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Subanalytic version of Whitney's extension theorem

by

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Dedicated to Professor Stanisław Łojasiewicz

Abstract. For any subanalytic C^k-Whitney field (k finite), we construct its subanalytic C^k-extension to $\mathbb{R}^n$. Our method also applies to other o-minimal structures; e.g., to semialgebraic Whitney fields.

1. Introduction. Let E be a subanalytic subset of $\mathbb{R}^n$. In this article we adopt the following definition of a subanalytic function. Let $\tau : \mathbb{R} \ni t \mapsto \mathbb{R}(t,1) \in \mathbf{P}^1$. A function $f : E \to \mathbb{R}$ is called *subanalytic* if the graph of $\tau \circ f$ is subanalytic in $\mathbb{R}^n \times \mathbf{P}^1$. A mapping $f = (f_1, \ldots, f_m) : E \to \mathbb{R}^m$ is called *subanalytic* if $f_1, \ldots, f_m$ are subanalytic. For the properties of subanalytic mappings the reader is referred to [2, Sect. 3] or/and [9, Sect. 2]. Other fundamental results of the theory of subanalytic sets can be found in [1, 4, 5, 6, 8, 9, 12].

We shall prove the following:

THEOREM 1. *Let E be a closed subanalytic subset of $\mathbb{R}^n$, and let p and q be positive integers, $p \le q$. Let*

$$F(x,X) = \sum_{|\kappa| \le p} \frac{1}{\kappa!} F^\kappa(x) X^\kappa \quad (X = (X_1, \ldots, X_n))$$

be a C^p-Whitney field subanalytic on E (i.e., F^κ are subanalytic functions on E). Then there exists a subanalytic C^p-function $f : \mathbb{R}^n \to \mathbb{R}$, C^q on $\mathbb{R}^n \setminus E$, such that $D^\kappa f = F^\kappa$ on E whenever $\kappa \in \mathbb{N}^n$, $|\kappa| \le p$.

Whitney's construction [18] does not give subanalyticity. Our method also applies to the semialgebraic case (the corresponding results on semialgebraic sets are found in [3, 11]). Actually, it can even be used in a much

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more general setting; viz., in any o-minimal structure on the real field (see [17]).

The proof of Theorem 1 is based on subtle differential properties of subanalytic sets described in the next two sections.

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2. An estimate for the derivatives of a subanalytic function. In this section we will follow an idea from Gromov [7].

LEMMA 1. *Let $\lambda : \Delta \to \mathbb{R}$ be a C^2-function of one variable such that either $\lambda'' \ge 0$ on Δ or $\lambda'' \le 0$ on Δ. Then, for any interval $[t-r, t+r] \subset \Delta$, $|\lambda'(t)| \le 2\sup_{[t-r,t+r]} |\lambda|/r$.*

Proof. Suppose that $\lambda'' \le 0$. Then λ is concave and

$$(\lambda(t) - \lambda(s))/(t-s) \le (\lambda(t) - \lambda(t-r))/r \le 2 \sup_{[t-r,t+r]} |\lambda|/r$$

whenever $t - r < s < t$. It follows that $\lambda'(t) \le 2\sup_{[t-r,t+r]} |\lambda|/r$. Applying this to $\lambda(-t)$, we obtain $-\lambda'(t) \le 2\sup_{[t-r,t+r]} |\lambda|/r$.

Lemma 1 generalizes easily by induction:

LEMMA 2. *Let $\lambda : \Delta \to \mathbb{R}$ be a C^{p+1}-function $(p \ge 1)$ of one variable such that, for $i = 2, \ldots, p+1$, $\lambda^{(i)} \ge 0$ on Δ or $\lambda^{(i)} \le 0$ on Δ. Then, for any interval $[t-r, t+r] \subset \Delta$, $|\lambda^{(p)}(t)| \le 2^{\binom{p+2}{2}-2} \sup_{[t-r,t+r]} |\lambda|/r^p$.*

PROPOSITION 1. *Let $\phi : \Omega \to \mathbb{R}$ be a subanalytic function on an open subanalytic subset Ω of $\mathbb{R}^m$. Let $\alpha \in \mathbb{N}^m$. Then there exists a closed nowhere dense subset Z of Ω, subanalytic in $\mathbb{R}^m$, such that, for any open ball $K = K(u,r) \subset \Omega \setminus Z$, $|D^\alpha \phi(u)| \le C_\alpha \sup_K |\phi|/r^{|\alpha|}$, where C_α is a constant depending only on α.*

Proof. Clearly, we can assume that Ω is connected. Put $p = |\alpha|$ and $q = \binom{m+p-1}{m}$. Then we have

$$D^\alpha = \sum_{\nu=1}^{q} c_\nu \partial^p/\partial e_\nu^p,$$

where $\{e_\nu\}$ are suitably chosen unit vectors in $\mathbb{R}^m$, $\{c_\nu\}$ are real coefficients and $\partial^p/\partial e_\nu^p$ stands for the directional derivative. Let Z be the union of the zero-sets of all those functions $\partial^i \phi/\partial e_\nu^p$ $(i = 2, \ldots, p+1; \nu = 1, \ldots, q)$ which do not vanish identically. Suppose that $K = K(u,r) \subset \Omega \setminus Z$. Put $\lambda_\nu(t) = \phi(u + te_\nu)$. Then $\lambda^{(i)}(t) = (\partial^i\phi/\partial e_\nu^i)(u + te_\nu)$. Applying Lemma 2 to λ_ν, we obtain the needed inequality.

