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# Averaging theorems for linear operators in compact groups and semigroups

by

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**Abstract.** The Weyl criterion for uniform distribution of a sequence has an especially simple form in compact abelian groups. The authors use this and the structure of compact monothetic groups and semigroups to generalise the convergence, under certain compactness conditions, of the operator averages:

$$n^{-1}\sum_{k=1}^{n} T^k \to P \quad (n\to\infty)$$

where $P$ is a projection associated with the eigenvalue one of $T$.

**1. Introduction.** Let $T$ be a bounded linear operator acting on a complex Banach space $X$ and denote by $B(X)$ the Banach algebra of all bounded linear operators acting on $X$. If $\tau$ denotes one of the operator topologies on $B(X)$ let

$$S_\tau(T)=\tau\text{-cl}\{T^n : n=1,2,\ldots\}$$

in $B(X)$. The set of positive powers of $T$ is an abelian semigroup and, in the cases which we consider, so is $S_\tau(T)$.

Using the structure of compact monothetic (singly generated) groups and semigroups we seek to generalise well-known convergence theorems which state that, under certain conditions often including the $\tau$-compactness of the semigroup $S_\tau(T)$, the averages

$$n^{-1}\sum_{k=1}^{n} T^k \xrightarrow{\tau} P \quad (n\to\infty) \tag{1}$$

where $P$ is a projection in $B(X)$ related to the spectral properties of the eigenvalue one of the operator $T$ (see, for example, [6], Corollary 2, and [4], Theorem 2.2).

The topology $\tau$ will denote either the uniform operator topology or the weak operator topology and we shall exploit the fact that the minimal ideal

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of a compact abelian semigroup is a compact group whose identity is the minimal idempotent in the semigroup [6], [7]. This fact, particularly in the uniform case, points to a reduction of the semigroup problem to compact monothetic (singly-generated) groups and it is in this context that we obtain our strongest results. The theory of uniformly distributed sequences, originally due to Weyl [5], has a natural setting in compact abelian groups (§2) which allows us to obtain a considerable generalisation of the convergence theorem (1) in the case of compact monothetic operator groups (§3). The simple structure of uniformly compact monothetic semigroups of operators [6] leads to an analogous theorem for compact monothetic semigroups (§4) generalising (1). However, in the weakly compact case the structure of a monothetic operator semigroup may be much more complicated [1], [2], [9] and our group results do not seem capable of such straightforward generalisation.

**2. Uniformly distributed sequences in compact groups.** Let $G$ be a compact topological group and let $C(G)$ denote the Banach algebra of continuous complex-valued functions on $G$ in the supremum norm.

DEFINITION. A sequence $\{g_n\}_{n=1}^{\infty}$ of elements of the compact group $G$ is *uniformly distributed* in $G$ if

$$\forall f \in C(G), \quad n^{-1}\sum_{k=1}^{n} f(g_k) \to \int_G f(g)\,dg \quad (n \to \infty),$$

where integration is with respect to normalised Haar measure in $G$.

In the case of abelian groups there is a simple criterion for uniform distribution. If $G$ is a compact abelian group denote by $\widehat{G}$ the dual group of $G$ and by $\chi_1$ the unit character in $\widehat{G}$.

WEYL CRITERION. *If $G$ is a compact abelian group, a sequence $\{g_n\}_{n=1}^{\infty}$ of elements in $G$ is uniformly distributed in $G$ if, and only if,*

$$\forall \chi(\neq \chi_1) \in \widehat{G}, \quad n^{-1}\sum_{k=1}^{n} \chi(g_k) \to 0 \qquad (n \to \infty).$$

The proof follows at once from the observation that if $\chi(\neq \chi_1) \in \widehat{G}$ then

$$\int_G \chi(g)\,dg = 0,$$

and the fact that the linear space of the continuous characters on $G$ is dense in $C(G)$.

DEFINITION. An element $g$ of a compact topological group $G$ is a *generator* of $G$ if $G = \mathrm{cl}\{g^n\}_{n=1}^{\infty}$ which is equivalent here, on account of compactness, to $G = \mathrm{cl}\{g^n\}_{n=-\infty}^{\infty}$. In this case the group is abelian and monothetic.



A generator $g$ of a compact monothetic group $G$ is *irrational* if $\chi(g)$ is not a root of unity for each $\chi(\neq \chi_1)$ in $\widehat{G}$.

The following result is due to Eckmann [3].

