[15] A. Parusiński, *Lipschitz stratification of subanalytic sets*, Ann. Sci. Ecole Norm. Sup. 27 (1994), 661–696.

[16] J. C. Tougeron, *Idéaux de Fonctions Différentiables*, Springer, 1972.

[17] L. van den Dries and C. Miller, *Geometric categories and o-minimal structures*, Duke Math. J. 84 (1996), 497–540.

[18] H. Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Trans. Amer. Math. Soc. 36 (1934), 63–89.

[19] —, *Functions differentiable on the boundaries of regions*, Ann. of Math. 35 (1934), 482–485.

Laboratoire de Mathématiques
Université de Savoie
Campus Scientifique
73 376 Le Bourget-du-Lac Cedex, France
E-mail: kurdyka@univ-savoie.fr

Institute of Mathematics
Jagiellonian University
Reymonta 4
30-059 Kraków, Poland
E-mail: kurdyka@im.uj.edu.pl
pawlucki@im.uj.edu.pl

Received March 11, 1996
Revised version November 18, 1996 (3628)

STUDIA MATHEMATICA 124 (3) (1997)

On the Yosida approximation and the Widder–Arendt representation theorem

by

ADAM BOBROWSKI (Lublin)

Abstract. The Yosida approximation is treated as an inversion formula for the Laplace transform.

0. Introduction. The Yosida approximation is a standard tool in proving generation theorems for semigroups ([7], [9], [12]). In [10], a related power series was introduced and proven to yield an inversion formula for the Laplace transform ([10], Theorems 2.2–2.3, or [7], pp. 221–223, Theorems 6.3.3–6.3.6). Namely it was shown that the power series of the image function converges to the original function. In this article we shall show that this formula leads to a much simpler proof of a classical theorem of Widder characterizing the Laplace transform of a bounded complex-valued function. Furthermore, we shall provide a power-series-approximation formula for integrated Lipschitz continuous semigroups.

1. The Yosida approximation in Banach spaces. Let us start with a definition.

Definition 1. Fix $\omega \in \mathbb{R}$. Let L be a Banach space and let $(\omega,\infty) \ni \lambda \to f(\lambda)$ be an infinitely differentiable function with values in L, satisfying

$$\|f^{(n)}(\lambda)\| \le \frac{Mn!}{(\lambda-\omega)^{n+1}}, \tag{1.1}$$

where $M > 0$ is a constant. Put

$$g_\mu(t) = e^{-\mu t} t\mu^2 \sum_{n=0}^{\infty} \frac{t^n \mu^{2n} (-1)^n f^{(n)}(\mu)}{n!(n+1)!} \quad \text{for } \mu > \omega. \tag{1.2}$$

The functions $g_\mu(t)$ will be called the *Yosida approximation* of f.

1991 *Mathematics Subject Classification*: 44A10, 47D03, 41A65.

I would like to thank Professor J. Kisyński for valuable discussions as well as pointing out and correcting a gap in the proof of Proposition 1.

Remark 1. The above definition is justified by the fact that when $L = \mathcal{L}(X,X)$ is the Banach algebra of bounded linear operators acting in a Banach space X, and $f(\lambda) = R_\lambda = (\lambda - A)^{-1}$ is the resolvent of a densely defined operator A satisfying (1.1), then, by $R_\lambda^{(n)} = (-1)^n n! R_\lambda^{n+1}$, we have

$$e^{-\mu t} I + g_\mu(t) = e^{-\mu t} \sum_{n=0}^{\infty} \frac{(t\mu^2 R_\mu)^n}{n!} = e^{(\mu^2 R_\mu - \mu)t},$$

which is just the well-known Yosida approximation of the semigroup generated by A. The absence of the term $e^{-\mu t} I$ in our definition is due to the fact that instead of working in the Banach algebra $\mathcal{L}(X,X)$ we want to deal with functions in an arbitrary Banach space. Since in what follows we shall be interested in the mean convergence of $g_\mu(t)$ as $\mu \to \infty$ (see (2.2), (2.3) below), and for such a limit the factor in question is irrelevant, our results apply equally to the classical definition.

The same power series appears in [7], p. 223, and [10]; compare also [3], p. 236.

