x_{j-1}$ as for $x_0$ to obtain

$$x_2\in J_{j-2}^1\setminus\Delta_{j-1},\ \ldots,\ x_{j-1}\in J_1^1\setminus\Delta_2$$

and the corresponding sums are integers. It remains to prove that $x_j=T^{Q_{2k_1}K_1}x_{j-1}\in Y$. Obviously $x_j\notin B_1$ since $x_j\in J_1^{K_1+1}$. Suppose that $x_j\in B_2$. Then automatically $T^{-k}x_j\in B_2$ for all $k=0,1,\ldots,p_2$, where $p_2$

is greater than $Q_{2k_1}K_1$, in particular $x_{j-1} \in B_2$, which is a contradiction. Similarly if $x_j \in B_3$ then $T^{-k}x_j \in B_3$ for all $k = 0, 1, \ldots, p_3$, where $p_3$ is greater than $Q_{2k_2}K_2 + Q_{2k_1}K_1$; in particular $x_{j-2} \in B_3$, which is a contradiction. In the same way we exclude the possibility $x_j \in B_4, \ldots, x_j \in B_j$. If $x_j \in B_{j+r}$, $r \ge 1$ then we still obtain a contradiction by similar arguments, though this time to the fact that $T^n x \notin B_{j+r}$ for all $r \ge 1$. ■

Define

$$h_n = 1/L_{n-1}, \quad p_n = |J_n^1| - |\Delta_{n+1}|, \quad n = 1, 2, \ldots$$

Note that $h_n$ represents the distance between consecutive values of constancy of $f_n$, while $p_n$ represents the length of the shortest interval of constancy for $f_{n+1}$.

PROPOSITION. *If there exist $C > 0$ and $0 < \delta < 1$ such that for all $n \ge 0$,*

$$h_n \le C p_n^{\delta} \tag{6}$$

*then the function $f$ defined in the Theorem is Hölder continuous.*

Proof. Take $x, x' \in [0, 1)$. We want to prove that

$$|f(x) - f(x')| \le C|x - x'|^{\delta}. \tag{7}$$

If there exists $n \ge 1$ such that $x, x'$ belong to the same interval of constancy of $f_n$ then (7) is satisfied. Suppose that this is not the case. Then there exists a smallest $n$ with the property that either

(i) between $x$ and $x'$ there is at least one interval of constancy of $f_{n+1}$ or
(ii) $x$ and $x'$ belong to two consecutive intervals of constancy of $f_{n+1}$ (these two intervals can be of different size).

In case (i) we have

$$|x - x'| \ge p_n, \quad |f(x) - f(x')| \le h_n$$

so (7) follows immediately from (6).

In case (ii) we have

$$|x - x'| > p_{n+1}, \quad |f(x) - f(x')| = h_{n+1}$$

and again (7) follows from (6).

Set $i_n = |I_n|$ and $j_n = |J_n^1|$. In view of (1) and (2), for each $n \ge 1$ we have

$$\frac{a_{2k_{n+1}+1}}{Q_{2k_{n+1}+1} + Q_{2k_{n+1}}} < i_{n+1} < \frac{a_{2k_{n+1}+1}}{Q_{2k_{n+1}}}$$

so that

$$\frac{1}{3Q_{2k_{n+1}}} < i_{n+1}.$$



Moreover,

$$\frac{1}{Q_{2k_n+1}} > j_n > \frac{1}{Q_{2k_n+1} + Q_{2k_n}}.$$

Therefore $J_{n+1} \subset I_{n+1} \subset J_n$. Furthermore,

$$j_{n+1} = \frac{i_{n+1}}{a_{2k_{n+1}+1}} \ge \frac{1}{3Q_{2k_{n+1}}a_{2k_{n+1}+1}} \ge \frac{1}{3 \cdot 2Q_{2k_{n+1}-1}a_{2k_{n+1}}a_{2k_{n+1}+1}}$$
$$\ge \frac{1}{3 \cdot 2^2 Q_{2k_{n+1}-2}a_{2k_{n+1}-1}a_{2k_{n+1}}a_{2k_{n+1}+1}} \ge \ldots$$

By continuing, we see that for each $n \ge 1$,

$$j_n \ge \frac{1}{3 \cdot 2^{2k_n - 1}\prod_{s=1}^{2k_n+1} a_s} \tag{8}$$

Now, $|\Delta_{n+1}| = K_{n+1}j_{n+1} \le \varepsilon_n i_{n+1} < \varepsilon_n j_n$. Thus $p_n \ge (1 - \varepsilon_n)j_n$ and by (8),

$$p_n \ge \frac{1 - \varepsilon_n}{3 \cdot 2^{2k_n - 1}\prod_{s=1}^{2k_n+1} a_s}. \tag{9}$$ ■

COROLLARY 1. *If for $\alpha = [0 : a_1, a_2, \ldots]$ there exists a sequence $(k_n)$ such that for each $n \ge 1$,*

