STUDIA MATHEMATICA 124 (3) (1997)

Besov spaces on symmetric manifolds—the atomic decomposition

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Abstract. We give the atomic decomposition of the inhomogeneous Besov spaces defined on symmetric Riemannian spaces of noncompact type. As an application we get a theorem of Bernstein type for the Helgason–Fourier transform.

Several function spaces on $\mathbb{R}^n$ admit an atomic decomposition in the sense that every member of the space can be decomposed into a sum of simple building blocks, called atoms. The decomposition, which comes from the theory of Hardy spaces, proved to be useful for function spaces defined not necessarily on $\mathbb{R}^n$ (see, e.g., [14], [15]). In this paper we describe the atomic decomposition of inhomogeneous Besov spaces on symmetric spaces of noncompact type. Both the function spaces and atoms are defined in terms of the Helgason–Fourier transform. Although our approach is non-Euclidean, our Besov spaces coincide with the Besov spaces defined on Riemannian manifolds by H. Triebel [20] via uniform localization and our decomposition is analogous to that given by M. Frazier and B. Jawerth [7] for Besov spaces on $\mathbb{R}^n$. This is closely related to the inhomogeneity of the spaces. On the other hand, our theorem of Bernstein type differs from the Euclidean one.

The paper is organized as follows. In the first section some basic facts about symmetric manifolds of noncompact type and the non-Euclidean harmonic analysis are recalled. Section 2 contains definitions of atoms and Besov spaces. In particular, we prove a formula of Calderón type that takes a crucial part in our construction. The main result is proved in Section 3. In the last section we prove several simple applications, including the theorem of Bernstein type.

Throughout the paper we use the term "symmetric Riemannian manifold" instead of "symmetric Riemannian space" and reserve the term "space" for function spaces.

1991 *Mathematics Subject Classification*: 46E35, 43A85.

1. Preliminaries. We first recall the basic notation of Fourier analysis on Riemannian symmetric spaces of noncompact type. General references are [12] and [10]. Let G be a real semisimple Lie group, connected, noncompact, with finite center, and K be a maximal compact subgroup of G. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ be the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G and ϑ the corresponding Cartan involution. The point $o = \{eK\}$ is called the *origin* in the coset space $X = G/K$. We have a natural identification between $\mathfrak{s}$ and the tangent space of G/K at o. The Killing form of $\mathfrak{g}$ induces a K-invariant scalar product on $\mathfrak{s}$, and hence a G-invariant Riemannian metric on G/K. The map $Y \mapsto y = (\exp Y) \cdot o$ is a diffeomorphism of $\mathfrak{s}$ onto G/K. Set $|y| = |Y|$. It is the distance to the origin in G/K.

Fix a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{s}$. Let M be the centralizer of $\mathfrak{a}$ in K. Denote by $\mathfrak{a}^\star$ (resp. $\mathfrak{a}^\star_{\mathbb{C}}$) the real (resp. complex) dual of $\mathfrak{a}$. The Killing form of $\mathfrak{g}$ induces a scalar product on $\mathfrak{a}^\star$ and a $\mathbb{C}$-bilinear form on $\mathfrak{a}^\star_{\mathbb{C}}$. Denote by Σ the root system of $(\mathfrak{g}, \mathfrak{a})$. Let W be the Weyl group associated with Σ and let m_α denote the multiplicity of the root $\alpha \in \Sigma$. Choose a Weyl chamber $\mathfrak{a}_+$ in $\mathfrak{a}$ and the corresponding set Σ_+ of positive roots. Let $\overline{\mathfrak{a}}_+$ be the closure of $\mathfrak{a}_+$, and $\mathfrak{a}^\star_+$, $\overline{\mathfrak{a}^\star_+}$ the similar cones in $\mathfrak{a}^\star$.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be the Iwasawa decomposition of $\mathfrak{g}$. We have the corresponding decompositions of the group G: the Cartan decomposition $G = K(\overline{\exp \mathfrak{a}_+})K$ and the Iwasawa decomposition $G = K \exp \mathfrak{a} N$. Here N is the analytic subgroup of G corresponding to the nilpotent subalgebra $\mathfrak{n}$. Denote by $k(x)$ and $H(x)$ the Iwasawa components of $x \in G$ in K and $\mathfrak{a}$. We put

$$A(gK, kM) = -H(g^{-1}k).$$

The value $\exp A(x,b)$ is the complex distance from o to the horocycle in X through x with normal b. Finally, let

$$\varrho(H) = \frac{1}{2} \sum_{\alpha \in \Sigma_+} m_\alpha \alpha(H) \qquad (H \in \mathfrak{a})$$

and let $n = \dim X$, $\alpha = \dim \mathfrak{a}$, $d = n - \alpha$.

We identify functions on X with functions on G, which are K-invariant on the right. The homogeneous space $B = K/M = G/MAN$ is called the *boundary* of X. We denote the action of G on X and B by $(g, x) \mapsto g \cdot x$ and $(g, b) \mapsto g(b)$. If f is a suitable function, e.g. continuous with compact support, then its *Helgason–Fourier transform* is the function on $\mathfrak{a}^\star \times K/M$ given by

$$\begin{aligned}\mathcal{H}f(\lambda, b) &= \int_X f(x) e^{(-\sqrt{-1}\lambda + \varrho)A(x,b)}\, dx \\ &= \int_G f(g) e^{(\sqrt{-1}\lambda - \varrho)H(g^{-1}k)}\, dg, \qquad b = kM,\ (\lambda, kM) \in \mathfrak{a}^\star \times K/M,\end{aligned}$$

dx being a suitable normalized G-invariant measure on X and dg the Haar measure on G. The map $f \mapsto \mathcal{H}f$ can be extended to an isometry of $L_2(G/K)$ onto $L_2(\mathfrak{a}^\star_+ \times B, d\mu(\lambda)db)$. Here $db = dkM$ denotes the G-invariant measure on B such that $db(B) = 1$. The Plancherel measure $d\mu$ is given by $d\mu(\lambda) = |c(\lambda)|^{-2} d\lambda$ where $c(\lambda)$ is the *Harish-Chandra c-function*. One has the inversion formula

$$f(x) = \text{const} \cdot \int_{\mathfrak{a}^\star \times K/M} \mathcal{H}f(\lambda, kM) e^{(\sqrt{-1}\lambda + \varrho)H(x^{-1}k)}\, d\mu(\lambda)\, dkM.$$

The Helgason–Fourier transform can be represented as the composition $\mathcal{H}f = \mathcal{F}(\mathcal{R}(f))$ of the *Radon transform*

$$\mathcal{R}f(H, kM) = e^{\varrho(H)} \int_N f(k(\exp H)nK)\, dn \qquad (H \in \mathfrak{a},\ k \in K)$$

and the *Euclidean Fourier transform* on $\mathfrak{a}$,

$$\mathcal{F}g(\lambda) = \int_{\mathcal{A}} g(H) e^{-\sqrt{-1}\lambda(H)}\, dH \qquad (\lambda \in \mathfrak{a}^\star).$$

If the function f is bi-K-invariant then its Helgason–Fourier transform is kM-independent and in consequence it is a W-invariant function on $\mathfrak{a}^\star$. Moreover, it can be represented as the composition $\mathcal{H}f = \mathcal{F}(\mathcal{A}(f))$, where $\mathcal{A}$ is the *Abel transform*

$$\mathcal{A}f(H) = e^{\varrho(H)} \int_N f((\exp H)n)\, dn \qquad (H \in \mathfrak{g}).$$

The *L_p-Schwartz space $\mathcal{C}_p(X)$* on X, $0 \le p \le 2$, is defined as follows:

$$\begin{aligned}\mathcal{C}_p(X) = \{f \in C^\infty(X) :& \\ &\sup_{\substack{k_1,k_2 \in K\\ H \in \mathfrak{a}}} \langle H \rangle^r \Xi^{-2/p}(H) |f(D_1 : k_1(\exp H)k_2 : D_2)| < \infty \\ &\qquad\qquad (D_1, D_2 \in U(\mathfrak{g});\ r \ge 0)\}.\end{aligned}$$

Here, $f(D_1 : k_1(\exp H)k_2 : D_2)$ denotes the natural action of $D_1, D_2 \in U(\mathfrak{g})$ (the universal enveloping algebra of $\mathfrak{g}$) on $f \in C^\infty(G)$ and

$$\Xi(gK) = \int_K e^{-\varrho H(gk)}\, dk.$$

The space $\mathcal{C}_p(X)$ equipped with the natural topology becomes a metrizable Fréchet space. The space $C_0^\infty(X)$ of test functions is a dense subspace of $\mathcal{C}_p(X)$ and $\mathcal{C}_p(X) \subset \mathcal{C}_q(X)$ if $p \le q$. The space $\mathcal{C}_p(X)$ is contained in $L_q(X)$ for $q \ge p$ but not for $q < p$. The space $\mathcal{C}'_p(X)$ dual to $\mathcal{C}_p(X)$ consists of those distributions on X that can be extended to continuous functionals on $\mathcal{C}_p(X)$. The spaces $\mathcal{C}'_p(X)$ become locally convex topological vector spaces when equipped with weak topology. The image of the space $\mathcal{C}_p(X)$ in the

Helgason–Fourier transform can be explicitly described (see [4]). In particular, $\mathcal{H}f(\cdot,b) \in \mathcal{S}(\mathfrak{a}^\star)$ (the Schwartz space on $\mathfrak{a}^\star$) for any $f \in \mathcal{C}_p(X)$ and any $b \in B$. Moreover, the spaces $\mathcal{C}_p(X)$, $p < 2$, do not contain functions with compactly supported Helgason–Fourier transform.