COROLLARY. *For each $u \in \Omega \setminus Z$,*

$$|D^\alpha\phi(u)| \le C_\alpha \sup\{|\phi(v)| : |u - v| < \operatorname{dist}(u, Z \cup \partial\Omega)\}/\operatorname{dist}(u, Z \cup \partial\Omega)^{|\alpha|}.$$

PROPOSITION 2. *Let $\phi : \Omega \to \mathbb{R}$ be a subanalytic, analytic function on an open subanalytic subset Ω of $\mathbb{R}^m$. Suppose that $|\partial\phi/\partial x_j| \le M$ on Ω, $j = 1, \ldots, m$. Let $p \in \mathbb{N}$, $p > 0$. Then there exists a closed nowhere dense subset Z of Ω, subanalytic in $\mathbb{R}^m$ and such that*

$$|D^\alpha\phi(u)| \le C(m,p)M \operatorname{dist}(u, Z \cup \partial\Omega)^{1-|\alpha|}$$

whenever $u \in \Omega \setminus Z$, $\alpha \in \mathbb{N}^m$, $1 \le |\alpha| \le p$, $C(m,p)$ being a constant depending only on m and p.

Proof. Apply Proposition 1 to the derivatives $\partial\phi/\partial x_j$.

Remark 1. In the subanalytic case (but not in the general o-minimal case) our Proposition 2 follows from Parusiński's [15, Prop. 3.1] (compare also [14, §4]). (One should consider as vector fields the products of a unit vector and the function of distance from the union of strata of smaller dimension.)

3. Λ_p-regular mappings. Let Ω be an open bounded subset of $\mathbb{R}^k$. Let $\phi : \Omega \to \mathbb{R}^n$ be a C^p-mapping. We will call ϕ *Λ_p-regular* (in Ω) if there exists a constant $C > 0$ such that

$$|D^\alpha\phi(y)| \le C/\operatorname{dist}(y, \partial\Omega)^{|\alpha|-1} \quad \text{for } \alpha \in \mathbb{N}^k, 1 \le |\alpha| \le p;$$

in other words, $D^\alpha\phi(y) = O(\operatorname{dist}(y, \partial\Omega)^{1-|\alpha|})$ as $\operatorname{dist}(y, \partial\Omega) \to 0$, for all $\alpha \in \mathbb{N}^k$ with $1 \le |\alpha| \le p$.

Remark 2. Let ϕ be Λ_1-regular and let $A \subset \Omega$. Suppose that A has the following *Whitney arc property with exponent* 1 (WAP(1)): there exists a constant $C' > 0$ such that any two points $a_1, a_2 \in A$ can be joined in A by an arc of length $\le C'|a_1 - a_2|$. Then ϕ is a Lipschitz mapping on A and thus ϕ_A extends continuously to $\overline{A}$.

We shall use the following theorem of Whitney [19]:

THEOREM 2. *Let A be a locally closed subset of $\mathbb{R}^k$ having* WAP(1). *If*

$$G(y, Y) = \sum_{|\alpha| \le p} \frac{1}{\alpha!} G^\alpha(y) Y^\alpha$$

is a C^p-Whitney field on A, $A \subset B \subset \overline{A}$ and all the G^α's have continuous extensions $\overline{G}^\alpha$ to B, then

$$\overline{G}(y, Y) = \sum_{|\alpha| \le p} \frac{1}{\alpha!} \overline{G}^\alpha(y) Y^\alpha$$

is a C^p-Whitney field on B.

Proof. It is enough to repeat the argument from p. 76 in [16, Rem. 25].

We say that two closed subsets K and L of $\mathbb{R}^m$ are *regularly separated with exponent* 1 if there exists a constant $C > 0$ such that $\operatorname{dist}(u, K \cap L) \le C \operatorname{dist}(u, L)$ for each $u \in K$.

The following proposition motivates our interest in Λ_p-regular mappings.

PROPOSITION 3. *Suppose that $\Phi : \Omega \to \mathbb{R}^n$ is Λ_p-regular, and A is a closed subset of Ω having* WAP(1) *and such that $\overline{A}$ and $\partial\Omega$ are regularly separated with exponent* 1. *Let B be a compact subset of $\mathbb{R}^n$ such that $\Phi(A) \subset B$ and let F be a $\mathcal{C}^p$-Whitney field on B flat on $\overline{\Phi}(\overline{A} \setminus A)$. Let G be a $\mathcal{C}^p$-Whitney field on A defined by the formula*

$$G(y, Y) = F(\Phi(y), \widetilde{T}_y^p\Phi(Y)) \bmod (Y)^{p+1}, \quad \text{where } Y = (Y_1, \ldots, Y_k),$$

and

$$\widetilde{T}_y^p\Phi(Y) = \sum_{1 \le |\alpha| \le p} \frac{1}{\alpha!} D^\alpha\Phi(y) Y^\alpha.$$

Then G extends to a $\mathcal{C}^p$-Whitney field on $\overline{A}$ flat on $\overline{A} \setminus A$.