LEMMA 1. *The functions $g_\mu(t)$ are continuous and bounded by $Me^{\omega\mu t/(\mu-\omega)}$.*

Proof. Fix $\mu > \omega$. The estimate

$$\sum_{n=0}^{\infty} \left\| \frac{t^{n+1}\mu^{2n+2}(-1)^n f^{(n)}(\mu)}{n!(n+1)!} \right\| \le \sum_{n=0}^{\infty} \frac{t^{n+1}\mu^{2(n+1)}}{(n+1)!(\mu-\omega)^{n+1}} M = M(e^{\mu^2 t/(\mu-\omega)} - 1),$$

which is valid by (1.1), proves that the series appearing in (1.2) is convergent absolutely and almost uniformly. Its sum is therefore a continuous function. Furthermore,

$$\|g_\mu(t)\| \le M(e^{\omega\mu t/(\mu-\omega)} - e^{-\mu t}) < Me^{\omega\mu t/(\mu-\omega)}, \tag{1.3}$$

as desired.

PROPOSITION 1. *The Laplace transform of $g_\mu(t)$ is given by*

$$\int_0^\infty e^{-\lambda t} g_\mu(t)\, dt = \frac{\mu^2}{(\lambda+\mu)^2} f\left(\frac{\lambda\mu}{\lambda+\mu}\right), \quad \mu > \omega, \lambda > \frac{\omega\mu}{\mu-\omega}.$$

Proof. For $\omega < \nu < \mu$ and $n \ge 1$,

$$f(\nu) = \sum_{k=0}^{n} f^{(k)}(\mu) \frac{(\nu-\mu)^k}{k!} + \int_\mu^\nu f^{(n+1)}(\xi) \frac{(\nu-\xi)^n}{n!}\, d\xi.$$

By (1.1),

$$\left\| \int_\mu^\nu f^{(n+1)}(\xi) \frac{(\nu-\xi)^n}{n!}\, d\xi \right\| \le M(n+1) \int_\mu^\nu \left(\frac{\xi-\nu}{\xi-\omega}\right)^n \frac{1}{(\xi-\omega)^2}\, d\xi$$
$$= \frac{M(n+1)}{\omega-\nu} \int_0^{(\mu-\nu)/(\mu-\omega)} \alpha^n\, d\alpha = \frac{M}{\omega-\nu}\left(\frac{\mu-\nu}{\mu-\omega}\right)^{n+1} \xrightarrow[n\to\infty]{} 0$$

($\alpha = (\xi-\nu)/(\xi-\omega)$). Thus $f(\nu) = \sum_{n=0}^\infty f^{(n)}(\mu)(\nu-\mu)^n/n!$.

If $\lambda > \omega\mu/(\mu-\omega)$, then $\mu > \nu := \lambda\mu/(\lambda+\mu) > \omega$ and

$$f\left(\frac{\lambda\mu}{\lambda+\mu}\right) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\mu)}{n!}\left(\frac{\lambda\mu}{\lambda+\mu} - \mu\right)^n.$$

Consequently,

$$\int_0^\infty e^{-\lambda t} g_\mu(t)\, dt$$
$$= \int_0^\infty \left\{ e^{-(\mu+\lambda)t} t\mu^2 \sum_{n=0}^\infty \frac{t^n \mu^{2n}(-1)^n f^{(n)}(\mu)}{n!(n+1)!} \right\} dt$$
$$= \sum_{n=0}^\infty \frac{\mu^{2n+2}(-1)^n f^{(n)}(\mu)}{n!(n+1)!} \int_0^\infty e^{-(\lambda+\mu)t} t^{n+1}\, dt$$
$$= \sum_{n=0}^\infty \frac{\mu^{2n+2}(-1)^n f^{(n)}(\mu)}{n!(\lambda+\mu)^{n+2}} = \left(\frac{\mu}{\lambda+\mu}\right)^2 \sum_{n=0}^\infty \left(\frac{-\mu^2}{\lambda+\mu}\right)^n \frac{f^{(n)}(\mu)}{n!}$$
$$= \left(\frac{\mu}{\lambda+\mu}\right)^2 \sum_{n=0}^\infty \left(\frac{\lambda\mu}{\lambda+\mu} - \mu\right)^n \frac{f^{(n)}(\mu)}{n!} = \left(\frac{\mu}{\lambda+\mu}\right)^2 f\left(\frac{\lambda\mu}{\lambda+\mu}\right). \ \blacksquare$$

Observe that $f(\cdot)$ is a function of a *real* variable, so that the proof of the fact that it can be developed in a Taylor series was necessary.