PROPOSITION 1. *If $g$ is a generator of a compact monothetic group $G$ and $h \in G$, then the sequence $\{g^n h\}_{n=1}^{\infty}$ is uniformly distributed in $G$.*

Proof. Let $\chi \in \widehat{G}$. Then by summing a geometric series,

$$n^{-1}\sum_{k=1}^{n} \chi(g^k h) = n^{-1}\chi(h)\sum_{k=1}^{n} \chi(g)^k \to \begin{cases} 0, & \chi(g) \neq 1, \\ \chi(h), & \chi(g) = 1 \end{cases} \qquad (n \to \infty).$$

But $\chi(g) = 1 \Leftrightarrow \chi = \chi_1$ since $g$ generates $G$, and the result follows from the Weyl criterion.

We now examine compact monothetic groups with irrational generators. The obvious example is the circle group $\mathbb{T}$ all of whose generators are irrational. This is not an isolated phenomenon.

PROPOSITION 2. *If $G$ is a compact monothetic group, the following statements are equivalent:*

(i) *$G$ has an irrational generator;*
(ii) *all generators of $G$ are irrational;*
(iii) *$G$ is connected.*

Proof. Let $g$ be a generator of $G$, $\chi \in \widehat{G}$, and $n$ be a positive integer. Then

$$\chi(g)^n = 1 \Leftrightarrow \chi^n(g) = 1 \Leftrightarrow \chi^n = \chi_1,$$

since $g$ generates $G$. Thus $\chi(g)^n = 1 \Leftrightarrow \chi$ is of finite order in $\widehat{G}$. It follows that

$g$ is an irrational generator of $G \Leftrightarrow \widehat{G}$ is torsion free
$\Leftrightarrow G$ is connected ([8], Theorem 2.5.6).

It is now obvious that if $G$ has an irrational generator, all generators must be irrational.

PROPOSITION 3. *If $g$ is an irrational generator of the compact monothetic group $G$ and $h \in G$, then the sequence $\{(g)^{n^p} h\}_{n=1}^{\infty}$ is uniformly distributed in $G$ for each positive integer $p$.*

Proof. If $\chi \in \widehat{G}$ and $\chi \neq \chi_1$ then $\chi(g)$ is not a root of unity by hypothesis. A simple consequence of a result of Weyl ([5], p. 33) is that, for each positive integer $p$, the fractional parts of $(n^p\alpha)_{n=1}^{\infty}$ are uniformly distributed in $[0,1]$ if $\alpha$ is irrational. Thus by Weyl's criterion for uniform distribution ([5], p. 9),

$$n^{-1}\sum_{k=1}^{n} e^{2\pi i k^p \alpha} = n^{-1}\sum_{k=1}^{n}(e^{2\pi i\alpha})^{k^p} \to 0 \quad \text{as } n\to\infty.$$

Now for $\chi \neq \chi_1$ in $\widehat{\boldsymbol{G}}$ and $h \in \boldsymbol{G}$,

$$n^{-1}\sum_{k=1}^{n} \chi[(g)^{k^p}h] = n^{-1}\chi(h)\sum_{k=1}^{n}[\chi(g)]^{k^p} \to 0 \quad \text{as } n\to\infty.$$

Hence $\{(g)^{n^p}h\}_{n=1}^{\infty}$ is uniformly distributed in $\boldsymbol{G}$.

**3. Compact operator groups.** Let $w$ denote the weak operator topology in $B(X)$. If $T \in B(X)$ is such that the semigroup $S_w(T)$ is a $w$-compact group with unit $I$, the identity operator, we call $T$ a *$G$-operator* and write $\boldsymbol{G} = \boldsymbol{G}(T) = S_w(T)$. These operators have been studied in detail by Gillespie and West [4]. They have been characterised by Kaashoek and Wolff ([4], Theorem 1.2) as those operators with bounded iterates whose eigenvectors corresponding to unimodular eigenvalues span a dense subspace of $X$.