The convolution on G of a bi-K-invariant function with a K-right invariant function is K-right invariant and therefore it induces the "convolution" on X of two functions one of which is K-invariant on X. We will denote this "convolution" in the same way as the convolution on G. It is known that the Helgason–Fourier transform of the "convolution" of suitable functions is equal to the product of the corresponding Helgason–Fourier transforms. The definition can be extended to distributions (see [12, Ch. II, §5]).

2. Besov spaces and atoms—definitions. In this section we define Besov spaces and atoms we will work with. Moreover, we prove their simple properties needed in the next section. From our point of view the most significant difference between the Euclidean spaces and the symmetric Riemannian manifolds of noncompact type is the exponential growth of volumes of balls in the second case, if the radius tends to infinity. This is why it is not reasonable to define spaces of Besov type via behavior of functions with compactly supported Fourier transform. So we choose the approach via "non-Euclidean local means".

We start with the construction of a continuous resolution of unity on $\mathfrak{a}^\star$. Let Δ be the Laplace–Beltrami operator on X and let $\Gamma = -\Delta - |\varrho|^2$. Let k be a K-invariant C^∞-function on X and let $\kappa = \mathcal{A}(k)$. We assume that κ is a radial real-valued function and that

$$\operatorname{supp} k \subset \Omega(o,1), \qquad \mathcal{H}k(0) \neq 0.$$

Here $\Omega(x,r)$ is a geodesic ball. Let $k^N = \Gamma^N k$ and $\kappa^N = \mathcal{A}(k^N)$. Then $\mathcal{H}(k^N)(\lambda) = |\lambda|^{2N}\mathcal{H}k(\lambda)$ and $\kappa^N = \Delta_{\mathrm{e}}^N(\kappa)$, where Δ_{e} denotes the euclidean Laplace operator. We recall that $\mathcal{H}(k^N) = \mathcal{F}(\kappa^N)$. The function $\mathcal{F}(\kappa^N)$ is radial and admits real values on $\mathfrak{a}^\star$ since κ^N is real-valued and radial. Let $\lambda_0 \in \mathfrak{a}^\star$ be different from 0. We may assume that

$$\int_0^\infty (\mathcal{F}\kappa^N)^2(t\lambda_0)\,\frac{dt}{t} = 1. \tag{1}$$

The last integral is absolutely convergent. By dilations and rotations the identity (1) is valid for any $\lambda \in \mathfrak{a}^\star$, $\lambda \neq 0$. We define

$$(\mathcal{F}\kappa_{0,N})(\lambda) = 1 - \int_0^1 (\mathcal{F}\kappa^N)^2(t\lambda)\,\frac{dt}{t}. \tag{2}$$

Then $(\mathcal{F}\kappa_{0,N})(0) = 1$ and

$$(\mathcal{F}\kappa_{0,N})(\lambda) = \int_1^\infty (\mathcal{F}\kappa^N)^2(t\lambda)\,\frac{dt}{t}. \tag{3}$$

The identities (1) and (2) imply $\mathcal{F}(\kappa_{0,N}) \in \mathcal{S}(\mathfrak{a}^\star)$ and $\kappa_{0,N} \in C_0^\infty(\mathfrak{a})$. Thus

$$\mathcal{H}k_{0,N} + \int_0^1 (\mathcal{H}k^N)^2(t\lambda)\,\frac{dt}{t} = 1, \tag{4}$$

where $k_{0,N} = \mathcal{A}^{-1}\kappa_{0,N}$. If $f \in L_2(X)$ then the following formula of Calderón type holds:

$$f(x) = (f \star k_{0,N})(x) + \int_0^1 (f \star k_t^N \star k_t^N)(x)\,\frac{dt}{t} \tag{5}$$

(convergence in $L_2(X)$; see [17]), where $k_t^N = \mathcal{H}^{-1}(\mathcal{H}k^N(t\cdot))$.

Definition 1. Let $1 \le p \le \infty$, $1 \le q \le \infty$, and $s \in \mathbb{R}$. Let N be a positive integer such that $2N > |s|$. Let $\{k_{0,N}, k_t^N\}_{0<t\le 1}$ be the system of functions defined above. Then

$$\mathcal{B}^s_{p,q}(X) = \Big\{ f \in \mathcal{C}_1' : \|f \mid \mathcal{B}^s_{p,q}(X)\|^{\{k^N\}} = \|f \star k_{0,N} \mid L_p(X)\| + \Big(\int_0^1 t^{-sq}\|f \star k_t^N \mid L_p(X)\|^q\,\frac{dt}{t}\Big)^{1/q} < \infty \Big\}. \tag{6}$$

Remarks. 1. The expressions in (6) are of course norms that depend on N and the given system of functions k_t^N. It will be proved that different systems define equivalent norms.

2. In the Euclidean case the above approach to the Besov spaces is due to H. Triebel [21].

3. It was prove in [16] that for $1 < p < \infty$,

$$\mathcal{B}^s_{p,q}(X) = (H^{s_0}_p(X), H^{s_1}_p(X))_{\theta,q} \tag{7}$$

where $H^s_p(X)$ is the Bessel-potential space and $(\ ,\ )_{\theta,q}$ denotes the real interpolation method. In consequence the above defined Besov spaces coincide with the Besov spaces defined on X by uniform localization (cf. [20]).

In the next lemma we prove the properties of the system of functions k_t^N that will be needed in our constructions of atoms.

Lemma 1. *Let k_t^N, $0 \le t \le 1$, be the system of functions described at the beginning of this section.*

(a) *There is a constant C depending on k and N but independent of t such that*

$$\|k_t^N\|_\infty \le Ct^{-n} \quad \text{and} \quad \|k_t^N\|_1 \le C \ (0 < t \le 1). \tag{8}$$

(b) *For any $f \in \mathcal{C}_1'(X)$ we have*

$$f = f \star k_{0,N} \star k_{0,N} + \int_0^1 (f + f \star k_{0,N}) \star k_t^N \star k_t^N \frac{dt}{t}. \tag{9}$$

The convergence of the integral is understood in the $\mathcal{C}_1'(X)$ sense.

Proof. To prove the first inequality we use the well known estimates

$$|c(\lambda)|^{-2} \le C(1+|\lambda|^2)^d, \qquad d = n-\alpha,\ \alpha = \dim \mathfrak{a}. \tag{10}$$

We have

$$\begin{aligned}|k_t^N(x)| &\le \sup_{x\in\Omega(o,1),\, b\in B} e^{\varrho A(x,b)} \int_{\mathfrak{a}^*} |\mathcal{H}k^N(t\lambda)| \cdot |c(\lambda)|^{-2}\, d\lambda \\ &\le Ct^{-\alpha} \int_{\mathfrak{a}^*} |\mathcal{H}k^N(\lambda)|(1+|t^{-1}\lambda|^2)^{d/2}\, d\lambda \\ &\le Ct^{-n} \int_{\mathfrak{a}^*} |\mathcal{H}k^N(\lambda)|(t^2+|\lambda|^2)^{d/2}\, d\lambda \le Ct^{-n},\end{aligned}$$

since $0 < t \le 1$ and $\mathcal{H}k^N \in \mathcal{S}(\mathfrak{a}^*)$. The second inequality is a consequence of the first one and the polynomial growth of balls near the origin.

The proof of formula (9) is similar to the proof of formula (47) in [17]. Therefore we sketch it only. A standard argument with a Dirac sequence consisting of compactly supported smooth bi-K-invariant functions shows that the smooth functions are dense in $\mathcal{C}_1'(X)$. On the other hand, if $f \in C^\infty(X) \cap \mathcal{C}_1'(X)$ then using the dominated convergence theorem one can easily see that there is a sequence $\{f_n\} \subset C_0^\infty(X)$ such that $f_n \to f$ in $\mathcal{C}_1'(X)$. Thus it is sufficient to prove (9) for $f \in C_0^\infty(X)$ and the L_2-convergence. Let

$$f_\varepsilon(x) = \int_\varepsilon^1 [(f + f \star k_{0,N}) \star k_t^N \star k_t^N](x) \frac{dt}{t}, \qquad \varepsilon > 0.$$

Then

$$\mathcal{H}f_\varepsilon(\lambda, b) = (\mathcal{H}f + \mathcal{H}f\mathcal{H}k_{0,N})(\lambda, b) \int_\varepsilon^1 \mathcal{H}k^N(t\lambda) \frac{dt}{t}$$

and by (8), $\|f_\varepsilon\|_2 \le C\|f + f \star k_t^N\|_2 \log(1/\varepsilon)$. So, as $\varepsilon \to 0$,

$$\|f - f \star k_{0,N} \star k_0^N - f_\varepsilon\| \le \left\| \mathcal{H}f - \mathcal{H}f\mathcal{H}k_{0,N}\left(\mathcal{H}k_{0,N} + \int_\varepsilon^1 (\mathcal{H}k_t^N)^2(t\lambda) \frac{dt}{t}\right) - \mathcal{H}f \int_\varepsilon^1 (\mathcal{H}k_t^N)^2(t\lambda) \frac{dt}{t} \right\|_2 \to 0.$$

COROLLARY 1. *Let $-\infty < s < \infty$, $1 \le p \le \infty$, $1 \le q \le \infty$. Then*

(a) *$C_0^\infty(X) \subset \mathcal{B}_{p,q}^s(X)$ (topological embedding),*

(b) *if $s > 0$ then (6) with $\|f\|_p$ instead of $\|f \star k_{0,N}\|$ is an equivalent norm in $\mathcal{B}_{p,q}^s(X)$.*

Proof. First we note that $\Gamma k_t^N = t^{-2} k_t^{N+1}$ because $\mathcal{A}(\Gamma k_t^N) = -\Delta_e(\kappa_t^N)$ and $-\Delta_e(\kappa_t^N) = t^{-2}\kappa_t^{N+1}$. Let $f \in C_0^\infty(X)$. By the above remark and (8), we have

$$\begin{aligned}\|f \mid \mathcal{B}_{p,q}^s\| &\le C\|f\|_\infty + \left(\int_0^1 t^{q(2N-s)} \|f \star k_t\|_p^q \frac{dt}{t}\right)^{1/p} \\ &\le C(\|f\|_\infty + \|\Gamma^N f\|_\infty)\end{aligned}$$

since $2N - s > 0$. If $s > 0$ then

$$\begin{aligned}\|f\|_p &\le \|f \star k_{0,N}\|_p + \int_0^1 \|f \star k_t^N \star k_t^N\|_p \frac{dt}{t} \\ &\le \|f \star k_{0,N}\|_p + C\left(\int_0^1 t^{-sq} \|f \star k_t^N\|_p^q \frac{dt}{t}\right)^{1/q},\end{aligned}$$

where the last inequality follows from the Hölder inequality.