Proof. By the Newton formula, we have

$$\begin{aligned}
&G(y, Y) \\
&= \sum_{|\kappa| \le p} \frac{1}{\kappa!} F^\kappa(\Phi(y)) \Big(\sum_{1 \le |\alpha| \le p} \frac{1}{\alpha!} D^\alpha \Phi(y) Y^\alpha \Big)^\kappa \bmod (Y)^{p+1} \\
&= \sum_{|\kappa| \le p} F^\kappa(\Phi(y)) \sum_{\sum_\alpha \kappa_\alpha = \kappa} \prod_\alpha \frac{1}{\kappa_\alpha!} \prod_\alpha \frac{1}{\alpha!^{|\kappa_\alpha|}} (D^\alpha \Phi(y))^{\kappa_\alpha} Y^{\alpha|\kappa_\alpha|} \bmod (Y)^{p+1} \\
&= \sum_{\sum_\alpha |\kappa_\alpha| \le p} \Big[\prod_\alpha 1/(\kappa_\alpha! \alpha!^{|\kappa_\alpha|}) \Big] F^{\sum \kappa_\alpha}(\Phi(y)) \\
&\quad \times \prod_\alpha (D^\alpha \Phi(y))^{\kappa_\alpha} Y^{\alpha|\kappa_\alpha|} \bmod (Y)^{p+1}.
\end{aligned}$$

Hence, for any $\sigma \in \mathbb{N}^m$ such that $|\sigma| \le p$,

$$G^\sigma(y) = \sigma! \sum_{\sum \alpha|\kappa_\alpha| = \sigma} [\cdot] F^{\sum \kappa_\alpha}(\Phi(y)) \prod_\alpha (D^\alpha \Phi(y))^{\kappa_\alpha}.$$

Thus

$$\begin{aligned}
|G^\sigma(y)| &\le C_1 \sum_{\sum \alpha|\kappa_\alpha| = \sigma} |F^{\sum \kappa_\alpha}(\Phi(y))| \operatorname{dist}(y, \partial\Omega)^{\sum|\kappa_\alpha| - |\sigma|} \\
&\le C_2 \sum_{\sum \alpha|\kappa_\alpha| = \sigma} |F^{\sum \kappa_\alpha}(\Phi(y))| \operatorname{dist}(y, \overline{A} \cap \partial\Omega)^{\sum|\kappa_\alpha| - |\sigma|} \\
&= C_3 \varepsilon(\Phi(y), \overline{\Phi}(a)) |\Phi(y) - \overline{\Phi}(a)|^{p - \sum|\kappa_\alpha|} |y - a|^{\sum|\kappa_\alpha| - |\sigma|},
\end{aligned}$$

where $a \in \overline{A} \cap \partial\Omega$ and $|y - a| = \operatorname{dist}(y, \overline{A} \cap \partial\Omega)$.

Consequently, $G^\sigma(y) \to 0$ as $\operatorname{dist}(y, \overline{A} \cap \partial\Omega) \to 0$, and now Theorem 2 completes the proof.

Remark 3. Observe that if $r : B \to [0, \infty)$, $a \in \overline{A} \setminus A$ and $F^\kappa(x) = o(r(x)^{p-|\kappa|})$ as $x \to \Phi(a)$, for any $\kappa \in \mathbb{N}^n$ such that $|\kappa| \le p$, then $G^\sigma(y) = o(r(\Phi(y))^{p-|\sigma|})$ as $y \to a$, for any $\sigma \in \mathbb{N}^k$ such that $|\sigma| \le p$.

Remark 4. If Ω satisfies WAP(1), we can take $A = \Omega$ in Proposition 3.

4. Λ_p-regular cells. It appears that subanalytic sets are stratifiable into graphs of Λ_p-regular mappings. In fact, we shall prove more.

Let us recall that a *subanalytic stratification* of a subanalytic subset E of $\mathbb{R}^n$ is a locally finite (in $\mathbb{R}^n$) decomposition $\mathcal{T}$ of E into subanalytic, connected, analytic submanifolds of $\mathbb{R}^n$, called *strata*, such that, for each $\Gamma \in \mathcal{T}$, its *boundary* $(\overline{\Gamma} \setminus \Gamma) \cap E$ is the union of some strata of dimensions smaller than $\dim \Gamma$.

We shall say that S is an *open Λ_p-regular* (*subanalytic*) *cell* in $\mathbb{R}^n$ if

1) S is an open bounded interval in $\mathbb{R}$ when $n = 1$, and
2) $S = \{(x', x_n) : x' \in T, \phi_1(x') < x_n < \phi_2(x')\}$, where T is an open Λ_p-regular cell in $\mathbb{R}^{n-1}$, and $\phi_i : T \to \mathbb{R}$ $(i = 1, 2)$ are analytic, subanalytic, Λ_p-regular functions such that $\phi_1(x') < \phi_2(x')$ on T, when $n > 1$.

