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Sufficient conditions of optimality for multiobjective optimization problems with γ-paraconvex data

by

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Abstract. We study multiobjective optimization problems with γ-paraconvex multifunction data. Sufficient optimality conditions for unconstrained and constrained problems are given in terms of contingent derivatives.

1. Introduction. Many authors have studied multiobjective optimization problems in terms of some tangent derivative notions. Corley [4] has given optimality conditions for convex and nonconvex multiobjective problems in terms of the Clarke derivative. Luc [6] also gives optimality conditions when the data are upper semidifferentiable. Luc and Malivert [7] extend the concept of invex functions to invex multifunctions and study optimality conditions for multiobjective optimization with invex data in terms of contingent derivatives. Taa [12] gives optimality conditions with no assumption on the data but with the Shi derivative which is an enlarged version of contingent derivative.

In this paper we establish sufficient optimality conditions in terms of the contingent derivative for unconstrained and constrained multiobjective optimization problems when the data are γ-paraconvex or compactly γ-paraconvex with $\gamma > 1$. It is shown that the γ-paraconvexity data considerably simplify the assumptions in the optimality conditions. The notion of γ-paraconvex multifunctions has been introduced by Rolewicz [10] and openness and metric regularity of such multifunctions are studied in Jourani [5] (see also Allali and Amahroq [1] for another proof).

2. Preliminaries. Let X and Y be two Banach spaces and let F be a multifunction from X into Y. In the sequel we denote the effective domain

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and the graph of F respectively by

$$\operatorname{dom}(F)=\{x\in X: F(x)\neq\emptyset\},\qquad \operatorname{gr}(F)=\{(x,y)\in X\times Y: y\in F(x)\}.$$

If V is a nonempty subset of X, then we set

$$F(V)=\bigcup_{x\in V}F(x).$$

Let $\gamma>0$. The multifunction F is said to be *γ-paraconvex* (see [12]) if there is a constant $c>0$ such that for all $x,x'\in X$ and all $\alpha\in\,]0,1[$ the following inclusion holds:

$$\alpha F(x)+(1-\alpha)F(x')\subset F(\alpha x+(1-\alpha)x')+c\|x-x'\|^{\gamma}B_Y, \tag{2.1}$$

where B_Y is the closed unit ball of Y. Rolewicz [10] proved that when $\gamma>1$, the relation (2.1) is equivalent to

$$\alpha F(x)+(1-\alpha)F(x')\subset F(\alpha x+(1-\alpha)x')+c\min(\alpha,1-\alpha)\|x-x'\|^{\gamma}B_Y. \tag{2.2}$$

Following this definition we shall say that the multifunction F is *compactly γ-paraconvex* with $\gamma>1$ if there exists a convex compact subset S of Y containing 0 such that

$$\alpha F(x)+(1-\alpha)F(x')\subset F(\alpha x+(1-\alpha)x')+\min(\alpha,1-\alpha)\|x-x'\|^{\gamma}S. \tag{2.3}$$

It is obvious that any convex multifunction (i.e. whose graph is a convex subset of the product space $X\times Y$) is compactly γ-paraconvex and then γ-paraconvex but the converse may be false. For counterexamples see Rolewicz [10] and Jourani [5].

A multifunction F from X into Y is said to be *B-tangentially compact* at $(\overline{x},\overline{y})\in\operatorname{gr}(F)$ in the sense of Penot [8] if for any sequences $(t_n)\to 0$, (x_n) converging in X and any bounded sequence $(y_n)\subset Y$ with $\overline{y}+t_ny_n\in F(\overline{x}+t_nx_n)$ for all $n\in\mathbb{N}$, the set of cluster points of (y_n) is nonempty.

It is obvious that if Y is finite-dimensional then any multifunction defined from X into Y is B-tangentially compact at each point of its graph.

Let f be a single-valued map from X into Y and let $\overline{y}=f(\overline{x})$. If the Hadamard directional derivative $df(\overline{x},\cdot)$ of f exists at $\overline{x}$, that is, for each $\overline{v}\in X$ the limit

$$df(\overline{x},\overline{v}):=\lim_{(t,v)\to(0^+,\overline{v})}t^{-1}[f(\overline{x}+tv)-\overline{y}]$$

exists, then f is B-tangentially compact at $(\overline{x},\overline{y})$.