Now we define atoms that will be used in the decomposition of the Besov spaces.

DEFINITION 2. Let $\Omega = \Omega(x, r)$, $0 \le r \le 1$, be a geodesic ball in X. Let $s \in \mathbb{R}$ and $1 \le p \le \infty$. Let L and M be integers with

$$L \ge ([s]+1)_+ \quad \text{and} \quad M \ge \max([-s], -1), \tag{11}$$

where $d = n - \alpha$, $n = \dim X$, $\alpha = \dim \mathfrak{a}$ and $(t)_+ = \max(t, 0)$.

(a) A smooth function a is called an *s-atom centered in Ω* if

$$\operatorname{supp} a \subset \Omega(x, 2r), \tag{12}$$

$$\sup_{x\in X} |(\Gamma^m a)(x)| \le 1 \qquad \text{for any } m \le L. \tag{13}$$

(b) A smooth function a is called an *(s,p)-atom centered in Ω* if

$$\operatorname{supp} a \subset \Omega(x, 2r), \tag{14}$$

$$\sup_{x\in X} |(\Gamma^m a)(x)| \le r^{s-2m-n/p} \qquad \text{for any } m \le L, \tag{15}$$

$$D^\beta(\mathcal{H}a)(0, b) = 0 \qquad \text{for any } \beta,\ |\beta| \le M, \text{ and any } b \in B. \tag{16}$$

If $M = -1$ then (16) means that no moment conditions are required.

The following lemma is a simple consequence of the definition.

LEMMA 2. *Let a be an (s,p)-atom centered in $\Omega(x,r)$ (an s-atom centered in $\Omega(x,r)$). Then the function a_g, $g \in G$, defined by $a_g(x) = a(g^{-1}x)$ is an (s,p)-atom centered at $\Omega(g\cdot x, r)$ (an s-atom centered in $\Omega(g \cdot x, r)$).*

Proof. The estimates of $\Gamma^m a_g$ are obvious since Γ is an invariant differential operator. The formula

(17) $$A(g\cdot x, g(b)) = A(x,b) + A(g\cdot o, g(b)), \quad g \in G,\ b \in B,\ o = eK,$$

(see [13]) implies

$$(\mathcal{H}a_g)(\lambda, b) = e^{(-\sqrt{-1}\lambda+\varrho)A(g\cdot o, b)}(\mathcal{H}a)(\lambda, g^{-1}b).$$

Thus the moment condition is a consequence of the Leibniz rule.

The atomic decomposition of Besov spaces with $p \ge 1$ requires a rigid control on the location of the support of the atoms, therefore we need some coverings of the manifold X. Let r_j, $j = 0,1,2,\ldots$, be a sequence of positive numbers decreasing to zero. Let $\Omega_j = \{\Omega(x_{j,i}, r_j)\}_{i=1}^{\infty}$ be a uniformly locally finite covering of X by balls of radius r_j. The sequence (Ω_j), $j = 0,1,\ldots$, of coverings is said to be *uniformly locally finite* if there is a positive constant C such that for every j and every $x \in X$ the point x is an element of at most C balls of the covering Ω_j.

LEMMA 3. *Let X be a symmetric manifold of noncompact type. There is a uniformly locally finite sequence $\{\Omega_j\}$ of coverings of X by geodesic balls of radius $r_j = 2^{-j}$, $\Omega_j = \{\Omega(x_{j,i}, r_j)\}_{i\in\mathbb{N}}$, $j = 0,1,\ldots$ Moreover, if $l \in \mathbb{N}$ and $\Omega_{j,l} = \{\Omega(x_{j,i}, lr_j)\}_{i\in\mathbb{N}}$ then the sequence $\{\Omega_{j,l}\}$, $j = 0,1,\ldots$, is also uniformly locally finite.*

Proof. We sketch the proof only since the arguments are standard. Let $0 < r \le 1$. We show that there is a uniformly locally finite covering $\{\Omega(x_i, r)\}$ and a positive constant c independent of r such that $\Omega(x_i, cr) \cap \Omega(x_j, cr) = \emptyset$ if $i \ne j$. Let $\{(\Omega(y_i, 3), \exp_{y_i}^{-1})\}$, $i = 1,2,\ldots$, be a covering of X by exponential charts. Since the group of isometries acts transitively on X and the exponential mapping commutes with isometries there are constants C_1, C_2 independent of i such that

$$C_1 d_{\mathrm{e}}(\exp_{y_i}(z_1), \exp_{y_i}(z_2)) \le d_X(z_1,z_2) \le C_2 d_{\mathrm{e}}(\exp_{y_i}(z_1), \exp_{y_i}(z_2)),$$

$z_1, z_2 \in \Omega(y_i, 4)$, where d_{e} (resp. d_X) denotes the euclidean distance (resp. the distance in X). We can find points $x_1, \ldots, x_{k_1} \in \Omega(y_1, 3)$ such that

(18) $$\inf\{d_X(x,x_i) : i = 1,\ldots,k)\} < r \quad \text{for any } x \in \Omega(y_1, 3)$$

and

(19) $$d_X(x_i,x_j) > r_1 \quad \text{for } i,j = 1,\ldots,k,\ i\ne j, \text{ where } \quad r_1 = \min\left(\frac{1}{2}, \frac{C_1}{C_2\sqrt{n}}\right) r.$$

Next we can find points $x_{k_1+1}, \ldots, x_{k_2} \in \Omega(y_2, 3) \setminus \bigcup_{i=1}^{k_1} \Omega(x_i, r)$ such that (18) and (19) are satisfied for $k_1 < i, j \le k_2$. Since $r_1 < r/2$ we also have $d_X(x_i, x_j) > r_1$ for $i \le k_1 < j \le k_2$. Continuing this process we construct the covering we are looking for, with $c = \min(1/2, C_1/(C_2\sqrt{n}))$.

Let $x \in X$ and $J_x = \{i : x_i \in \Omega(x, (2l+1)r)\}$. Then

$$C_3|J_x| r_1^n \le \sum_{j\in J_x} \operatorname{vol} \Omega(x, (2l+1)r) \le \operatorname{vol} \Omega(x,(2l+1)r) \le C_4(2L+1)^n r^n.$$

Thus

$$|J_x| \le C_5 (2l+1)^n c^{-n},$$

C_5 being a constant independent of x and l. This finishes the proof.

3. The atomic decomposition of Besov spaces

THEOREM 1. *Let $s \in \mathbb{R}$, $1 \le p \le \infty$, $1 \le q \le \infty$. Let L and M be fixed integers satisfying (11). Let $\{\Omega_j\}_{j=0}^{\infty}$ be a uniformly locally finite sequence of coverings of X, $\Omega_j = \{\Omega(x_{j,i}, 2^{-j})\}$.*

(a) *Each $f \in B^s_{p,q}(X)$ can be decomposed as*

(20) $$f = \sum_{i\in\mathbb{N}} s_i a_i + \sum_{j=0}^{\infty}\sum_{i\in\mathbb{N}} s_{j,i} a_{j,i} \quad (\text{convergence in } \mathcal{C}_1(X))$$

where a_i is an s-atom centered in the ball $\Omega(x_{1,i}, 1)$, $a_{j,i}$ is an (s,p)-atom centered in the ball $\Omega(x_{j,i}, 2^{-j})$, and s_i and $s_{j,i}$ are complex numbers with

(21) $$\Big(\sum_{i\in\mathbb{N}} |s_i|^p\Big)^{1/p} + \Big(\sum_{j=0}^{\infty}\Big(\sum_{i\in\mathbb{N}} |s_{j,i}|^p\Big)^{q/p}\Big)^{1/q} < \infty$$

(with the usual modification if either $p = \infty$ or $q = \infty$).

(b) *Conversely, suppose that $f \in \mathcal{C}_1(X)$ can be represented as in (20) and (21). Then $f \in B^s_{p,q}(X)$.*

Furthermore, the infimum of the left hand side of (21) over all admissible representations (for a fixed sequence of coverings and fixed integers L, M) is an equivalent norm in $B^s_{p,q}(X)$.