Remark 5. Then S has WAP(1), ϕ_1, ϕ_2 extend continuously to $\overline{T}$ and $\overline{S}$ is compact.

We extend the last definition. If $m \in \mathbb{Z}$ and $0 \le m < n$, we shall say that S is an *m-dimensional Λ_p-regular cell* in $\mathbb{R}^n$ if $S = \{(u, \phi(u)) : u \in T\}$, where T is an open Λ_p-regular cell in $\mathbb{R}^m = \mathbb{R}^m \times 0^{n-m}$ and $\phi : T \to \mathbb{R}^{n-m}$ is an analytic subanalytic Λ_p-regular mapping.

PROPOSITION 4. *Any compact subanalytic subset E of $\mathbb{R}^n$ has a finite stratification $E = S_1 \cup \ldots \cup S_r$ such that each S_j is a Λ_p-regular cell in $\mathbb{R}^n$ in some linear coordinate system. Moreover, if $A_1, \ldots, A_k$ are any subsets of E subanalytic in $\mathbb{R}^n$, we can have each S_j compatible with each A_i in the following sense: if $A_i \cap S_j \ne \emptyset$, then $S_j \subset A_i$.*

Proof. Put $m = \dim E$. It suffices to prove that there exists a finite family $T_1, \ldots, T_r$ such that each T_j is an m-dimensional Λ_p-regular cell in $\mathbb{R}^n$ in some linear coordinate system, T_j's are open, pairwise disjoint subsets of E, compatible with each A_i, and $\dim(E \setminus \bigcup_j T_j) < m$; because then we shall use the induction hypothesis to $E' = E \setminus \bigcup_j T_j$ with the subsets $A_i \cap E'$ and $(\overline{T}_j \setminus T_j) \cap E'$.

By the main result of [10], we first find Λ_1-regular cells $T_1, \ldots, T_r$ in appropriate linear coordinate systems in $\mathbb{R}^n$. By Proposition 2 and using induction on n, we easily see that any Λ_1-regular cell can be represented as a finite disjoint union of Λ_p-regular cells and some nowhere dense subset. This completes the proof.

Assume now that S is an open Λ_p-regular cell in $\mathbb{R}^n$. We define by induction on n a sequence $\varrho_1, \ldots, \varrho_{2n}$ of the *functions associated with the cell* S:

1) When $n = 1$ and $S = (a_1, a_2)$, we put $\varrho_1(x) = x - a_1$ and $\varrho_2(x) = a_2 - x$.

2) When $n > 1$ and $S = \{(x', x_n) : x' \in T,\ \phi_1(x') < x_n < \phi_2(x')\}$, let $\sigma_1, \ldots, \sigma_{2n-2}$ be the functions associated with T. Then we put, for any $x \in \overline{S}$,

$$\varrho_j(x) = \varrho_j(x', x_n) = \sigma_j(x') \quad \text{for } j = 1, \ldots, 2n-2,$$
$$\varrho_{2n-1}(x) = x_n - \overline{\phi}_1(x') \quad \text{and} \quad \varrho_{2n}(x) = \overline{\phi}_2(x') - x_n.$$

The functions ϱ_j are subanalytic, continuous on $\overline{S}$ and analytic on S.

LEMMA 3. *There exists a constant $M > 0$ such that*

$$M \min_j \varrho_j(x) \le \operatorname{dist}(x, \partial S) \le \min_j \varrho_j(x) \quad \textit{for } x \in \overline{S}.$$

Proof. This follows easily from the fact that the faces of S are Lipschitz maps.

It is also easy to check the following:

LEMMA 4. *The functions associated with an open Λ_p-regular cell S are Λ_p-regular on S.*

We shall need the following consequence of Lemmas 3 and 4.

LEMMA 5. $D^\alpha(1/\varrho_j)(x) = O(\operatorname{dist}(x, \partial S)^{-|\alpha|-1})$ *as* $\operatorname{dist}(x, \partial S) \to 0$, $x \in S$, *for all* $\alpha \in \mathbb{N}^n$ *with* $|\alpha| \le p$ *and* $j = 1, \ldots, 2n$.

Proof. For any $\alpha \ne 0$, we have

$$D^\alpha(1/\varrho_j) = \sum_{\nu=1}^{\alpha} \Big(\sum_{\substack{\lambda_1 + \ldots + \lambda_\nu = \alpha \\ \lambda_1 \ne 0, \ldots, \lambda_\nu \ne 0}} a^\alpha_{\lambda_1 \ldots \lambda_\nu} (D^{\lambda_1} \varrho_j) \ldots (D^{\lambda_\nu} \varrho_j) \Big) \cdot \varrho_j^{-1-\nu},$$

where the coefficient $a^\alpha_{\lambda_1 \ldots \lambda_\nu}$ depends only on $\alpha, \lambda_1, \ldots, \lambda_\nu$. The lemma follows from Lemmas 4 and 3.