If F is upper semidifferentiable at $(\overline{x},\overline{y})\in\operatorname{gr}(F)$ in the sense of Luc [6], that is, for each sequence $((x_n,y_n))\subset\operatorname{gr}(F)$ not coinciding with $(\overline{x},\overline{y})$ and converging to it, there is an infinite subset $I\subset\mathbb{N}$ and a sequence $(\beta_n)_{n\in I}$ of positive numbers such that the sequence

$$(\beta_n(x_n-\overline{x},y_n-\overline{y}))_{n\in I}$$

converges to some nonzero vector of the product space $X\times Y$, then F is B-tangentially compact at $(\overline{x},\overline{y})$. Indeed, let $(t_n)\to 0$, $(x_n)\to x\in X$ and a bounded sequence $(y_n)\subset Y$ such that

$$\overline{y}+t_ny_n\in F(\overline{x}+t_nx_n)\quad\text{ for all } n.$$

By upper semidifferentiability of F at $(\overline{x},\overline{y})$, there exist an infinite subset $I\subset\mathbb{N}$ and a sequence $(\beta_n)_{n\in I}$ of positive numbers such that the sequence

$$(\beta_nt_n(x_n,y_n))_{n\in I}$$

converges to some nonzero vector of $X\times Y$. For all $n\in I$ put

$$w_n:=\beta_nt_nx_n\quad\text{and}\quad z_n:=\beta_nt_ny_n.$$

Since (w_n) and (z_n) do not both converge to 0, it is clear that the sequence $(\beta_nt_n)^{-1}$ is bounded. Hence (y_n) has a converging subsequence.

In order to recall the definition of the contingent derivative, let us first recall the definition of the contingent cone to a subset A of X at a point $\overline{x}$ in the closure $\operatorname{CL}(A)$ of A.

Definition 2.1. The *contingent cone* $K(A;\overline{x})$ of A at $\overline{x}\in\operatorname{CL}(A)$ is the set of all $v\in X$ such that there exist $(t_n)\to 0$ and $(v_n)\to v$ with

$$\overline{x}+t_nv_n\in A\quad\text{ for all } n.$$

It is known that when A is a convex subset, then $K(A;\overline{x})$ is also a convex subset of X. The contingent derivative of F is defined by considering the contingent cone to the graph of F (see [3] and [2]).

Definition 2.2 [3]. Let $(\overline{x},\overline{y})\in\operatorname{gr}(F)$. The *contingent derivative* $DF(\overline{x},\overline{y})$ of F at $(\overline{x},\overline{y})$ is the multifunction whose graph is the contingent cone $K(\operatorname{gr}(F);(\overline{x},\overline{y}))$ to the graph of F at $(\overline{x},\overline{y})$.

$DF(\overline{x},\overline{y})$ is a positively homogeneous multifunction with closed graph.

Due to Definition 2.1, $y\in DF(\overline{x},\overline{y})(x)$ if and only if there exist sequences $(t_n)\to 0$, $(y_n)\to y$ and $(x_n)\to x$ such that

$$\overline{y}+t_ny_n\in F(\overline{x}+t_nx_n)\quad\text{ for all } n\in\mathbb{N}.$$

Recall that for a family $(A_\alpha)_{\alpha>0}$ of subsets of Y, $\limsup_{\alpha\to0+}A_\alpha$ is defined by

$$\limsup_{\alpha\to0+}A_\alpha:=\{a\in Y:\exists(\alpha_n,a_n)\to(0,a),\ a_n\in A_{\alpha_n}\ \forall n\}.$$

The following results will be crucial in the sequel.

Proposition 2.1. *Let $\gamma>1$ and $(\overline{x},\overline{y})\in\operatorname{gr}(F)$. Then:*

(i) *If F is compactly γ-paraconvex then for all $x\in X$,*

$$F(x)-\overline{y}\subset DF(\overline{x},\overline{y})(x-\overline{x})+\|x-\overline{x}\|^{\gamma}S,$$

where S is given by (2.3).