EXAMPLE 1. If $L = \mathcal{L}(X,X)$, $B \in \mathcal{L}(X,X)$ and $f(\lambda) = R_B(\lambda)$ satisfying (1.1) is a B-resolvent of a closed and densely defined operator $A : D(A) \to X$ (see [3], p. 230), then

$$BR_B(\lambda) - BR_B(\mu) = (\mu - \lambda)R_B(\lambda)R_B(\mu) \tag{1.4}$$

and $H_\mu(t) = g_\mu(t) + e^{-\mu t}B$ is the Yosida approximation of the regularized semigroup generated by A ([3], p. 236). Its Laplace transform is given by

$$\int_0^\infty e^{-\lambda t} H_\mu(t)\,dt = \frac{B}{\lambda+\mu} + \left(\frac{\mu}{\lambda+\mu}\right)^2 R_B\left(\frac{\lambda\mu}{\lambda+\mu}\right)$$

([3], p. 237, Lemma 4V); the assumption that A is densely defined is inessential. In particular, if $B = I$, we get the well-known result appearing in [4], p. 312, and [9], p. 26, eq. (7.7).

2. The Widder–Arendt representation theorem and integrated semigroups. Let $L^1(\mathbb{R}^+)$ be the convolution algebra of (equivalence classes of) Lebesgue integrable functions $\psi : \mathbb{R}^+ \to \mathbb{R}$. We shall use the following notation: $L^1(\mathbb{R}^+) \ni \psi = \{\psi(x)\}$ means that ψ is the equivalence class of the function $\mathbb{R}^+ \ni x \to \psi(x) \in \mathbb{R}$. In such cases, to avoid misunderstanding, we shall always use x as an independent variable of the function ψ. Other variables should be considered as parameters.

Let $\omega \in \mathbb{R}$. The linear space $L^1_\omega(\mathbb{R}^+)$ of all (equivalence classes of) functions ψ such that $\widehat{\psi} = \{e^{\omega x}\psi(x)\}$ belongs to $L^1(\mathbb{R}^+)$, is a Banach space when considered with the norm $\|\psi\|_{L^1_\omega(\mathbb{R}^+)} = \|\widehat{\psi}\|_{L^1(\mathbb{R}^+)}$. Furthermore, since

$$\begin{aligned}\|\psi * \varphi\|_{L^1_\omega(\mathbb{R}^+)} &= \int_0^\infty e^{\omega x}\left|\int_0^x \psi(x-y)\varphi(y)\,dy\right| dx \\ &\le \int_0^\infty\int_y^\infty e^{\omega x}|\psi(x-y)\varphi(y)|\,dx\,dy \\ &= \int_0^\infty e^{\omega x}\int_0^\infty e^{\omega y}|\psi(y)|\,dy|\varphi(x)|\,dx \\ &= \|\psi\|_{L^1_\omega(\mathbb{R}^+)}\|\varphi\|_{L^1_\omega(\mathbb{R}^+)},\end{aligned}$$

$L^1_\omega(\mathbb{R}^+)$ is also a Banach algebra, multiplication being the standard convolution

$$(\psi * \varphi)(x) = \int_0^x \psi(x-y)\varphi(y)\,dy.$$

The function $(\omega,\infty) \ni \lambda \to e_\lambda = \{e^{-\lambda x}\} \in L^1_\omega(\mathbb{R}^+)$ satisfies the Hilbert equation

$$(\lambda-\mu)e_\lambda * e_\mu = e_\mu - e_\lambda.$$

Therefore, it is infinitely differentiable, and

$$\begin{aligned}(2.1)\qquad \left\|\frac{d^n}{d\lambda^n}e_\lambda\right\|_{L^1_\omega(\mathbb{R}^+)} &= \left\|n!e_\lambda^{*(n+1)}\right\|_{L^1_\omega(\mathbb{R}^+)} = \|\{e^{-\lambda x}x^n\}\|_{L^1_\omega(\mathbb{R}^+)} \\ &= \int_0^\infty e^{-\lambda x}e^{\omega x}x^n\,dx = \frac{n!}{(\lambda-\omega)^{n+1}}.\end{aligned}$$