5. Two lemmas on C^p-functions

LEMMA 6. *Let Γ be an open subset of $\mathbb{R}^n$, $a \in \overline{\Gamma}$ and $r : \Gamma \to \mathbb{R}$. Let $g, h : \Gamma \to \mathbb{R}$ be C^p-functions such that $D^\kappa g(x) = o(r(x)^{p-|\kappa|})$ and $D^\kappa h(x) = O(r(x)^{-|\kappa|})$ as $x \to a$, for $|\kappa| \le p$. Then $D^\kappa(gh)(x) = o(r(x)^{p-|\kappa|})$ as $x \to a$, for $|\kappa| \le p$.*

Proof. Immediate by Leibniz's formula.

LEMMA 7. *Let $\chi : \Omega \to \mathbb{R}$ be a C^p-function on an open subset Ω of $\mathbb{R}^m$ $(m < n)$ and $r : \Omega \to (0, \infty)$, $c \in \overline{\Omega}$. Assume that $D^\alpha \chi(u) = O(r(u)^{-|\alpha|-1})$ as $u \to c$, for all α with $|\alpha| \le p$. Let $\psi : \mathbb{R} \to \mathbb{R}$ be any C^p-function. Let Γ be an open subset of $\mathbb{R}^m \times \mathbb{R}^{n-m} = \mathbb{R}^n$ contained in $\{(u, w) \in \Omega \times \mathbb{R}^{n-m} : |w_i| \le Cr(u)\}$, where C is a constant. Define $g : \Gamma \to \mathbb{R}$ by*

$$g(u, w) = \psi(\chi(u) w_1) \cdot \ldots \cdot \psi(\chi(u) w_{n-m}).$$

Then $D^{(\alpha,\beta)} g(u, w) = O(r(u)^{-|\alpha|-|\beta|})$ as $(u, w) \to (c, 0)$, for all (α, β) with $|\alpha| + |\beta| \le p$.

Proof. It suffices to prove this for each of the functions $g_i(u, w) = \psi(\chi(u) w_i)$ separately, so we can assume $n - m = 1$. We have

$$D^{(\alpha,\beta)} g(u, w) = \sum_{\gamma_1 + \ldots + \gamma_\beta + \sigma = \alpha} \frac{\alpha!}{\gamma_1! \ldots \gamma_\beta! \sigma!} D^{\gamma_1} \chi(u) \ldots D^{\gamma_\beta} \chi(u) \cdot D^{(\sigma,0)} [\psi^{(\beta)}(\chi(u) w)],$$

and, for $\sigma \ne 0$,

$$D^{(\sigma,0)} [\psi^{(\beta)}(\chi(u) w)] = \sum_{s=1}^{|\sigma|} w^s \sum_{\lambda_1 + \ldots + \lambda_s = \sigma} A^\sigma_{\lambda_1 \ldots \lambda_s} D^{\lambda_1} \chi(u) \ldots D^{\lambda_s} \chi(u) \psi^{(\beta+s)}(\chi(u) w),$$

where $A^\sigma_{\lambda_1 \ldots \lambda_s}$ depends only on $\sigma, \lambda_1, \ldots, \lambda_s$. This, together with the boundedness of $\chi(u) w$ at $(c, 0)$, gives the required inequality.

6. Proof of Theorem 1.

By a subanalytic C^q-partition of unity, we reduce the general case to that with E compact. Let A denote the closure of $\bigcup_\kappa \{x \in E : F^\kappa(x) \ne 0\}$. We will prove by induction on $m = \dim A$ that there exists a function f satisfying the conclusion of Theorem 1 and, in addition, C^q on $\mathbb{R}^n \setminus A$.

The case $m = 0$ being obvious, we assume $m > 0$. Take a stratification $A = S_1 \cup \ldots \cup S_r$ of A as in Proposition 4 such that S_j are compatible with the set $\bigcup_\kappa \{x \in E : F^\kappa(x) \ne 0\}$, which is open in E, and F^κ are analytic on each S_j. Let $\dim S_j = m$ for $j = 1, \ldots, k$, and $\dim S_j < m$ for $j = k+1, \ldots, r$. By the induction hypothesis, we can assume that F is flat on $\bigcup_{j>k} S_j$. Next, using induction on k, we can assume that $k = 1$, and so $\bigcup_\kappa \{x \in E : F^\kappa(x) \ne 0\} = S$ is an m-dimensional Λ_p-regular cell in $\mathbb{R}^n$ and F is flat on $\overline{S} \setminus S$. In the case $m = n$ (i.e. S is open in $\mathbb{R}^n$), it suffices to define $f(x) = F^0(x)$ for $x \in S$, and $f(x) = 0$ for $x \in \mathbb{R}^n \setminus S$ (Hestenes' Lemma [16, p. 80]), so let $1 \le m < n$. Then $S = \{(u, \phi(u)) : u \in T\}$, where $\phi : T \to \mathbb{R}^{n-m}$.

We will distinguish two cases.

Case I: $E = A = \overline{S}$, *and* $\phi = 0$. In this case $E = \overline{T} \times 0$. Put $\Gamma(T) = \{(u,w) \in T \times \mathbb{R}^{n-m} : |w| < \operatorname{dist}(u, \partial T)\}$.