(ii) *If F is γ-paraconvex and B-tangentially compact at $(\overline{x},\overline{y})$ then for all $x \in X$,*

$$F(x)-\overline{y} \subset DF(\overline{x},\overline{y})(x-\overline{x})+c\|x-\overline{x}\|^{\gamma}B_Y,$$

where c is given by (2.2).

Proof. (i) Let $\alpha \in]0,1[$. By compact γ-paraconvexity we have

$$\alpha F(x)+(1-\alpha)\overline{y} \subset \alpha F(x)+(1-\alpha)F(\overline{x}) \subset F(\alpha x+(1-\alpha)\overline{x})+\alpha\|x-\overline{x}\|^{\gamma}S,$$

thus

$$F(x)-\overline{y} \subset \alpha^{-1}[F(\overline{x}+\alpha(x-\overline{x}))-\overline{y}]+\|x-\overline{x}\|^{\gamma}S,$$

and hence

$$F(x)-\overline{y} \subset \limsup \alpha^{-1}[F(\overline{x}+\alpha(x-\overline{x}))-\overline{y}]+\|x-\overline{x}\|^{\gamma}S.$$

Now let $v \in \limsup \alpha^{-1}[F(\overline{x}+\alpha(x-\overline{x}))-\overline{y}]+\|x-\overline{x}\|^{\gamma}S$. Then there exist sequences $(\alpha_n,v_n) \to (0^+,v)$ and $(b_n) \subset S$ such that

$$\overline{y}+\alpha_n(v_n-\|x-\overline{x}\|^{\gamma}b_n) \in F(\overline{x}+\alpha_n(x-\overline{x})).$$

By compactness of S, there exist an infinite subset $I \subset \mathbb{N}$ and $b \in S$ such that

$$(v_n-\|x-\overline{x}\|^{\gamma}b_n)_{n\in I} \to v-\|x-\overline{x}\|^{\gamma}b.$$

This implies that

$$v \in DF(\overline{x},\overline{y})(x-\overline{x})+\|x-\overline{x}\|^{\gamma}S,$$

which completes the proof of (i). The proof of (ii) is similar.

The following corollary is a direct consequence of the above proposition.

COROLLARY 2.1. *Let $(\overline{x},\overline{y}) \in \operatorname{gr}(F)$. If Y is finite-dimensional and if F is γ-paraconvex with $\gamma>1$, then for all $x \in X$,*

$$F(x)-\overline{y} \subset DF(\overline{x},\overline{y})(x-\overline{x})+c\|x-\overline{x}\|^{\gamma}B_Y.$$

Consider now the multifunction G from X into a Banach space Z. In the sequel the couple (F,G) will be the multifunction from X into $Y\times Z$ defined by

$$(F,G)(x)=(F(x),G(x))=F(x)\times G(x) \quad \text{for } x \in \operatorname{dom}(F)\cap\operatorname{dom}(G).$$

As a direct consequence of (2.2), (2.3) and Proposition 2.1 we have the following lemma.

LEMMA 2.1. *Let $\overline{y} \in F(\overline{x})$ and $\overline{z} \in G(\overline{x})$. Then:*

(i) *If F and G are γ-paraconvex then (F,G) is γ-paraconvex.*

(ii) *If F and G are compactly γ-paraconvex then (F,G) is compactly γ-paraconvex.*

(iii) *Let $\gamma>1$ be a real number.*

(a) *If F and G are compactly γ-paraconvex, then for all $x \in X$,*

$$(F,G)(x) \subset (\overline{y},\overline{z})+(DF(\overline{x},\overline{y}),DG(\overline{x},\overline{y}))(x-\overline{x})+\|x-\overline{x}\|^{\gamma}(S_1\times S_2),$$

where S_1 and S_2 are compact subsets of Y and Z respectively.