PROPOSITION 2. *Suppose that f satisfies* (1.1) *with $\omega = 0$. Then for each $\phi \in L^1(\mathbb{R}^+)$ the limit*

$$(2.2)\qquad \mathcal{T}(\phi) = \lim_{\mu\to\infty}\int_0^\infty \phi(t)g_\mu(t)\,dt$$

exists, and $\mathcal{T} : L^1(\mathbb{R}^+) \to L$ is a bounded linear operator with $\|\mathcal{T}\| \le M$.

Proof. Consider the operators $\mathcal{T}_\mu : L^1(\mathbb{R}^+) \to L$ defined as $\mathcal{T}_\mu(\psi) = \int_0^\infty \psi(t)g_\mu(t)\,dt$. The estimate (1.3) asserts that $\|\mathcal{T}_\mu\| \le M$. Since by Proposition 1,

$$\mathcal{T}_\mu(e_\lambda) = \frac{\mu^2}{(\lambda+\mu)^2} f\left(\frac{\lambda\mu}{\lambda+\mu}\right) \to f(\lambda) \quad \text{as } \mu\to\infty,$$

and the e_λ, $\lambda > 0$, form a total set in $L^1(\mathbb{R}^+)$, the proof is complete. ∎

PROPOSITION 3. *Suppose that f satisfies* (1.1). *Then for each $\phi \in L^1_\omega(\mathbb{R}^+)$ the limit*

$$(2.3)\qquad \mathcal{T}(\phi) = \lim_{\mu\to\infty}\int_0^\infty e^{-\varepsilon(\mu)t}\phi(t)g_\mu(t)\,dt = \lim_{\mu\to\infty}\int_0^\infty e^{\omega t}\phi(t)\widehat{g}_\mu(t)\,dt$$

exists, where $\varepsilon(\mu) = \omega^2/(\mu-\omega)$ and $\widehat{g}_\mu(t)$ is the Yosida approximation of the function $\widehat{f}(\lambda) = f(\lambda+\omega)$. Furthermore, $\mathcal{T} : L^1_\omega(\mathbb{R}^+) \to L$ is a bounded linear operator with $\|\mathcal{T}\| \le M$.

Proof. Fix $\varepsilon > 0$ and $\omega \in \mathbb{R}$. By assumption, the function $(0,\infty) \ni \lambda \to \widehat{f}(\lambda) := f(\lambda+\omega)$ satisfies

$$(2.4)\qquad \|\widehat{f}^{(n)}(\lambda)\| \le \frac{Mn!}{\lambda^{n+1}}.$$

Let $\widehat{g}_\mu(t)$, $\mu > 0$, be its Yosida approximation. Define $h_\mu(t) : [0,\infty) \to L$ and $\mathcal{T}_\mu : L^1(\mathbb{R}^+) \to L$ by

$$h_\mu(t) = \widehat{g}_\mu(t) - e^{-(\omega+\varepsilon(\mu))t}g_\mu(t)$$

and $\mathcal{T}_\mu(\varphi) = \int_0^\infty \varphi(t)h_\mu(t)\,dt$, where $g_\mu(t)$ is the Yosida approximation of f. Estimate (1.3) yields $\|h_\mu(t)\| \le 2M$ and $\|\mathcal{T}_\mu\| \le 2M$. Using Proposition 1 we get

$$\lim_{\mu\to\infty} \mathcal{T}_\mu(e_\lambda) = \lim_{\mu\to\infty} \int_0^\infty e^{-\lambda t} h_\mu(t)\,dt$$
$$= \lim_{\mu\to\infty} \left[\frac{\mu^2}{(\lambda+\mu)^2} f\left(\frac{\mu\lambda}{\lambda+\mu}+\omega\right) - \frac{\mu^2}{(\lambda+\omega+\varepsilon+\mu)^2} f\left(\frac{\mu(\lambda+\omega+\varepsilon)}{\lambda+\varepsilon+\omega+\mu}\right)\right] = 0$$

where $\varepsilon = \varepsilon(\mu)$. Thus, as in Proposition 2,

$$\lim_{\mu\to\infty} \mathcal{T}_\mu(\varphi) = \lim_{\mu\to\infty} \int_0^\infty \varphi(t) h_\mu(t)\,dt = 0 \quad \text{for all } \varphi \in L^1(\mathbb{R}^+).$$