We shall construct a function f satisfying the conclusion of Theorem 1 *such that* $f = 0$ *on* $\mathbb{R}^n \setminus \Gamma(T)$. Since F is the sum of the C^p-Whitney fields

$$F_\beta(u,0;X) = F_\beta(u,0;U,W) = \sum_{|\alpha| \le p-|\beta|} \frac{1}{\alpha!\beta!} F^{(\alpha,\beta)}(u) U^\alpha W^\beta,$$

where $\beta \in \mathbb{N}^{n-m}$, $|\beta| \le p$, $U = (U_1, \ldots, U_m)$, $W = (W_1, \ldots, W_{n-m})$, we can assume F is equal to one of them; i.e., $F(u,0;X) = F_\beta(u,0;X)$, for a fixed β. By Corollary to Proposition 1 and Proposition 4, there exists a finite family $\{Q_\nu\}$ of pairwise disjoint open subsets of T such that each Q_ν is an open Λ_p-regular cell (in a suitable linear coordinate system), $Z = \overline{T} \setminus \bigcup_\nu Q_\nu$ has dimension $< m$, and

$$\begin{aligned}|D^\gamma F^{(\alpha,\beta)}(u)| \\ \le C \sup\{|F^{(\alpha,\beta)}(v)| : v \in Q_\nu,\ |u - v| < \operatorname{dist}(u, \partial Q_\nu)\}/\operatorname{dist}(u, \partial Q_\nu)^{|\gamma|}\end{aligned}$$

whenever $u \in Q_\nu$, $\alpha \in \mathbb{N}^m$, $|\alpha| \le p - |\beta|$, and $\gamma \in \mathbb{N}^m$, $|\gamma| \le p$.

Since $Z \times 0$ and $\mathbb{R}^n \setminus \Gamma(T)$ are regularly separated with exponent 1, the field G defined as the glueing of the restriction of F to Z with the zero field in $\mathbb{R}^n \setminus \Gamma(T)$ is a C^p-Whitney field (cf. [13, Chap. I, Rem. 5.6]). By applying the induction hypothesis to G, we can assume that F is flat on Z. Since $\Gamma(Q_\nu) \subset \Gamma(T)$, it suffices to construct a required function f_ν for each ν separately (as then we shall take $f = \sum f_\nu$). Hence, without any loss of generality, we can assume that $T = Q_\nu$.

Put $h(u,w) = F^{(0,\beta)}(u) w^\beta$ for $u \in \overline{T}$ and $w \in \mathbb{R}^{n-m}$. Clearly, h is an analytic function in $T \times \mathbb{R}^{n-m}$.

LEMMA 8. *Let* $\kappa = (\sigma, \tau) \in \mathbb{N}^m \times \mathbb{N}^{n-m}$, $|\kappa| \le p$, *and let* $a \in \partial T$. *Then* $D^\kappa h(u,w) = o(\operatorname{dist}(u, \partial T)^{p-|\kappa|})$ *as* $\Gamma(T) \ni (u,w) \to (a,0)$.

Proof. Obviously, we can assume $\tau \le \beta$.

Suppose first that $|\sigma| \le p - |\beta|$. Then

$$D^\kappa h(u,w) = [\beta!/(\beta-\tau)!] F^{(\sigma,\beta)}(u) w^{\beta-\tau},$$

and, by Whitney's regularity condition, we have

$$F^{(\sigma,\beta)}(u) = o(\operatorname{dist}(u,\partial T)^{p-|\sigma|-|\beta|})$$

as $u \to a$. Since $|w| < \operatorname{dist}(u, \partial T)$, the lemma follows in this case.

Suppose now that $|\sigma| > p - |\beta|$. Then $\sigma = \alpha + \gamma$, where $|\alpha| = p - |\beta|$, and

$$\begin{aligned}|D^\kappa h(u,w)| &= |[\beta!/(\beta-\tau)!] D^\gamma F^{(\alpha,\beta)}(u) w^{\beta-\tau}| \\ &\le [C\beta!/(\beta-\tau)!] \varepsilon(u) \operatorname{dist}(u,\partial T)^{-|\gamma|} \operatorname{dist}(u,\partial T)^{|\beta|-|\gamma|},\end{aligned}$$

where $\varepsilon(u) = \sup\{|F^{(\alpha,\beta)}(v)| : |v - u| < \operatorname{dist}(u, \partial T)\} \to 0$ as $u \to a$. Since $|\beta| - |\tau| - |\gamma| = |\beta| - |\tau| + |\alpha| - |\sigma| = p - |\tau| - |\sigma| = p - |\kappa| \ge 0$, the proof of the lemma is complete.