We now construct examples of $G$-operators $T$. Let $\mathbb{T}$ denote the circle group and let $X$ be the space $C(\mathbb{T})$ or the space $L^p(\mathbb{T})$ $(1 \le p < \infty)$. Define the translation operator $T_{t_0}$ on $X$ by

$$T_{t_0}f(t) = f(t_0 t) \quad (t \in \mathbb{T}),$$

where in the second space this equation is to be interpreted almost everywhere. Then $T_{t_0}$ is a $G$-operator and if we assume further that $t_0$ is not a root of unity, we see that

$$\boldsymbol{G} = \boldsymbol{G}(T_{t_0}) = \{T_t : t \in \mathbb{T}\}.$$

Then it is simple to check as in Examples 5.1 and 5.4 of [4] that the mapping

$$t \to T_t : \mathbb{T} \to \boldsymbol{G}$$

is a group isomorphism and a homeomorphism. Since $t_0$ is an irrational generator of $\mathbb{T}$, it follows that $T_{t_0}$ is an irrational generator of $\boldsymbol{G}$. Thus, by Proposition 3, the sequence $\{[T_{t_0}]^{n^p}\}_{n=1}^{\infty}$ is uniformly distributed in $\boldsymbol{G}$ for each positive integer $p$.

Another example of a uniformly distributed sequence in $\boldsymbol{G} = \boldsymbol{G}(T_{t_0})$ where $T_{t_0}$ is chosen as above may be constructed by means of Weyl's theorem already referred to using a polynomial $q$ with real coefficients such that $q(x) - q(0)$ has at least one irrational coefficient. Then the sequence

$$\{s_n = e^{2\pi i q(n)} : n = 1, 2, \ldots\}$$

is uniformly distributed in $\mathbb{T}$, thus it follows that the sequence

$$\{T_{s(n)} : n = 1, 2, \ldots\}$$



is uniformly distributed in the group $\boldsymbol{G} = \boldsymbol{G}(T_{t_0})$ in each of the above examples.

Let $T$ be a $G$-operator on a Banach space $X$. If $\{S_n\}_{n=1}^{\infty}$ is a uniformly distributed sequence in $\boldsymbol{G}(T)$ we prove that, in the weak operator topology, the averages

$$n^{-1}\sum_{k=1}^{n} S_k \xrightarrow{w} P \quad (n\to\infty),$$

where $P$ is an idempotent in $B(X)$ independent of the particular sequence $\{S_n\}_{n=1}^{\infty}$. Since this result holds in a more general setting we deal with groups of invertible operators (not necessarily abelian) which are compact in the weak operator topology.

Let $\boldsymbol{G}$ be a group of invertible operators in $B(X)$ which is compact in the weak operator topology. If $\boldsymbol{S}$ is the closure in this topology of the convex hull of $\boldsymbol{G}$, it is well known that $\boldsymbol{S}$ is a $w$-compact convex (multiplicative) semigroup in $B(X)$. A *two-sided ideal* of the semigroup $\boldsymbol{S}$ is a subset $J$ of $\boldsymbol{S}$ such that the subsets $J\boldsymbol{S}$ and $\boldsymbol{S}J$ are contained in $J$.

We now introduce the idempotent $P$.

PROPOSITION 4. *The integral*

$$P = \int_{\boldsymbol{G}} T\,dT$$

*exists in the weak operator topology and determines an idempotent in $B(X)$. Further,*

$$P \in \boldsymbol{S}; \quad PS = SP = P \quad (S \in \boldsymbol{S}); \tag{2}$$

*and $\{P\}$ is the minimal closed two-sided ideal of the compact convex semigroup $\boldsymbol{S}$.*

*The range of $P$ is*

$$P(X) = \bigcap_{T\in\boldsymbol{G}} \ker(I - T),$$

*while the kernel of $P$ is the closed subspace of $X$ spanned by the subspaces*

$$\{(I-T)X : T \in \boldsymbol{G}\}.$$

Proof. The integral exists in the weak operator topology ([7], p. 76), that is, the equation

$$\langle f, Px\rangle = \int_{\boldsymbol{G}} \langle f, Tx\rangle\,dT \quad (x \in X,\ f \in X^*)$$

defines a linear operator $P$ whose boundedness follows from the uniform boundedness of $\|T\|$ for $T \in \boldsymbol{G}$.