On the other hand, by (2.4) and Proposition 2, for all $\varphi \in L^1(\mathbb{R}^+)$, the limit $\lim_{\mu\to\infty} \int_0^\infty \varphi(t)\widehat{g}_\mu(t)\,dt$ exists. Let ϕ belong to $L^1_\omega(\mathbb{R}^+)$. Then $\varphi := \{e^{\omega x}\phi(x)\}$ belongs to $L^1(\mathbb{R}^+)$. Thus, the limit

$$\lim_{\mu\to\infty} \int_0^\infty e^{-\varepsilon(\mu)t}\phi(t) g_\mu(t)\,dt \tag{2.5}$$
$$= \lim_{\mu\to\infty} \int_0^\infty e^{-(\varepsilon(\mu)+\omega)t}\varphi(t) g_\mu(t)\,dt$$
$$= \lim_{\mu\to\infty} \int_0^\infty \varphi(t)\widehat{g}_\mu(t)\,dt - \lim_{\mu\to\infty} \int_0^\infty \varphi(t) h_\mu(t)\,dt$$
$$= \lim_{\mu\to\infty} \int_0^\infty \varphi(t)\widehat{g}_\mu(t)\,dt = \lim_{\mu\to\infty} \int_0^\infty e^{\omega t}\phi(t)\widehat{g}_\mu(t)\,dt$$

also exists. To complete the proof it is enough to show that $\|\mathcal{T}\| \le M$. But $\|\mathcal{T}(\phi)\| = \|\lim_{\mu\to\infty} \int_0^\infty \phi(t)e^{\omega t}\widehat{g}_\mu(t)\,dt\| \le M\|\phi\|_{L^1_\omega(\mathbb{R}^+)}$. ■

Remark 2. The reason why in the case $\omega \ne 0$ one has to introduce additionally the factor $e^{-\varepsilon t}$ in (2.3) is clearly explained by estimate (1.3) and the fact that $\omega\mu/(\mu-\omega) > \omega$ for $\omega \ne 0$.

THEOREM 1. *Fix $\omega \in \mathbb{R}$. Let $f(\lambda)$ be an L-valued function. The following statements are equivalent.*

(i) *f is infinitely differentiable and there exists a constant $M > 0$ such that for $\lambda > \omega$ and $n \in \mathbb{N}$,*

$$\|f^{(n)}(\lambda)\| \le \frac{n!M}{(\lambda-\omega)^{n+1}}. \tag{2.6}$$

(ii) *There exists a bounded linear operator $\mathcal{T} : L^1_\omega(\mathbb{R}^+) \to L$ such that $f(\lambda) = \mathcal{T}(e_\lambda)$ where $e_\lambda = \{e^{-\lambda x}\}$ for $\lambda > \omega$.*

The operator $\mathcal{T}$ is given by (2.3).

Proof. The implication (i)⇒(ii) was actually proven in Proposition 3, except for the formula $f(\lambda) = \mathcal{T}(e_\lambda)$. But, by (2.3) and Proposition 1,

$$\mathcal{T}(e_\lambda) = \lim_{\varepsilon\to 0+}\lim_{\mu\to\infty} \int_0^\infty e^{-\lambda t}e^{-\varepsilon t} g_\mu(t)\,dt$$
$$= \lim_{\varepsilon\to 0+}\lim_{\mu\to\infty} \frac{\mu^2}{(\lambda+\varepsilon+\mu)^2} f\left(\frac{(\lambda+\varepsilon)\mu}{\lambda+\varepsilon+\mu}\right) = f(\lambda),$$

as desired.