Proof. (a) Step 1. Let $\mathcal{E}_j = \{E_{j,i}\}$ be a decomposition of X into a sum of disjoint sets such that $E_{j,i} \subset \Omega(x_{j,i}, 2^{-j})$. Let $GE_{j,i} = \pi^{-1}(E_{j,i})$ and $TE_{j,i} = GE_{j,i} \times (2^{-j-1}, 2^{-j})$, where $\pi : G \to X$ is the natural projection. Let (k_t^N), $2N > \max(s,0)$, $0 \le t \le 1$, be the system of functions described at the beginning of Section 2. Let N be an integer such that $2N > |s|$. We choose integers N_1 and N_2 satisfying $2N_2 > M$, $2N_1 = N + N_2$. Let $\{k_t^{N_1}\}$,

$0 \le t \le 1$, be the system of functions described above. We put

$$s_{j,i} = 2^{js} C_{N,L} \Big(\int_{GE_{j,i}} \Big(\int_{2^{-j-1}}^{2^{-j}} |(f + f \star k_{0,N_1}) \star k_t^N|(g) \frac{dt}{t} \Big)^p dg \Big)^{1/p}, \tag{22}$$

$$a_{j,i}(x) = s_{j,i}^{-1} \int \int_{TE_{j,i}} [(f + f \star k_{0,N_1}) \star k_t^N](g) k_t^{N_2}(g^{-1}x) \frac{dt}{t}\, dg, \tag{23}$$

$$s_i = c_{N,L} \Big(\int_{GE_{0,i}} |f \star k_{0,N_1}|^p(g)\, dg \Big)^{1/p}, \tag{24}$$

$$a_i(x) = s_i^{-1} \int_{GE_{0,i}} (f \star k_{0,N_1})(g) k_{0,N_1}(g^{-1}x)\, dg, \tag{25}$$

where $x \in G$ (with the usual modification if $p = \infty$). Here $C_{N,L}$ and $c_{N,L}$ are constants independent of f and j, i that will be described later. We note that $a_{j,i}$ are smooth K-right invariant functions on G and in consequence they are well defined on X. The K-right invariance of $a_{j,i}$ follows easily from bi-K-invariance of k_t^N and (23).

Step 2. We prove that $a_{j,i}$ is an (s,p)-atom centered in the ball $\Omega(x_{j,i}, 2^{-j})$. The proof that a_i is an s-atom is similar and therefore it is omitted here.

Let $xK \notin \Omega(x_{j,i}, 2^{-j+1})$, $x \in G$. If $gK \in E_{j,i}$, $g \in G$ then

$$\begin{aligned} 2^{-j+1} < d_X(xK, x_{j,i}) &\le d_X((g^{-1}x)K, o) + d_X(o, g^{-1} \cdot x_{j,i}) \\ &\le d_X((g^{-1}x)K, o) + 2^{-j}. \end{aligned}$$

Thus $(g^{-1}x)K \notin \Omega(o,t)$ for $t \in (2^{-j-1}, 2^{-j})$. This and (23) imply $a_{j,i}(x) = 0$. So (14) is satisfied.

Since the operator Γ^m is invariant and $a_{j,i}$ can be interpreted as a convolution we have

$$\begin{aligned} (\Gamma^m a_{j,i})(x) &= s_{j,i}^{-1} \Gamma^m \Big(\int_{2^{-j-1}}^{2^{-j}} [\chi_{GE_{j,i}}((f + f \star k_{0,N_1}) \star k_t^N)] \star k_t^{N_2} \frac{dt}{t} \Big)(x) \\ &= s_{j,i}^{-1} \int_{2^{-j-1}}^{2^{-j}} \int_{GE_{j,i}} [(f + f \star k_{0,N_1}) \star k_t^N](g) \Gamma^m k_t^{N_2}(g^{-1}x)\, dg \frac{dt}{t} \end{aligned}$$

(cf. [13]). But

$$(\Gamma^m k_t^{N_2})(x) = t^{-2m} k_t^{N_2+m}(x).$$

Therefore Lemma 1 implies

$$\sup_{2^{-j-1} \le t \le 2^{-j}} |\Gamma^m k_t^{N_2}(x)| \le C_0 (2^{-j})^{-2m-n}$$

where C_0 is a constant depending on N_2 and on the system of functions $k_t^{N_2}$. In consequence,

$$\begin{aligned} |\Gamma^m a_{j,i}(x)| &\le s_{j,i}^{-1} \int_{GE_{j,i}} \sup_{2^{-j-1} \le t \le 2^{-j}} |\Gamma^m k_t^{N_2}(g^{-1}x)| \\ &\quad \times \Big| \int_{2^{-j-1}}^{2^{-j}} [(f + f \star k_{0,N_1}) \star k_t^N](g) \frac{dt}{t} \Big|\, dg \\ &\le s_{j,i}^{-1} \Big(\int_{GE_{j,i}} \Big| \int_{2^{-j-1}}^{2^{-j}} [(f + f \star k_{0,N_1}) \star k_t^N](g) \frac{dt}{t} \Big|^p dg \Big)^{1/p} \\ &\quad \times \Big(\int_{GE_{j,i}} \big(\sup_{2^{-j-1} \le t \le 2^{-j}} |\Gamma^m k_t^{N_2}(g^{-1}x)| \big)^{p'} dg \Big)^{1/p'} \\ &\le C_N^{-1} C_0 (2^{-j})^{s-2m-n} \operatorname{vol}(E_{j,i})^{1/p'} \le (2^{-j})^{s-2m-n/p}. \end{aligned}$$

for any $m \le L$, after an obvious definition of $C_{N,L}$.

Now we check the moment condition. We see at once that

$$D^\beta(\mathcal{H} k_t^{N_2})(0,b) = D^\beta(\mathcal{F} \kappa_t^{N_2})(0) = 0 \tag{26}$$

for any multi-index β, $|\beta| < 2N_2$, and any $b \in B$. On the other hand, by the Fubini theorem and the formula (17),

$$\begin{aligned} (\mathcal{H} a_{j,i})(\lambda, B) &= s_{j,i}^{-1} \int \int_{TE_{j,i}} [(f + f \star k_{0,N_1}) \star k_t^N](g) \\ &\quad \times \Big(\int_X k_t^{N_2}(g^{-1} \cdot x) e^{(-\sqrt{-1}\lambda + \varrho) A(x,b)}\, dx \Big) \frac{dt}{t}\, dg \\ &= s_{j,i}^{-1} \int \int_{TE_{j,i}} [(f + f \star k_{0,N_1}) \star k_t^N](g) \\ &\quad \times e^{(-\sqrt{-1}\lambda + \varrho) A(g \cdot o, b)} (\mathcal{H} k_t^{N_2})(\lambda, g^{-1}(b)) \frac{dt}{t}\, dg. \end{aligned}$$

Combining the last identity with (26) we get

$$D^\beta(\mathcal{H} a_{j,i})(0,b) = 0 \tag{27}$$

for any $b \in B$ and any β with $|\beta| < M$.

Step 3. Now we decompose $f \in \mathcal{B}_{p,q}^s$ into a sum of atoms. Easy computations show that

$$k_t^{N_1} \star k_t^{N_1} = k_t^N \star k_t^{N_2}.$$

Thus we can write the formula from Lemma 1 in the following way:

$$f = f\star k_{0,N_1}\star k_{0,N_1} + \sum_{j=0}^{\infty}\int\limits_{2^{-j-1}}^{2^{-j}} (f + f\star k_{0,N_1})\star k_t^N \star k_t^{N_2}\,\frac{dt}{t}$$
$$= \sum_{i=1}^{\infty} s_i a_i + \sum_{j=0}^{\infty}\sum_{i=1}^{\infty} s_{j,i} a_{j,i}.$$

The last identity leads to the decomposition of f into atoms that are described in Step 1.

Moreover,

$$\Big(\sum_{j=0}^{\infty}\Big(\sum_{i=1}^{\infty}|s_{j,i}|^p\Big)^{q/p}\Big)^{1/q}$$
$$\le C\Big(\sum_{j=0}^{\infty}2^{jsq}\Big(\int\limits_G\Big(\int\limits_{2^{-j-1}}^{2^{-j}}|(f+f\star k_{0,N_1})\star k_t^N|(g)\,\frac{dt}{t}\Big)^p dg\Big)^{q/p}\Big)^{1/q}$$
$$\le C\Big(\sum_{j=0}^{\infty}2^{jsq}\Big(\int\limits_{2^{-j-1}}^{2^{-j}}\Big(\int\limits_G|(f+f\star k_{0,N_1})\star k_t^N(g)|^p\,dg\Big)^{1/p}\frac{dt}{t}\Big)^q\Big)^{1/q}$$
$$\le C\Big(\sum_{j=0}^{\infty}2^{jsq}\int\limits_{2^{-j-1}}^{2^{-j}}\|(f+f\star k_{0,N_1})\star k_t^N\|_p^q\,\frac{dt}{t}\Big)^{1/q}$$
$$\le C\Big(\int\limits_0^1 t^{-sq}\|(f+f\star k_{0,N_1})\star k_t^N\|_p^q\,\frac{dt}{t}\Big)^{1/q}$$
$$\le C\Big(\int\limits_0^1 t^{-sq}\|f\star k_t^{N_1}\|_p^q\,\frac{dt}{t}\Big)^{1/q}$$
$$+ C\Big(\int\limits_0^1 t^{-sq}\|f\star k_{0,N_1}\star k_t^N\|_p^q\,\frac{dt}{t}\Big)^{1/q}$$
$$\le C\Big(\int\limits_0^1 t^{-sq}\|f\star k_t^N\|_p^q\,\frac{dt}{t}\Big)^{1/q},$$

where the second inequality follows by the Minkowski inequality for integrals and the third one from the inequality $q \ge 1$. The proof will be completed by showing that

$$\Big(\sum_{i=0}^{\infty}|s_i|^p\Big)^{1/p} \le C\|f \mid \mathcal{B}^s_{p,q}(X)\|^{\{k^N\}}. \tag{28}$$

We have

$$C^{-1}\Big(\sum_{i=0}^{\infty}|s_i|^p\Big)^{1/p} \le \|f\star k_{0,N}\|_p + \int\limits_0^1\|f\star k_t^N\star k_{0,N_1}\star k_t^N\|_p\,\frac{dt}{t}$$
$$\le \|f\star k_{0,N}\|_p + \Big(\int\limits_0^1 t^{-sq}\|f\star k_t^{N_1}\|_p^q\,\frac{dt}{t}\Big)^{1/p}$$
$$\times\Big(\int\limits_0^1 t^{sq'}\|k_{0,N_1}\star k_t^N\|_1^{q'}\,\frac{dt}{t}\Big)^{1/q'}$$
$$\le \|f \mid \mathcal{B}^s_{p,q}(X)\|^{\{k^N\}}$$

(see Corollary 1). Thus

$$\Big(\sum_{i\in\mathbb{N}}|s_i|^p\Big)^{1/p} + \Big(\sum_{j=0}^{\infty}\Big(\sum_{i\in\mathbb{N}}|s_{j,i}|^p\Big)^{q/p}\Big)^{1/q} \le C\|f \mid \mathcal{B}^s_{p,q}(X)\|^{\{k^N\}}. \tag{29}$$

This finishes the proof of part (a) of the theorem.