By the invariance of Haar measure we see that, if $G \in \boldsymbol{G}$, then

$$PG = \int_{\boldsymbol{G}} TG\, dT = \int_{\boldsymbol{G}} T\, dT = P,$$

similarly $GP = P$. If $S$ is contained in the convex hull of $\boldsymbol{G}$, then

$$S = \sum_{i=1}^{n} \alpha_i G_i \quad \Big(\alpha_i \geq 0,\ \sum_{i=1}^{n} \alpha_i = 1,\ G_i \in \boldsymbol{G}\Big),$$

hence

$$PS = \sum_{i=1}^{n} \alpha_i P G_i = \sum_{i=1}^{n} \alpha_i P = P,$$

similarly $SP = P$. By $w$-continuity

$$PS = SP = P \quad (S \in \boldsymbol{S}).$$

If $M(\boldsymbol{G})$ denotes the space of complex regular Borel measures on $\boldsymbol{G}$, the Krein–Milman theorem, applied to the weak* compact convex subset of positive measures of unit norm shows that Haar measure is a weak* limit in $M(\boldsymbol{G})$ of a net of convex combinations of evaluation measures at points of $\boldsymbol{G}$. Thus, for $x \in X$, $f \in X^*$, if $\varepsilon_G$ is the point measure at $G$, then

$$\begin{aligned}\langle f, Px\rangle &= \int_{\boldsymbol{G}} \langle f, Tx\rangle\, dT = \lim_{\gamma} \int_{\boldsymbol{G}} \langle f, Tx\rangle\, d\Big(\sum_{j=1}^{n_\gamma} \alpha_j \varepsilon_{G_j}\Big) \\ &= \lim_{\gamma} \sum_{j=1}^{n_\gamma} \alpha_j \int_{\boldsymbol{G}} \langle f, Tx\rangle\, d\varepsilon_{G_j} = \lim_{\gamma} \Big\langle f, \sum_{j=1}^{n_\gamma} \alpha_j G_j x\Big\rangle,\end{aligned}$$

and so $P \in \boldsymbol{S}$. It follows from (2) that $P$ is an idempotent.

Now, the equation

$$P\boldsymbol{S}P = \{P\}$$

uniquely determines $P$ among elements of $\boldsymbol{S}$ and then, by [7], Theorem 7.2, the minimal closed two-sided ideal of $\boldsymbol{S}$ is $\{P\}$.

It remains to examine the range and kernel of the idempotent $P$. Let $T \in \boldsymbol{G}$. Since $(I - T)P = 0$ it follows that

$$P(X) \subset \bigcap_{T \in \boldsymbol{G}} \ker(I - T).$$

Conversely, let $x \in \bigcap_{T \in \boldsymbol{G}} \ker(I - T)$. Then

$$Tx = x \quad (T \in \boldsymbol{G}),$$

thus

$$Px = \int_{\boldsymbol{G}} Tx\, dT = \int_{\boldsymbol{G}} x\, dT = x,$$

and $x \in P(X)$ as required.



Turning to the kernel of $P$, since $P(I - T) = 0$ for $T \in \boldsymbol{G}$, we see that

$$(I - T)X \subset \ker(P) \quad (T \in \boldsymbol{G}),$$

hence $Y$, the closed span in $X$ of $\{(I - T)X : T \in \boldsymbol{G}\}$, is contained in $\ker(P)$.

Conversely, if $x \in \ker(P)\backslash Y$, by the Hahn–Banach theorem there exists $f \in X^*$ such that $f(x) \neq 0 = f(Y)$. Then

$$\langle f, x\rangle = \langle f, (I - P)x\rangle = \int_{\boldsymbol{G}} \langle f, (I - T)x\rangle\, dT = 0,$$

giving the required contradiction.

Note that $P$ is non-zero if, and only if, there exists a non-zero $x \in X$ such that $Tx = x$ for all $T \in \boldsymbol{G}$.

PROPOSITION 5. *Let $\boldsymbol{G}$ be a group of invertible operators in $B(X)$ which is compact in the weak operator topology. If $\{S_n\}_{n=1}^{\infty}$ is a uniformly distributed sequence in $\boldsymbol{G}$ then*

$$n^{-1} \sum_{k=1}^{n} S_k \xrightarrow{w} P \quad (n \to \infty),$$

*where $P$ is the idempotent of Proposition 4.*

Proof. If $x \in X$ and $f \in X^*$, the function

$$T \to \langle f, Tx\rangle : \boldsymbol{G} \to \mathbb{C}$$

is continuous in the weak operator topology. Hence, for a uniformly distributed sequence $\{S_n\}_{n=1}^{\infty}$ in $\boldsymbol{G}$,

$$n^{-1} \sum_{k=1}^{n} \langle f, S_k x\rangle \to \int_{\boldsymbol{G}} \langle f, Tx\rangle\, dT \quad (n \to \infty).$$