In order to prove the converse statement it is enough to note that since $\lambda \to e_\lambda$ is infinitely differentiable, so is $\mathcal{T}(e_\lambda)$, and, by (2.1),

$$\|f^{(n)}(\lambda)\| = \left\|\mathcal{T}\left(\frac{d^n}{d\lambda^n}e_\lambda\right)\right\| \le \|\mathcal{T}\| \left\|\frac{d^n}{d\lambda^n}e_\lambda\right\|_{L^1(\mathbb{R}^+)} \le \|\mathcal{T}\|\frac{n!}{(\lambda-\omega)^{n+1}}. \ ■$$

Remark 3. The above theorem has already appeared e.g. in [6], pp. 158–159 and [8], Theorems 2.1 and 2.3; it is the formula (2.3) that is new. In the above mentioned paper the Post–Widder approximation was employed. The same inversion formula was used by Widder himself to prove his famous result presented below [11]. Actually, the proof of Theorem 1, as presented in [6], [8], is nothing but repetition of Widder's argument [11]. Our argument seems to be completely new and leads to a substantial simplification of the proof. Moreover, it appears to be of importance that, in contrast to the case of the Post–Widder inversion formula, one is able to compute the Laplace transform of $g_\mu(t)$ explicitly.

Let us also note that, if $f(\lambda)$ is a Banach algebra-valued function and satisfies the Hilbert equation, then, as noted by J. Kisyński (personal communication), the corresponding $\mathcal{T}$ is a homomorphism of the convolution algebra $L^1_\omega(\mathbb{R}^+)$. Indeed, it is enough to note that $\mathcal{T}(e_\lambda)\mathcal{T}(e_\mu) = \mathcal{T}(e_\lambda * e_\mu)$, the set e_λ, $\lambda > \omega$, being total in $L^1_\omega(\mathbb{R}^+)$. The above formula follows from the Hilbert equation:

$$\mathcal{T}(e_\lambda)\mathcal{T}(e_\mu) = R_\lambda R_\mu = \frac{1}{\mu-\lambda}[R_\lambda - R_\mu]$$
$$= \mathcal{T}\left(\frac{1}{\mu-\lambda}(e_\lambda - e_\mu)\right) = \mathcal{T}(e_\lambda * e_\mu).$$

Analogously one proves that if f satisfies (1.4) then $B\mathcal{T}(e_\lambda)\mathcal{T}(e_\mu) = \mathcal{T}(e_\lambda)\mathcal{T}(e_\mu)B = \mathcal{T}(e_\lambda * e_\mu)$. This formula could be used as a starting point for the proof of Da Prato's generation theorem for regularized semigroups [3]. ■

COROLLARY 1. *If f satisfies* (1.1), *then the limit*

$$\lim_{\mu\to\infty}\int_0^t g_\mu(s)\,ds =: u(t) \tag{2.7}$$

exists (*almost uniformly in* $t\in[0,\infty)$) *and is a Lipschitz continuous function with Lipschitz constant* $Me^{\omega t_0}$ *on* $[0,t_0]$. *Furthermore*,

$$\int_0^\infty e^{-\lambda t}u(t)\,dt = \frac{f(\lambda)}{\lambda} \quad \textit{for } \lambda>\omega,\ \lambda\neq 0. \tag{2.8}$$

Proof. By Theorem 1 the limit $\lim_{\mu\to\infty}\int_0^t e^{-\varepsilon(\mu)s}g_\mu(s)\,ds =: u(t)$ exists, with $\varepsilon(\mu)=\omega/(\mu-\omega)$, and since the integrand is bounded by $Me^{\omega s}$ the limit is easily proven to be almost uniform in $[0,\infty)$. The function $u(t)$ is Lipschitz continuous by the same reason. Since for $0\le t\le t_0$,

$$\Big\|\int_0^t e^{-\varepsilon(\mu)s}g_\mu(s)\,ds-\int_0^t g_\mu(s)\,ds\Big\| \le M\{e^{\omega\mu/(\mu-\omega)t_0}\vee 1\}\int_0^{t_0}(1-e^{-\varepsilon(\mu)s})\,ds\to 0$$

as $\mu\to\infty$, we get (2.7). Formula (2.8) follows by Proposition 1 and integration by parts. ∎