(b) Step 1. Let $f = \sum_{i\in\mathbb{N}} s_i a_i + \sum_{j=0}^{\infty}\sum_{i\in\mathbb{N}} s_{j,i}a_{j,i}$ with

$$\Big(\sum_{i\in\mathbb{N}}|s_i|^p\Big)^{1/p} + \Big(\sum_{j=0}^{\infty}\Big(\sum_{i\in\mathbb{N}}|s_{j,i}|^p\Big)^{q/p}\Big)^{1/q} < \infty.$$

Then

$$\|f \mid \mathcal{B}^s_{p,q}(X)\|^{\{k^N\}}$$
$$\le \Big\|\sum_{i\in\mathbb{N}} s_i a_i\star k_{0,N}\Big\|_p + \Big(\int\limits_0^1 t^{-sq}\Big\|\sum_{i\in\mathbb{N}} s_i a_i\star k_t^N\Big\|_p^q\,\frac{dt}{t}\Big)^{1/q}$$
$$+ \Big\|\sum_{j=0}^{\infty}\sum_{i\in\mathbb{N}} s_{j,i}a_{j,i}\star k_{0,N}\Big\|_p + \Big(\int\limits_0^1 t^{-sq}\Big\|\sum_{j=0}^{\infty}\sum_{i\in\mathbb{N}} s_{j,i}a_{j,i}\star k_t^N\Big\|_p^q\,\frac{dt}{t}\Big)^{1/q}.$$

We estimate each summand separately. The estimates for the first and the second term are independent of s. Estimating the last two terms we consider two cases: $s > 0$ and $s \le 0$.

Step 2. We estimate the first and the second summand. The inequality

$$\Big\|\sum_{i\in\mathbb{N}} s_i a_i\star k_{0,N}\Big\|_p \le C\Big\|\sum_{i\in\mathbb{N}} s_i a_i\Big\|_p \le C\Big(\sum_{i\in\mathbb{N}}|s_i|^p\Big)^{1/p} \tag{30}$$

is obvious since the covering is uniformly locally finite and the functions a_j are uniformly bounded, (see (12), (13)). To estimate the second summand we note first that

$$\|a_i\star k_t^N\|_p = t^{2N}\|\Gamma^N a_i\star k_t\|_p \le Ct^{2N}.$$

So,

$$\Big(\int_0^1 t^{-sq}\Big\|\sum_{i\in\mathbb{N}} s_i a_i \star k_t^N\Big\|_p^q \frac{dt}{t}\Big)^{1/q}$$
$$\le C\Big(\int_0^1 t^{-sq}\Big(\int_X \sum_{i\in\mathbb{N}} |s_i|^p |a_i\star k_t^N|^p\,dx\Big)^{q/p}\frac{dt}{t}\Big)^{1/q}$$
$$\le C\Big(\int_0^1 t^{(2N-s)q}\Big(\sum_{i\in\mathbb{N}}|s_i|^p\Big)^{q/p}\frac{dt}{t}\Big)^{1/q} \le C\Big(\sum_{i\in\mathbb{N}}|s_i|^p\Big)^{1/p}.$$

Step 3. Let $0 < s < 2N$. The definition of the functions $a_{j,i}$ (see (14), (15)) implies

$$\Big\|\sum_{j=0}^\infty\sum_{i\in\mathbb{N}} s_{j,i}a_{j,i}\star k_{0,N}\Big\|_p \le C\sum_{j=0}^\infty\Big(\sum_{i\in\mathbb{N}}|s_{j,i}|^p\int_X |a_{j,i}|^p\,dx\Big)^{1/p} \tag{31}$$
$$\le C\sum_{j=0}^\infty\Big(\sum_{i\in\mathbb{N}}|s_{j,i}|^p\Big)^{1/p}2^{-js}$$
$$\le C\Big(\sum_{j=0}^\infty\Big(\sum_{i\in\mathbb{N}}|s_{j,i}|^p\Big)^{q/p}\Big)^{1/q},$$

where the last inequality follows from the Hölder inequality.

It remains to estimate the last summand. Let J be smallest integer such that $2J \ge ([s]+1)_+$. Then $J \le N$ and $J\le L$. From Lemma 1 and (14)–(15) we see that

$$\|a_{j,i}\star k_t^N\|_p \le t^{2J}\|\Gamma^J a_{j,i}\|_p\|k_t^{N-J}\|_1 \le Ct^{2J}(2^{-j})^{s-2J}, \tag{32}$$

and

$$\Big\|\sum_{i\in\mathbb{N}} s_{j,i}a_{j,i}\star k_t^N\Big\|_p \le C2^{-js}\Big(\sum_{i\in\mathbb{N}}|s_{j,i}|^p\Big)^{1/p}. \tag{33}$$

The last inequality can be proved in the following way:

$$\Big\|\sum_{i\in\mathbb{N}} s_{j,i}a_{j,i}\star k_t^N\Big\|_p \le C\Big\|\Big(\sum_{i\in\mathbb{N}}|s_{j,i}|^p|a_{j,i}|^p\Big)^{1/p}\star|k_t^N|\Big\|_p$$
$$\le C\int_G |k_t^N|(y)\Big(\int_G\sum_{i\in\mathbb{N}}|s_{j,i}|^p|a_{j,i}(xy^{-1})|^p\,dx\Big)^{1/p}dy$$
$$\le C\int_G |k_t^N|(y)\Big(\sum_{i\in\mathbb{N}}|s_{j,i}|^p\int_G|a_{j,i}(x)|^p\,dx\Big)^{1/p}dy$$
$$\le C2^{-js}\Big(\sum_{i\in\mathbb{N}}|s_{j,i}|^p\Big)^{1/p}.$$

The rest of the estimates runs as follows:

$$\Big(\int_0^1 t^{-sq}\Big\|\sum_{j=0}^\infty\sum_{i\in\mathbb{N}} s_{j,i}a_{j,i}\star k_t^N\Big\|_p^q\frac{dt}{t}\Big)^{1/q}$$
$$\le C\Big(\sum_{k=0}^\infty\int_{2^{-k-1}}^{2^{-k}} t^{-sq}\Big(\sum_{j=0}^\infty\Big\|\sum_{i\in\mathbb{N}} s_{j,i}a_{j,i}\star k_t^N\Big\|_p\Big)^q\frac{dt}{t}\Big)^{1/q}$$
$$\le C\Big(\sum_{k=0}^\infty\Big(\sum_{j=0}^k\Big(\sum_{i\in\mathbb{N}}|s_{j,i}|^p\sup_{2^{-k-1}\le t\le 2^{-k}} t^{-s}\|a_{j,i}\star k_t^N\|_p^p\Big)^{1/p}\Big)^q\Big)^{1/q}$$
$$+C\Big(\sum_{k=0}^\infty\Big(\sum_{j=k+1}^\infty\sup_{2^{-k-1}\le t\le 2^{-k}} t^{-s}\Big\|\sum_{i\in\mathbb{N}} s_{j,i}a_{j,i}\star k_t^N\Big\|_p\Big)^q\Big)^{1/q}$$
$$\le C\Big(\sum_{k=0}^\infty\Big(\sum_{j=0}^k\Big(\sum_{i\in\mathbb{N}}|s_{j,i}|^p\Big)^{1/p}2^{-(k-j)(2J-s)}\Big)^q\Big)^{1/q}$$
$$+C\Big(\sum_{k=0}^\infty\Big(\sum_{j=k+1}^\infty\Big(\sum_{i\in\mathbb{N}}|s_{j,i}|^p\Big)^{1/p}2^{-(j-k)s}\Big)^q\Big)^{1/q}$$
$$\le C\Big(\sum_{j=0}^\infty\Big(\sum_{i\in\mathbb{N}}|s_{j,i}|^p\Big)^{q/p}\Big)^{1/q}.$$