Thus

$$\Big\langle f, \Big(n^{-1} \sum_{k=1}^{n} S_k\Big)x\Big\rangle \to \langle f, Px\rangle \quad (n \to \infty),$$

that is,

$$n^{-1} \sum_{k=1}^{n} S_k \xrightarrow{w} P \quad (n \to \infty).$$

We remark that if $\boldsymbol{G}$ is a $w$-compact group of operators in $B(X)$, the weak and strong operator topologies coincide on $\boldsymbol{G}$ ([7], p. 93). Observe that results corresponding to Propositions 4 and 5, with obvious modifications, hold for operators which generate groups compact in the uniform operator topology. To see this let $\boldsymbol{G}$ be a $u$-compact group ($u$ denotes the uniform operator topology) of invertible operators in $B(X)$. By Mazur's theorem the $u$-closed convex hull $\Sigma$ of $\boldsymbol{G}$ in $B(X)$ is $u$-compact and, a fortiori, $w$-compact in $B(X)$; further, the $u$- and $w$-topologies coincide on $\Sigma$ (and on $G$). Thus the results for the $u$-topology can be deduced directly from Proposition 5.

**4. Compact operator semigroups.** Let $T \in B(X)$ and let $u$ denote the uniform operator topology. Assume that the semigroup $S_u(T)$ is $u$-compact. Then $S_u(T)$ contains a single idempotent $E$ which decomposes the Banach space $X$ into the direct sum of closed subspaces reducing $T$ as follows:

$$X = X_0 \oplus X_1, \quad E = 0_0 \oplus I_1, \quad T = T_0 \oplus T_1.$$

Further, if $r(T), \sigma(T)$ denote the spectral radius and spectrum of $T$ then $r(T_0) < 1$ and $\sigma(T_1)$ consists of a finite number of simple poles of $T$ of unit modulus ([6], Theorem 3). Note that

$$ES_u(T) = S_u(ET)$$

is a $u$-compact group with unit $E$. Thus $S_u(T_0)$ is a $u$-compact semigroup whose only accumulation point is $0_0$, while $S_u(T_1)$ is a $u$-compact group with unit $I_1$; in other words, $T_1$ is a $G$-operator (in the uniform topology) on the Banach space $X_1$.

Suppose now that $\{k_j\}$ is a strictly increasing sequence of positive integers such that the operator sequence

$$\{T_1^{k_j} : j = 1, 2, \ldots\}$$

is uniformly distributed in the $u$-compact group $S_u(T_1)$. Then, by the analogue of Proposition 5 in the uniform operator topology,

$$(3) \qquad n^{-1} \sum_{j=1}^{n} T_1^{k_j} \xrightarrow{u} P_1 \quad (n \to \infty).$$

Further, $EP_1 = P_1$ as $P_1$ is the spectral projection of the operator $T_1$ associated with the eigenvalue 1.

Now, $r(T_0) < 1$ so

$$T_0^n \xrightarrow{u} 0_0 \quad (n \to \infty).$$

Hence, given $\varepsilon > 0$ we may choose a positive integer $N$ such that for $n > N$,

$$\|T_0^n\| < \varepsilon.$$

Further, we may choose a positive integer $J$ such that $k_j > N$ if $j > J$. Hence for $n > J$,

$$\Big\|n^{-1} \sum_{j=1}^{n} T_0^{k_j}\Big\| \le n^{-1} \Big\|\sum_{j=1}^{J} T_0^{k_j}\Big\| + n^{-1} \sum_{j=J+1}^{n} \|T_0^{k_j}\| \le n^{-1} \Big\|\sum_{j=1}^{J} T_0^{k_j}\Big\| + \varepsilon,$$



and, since the right hand side may be made arbitrarily small by choice of $n$, we see that

$$(4) \qquad n^{-1} \sum_{j=1}^{n} T_0^{k_j} \xrightarrow{u} 0_0 \quad (n \to \infty).$$

Now combining (3) and (4) we have proved the following:

PROPOSITION 6. *Let $T$ be a bounded linear operator such that $S_u(T)$ is a $u$-compact monothetic semigroup with minimal idempotent $E$ and suppose that $E : T = T_0 \oplus T_1$ as above. If $\{k_j\}$ is a strictly increasing sequence of positive integers such that the operator sequence $\{T_1^{k_j} : j = 1, 2, \ldots\}$ is uniformly distributed in the compact group $S_u(T_1)$ then*

$$(5) \qquad n^{-1} \sum_{j=1}^{n} T^{k_j} \xrightarrow{u} P \quad (n \to \infty)$$

*where $P$ is the spectral projection of $T$ associated with the eigenvalue* 1.