THEOREM 2 (Widder [11], p. 315, Theorem 16a). *A necessary and sufficient condition for an infinitely differentiable complex-valued function* $f(\lambda)$ *to be the Laplace transform of a bounded measurable function is that there exist a constant* $M>0$ *such that*

$$|f^{(n)}(\lambda)|\le\frac{Mn!}{\lambda^{n+1}}. \tag{2.9}$$

Proof. It is evident that (2.9) is necessary. Conversely, if (2.9) holds, then, by Corollary 1, there exists a Lipschitz continuous function $u(t)$, $|u(t)-u(s)|\le M|t-s|$, satisfying (2.8). Since $u(t)$ is complex-valued, by the Lebesgue theorem it is a.e. differentiable, and by (2.8) one easily sees that $u'(t)$ is a function we were looking for.

As a by-product of our proof we also obtain an inversion formula for the Laplace transform, valid if the original function is bounded:

$$\mathcal{L}^{-1}(f)=\frac{d}{dt}\lim_{\mu\to\infty}\int_0^t g_\mu(s)\,ds.$$

Remark 4. As is now well known ([1]), the above theorem is also true if "complex-valued function" is replaced by "function with values in a Banach space with the Radon–Nikodym property". The proof remains the same.

THEOREM 3 (Arendt). *Suppose* $\omega<\lambda\to f(\lambda)$ *is an infinitely differentiable Banach space-valued function. Then the following two conditions are equivalent*:

(a) *there exists an* $M\ge 0$ *such that*

$$\|f^{(k)}(\lambda)\|\le\frac{Mk!}{(\lambda-\omega)^{k+1}},\quad k\ge 0,\ \lambda>\omega,$$

(b) *there exists a function* $u(t)$ *with values in the same Banach space, Lipschitz continuous with Lipschitz constant* $e^{\omega t_0}$ *on* $[0,t_0]$, *such that*

$$\frac{f(\lambda)}{\lambda}=\int_0^\infty e^{-\lambda t}u(t)\,dt\quad \textit{for } \lambda>\omega,\ \lambda\ne 0.$$

Furthermore,

$$u(t)=\lim_{\mu\to\infty}\int_0^t e^{-\mu s}s\mu^2\sum_{n=0}^\infty\frac{s^n\mu^{2n}(-1)^n}{n!(n+1)!}f^{(k)}(\mu)\,ds.$$

Proof. The proof of (b)⇒(a) is straightforward. The converse was proved in Corollary 1. ∎

The advantage of the proof we offered here in comparison to Arendt's original argument is that an explicit approximation of $u(t)$ is provided. This result reflects the spirit of [6] and [8], where the Hille (i.e. Post–Widder) approximation was considered.

Note that if $\omega\ge 0$ and $f(\lambda)=\lambda^{-(n-1)}R_\lambda$ where $n\ge 1$ and R_λ, $\lambda>\omega$, is a resolvent in a Banach space X, then the above result serves as a generation theorem for an n-times integrated semigroup, say $\{T_n(t), t\ge 0\}$, given by

$$T_n(t)=\lim_{\mu\to\infty}\int_0^t e^{-\mu s}s\mu^2\sum_{n=0}^\infty\frac{s^n\mu^{2n}(-1)^n}{n!(n+1)!}\Big(\frac{R_\mu}{\mu^{n-1}}\Big)^{(k)}ds$$

(the limit taken in the norm of $\mathcal{L}(X,X)$). In particular, if $n=0$, we have

$$T_1(t)=\lim_{\mu\to\infty}\int_0^t[e^{-\mu s}+g_\mu(s)]\,ds=\lim_{\mu\to\infty}\int_0^t e^{(\mu^2R_\mu-\mu)s}\,ds,$$

which proves that $T_1(t)$ is a limit of once-integrated semigroups generated by $\mu^2R_\mu-\mu$. This result was already established in [2] with the limit taken in the strong topology. Let us also note here that the Yosida approximation of distribution semigroups was considered by H. Fattorini [5]. His results are worth comparing with our formula (2.3).

References

[1] W. Arendt, *Vector-valued Laplace transforms and Cauchy problem*, Israel J. Math. 59 (1987), 327–352.