Let $\psi : [0,\infty) \to [0,1]$ be a semialgebraic C^q-function such that $\psi(t) = 1$ near 0, and $\psi(t) = 0$ for $t \ge 1$. Let $\varrho_1, \ldots, \varrho_{2m}$ denote the functions associated with T. We put

$$f(u,w) = \prod_{i=1}^{n-m} \prod_{j=1}^{2m} \psi(w_i \sqrt{n-m}/(M \varrho_j(u))) h(u,w).$$

This is a C^q-function on $T \times \mathbb{R}^{n-m}$. It follows from Lemmas 8, 5, 7 and Lemma 6 (where we put $r(u) = \operatorname{dist}(u, \partial T)$) that

$$D^\kappa f(u,w) \to 0 \quad \text{as } \Gamma(T) \ni (u,w) \to (a,0) \in \partial T \times 0,$$

for all $\kappa \in \mathbb{N}^n$ with $|\kappa| \le p$. On the other hand, $f(u,w) = 0$ if $(u,w) \in (T \times \mathbb{R}^{n-m}) \setminus \Gamma(T)$, due to Lemma 3, so f extends to a C^p-function on $\mathbb{R}^n$, equal to 0 on $\mathbb{R}^n \setminus \Gamma(T)$.

Case II: *general.* We define a subanalytic function $r : T \to (0, \infty)$ by the formula

$$r(u) = \begin{cases} \inf\{|w - \phi(u)| : (u,w) \in E \setminus S\} & \text{if } \{w : (u,w) \in E \setminus S\} \ne \emptyset, \\ 1 & \text{otherwise.}\end{cases}$$

By Proposition 4, there exists a finite family $\{Q_\nu\}$ of pairwise disjoint open subsets of T such that each Q_ν is a Λ_p-regular cell (in a suitable linear coordinate system), $Z = \overline{T} \setminus \bigcup_\nu Q_\nu$ has dimension $< m$, r is analytic on each Q_ν and, for each ν, either there is $i \in \{1, \ldots, m\}$ such that $|\partial r/\partial u_i| > 1$ on Q_ν, or $|\partial r/\partial u_i| \le 1$ on Q_ν for all $i \in \{1, \ldots, m\}$. In the second case, by an additional decomposition and Proposition 2, we can assume that the functions $|D^\alpha r(u)| \operatorname{dist}(u, \partial Q_\nu)^{|\alpha|-1}$ are bounded on Q_ν for $\alpha \in \mathbb{N}^m$ with $1 \le |\alpha| \le p$.

By the induction hypothesis, we can assume that F is flat on $\overline{S} \cap (Z \times \mathbb{R}^{n-m})$. Next, by using induction on the number of Q_ν, we can simply assume that $T = Q_\nu$ for some ν.

By Whitney's regularity condition for F, we have

$$F^\kappa(u, \phi(u)) = o(r(u)^{p-|\kappa|}) \quad \text{as } \operatorname{dist}(u, \partial T) \to 0,$$

for all $\kappa \in \mathbb{N}^n$ with $|\kappa| \le p$. By Proposition 3, the transformation G of the restriction of F to $\overline{S}$ by means of the Λ_p-regular automorphism

$$\Phi : T \times \mathbb{R}^{n-m} \ni (u,w) \mapsto (u, \phi(u) + w) \in T \times \mathbb{R}^{n-m}$$

is a C^p-Whitney field on $\overline{T} \times 0$ flat on $\partial T \times 0$.

By Remark 3,

$$(*) \qquad G^\kappa(u,0) = o(r(u)^{p-|\kappa|}) \quad \text{as } \operatorname{dist}(u, \partial T) \to 0,$$

for all $\kappa \in \mathbb{N}^n$ with $|\kappa| \le p$. It suffices to construct a subanalytic $\mathcal{C}^p$-function $g : \mathbb{R}^n \to \mathbb{R}$, $\mathcal{C}^q$ on $\mathbb{R}^n \setminus (\overline{T} \times 0)$, such that $D^\kappa g(u,0) = G^\kappa(u,0)$ as $u \in T$ for $|\kappa| \le p$, and $g(u,w)$ may not vanish only if $u \in T$ and $|w| < \min(r(u), \operatorname{dist}(u, \partial T))$; because then $g \circ \Phi$ will be the desired extension of F (again by Proposition 3).

(1) Assume first that all the functions $|D^\alpha r(u)| \operatorname{dist}(u,\partial T)^{|\alpha|-1}$, where $1 \le |\alpha| \le p$, are bounded on T. For any $P \subset T$, let

$$\Gamma_*(P) = \{(u,w) \in P \times \mathbb{R}^{n-m} : |w| < r(u)\}.$$

Let $Q = \{u \in T : r(u) < \operatorname{dist}(u,\partial T)\}$. Then the functions $|D^\alpha r(u)| r(u)^{|\alpha|-1}$, where $1 \le |\alpha| \le p$, are bounded on Q and, by the formula in the proof of Lemma 5, $|D^\alpha(1/r)(u)| r(u)^{|\alpha|+1}$ are bounded on Q. Let $g_0 : \mathbb{R}^n \to \mathbb{R}$ denote the extension of the field G constructed in Case I. Then, by the Taylor formula, $(*)$ implies

$$D^\kappa g_o(u,w) = o(r(u)^{p-|\kappa|}) \quad \text{as } \Gamma_*(T) \ni (u,w) \to (c,0) \in (\partial T) \times 0,$$

for all $\kappa \in \mathbb{N}^n$ with $|\kappa| \le p$. Now we define

$$g(u,w) = \prod_{i=1}^{n-m} \psi(w_i \sqrt{n-m}/r(u)) g_0(u,w) \quad \text{for } (u,w) \in T \times \mathbb{R}^{n-m}.$$