The difficulty in extending this result to the $w$-topology lies in the fact that if $S_w(T)$ is a $w$-compact semigroup then it may possess many distinct idempotents ([1], [2], [9]), thus the semigroup $S_w(T_0)$ may have many other accumulation points besides $0_0$ and we do not have the control over $T_0$ which we had in the $u$-case.

The hypothesis in Proposition 6 that the sequence $\{T_1^{k_j}\}_{j=1}^{\infty}$ be uniformly distributed in the $u$-compact group $S_u(T_1)$ is sufficient for the convergence in (3) and therefore in (5) but is far from being necessary. In fact we can give necessary and sufficient conditions for convergence in terms of the eigenvalues of $T_1$ which are the unimodular eigenvalues of $T$.

Let $\sigma(T_1) = \{\lambda_1, \ldots, \lambda_l\}$; then under the obvious identification of the operator $T_1$ with its set of eigenvalues, the $u$-compact monothetic group $S_u(T_1)$ is topologically and algebraically isomorphic to the closed subgroup of $\mathbb{T}^l$

$$\mathrm{cl}\{(\lambda_1^n, \ldots, \lambda_l^n) : n = 1, 2, \ldots\}$$

([7], p. 433). The following result now follows immediately.

PROPOSITION 7. *With the hypotheses and notation of Proposition* 6, *if $T_1$ is zero then*

$$n^{-1} \sum_{j=1}^{n} T^{k_j} \xrightarrow{u} 0 \quad (n \to \infty);$$

*if $T_1$ is not zero, let $\sigma(T_1) = \{\lambda_1, \ldots, \lambda_l\}$; then*

$$n^{-1} \sum_{j=1}^{n} T^{k_j} \xrightarrow{u} P \quad (n \to \infty)$$

*if, and only if,*

$$n^{-1}\sum_{j=1}^{n}\lambda_p^{k_j}\to 0\qquad (n\to\infty)$$

*for each $\lambda_p$ in $\sigma(T_1)$ which is not equal to one.*

A similar result holds for an operator $T$ which generates a $w$-compact monothetic group $\boldsymbol{G}(T)$ of invertible operators in $B(X)$ using the representation of $\boldsymbol{G}(T)$ in terms of the unimodular eigenvalues of $T$ given in [4], Theorem 3.3.

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# Cyclic space isomorphism of unitary operators

by

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**Abstract.** We introduce a new equivalence relation between unitary operators on separable Hilbert spaces and discuss a possibility to have in each equivalence class a measure-preserving transformation.

**Introduction.** Let $U$ be a unitary operator on a separable Hilbert space $H$. For any $x\in H$ we define the *cyclic space* generated by $x$ as $Z(x)=\operatorname{span}\{U^nx : n\in\mathbb{Z}\}$. By the *spectral measure* $\mu_x$ of $x$ we mean the Borel measure on the circle determined by the equalities

$$\widehat{\mu}_x(n)=\int_{\mathbb{T}} z^n\,d\mu_x(z)=(U^nx,x)$$

for every $n\in\mathbb{Z}$.

THEOREM 0.1 (spectral theorem, see [9]). *There exists in $H$ a sequence $x_1,x_2,\ldots$ such that*

$$H=\bigoplus_{n=1}^{\infty}Z(x_n)\quad\text{and}\quad \mu_{x_1}\gg\mu_{x_2}\gg\ldots \tag{1}$$

*Moreover, for any sequence $y_1,y_2,\ldots$ in $H$ satisfying* (1) *we have $\mu_{x_1}\equiv\mu_{y_1}$, $\mu_{x_2}\equiv\mu_{y_2},\ldots$*

One of the most important (still open) problems in ergodic theory is a classification of ergodic dynamical systems with respect to spectral equivalence, i.e. given a sequence

$$\mu_1\gg\mu_2\gg\ldots \tag{2}$$

of positive finite measures on the circle we ask if there exists an ergodic dynamical system $T:(X,\mathcal{B},\varrho)\to(X,\mathcal{B},\varrho)$ such that some spectral sequence (1) for $U=U_T$ ($U_T:L^2(X,\varrho)\to L^2(X,\varrho)$, $U_Tf=f\circ T$) coincides with (2).

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