Step 4. We turn to the case $s\le 0$. This is the case in which the moment condition is important. We begin by observing two simple facts about the Radon and inverse Fourier transforms. Let f be a smooth function and $\operatorname{supp} f\subset\Omega(o,r)$, $r>0$. Let ξ be the horocycle through $x\in X$ with normal kM. The Haar measure dn on N induces a $d\sigma$ on the horocycle $\xi_o = N\cdot o$ and by translations on ξ. Therefore,

$$\mathcal{R}f(H,kM) = e^{\varrho(H)}\int_\xi f(x)\,d\sigma(x)$$

(cf. [13]). It is not difficult to see that $d\sigma(\Omega(o,r)\cap\xi)\le Cr^d$ if only r is sufficiently small. Moreover, it was proved by Helgason [13] that the Radon transform maps $C_0^\infty(X)$ into $C_0^\infty(\mathfrak{a}\times B)$ and that $\operatorname{supp} f\subset\Omega(o,r)$ implies $\operatorname{supp}\mathcal{R}f\subset B(0,r)$ (the support conservation property.) Thus, there exists a constant C depending on r_o but independent of f such that the inequality

$$\sup_{(H,b)\in\mathfrak{a}\times B}|\mathcal{R}f(H,b)|\le Cr^d\sup_{x\in X}|f(x)| \tag{34}$$

holds for any smooth function f with $\operatorname{supp} f\subset\Omega(o,r)$, $r\le r_o$. On the other hand, since

$$f(x) = C \int_{\mathfrak{a}^*\times B} e^{(\sqrt{-1}\lambda+\varrho)A(x,b)}(1+|\lambda|^2)^{n'} \times \mathcal{H}f(\lambda,b)(1+|\lambda|^2)^{-n'}|c(\lambda)|^{-2}\,d\lambda\,db, \tag{35}$$

the estimate (10) implies

$$\sup_{x\in X}|f(x)| \le C\sum_{l=0}^{n'}\sup_{(\lambda,b)\in\mathfrak{a}^*\times B}|\mathcal{F}(\Delta_e^l(\mathcal{R}f))(\lambda,b)| \le C\sum_{l=0}^{n'}\|\Delta_e^l(\mathcal{R}f)\mid L_1(\mathfrak{a})\|, \tag{36}$$

where $n' = [n/2]+1$.

In terms of the Radon transform the moment condition (16) reads

$$\int_{\mathfrak{a}} H^\beta \mathcal{R}a_{j,i}(H,b)\,dH = 0, \tag{37}$$

for any $b\in B$ and any multi-index β with $|\beta|\le M$.

Now we estimate $a_{j,i}\star k_{0,N}$. Let $a(x) = a_{j,i}(g_{j,i}^{-1}x)$, $x_{j,i} = g_{j,i}K$, and $\kappa_l = \Delta_e^l\kappa_{0,N}$. Lemma 2 shows that a is an (s,p)-atom centered in $\Omega(o,2^{-j})$. From (37) it may be concluded that

$$\mathcal{R}a\star\kappa_l(H_0) = \int_{B(H_0,2^{-j})}\mathcal{R}a(H_0-H) \times\Big(\kappa_l(H) - \sum_{|\beta|\le M}\partial^\beta\kappa_l(H_0)(H-H_0)^\beta/\beta!\Big)\,dH. \tag{38}$$

By Taylor's formula, (34) and (15),

$$|\mathcal{R}a\star\kappa_l(H_0)| \le C\int_{B(H_0,2^{-j})}|\mathcal{R}a(H_0-H)|\cdot|H-H_0|^{M+1}\,dH \le C2^{-j(s-n/p+M+1+n)}. \tag{39}$$

Together with (36), this gives

$$\sup_{x\in X}|a\star k_{0,N}(x)| \le C2^{-j(s-n/p+M+1+n)}, \tag{40}$$

where C is a constant independent of j. But $f_g\star h = (f\star h)_g$, and therefore the inequality (40) is true after replacing a by any $a_{j,i}$ for given j.

The support of the function $a_{j,i}\star k_{0,N}$ is contained in $\Omega(x_{j,i},2^{-j}+r_o)$ if $\operatorname{supp} k_{0,N}\subset\Omega(o,r_o)$. The covering $\{\Omega(x_{j,i},2^{-j}+r_o)\}$ is uniformly locally finite and moreover there is a constant C independent of j such that at most $C2^{jn}$ balls of the covering have a nonempty intersection. This can be proved by the standard argument.

In consequence, similar arguments to (31) show that

$$\Big\|\sum_{j=0}^\infty\sum_{i\in\mathbb{N}} s_{j,i}a_{j,i}\star k_{0,N}\Big\|_p \le C\Big(\sum_{j=0}^\infty\Big(\sum_{i\in\mathbb{N}}|s_{j,i}|^p\Big)^{q/p}\Big)^{1/q}. \tag{41}$$

The estimate of $a_{j,i}\star k_t^N$ is a bit more difficult since we must take dilations into account. Let $\delta_t\psi(x) = t^{-\alpha}\psi(t^{-1}x)$, ψ being a suitable function on $\mathfrak{a}$. Then

$$\mathcal{R}a(\cdot,b)\star\kappa_t^N = \delta_t(\delta_{t^{-1}}(\mathcal{R}a(\cdot,b))\star\kappa^N), \tag{42}$$

and in consequence

$$\mathcal{H}(a\star k_t^N)(\lambda,b) = \mathcal{F}(\delta_{t^{-1}}(\mathcal{R}a(\cdot,b))\star\kappa^N)(t\lambda). \tag{43}$$

Changing the variable in the inversion formula we hence get

$$a\star k_t^N(x) = Ct^{-\alpha}\int_{\mathfrak{a}^*\times B} e^{(\sqrt{-1}t^{-1}\lambda+\varrho)A(x,b)} \times\mathcal{F}(\delta_{t^{-1}}(\mathcal{R}a(\cdot,b))\star\kappa^N)(\lambda)|c(t^{-1}\lambda)|^{-2}\,d\lambda\,db. \tag{44}$$

We divide the integral in (44) into two parts: near the origin and at infinity. We conclude from (10) that

$$\begin{aligned}\Big|\int_{B(0,1)\times B} & e^{(\sqrt{-1}t^{-1}\lambda+\varrho)A(x,b)} \times\mathcal{F}(\delta_{t^{-1}}(\mathcal{R}a(\cdot,b))\star\kappa^N)(\lambda)|c(t^{-1}\lambda)|^{-2}\,d\lambda\,db\Big| \\ &\le Ct^{-d}\int_{B(0,1)\times B}|\mathcal{F}(\delta_{t^{-1}}(\mathcal{R}a(\cdot,b))\star\kappa^N)(\lambda)|(t^2+|\lambda|^2)^{d/2}\,d\lambda\,db \\ &\le Ct^{-d}\sup_{\mathfrak{a}^*\times B}|\mathcal{F}(\delta_{t^{-1}}(\mathcal{R}a(\cdot,b))\star\kappa^N)(\lambda)|.\end{aligned} \tag{45}$$

Now we estimate the second integral. We have

$$\begin{aligned}\Big|\int_{(\mathfrak{a}^*\setminus B(0,1))\times B} & e^{(\sqrt{-1}t^{-1}\lambda+\varrho)A(x,b)} \times\mathcal{F}(\delta_{t^{-1}}(\mathcal{R}a(\cdot,b))\star\kappa^N)(\lambda)|c(t^{-1}\lambda)|^{-2}\,d\lambda\,db\Big| \\ &\le C\int_{(\mathfrak{a}^*\setminus B(0,1))\times B}|(\mathcal{F}(\delta_{t^{-1}}(\mathcal{R}a(\cdot,b))\star\kappa^N)(\lambda)| \times|\lambda|^{2n'-a-1-d}|c(t^{-1}\lambda)|^{-2}\,d\lambda\,db \\ &\le Ct^{-d}\sup_{\mathfrak{a}^*\times B}|\mathcal{F}(\delta_{t^{-1}}(\mathcal{R}a(\cdot,b))\star\kappa^{N+n'})(\lambda)|,\end{aligned} \tag{46}$$

where the last inequality follows from (10) since $0 < t \le 1$ and from the inequality $2n' \ge n + 1/2$.

The functions $\delta_{t^{-1}}(\mathcal{R}a)$ satisfy (37). So, in the same manner as in (38) and (39) we can see that

$$(47)\quad |\delta_{t^{-1}}(\mathcal{R}a) \star \kappa(H_0)| \le Ct^a \int_{B(H_0,t^{-1}2^{-j})} |\mathcal{R}a(t(H_0 - H))| \cdot |H - H_0|^{M+1}\,dH \le C2^{-j(s-n/p+n+M+1)}t^{-M-1}.$$

This gives for $t \ge 2^{-j}$ the inequality

$$(48)\quad \mathcal{F}(\delta_{t^{-1}}(\mathcal{R}a(\cdot,b)) \star \kappa)(t\lambda) \le C2^{-j(M+1+s-n/p+n)}t^{-M-1}.$$

The same is true if we take $\kappa^{N+n'}$ instead of κ^N.

So, the counterpart of (40) for $t \ge 2^{-j}$ looks as follows:

$$(49)\quad \sup_{x\in X} |a \star k_t^N(x)| \le Ct^{-M-1-n}2^{-j(s+n/p+n+M+1)}$$

(see (44)–(48)). We thus get, for $t \ge 2^{-j}$,

$$(50)\quad \|a_{j,i} \star k_t^N(x)\|_p \le Ct^{-M-1-n+n/p}2^{-j(s-n/p+n+M+1)}.$$

Using this estimate and the inequality (33) for $t \le 2^{-j}$ we can show in the same way as in the case $s > 0$ that

$$\Big(\int_0^1 t^{-sq}\Big\|\sum_{j=0}^{\infty}\sum_{i\in\mathbb{N}} s_{j,i}a_{j,i} \star k_t^N\Big\|_p^q \frac{dt}{t}\Big)^{1/q} \le \Big(\sum_{j=0}^{\infty}\Big(\sum_{i\in\mathbb{N}}|s_{j,i}|^p\Big)^{q/p}\Big)^{1/q}.$$

Thus we have proved that

$$(51)\quad \|f \mid \mathcal{B}_{p,q}^s(X)\|^{\{k^N\}} \le C\Big(\sum_{i\in\mathbb{N}}|s_i|^p\Big)^{1/p} + \Big(\sum_{j=0}^{\infty}\Big(\sum_{i\in\mathbb{N}}|s_{j,i}|^p\Big)^{q/p}\Big)^{1/q}.$$

Together with (29), this proves the theorem.