It follows from Lemmas 7 and 6 that, for any $c \in (\partial Q) \cap (\partial T)$ and $\kappa \in \mathbb{N}^n$ with $|\kappa| \le p$,

$$D^\kappa g(u,w) \to 0 \quad \text{as } \Gamma_*(Q) \ni (u,w) \to (c,0).$$

Moreover, if $|w| \ge r(u)$, then $g(u,w) = 0$, so

$$D^\kappa g(u,w) \to 0 \quad \text{as } Q \times \mathbb{R}^{n-m} \ni (u,w) \to (c,0).$$

Let $Q' = \{u \in T : r(u) \ge \operatorname{dist}(u,\partial T)\}$. By the proof of Lemma 5, the functions

$$|D^\alpha(1/r)(u)| \operatorname{dist}(u,\partial T)^{|\alpha|+1} \quad \text{for } \alpha \in \mathbb{N}^m,\ |\alpha| \le p,$$

are bounded on Q'. Since

$$D^\kappa g_0(u,w) = o(\operatorname{dist}(u,\partial T)^{p-|\kappa|})$$

as $T \times \mathbb{R}^{n-m} \ni (u,w) \to (c,0) \in (\partial T) \times 0$, for all $\kappa \in \mathbb{N}^n$ with $|\kappa| \le p$, it follows from Lemmas 7 and 6 that, for any $c \in (\partial Q') \cap (\partial T)$ and $\kappa \in \mathbb{N}^n$ with $|\kappa| \le p$,

$$D^\kappa g(u,w) \to 0 \quad \text{as } (Q' \times \mathbb{R}^{n-m}) \cap \Gamma(T) \ni (u,w) \to (c,0).$$

On the other hand, if $|w| \ge \operatorname{dist}(u,\partial T)$ and $u \in Q'$, then $g(u,w) = 0$, so $D^\kappa g(u,w) \to 0$ as $Q' \times \mathbb{R}^{n-m} \ni (u,w) \to (c,0)$.

Consequently, $D^\kappa g(u,w) \to 0$ as $\operatorname{dist}(u,\partial T) \to 0$, for all $\kappa \in \mathbb{N}^n$ with $|\kappa| \le p$; thus g extends by 0 to the required function.

(2) Assume now that there exists $i \in \{1,\dots,m\}$ such that $|\partial r/\partial u_i| > 1$ on T. We shall check that $r(u) \ge \operatorname{dist}(u,\partial T)$ for each $u \in T$. To see this, take any point $a = (a_1,\dots,a_m) \in T$. Then

$$\{t \in \mathbb{R} : (a_1,\dots,\underset{(i)}{t},\dots,a_m) \in T\} = (b_1,c_1) \cap \dots \cap (b_k,c_k),$$

where $b_1 < c_1 \le b_2 < \dots \le b_k < c_k$. For some $l \in \{1,\dots,k\}$, $a_i \in (b_l,c_l)$. It is now clear that, for any $u_i \in (b_l,c_l)$,

$$\begin{aligned} r(a_1,\dots,u_i,\dots,a_m) &\ge \max(|u_i - b_l|, |u_i - c_l|) \\ &\ge \operatorname{dist}((a_1,\dots,u_i,\dots,a_m),\partial T), \end{aligned}$$

hence $r(a) \ge \operatorname{dist}(a,\partial T)$.

It suffices to put $g(u,w) = g_0(u,w)$ to obtain the required function. This completes the proof of the theorem.

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On the Yosida approximation and the Widder–Arendt representation theorem

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Abstract. The Yosida approximation is treated as an inversion formula for the Laplace transform.

0. Introduction. The Yosida approximation is a standard tool in proving generation theorems for semigroups ([7], [9], [12]). In [10], a related power series was introduced and proven to yield an inversion formula for the Laplace transform ([10], Theorems 2.2–2.3, or [7], pp. 221–223, Theorems 6.3.3–6.3.6). Namely it was shown that the power series of the image function converges to the original function. In this article we shall show that this formula leads to a much simpler proof of a classical theorem of Widder characterizing the Laplace transform of a bounded complex-valued function. Furthermore, we shall provide a power-series-approximation formula for integrated Lipschitz continuous semigroups.

1. The Yosida approximation in Banach spaces. Let us start with a definition.

DEFINITION 1. Fix $\omega \in \mathbb{R}$. Let L be a Banach space and let $(\omega,\infty) \ni \lambda \to f(\lambda)$ be an infinitely differentiable function with values in L, satisfying

$$\|f^{(n)}(\lambda)\| \le \frac{Mn!}{(\lambda-\omega)^{n+1}}, \tag{1.1}$$

where $M > 0$ is a constant. Put

$$g_\mu(t) = e^{-\mu t} t\mu^2 \sum_{n=0}^{\infty} \frac{t^n \mu^{2n} (-1)^n f^{(n)}(\mu)}{n!(n+1)!} \quad \text{for } \mu > \omega. \tag{1.2}$$

The functions $g_\mu(t)$ will be called the *Yosida approximation* of f.

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