(b) *If F and G are γ-paraconvex and B-tangentially compact at $(\overline{x},\overline{y}) \in \operatorname{gr}(F)$ and $(\overline{x},\overline{z}) \in \operatorname{gr}(G)$ respectively, then there exists $c>0$ such that for all $x \in X$,*

$$(F,G)(x) \subset (\overline{y},\overline{z})+(DF(\overline{x},\overline{y}),DG(\overline{x},\overline{z}))(x-\overline{x})+c\|x-\overline{x}\|^{\gamma}(B_Y\times B_Z).$$

Now let us recall some basic definitions in multiobjective optimization problems. Let Y^+ be a pointed $(Y^+\cap(-Y^+)=\{0\})$ closed convex cone of Y with nonempty interior $\operatorname{Int}(Y^+)$. Let A be a nonempty subset of Y and $\overline{y} \in A$. Then $\overline{y}$ is said to be a *Pareto minimal point* (respectively a *weak Pareto minimal point*) of A with respect to Y^+ if

$$(A-\overline{y})\cap(-Y^+)=\{0\} \quad (\text{resp. } (A-\overline{y})\cap(-\operatorname{Int}(Y^+))=\emptyset).$$

We denote by $\operatorname{Min}(A)$ the set of all Pareto minimal points of A and by $\operatorname{W.Min}(A)$ the set of all weak Pareto minimal points of A. Let C be a nonempty subset of X and consider the multiobjective optimization problem

$$(P) \quad \text{Minimize } F(x) \text{ subject to } x \in C.$$

A point $(\overline{x},\overline{y}) \in \operatorname{gr}(F)$ is said to be a *local* (respectively a *weak local*) *Pareto minimal point* of (P) with respect to Y^+ if there exists a neighborhood V of $\overline{x}$ such that $\overline{y} \in \operatorname{Min} F(V\cap C)$ (respectively $\overline{y} \in \operatorname{W.Min}F(V\cap C)$). This means that for all $x \in V\cap C$,

$$F(x) \subset \overline{y}+(Y\setminus(-Y^+))\cup\{0\} \quad (\text{resp. } F(x) \subset \overline{y}+Y\setminus(-\operatorname{Int}(Y^+))).$$

In the following section we study the constrained problem

$$(P_1) \quad \text{Minimize } F(x) \text{ subject to } x \in G^-(-Z^+)$$

where G is a multifunction from X into a Banach space Z, Z^+ is a pointed closed convex cone of Z with nonempty interior and $G^-(-Z^+)$ is the subset of X defined by

$$G^-(-Z^+):=\{x \in X : G(x)\cap(-Z^+)\neq\emptyset\}.$$

Also we shall derive the optimality conditions for the unconstrained problem

$$(P_1)' \quad \text{Minimize } F(x).$$

3. Optimality conditions. In this section we consider a pointed closed convex cone Y^+ of Y with nonempty interior and we study optimality conditions for the constrained problem (P_1) with respect to Y^+. This problem is defined in the preliminaries by

$$(P_1) \quad \text{Minimize } F(x) \text{ subject to } G(x)\cap(-Z^+)\neq\emptyset.$$

Let us start by some recalls. The following definition has been introduced in [13].

DEFINITION 3.1. A *base* Q of Y^+ is a nonempty subset of Y^+ with $0 \notin \mathrm{CL}(Q)$ such that every $c \in Y^+ \setminus \{0\}$ has a unique representation

$$c = rq \quad \text{with } r > 0 \text{ and } q \in \mathrm{CL}(Q).$$

Remark 3.1. (i) If Q is compact one says that Y^+ has a compact base.
(ii) If Y is finite-dimensional then Y^+ has a compact base (see [13]).

In the sequel we shall us the following lemma which has been proved in [11].

LEMMA 3.1 [11]. *The cone Y^+ has a compact base if and only if $Y^+ \cap E$ is compact, where $E := \{y \in Y : \|y\| = 1\}$.*

Now we are able to state sufficient optimality conditions for (P_1). The proof of the following theorem uses some ideas of Luc [6] and Taa [12].

THEOREM 3.1 (Sufficient optimality conditions). *Let $(\overline{x}, \overline{y}) \in \mathrm{gr}(F)$, $(\overline{x}, \overline{z}) \in \mathrm{gr}(G)$ and $\overline{z} \in -Z^+$. Suppose that F and G are compactly γ-paraconvex (or F and G are γ-paraconvex and B-tangentially compact at $(\overline{x}, \overline{y})$ and $(\overline{x}, \overline{z})$ respectively) with $\gamma > 1$, X is finite-dimensional and $Y^+ \times Z^+$ has a compact base. Moreover, let the following conditions be satisfied:*

(i) $(DF(\overline{x}, \overline{y}), DG(\overline{x}, \overline{z}))(0) \cap -(Y^+, K(Z^+; -\overline{z})) = \{(0,0)\}$,
(ii) $(DF(\overline{x}, \overline{y}), DG(\overline{x}, \overline{z}))(x) \cap -(Y^+, K(Z^+; -\overline{z})) = \emptyset$ for every $x \in \mathrm{dom}(DF(\overline{x}, \overline{y})) \cap \mathrm{dom}(DG(\overline{x}, \overline{z})) \setminus \{0\}$.