(a) $\dfrac{K_n}{a_{2k_n+1}} \le \varepsilon_n$,

(b) $\dfrac{1}{K_1 \cdot \ldots \cdot K_{n-1}} \le C\left(\dfrac{1 - \varepsilon_n}{3 \cdot 2^{2k_n - 1}\prod_{s=1}^{2k_n+1} a_s}\right)^{\delta}$

*for some $C > 0$ and $0 < \delta < 1$ then there exists a coboundary cocycle $\psi : \mathbb{S}^1 \to \mathbb{S}^1$ which is of bounded variation, Hölder continuous and has degree 1.* ■

COROLLARY 2. *There exists an irrational number $\alpha$ and a bounded variation coboundary cocycle which is homotopic to the identity and Hölder continuous with an arbitrary Hölder exponent $0 < \delta < 1$.*

Proof. Let $0 < \lambda_n \to 0$ and define $\widetilde{p}_n = 2^{\lambda_1 + \ldots + \lambda_n}\widetilde{p}_0$, where $\widetilde{p}_0 > 0$. We will assume that

$$(\forall n \ge 1) \quad \sum_{j=1}^{n} \lambda_j \ge \frac{1}{2}\sqrt{n}. \tag{10}$$

Therefore

$$\widetilde{p}_n \ge 2^{\frac{1}{2}\sqrt{n}}\widetilde{p}_0. \tag{11}$$

Choose $-1 < \eta_i < 1$ so that $p_i = \widetilde{p}_i + \eta_i$ is odd. Note that

$$\Big|\sum_{i=0}^{n-1}\widetilde{p}_i - \sum_{i=0}^{n-1} p_i\Big| \le n.$$

We have

(12) for each $0 < \delta < 1$ there exists $n_0$ such that for all $n \geq n_0$,

$$\sum_{i=0}^{n-1} p_i \geq n\delta + \frac{(n+1)(n+2)}{2}\delta + \Big(\sum_{i=0}^{n} p_i\Big)\delta.$$

Indeed, first notice that

$$\frac{n^2}{\sum_{i=0}^{n-1} p_i} \to 0$$

by (11). Then observe that $\lim p_n/p_{n-1} = \lim \widetilde{p}_n/\widetilde{p}_{n-1} = 1$, so

$$\lim \frac{p_n}{\sum_{i=0}^{n-1} p_i} = 0.$$

Therefore (12) holds true since $\delta < 1$.

Put $a_1 = 5 \cdot 5^{p_0}$, $a_{2n+1} = 5^{n+1} \cdot 5^{p_n}$ and $a_{2t} = 1$ for all $t \geq 1$. So we let $\varepsilon_n = 1/5^n$ and $K_n = 5^{p_n}$. Now set $k_n = n$. By Corollary 1 it suffices to show that for every $0 < \delta < 1$ there exists $C > 0$ such that for all $n \geq 1$,

$$\frac{1}{5^{p_1} \cdot \ldots \cdot 5^{p_{n-1}}} \leq C\bigg(\frac{1 - 1/5^n}{5 \cdot 3^{n-1} \cdot (5 \cdot 5^{p_0}) \cdot \ldots \cdot (5^{n+1} \cdot 5^{p_n})}\bigg)^{\delta}$$

whence it is enough to show that

$$C \cdot 5^{p_1+\ldots+p_{n-1}}\Big(1 - \frac{1}{5^n}\Big)^{\delta} \geq 5^{n\delta} \cdot 5^{(n+1)(n+2)\delta/2} \cdot 5^{(p_0+\ldots+p_n)\delta}.$$

Therefore our assertion follows directly from (12) for an appropriate choice of $C$. ■

Remark. In [8] it is shown that if $f : \mathbb{S}^1 \to \mathbb{R}$ and $g \in L^2(\mathbb{S}^1)$ with $g(x) = \sum_{n\neq 0} g_n e^{2\pi i n x}$ and $g_n = o(1/n)$ then for every irrational $\alpha$ there exists a subsequence $(Q_{n_j})$ of denominators of $\alpha$ such that

(13) $$g^{(Q_{n_j})} \to 0 \quad \text{in} \quad L^2(\mathbb{S}^1),$$

generalizing the previously known similar result for absolutely continuous functions (see [4]). As noticed in [3], the condition (13) says in particular that for each nonzero $d \in \mathbb{Z}$ the cocycle $e^{2\pi i(dx+g(x))}$ is ergodic with respect to every irrational rotation.

The result of this section says then that (13) is not satisfied for the cocycle $g(x) = f(x) - x$, where $f$ comes from Corollary 2; the condition (13) is not satisfied though the Fourier coefficients of $g$ are absolutely summable and $g_n = O(1/n)$ with $g_n = o(1/n)$ for $n$ from a set of density 1.



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