[2] A. Bobrowski, *Integrated semigroups and the Trotter–Kato theorem*, Bull. Polish Acad. Sci. Math. 41 (1994), 297–304.

[3] G. Da Prato, *Semigruppi regolarizzabili*, Ricerche Mat. 15 (1966), 223–248.
[4] C. Dellacherie and P. A. Meyer, *Probabilities and Potential. C*, North-Holland Math. Stud. 151, North-Holland, Amsterdam, 1988.
[5] H. O. Fattorini, *A representation theorem for distribution semigroups*, J. Funct. Anal. 6 (1970), 83–96.
[6] B. Hennig and F. Neubrander, *On representations, inversions, and approximations of Laplace transforms in Banach spaces*, Appl. Anal. 49 (1993), 151–170.
[7] E. Hille and R. S. Phillips, *Functional Analysis and Semi-Groups*, Amer. Math. Soc. Colloq. Publ. 31, Providence, R.I., 1957.
[8] F. Neubrander, *The Laplace–Stieltjes transform in Banach spaces and abstract Cauchy problems*, in: Proc. 3rd International Workshop Conference in Evolution Equations, Control Theory and Biomathematics, Han-sur-Lesse, P. Clément and G. Lumer (eds.), Lecture Notes in Pure and Appl. Math. 155, Dekker, 1994, 417–431.
[9] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Appl. Math. Sci. 44, 1983, Springer, New York, 1983.
[10] R. S. Phillips, *An inversion formula for the Laplace transform and semigroups of linear operators*, Ann. of Math. 59 (1954), 325–356.
[11] D. V. Widder, *The Laplace Transform*, Princeton Univ. Press, 1946.
[12] K. Yosida, *Functional Analysis*, Springer, Berlin, 1968.

Chair of Mathematics
Department of Electrical Engineering
Lublin Technical University
Nadbystrzycka 38A
20-618 Lublin, Poland

Received May 13, 1996 (3668)

STUDIA MATHEMATICA 124 (3) (1997)

Quasi-multipliers of the algebra of approximable operators and its duals

by

MICHAEL GROSSER (Wien)

Abstract. Let A be the Banach algebra $K_0(X)$ of approximable operators on an arbitrary Banach space X. For the spaces of all bilinear continuous quasi-multipliers of A resp. its dual A^* resp. its bidual A^{**}, concrete representations as spaces of operators are given.

1. Introduction. Let X be an arbitrary Banach space (we do not assume any kind of approximation property for X) and denote by A the algebra $K_0(X)$ of approximable operators on X (i.e. of all operators which are uniform limits of continuous linear operators from X to X having finite rank), equipped with the usual operator norm. A can be considered as A-A-bimodule in the natural way; therefore, the first resp. second Banach duals A^* resp. A^{**} of A become A-A-bimodules by the first resp. second adjoints of the actions of A on A.

In this article, we shall give representations of the quasi-multiplier spaces of A, A^* and A^{**}, respectively. The result for A itself is known already for at least 17 years ([G1, 3.24 and 3.26]): $QM(A)$ is isometrically isomorphic to $L(X^*)$ where $g \in L(X^*)$ corresponds to the quasi-multiplier ϕ_g determined by $\iota_X \circ \phi_g(a,b) := a^{**} \circ g^* \circ b^{**} \circ \iota_X$. (The notation is explained in detail in the following section.) For the special case where X^* satisfies the bounded approximation property, this result was restated and proved recently in [AR, Corollary 4.3].

A^*, being isometrically isomorphic to $I(X^*)$, can be considered either as an A-A-bimodule in the natural way or, with multiplication defined by composition of operators, as a Banach algebra in its own right. Adopting the first point of view (as is done in Section 3), $QM(A^*)$ is given as $QM_A(A^*) := B_A^A(A, A; A^*)$ while in the second case (treated in Section 4), $QM_{A^*}(A^*) = B_{A^*}^{A^*}(A^*, A^*; A^*)$ (the subscripts to QM are meant to specify which of the two variants is intended).

1991 *Mathematics Subject Classification*: Primary 46H35; Secondary 47D35, 46H25.