Then $(\overline{x}, \overline{y})$ is a local Pareto minimal point of (P_1) with respect to Y^+.

Proof. We prove only the case where F and G are compactly γ-paraconvex. In an analogous way one can prove the other case. Suppose that $(\overline{x}, \overline{y})$ is not a local Pareto minimal point of (P_1). Then there exist sequences $(x_n) \to \overline{x}$, $(y_n) \subset Y$ and $(z_n) \subset Z$ such that for all $n \in \mathbb{N}$ one has

$$(3.3.1) \qquad (\overline{y} - y_n, \overline{z} - z_n) \in (Y^+ \times (Z^+ + \overline{z})) \setminus \{(0,0)\},$$

$$(3.3.2) \qquad z_n \in G(x_n) \quad \text{and} \quad y_n \in F(x_n).$$

For all $n \in \mathbb{N}$ put

$$(3.3.3) \qquad \alpha_n (b_n, q_n) := (\overline{y} - y_n, -z_n),$$

where

$$\alpha_n = \|(\overline{y} - y_n, -z_n)\|, \quad b_n = \alpha_n^{-1}(\overline{y} - y_n) \quad \text{and} \quad q_n = \alpha_n^{-1}(-z_n).$$

Since Z^+ is convex and since $-\overline{z} \in Z^+$, we have

$$Z^+ \subset -\overline{z} + K(Z^+; -\overline{z}),$$

and by (3.3.1) and (3.3.3) we get

$$(3.3.4) \qquad \overline{z} + \alpha_n q_n \in K(Z^+; -\overline{z}) \quad \text{for all } n \in \mathbb{N}.$$

By Lemma 2.1(iii)(a), (3.3.2) and (3.3.3) there are sequences $(a_n) \subset S_1$ and $(a'_n) \subset S_2$ such that for all n,

$$(3.3.5) \qquad -\alpha_n b_n - \|x_n - \overline{x}\|^\gamma a_n \in DF(\overline{x}, \overline{y})(x_n - \overline{x})$$

and

$$(3.3.6) \qquad -\overline{z} - \alpha_n q_n - \|x_n - \overline{x}\|^\gamma a'_n \in DG(\overline{x}, \overline{z})(x_n - \overline{x}).$$

Since $Y^+ \times Z^+$ has a compact base, Lemma 3.1 implies that by extracting subsequences if necessary we can assume that there are $b \in Y^+$ and $q \in Z^+$ with $\|(b, q)\| = 1$ such that

$$(3.3.7) \qquad (b_n) \to b \quad \text{and} \quad (q_n) \to q.$$

There are two cases to consider.

First case: (α_n) has no convergent subsequence. Then $(\alpha_n) \to \infty$. Dividing (3.3.5) and (3.3.6) by α_n for all n and taking the limit as $n \to \infty$, it follows from (3.3.7) that

$$(3.3.8) \qquad -(b, q) \in (DF(\overline{x}, \overline{y}), DG(\overline{x}, \overline{z}))(0).$$

By (3.3.1), (3.3.3), (3.3.4) and (3.3.7) we get

$$(3.3.9) \qquad -(b, q) \in (-Y^+, -K(Z^+; -\overline{z})) \setminus \{(0,0)\}.$$

Hence, (3.3.8) and (3.3.9) contradict (i).

Second case: The sequence (α_n) has a convergent subsequence, which we denote also by (α_n), with limit $\alpha \in [0, \infty[$. Here, we have two cases: $\alpha = 0$ or $\alpha \neq 0$.