Remarks. 1. It follows immediately from the above theorem that the definition of the spaces $\mathcal{B}_{p,q}^s(X)$ is independent of the function k and the integer N with $2N > |s|$.

2. There is a group-theoretical interpretation of atomic decompositions due to H.-G. Feichtinger and K. H. Gröchenig [5, 6]. This approach seems to be more suitable for homogeneous function spaces. Moreover, it requires the integrability of the representation considered. In contrast to the Euclidean case the representations that built up the Helgason–Fourier transform are not only nonintegrable but even not square-integrable as elements of the principal series of the group G.

4. Some applications. In this section we prove several simple consequences of the main theorem. We concentrate on the cases $p = 1$ and $p = \infty$. If $1 < p < \infty$ then the Besov spaces are interpolation spaces of the corresponding Bessel-potential spaces. So, in that case some of the results below follow by the real interpolation method.

PROPOSITION 1. *Let* $1 \le p, q, q_0, q_1 \le \infty$ *and* $s, s_0, s_1 \in \mathbb{R}$.

(i) (*Elementary embeddings*)

$$(52)\quad \mathcal{B}_{p,q_0}^s(X) \subset \mathcal{B}_{p,q_1}^s(X) \quad \text{if } q_0 \le q_1,$$

$$(53)\quad \mathcal{B}_{p,q_0}^{s_0}(X) \subset \mathcal{B}_{p,q_1}^{s_1}(X) \quad \text{if } s_1 \le s_0.$$

(ii) (*Embeddings with different metrics*)

$$(54)\quad \mathcal{B}_{p,q}^{s_0}(X) \subset \mathcal{B}_{\infty,q}^{s_1}(X) \quad \text{if } s_1 = s_0 - n/p,$$

$$(55)\quad \mathcal{B}_{1,q}^{s_0}(X) \subset \mathcal{B}_{p,q}^{s_1}(X) \quad \text{if } s_0 - n = s_1 - n/p,$$

$$(56)\quad \mathcal{B}_{1,1}^{0}(X) \subset L_1(X) \subset \mathcal{B}_{1,\infty}^{0}(X),$$

$$(57)\quad \mathcal{B}_{\infty,1}^{0}(X) \subset C(X) \subset \mathcal{B}_{\infty,\infty}^{0}(X),$$

where $C(X)$ *denotes the space of bounded continuous functions on* X.

Proof. The embedding (52) follows immediately from Theorem 1 and the monotonicity of the sequence spaces l_q. Now, (52) implies the embeddings $\mathcal{B}_{p,q_0}^{s_0}(X) \subset \mathcal{B}_{p,\infty}^{s_0}(X)$ and $\mathcal{B}_{p,1}^{s_1}(X) \subset \mathcal{B}_{p,q_1}^{s_1}(X)$. Thus it is sufficient to prove that $\mathcal{B}_{p,\infty}^{s_0}(X) \subset \mathcal{B}_{p,1}^{s_1}(X)$. But this is a simple consequence of the definition of the spaces provided that $s_0 > s_1$.

If we take the constants M and L in the definition of atoms sufficiently large then every s_0-atom is an s_1-atom and every (s_0, p)-atom is an (s_1, ∞)-atom if $s_1 = s_0 - n/p$. This implies (54). The proof of (55) is the same.

Both embeddings in (56) and the right embedding in (57) follow easily from the Calderón formula (5), the definition of the Besov spaces and (8). We prove the right embedding in (57). Let $f = \sum_i \lambda_i a_i + \sum_j \sum_i \lambda_{j,i}a_{j,i}$ be a decomposition of f into atoms. Since we deal with a uniformly locally finite sequence of coverings there is a constant C such that every point $x \in X$ is an element of at most C balls $\Omega(x_{j,i}, 2^{-j})$. Thus the series representing $f \in \mathcal{B}_{\infty,1}^0(X)$ is absolutely convergent in the sup-norm. In consequence, f is a continuous function and

$$|f(x)| \le \sum_{i\in I_x}|\lambda_i| + \Big(\sum_j\Big(\sum_{i\in I_{j,x}}|\lambda_{j,i}|\Big)^q\Big)^{1/q} \le C\sup_i|\lambda_i| + C\Big(\sum_j(\sup_i|\lambda_{j,i}|)^q\Big)^{1/q}.$$

The constant C is independent of the given decomposition, and therefore taking the supremum over x and then the infimum over all representations of f we get (57).

PROPOSITION 2. *If $s \geq 0$ and $1 \leq q \leq \infty$ then the space $\mathcal{B}^s_{1,q}(X)$ is a convolution algebra.*

Proof. Let ϑ, θ be functions satisfying the conditions (1)–(3) and moreover let $\operatorname{supp}\vartheta \subset \Omega(o,1/2)$ and $\operatorname{supp}\theta \subset \Omega(o,1/2)$. Then the function $k = \vartheta \star \theta$ satisfies the conditions (1)–(3) as well. Moreover, if $N = N_1 + N_2$ then

$$k_t^N = \vartheta_t^{N_1} \star \theta_t^{N_2}$$

because Γ is an invariant differential operator and the functions are bi-K-invariant [12]. Using these functions we get

$$\Big(\int_0^1 t^{-sq}\|f\star g\star k_t^N\|_1^q \frac{dt}{t}\Big)^{1/q} \leq C\Big(\int_0^1 t^{-sq}\|f\star\vartheta_t^{N_1}\|_1^q \frac{dt}{t}\Big)^{1/q} \sup_{0<t\leq 1}\|g\star\theta_t^{N_2}\|_1$$
$$\leq \|f \mid \mathcal{B}^s_{1,q}(X)\|\cdot\|g \mid \mathcal{B}^0_{1,\infty}(X)\|.$$

It remains to estimate the term $\|f\star g\star k_{N,0}\|_p$. If $s > 0$ then

$$\|f\star g\star k_{N,0}\|_1 \leq C\|f\|_1\cdot\|g\star k_{N,0}\|_1 \leq C\|f \mid \mathcal{B}^s_{1,q}(X)\|\cdot\|g \mid \mathcal{B}^0_{1,\infty}(X)\|.$$

Let $s = 0$ and $2L > N$. Then

$$\begin{aligned}
&\|f\star g\star k_{N,0}\|_1\\
&\quad\leq C\|f\star(I-\Delta)^L k_{N,0}\|_1\|(I-\Delta)^{-L}g\|_1\\
&\quad\leq C\|f \mid \mathcal{B}^0_{1,q}(X)\|\\
&\qquad\times\Big(\|g\star k_{N,0}\|_1 + \Big\|\int_0^1 g\star k_t^N\star(I-\Delta)^{-L}k_N^t\frac{dt}{t}\Big\|_1\Big)\\
&\quad\leq C\|f \mid \mathcal{B}^0_{1,q}(X)\|\\
&\qquad\times\Big(\|g\star k_{N,0}\|_1 + \sup_{0<t\leq 1}\|g\star k_t^N\|_1\int_0^1 t^{2N}\|(I-\Delta)^{-L}\Gamma^N k_t\|_1\frac{dt}{t}\Big)\\
&\quad\leq C\|f \mid \mathcal{B}^0_{1,q}(X)\|\cdot\|g \mid \mathcal{B}^0_{1,\infty}(X)\|.
\end{aligned}$$

This finishes the proof of the proposition.

We also have the following Fourier embedding theorems of Bernstein type.

PROPOSITION 3. *Let $1 \leq q \leq 2$. Then the Helgason–Fourier transform $\mathcal{H}$ maps $\mathcal{B}^{n(1/q-1/2)}_{2,q}(X)$ continuously into $L^q(\mathfrak{a}^\star\times B, |c(\lambda)|^{-2}d\lambda db)$.*

Proof. For $p = 2$ and $q = 2$ the theorem is obvious since $\mathcal{B}^0_{2,2}(X) = L_2(X)$ (see [16]). If $p = 2$ and $q = 1$ then the Calderón formula implies

$$\begin{aligned}
\|\mathcal{H}f\|_1 &\leq \|\mathcal{H}f\mathcal{H}k_{0,N}\|_1 + \int_0^1\|\mathcal{H}f\mathcal{H}k_t^N\mathcal{H}k_t^N\|_1\frac{dt}{t}\\
&\leq \|\mathcal{H}f\|_2\|\mathcal{H}k_{0,N}\|_2 + \int_0^1\|\mathcal{H}f\mathcal{H}k_t^N\|_2\|\mathcal{H}k_t^N\|_2\frac{dt}{t}\\
&\leq \|f\|_2\|k_{0,N}\|_2 + \int_0^1 t^{-n/2}\|f\star k_t^N\|_2\frac{dt}{t}\sup_{t\in(0,1)}t^{n/2}\|k_t^N\|_2\\
&\leq C\Big(\|f\|_2 + \int_0^1 t^{-n/2}\|f\star k_t^N\|_2\frac{dt}{t}\Big),
\end{aligned}$$

where the last inequality follows from (8). For $1 < q < 2$ the theorem follows now by real interpolation.

To formulate our main result of Bernstein type we introduce the following notation. Let $\mathcal{S}_1$ be the convex hull of the set $\{w(\varrho) : w \in W\}$. We put $\mathcal{T}_1 = \mathfrak{a}^\star + \sqrt{-1}\,\mathcal{S}_1$. The set $\mathcal{T}_1$ is the tube in $\mathfrak{a}^\star_{\mathbb{C}}$ with base $\mathcal{S}_1$ around $\mathfrak{a}^\star$.