(a) $\alpha \neq 0$. By dividing (3.3.5) and (3.3.6) by α_n and taking the limit as $n \to \infty$, it follows from (3.3.7) that

$$(3.3.10) \qquad (-b, -\alpha^{-1}\overline{z} - q) \in (DF(\overline{x}, \overline{y}), DG(\overline{x}, \overline{z}))(0).$$

From (3.3.4) and (3.3.7) we get

$$(-b, -\alpha^{-1}\overline{z} - q) \in (-Y^+, -K(Z^+; -\overline{z})) \setminus \{(0,0)\},$$

which, together with (3.3.10), contradicts (i).

(b) $\alpha = 0$. In this case we have $\overline{z} = 0$ or $\overline{z} \neq 0$.

(b1) Suppose $\overline{z} \neq 0$. By taking the limit as $n \to \infty$ in (3.3.4) and (3.3.6) we get

$$-z \in DG(\overline{x}, \overline{z})(0) \cap -K(Z^+; -\overline{z}),$$

which contradicts (i).

(b2) Suppose $\overline{z} = 0$. Also we have two cases.

First case: $(\alpha^{-1}(x_n - \overline{x}))$ has a convergent subsequence, denoted again by $(\alpha_n^{-1}(x_n - \overline{x}))$, to some point $x_0 \in X$. Dividing (3.3.5) and (3.3.6) by α_n, it follows that

$$-b_n - \|x_n - \overline{x}\|^{\gamma-1}\alpha_n^{-1}\|x_n - \overline{x}\|a_n \in DF(\overline{x},\overline{y})(\alpha_n^{-1}(x_n - \overline{x}))$$

and

$$-q_n - \|x_n - \overline{x}\|^{\gamma-1}\alpha_n^{-1}\|x_n - \overline{x}\|a_n' \in DG(\overline{x},\overline{z})(\alpha_n^{-1}(x_n - \overline{x})).$$

By taking the limit as $n \to \infty$ we get

$$-(b,q) \in (DF(\overline{x},\overline{y}), DG(\overline{x},\overline{z}))(x_0). \tag{3.3.11}$$

Moreover, by (3.3.4) and (3.3.7) we have

$$-(b,q) \in -Y^+ \times (-K(Z^+; -\overline{z})),$$

which together with (3.3.11) contradicts (i) if $x_0 = 0$ or (ii) if $x_0 \neq 0$.

Second case: $(\alpha^{-1}(x_n - \overline{x}))$ has no convergent subsequence, that is,

$$(\alpha_n\|x_n - \overline{x}\|^{-1}) \to 0.$$

Since X is finite-dimensional, by extracting a subsequence if necessary we can assume that

$$((x_n - \overline{x})\|x_n - \overline{x}\|^{-1}) \to v,$$

where $v \in X$ and $\|v\| = 1$. Then by dividing (3.3.5) by $\|x_n - \overline{x}\|$ and taking the limit as $n \to \infty$, we get $0 \in DF(\overline{x},\overline{y})(v)$; this contradicts (ii). The proof is complete.

COROLLARY 3.2. *Let $(\overline{x},\overline{y}) \in \mathrm{gr}(F)$ and F be a compactly γ-paraconvex multifunction with $\gamma > 1$. Assume that X is finite-dimensional, Y^+ has a compact base and*

(i) $DF(\overline{x},\overline{y})(0) \cap (-Y^+) = \{0\}$,
(ii) $DF(\overline{x},\overline{y})(x) \cap (-Y^+) = \emptyset$ *for each* $x \in \mathrm{dom}(DF(\overline{x},\overline{y})) \setminus \{0\}$.

Then $(\overline{x},\overline{y})$ is a local Pareto minimal point of $(P_1)'$ with respect to Y^+.

Proof. The result is a direct consequence of the above theorem by considering the multifunction G whose graph is $X \times \{0\}$.

We also have the next result.

COROLLARY 3.3. *Let $(\overline{x},\overline{y}) \in \mathrm{gr}(F)$ and F be γ-paraconvex with $\gamma > 1$. Assume that F is B-tangentially compact at $(\overline{x},\overline{y})$, X is finite-dimensional, Y has a compact base and the conditions* (i) *and* (ii) *of Theorem* 3.2 *hold. Then $(\overline{x},\overline{y})$ is a local Pareto minimal point of $(P_1)'$ with respect to Y^+.*

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