THEOREM 2. *Let $\mathcal{T}_1 = \mathfrak{a}^\star + \sqrt{-1}\,\mathcal{S}_1$ be the tube defined above. Then for any $f \in \mathcal{B}^n_{1,1}(X)$ and any $b \in B$ the Helgason–Fourier transform $\mathcal{H}f(\cdot,b)$ can be extended to a measurable function in $\mathcal{T}_1$. Moreover, there is a positive constant C such that*

$$\int_{\mathfrak{a}^\star\times B}|\mathcal{H}f(\lambda+\sqrt{-1}\,\eta,b)|\cdot|c(\lambda)|^{-2}\,d\lambda\,db \leq C\|f \mid \mathcal{B}^n_{1,1}(X)\| \tag{58}$$

for every $\eta \in \mathcal{S}_1$.

Proof. To prove the theorem we use the atomic decomposition.

Step 1. According to Theorem 1 the constants L and M satisfying (11) are at our disposal. We take $M = -1$. The constant L will be described later on.

Let a be an s-atom centered in $\Omega(o,1)$. Then the Eguchi Theorem (cf. [4, Lemma 4.1.1]), Sobolev embedding theorem for Riemannian manifolds and (12) and (13) imply

$$\begin{aligned}
\sup_{(\lambda,b)\in\mathcal{T}\times B}(1+|\lambda|)^m|\mathcal{H}a(\lambda,b)| &\leq C\sup_{x\in\Omega(o,1)}|f(D_1 : x : D_2)|\\
&\leq C\sum_{j=0}^L\|\Gamma^j a\|_1 \leq C,
\end{aligned}\tag{59}$$

where $D_1, D_2 \in U(\mathfrak{g})$ and C is the constant depending on L but independent of a.

In the same manner we can see that the inequality

$$\sup_{(\lambda,b)\in\mathcal{T}\times B}(1+|\lambda|)^m|\mathcal{H}a(\lambda,b)|\le C \tag{60}$$

holds for any $(s,1)$-atom a centered in $\Omega(o,1)$ (cf. (14)–(16)).

Now, let a_j be an $(s,1)$-atom centered in $\Omega(o,2^{-j})$. For simplicity of notation we introduce the operator δ_t acting on continuous functions on $\mathfrak{a}$ by $\delta_t f(H)=f(tH)$. We put

$$a=2^{j(s-\alpha)}\mathcal{R}^{-1}\circ\delta_{2^{-j}}\circ\mathcal{R}(a_j).$$

The support conservation property implies $\operatorname{supp} a\subset\Omega(o,2)$. On the other hand, (34) and (36) imply

$$\sup_{x\in X}|(\Gamma^m a)(x)|\le C\qquad\text{for any } m\le L',$$

where $L'=L-[n/2]-1$ and the constant C is independent of j and a_j. Thus a is an (s,p)-atom centered in $\Omega(o,1)$ after multiplication by a positive constant independent of j and a_j. Thus

$$|\mathcal{H}a_j(\lambda,b)|\le C2^{-j(s-\alpha)}2^{-j\alpha}|\mathcal{H}a(2^{-j}\lambda,b)|\le C2^{-js}(1+|2^{-j}\lambda|)^{-m}.$$

In consequence,

$$\sup_{b\in B}|\mathcal{H}a_j(\lambda,b)|\le C2^{-js}(1+|2^{-j}\lambda|)^{-m} \tag{61}$$

and the constant C is independent of a_j and λ.

Step 2. If $a_{j,i}$ is an $(s,1)$-atom centered in $\Omega(x_{j,i},2_{-j})$, then $a(x)=a_{j,i}(g_{j,i}x)$, $x_{j,i}=g_{j,i}K$, is an $(s,1)$-atom centered in $\Omega(o,2^{-j})$ and

$$\mathcal{H}(a_{j,i})(\mu,b)=e^{(-\sqrt{-1}\mu+\varrho)A(g_{j,i}\cdot o,b)}\mathcal{H}(a)(\lambda,g_{j,i}^{-1}(b)),\quad \mu\in\mathcal{T}. \tag{62}$$

Let $\mu=\lambda+\sqrt{-1}\,\eta$, $\eta\in\mathcal{S}$. Then

$$\int_B|\mathcal{H}a_{j,i}(\mu,b)|\,db\le C\int_K e^{-(\eta+\varrho)H(g_{j,i}k)}\,dk\sup_{b\in B}|\mathcal{H}(a)(\mu,b)|.$$

The integral on the right hand side is equal to the value $\varphi_{-\sqrt{-1}\eta}(x_{j,i})$ of the spherical function $\varphi_{-\sqrt{-1}\eta}$ at $x_{j,i}$.

But the spherical functions $\varphi_{-\sqrt{-1}\eta}$ are bounded for $\eta\in\mathcal{S}_1$ (cf. [12, Theorem IV.8.1]). In consequence,

$$\int_B|\mathcal{H}a_{j,i}(\lambda+\sqrt{-1}\,\eta,b)|\,db\le C2^{-js}(1+|2^{-j}\lambda|)^{-m}. \tag{63}$$

In the same way we can prove that the inequality

$$\int_B|\mathcal{H}a_i(\lambda+\sqrt{-1}\,\eta,b)|\,db\le C(1+|\lambda|)^{-m} \tag{64}$$

holds for any s-atom a_i.

Step 3. Let $f\in\mathcal{B}^n_{1,1}(X)$ and $f=\sum_{i\in\mathbb{N}}s_ia_i+\sum_{j=0}^\infty\sum_{i\in\mathbb{N}}s_{j,i}a_{j,i}$ be the atomic decomposition of f. Then

$$\int_{\mathfrak{a}^*\times B}\Big|\mathcal{H}\Big(\sum_{i\in\mathbb{N}}s_ia_i(\lambda+\sqrt{-1}\,\eta,b)\Big)\Big|\cdot|c(\lambda)|^{-2}\,d\lambda\,db$$
$$\le C\sum_{i\in\mathbb{N}}|s_i|\int_{\mathfrak{a}^*}(1+|\lambda|)^{-m}|c(\lambda)|^{-2}\,d\lambda\le C\sum_{i\in\mathbb{N}}|s_i|.$$

In a similar way

$$\sum_{j=0}^\infty\sum_{i\in\mathbb{N}}|s_{j,i}|\int_{\mathfrak{a}^*}\int_B|\mathcal{H}a_{j,i}(\lambda+\sqrt{-1}\,\eta,b)|\,db|c(\lambda)|^{-2}\,d\lambda$$
$$\le C\sum_{j=0}^\infty\Big(\sum_{i\in\mathbb{N}}|s_{j,i}|\Big)\int_{\mathfrak{a}^*}(1+|2^{-j}\lambda|)|c(\lambda)|^{-2}\,d\lambda$$
$$\le C\sum_{j=0}^\infty\sum_{i\in\mathbb{N}}|s_{j,i}|\sup_j 2^{-js}\sum_{i\in\mathbb{N}}2^{jn}\le C\sum_{j=0}^\infty\sum_{i\in\mathbb{N}}|s_{j,i}|$$

if $s=n$. This finishes the proof.

Remark. One can also consider the Fourier image of $\mathcal{B}^s_{1,1}(X)$ under the action of the spherical transform

$$\widetilde{f}(\lambda)=\int_X\varphi_{-\lambda}(x)f(x)\,dx,\qquad\lambda\in\mathfrak{a}^\star.$$

The last theorem is true for the spherical transform since we have $\widetilde{f}(\lambda)=\int_B\mathcal{H}f(\lambda,b)\,db$ if $f\in\mathcal{C}_1$. Moreover, using the standard arguments with boundedness of the spherical functions φ_λ with $\lambda\in\mathcal{T}_1$, one can prove that the function $\widetilde{f}$ is holomorphic inside $\mathcal{T}_1$ if $f\in\mathcal{B}^n_{1,1}(X)$.

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Received January 26, 1995 (3411)
Revised version December 12, 1996

STUDIA MATHEMATICA 124 (3) (1997)

Sufficient conditions of optimality for multiobjective optimization problems with γ-paraconvex data

by

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Abstract. We study multiobjective optimization problems with γ-paraconvex multifunction data. Sufficient optimality conditions for unconstrained and constrained problems are given in terms of contingent derivatives.

1. Introduction. Many authors have studied multiobjective optimization problems in terms of some tangent derivative notions. Corley [4] has given optimality conditions for convex and nonconvex multiobjective problems in terms of the Clarke derivative. Luc [6] also gives optimality conditions when the data are upper semidifferentiable. Luc and Malivert [7] extend the concept of invex functions to invex multifunctions and study optimality conditions for multiobjective optimization with invex data in terms of contingent derivatives. Taa [12] gives optimality conditions with no assumption on the data but with the Shi derivative which is an enlarged version of contingent derivative.

In this paper we establish sufficient optimality conditions in terms of the contingent derivative for unconstrained and constrained multiobjective optimization problems when the data are γ-paraconvex or compactly γ-paraconvex with $\gamma > 1$. It is shown that the γ-paraconvexity data considerably simplify the assumptions in the optimality conditions. The notion of γ-paraconvex multifunctions has been introduced by Rolewicz [10] and openness and metric regularity of such multifunctions are studied in Jourani [5] (see also Allali and Amahroq [1] for another proof).

2. Preliminaries. Let X and Y be two Banach spaces and let F be a multifunction from X into Y. In the sequel we denote the effective domain

1991 *Mathematics Subject Classification*: Primary 90C29; Secondary 49K30.
Key words and phrases: contingent derivative, γ-paraconvex multifunction, optimality conditions, B-tangentially compact, compactly γ-paraconvex multifunction, Pareto minimal point.