[2] J. P. Conze, *Entropie d'un groupe abélien de transformations*, Z. Wahrsch. Verw. Gebiete 25 (1972), 11–30.
[3] S. Goldstein and O. Penrose, *A non-equilibrium entropy for dynamical systems*, J. Statist. Phys. 24 (1981), 325–343.
[4] S. A. Kalikow, *T, T^{-1} transformation is not loosely Bernoulli*, Ann. of Math. 115 (1982), 393–409.
[5] B. Kamiński, *Mixing properties of two-dimensional dynamical systems with completely positive entropy*, Bull. Acad. Polon. Sci. Sér. Sci. Math. 28 (1980), 453–463.
[6] —, *The theory of invariant partitions for $\mathbb{Z}^d$-actions*, ibid. 29 (1981), 349–362.
[7] —, *A representation theorem for perfect partitions for $\mathbb{Z}^2$-actions with finite entropy*, Colloq. Math. 56 (1988), 121–127.
[8] —, *Decreasing nets of σ-algebras and their applications to ergodic theory*, Tôhoku Math. J. 43 (1991), 263–274.
[9] Z. S. Kowalski, *A generalized skew product*, Studia Math. 87 (1987), 215–222.
[10] W. Krieger, *On generators in exhaustive σ-algebras of ergodic measure-preserving transformations*, Z. Wahrsch. Verw. Gebiete 20 (1971), 75–82.
[11] I. Meilijson, *Mixing properties of a class of skew-products*, Israel J. Math. 19 (1974), 266–270.
[12] V. A. Rokhlin, *Lectures on the entropy theory of measure-preserving transformations*, Uspekhi Mat. Nauk 22 (5) (1967), 3–56 (in Russian).
[13] T. Shimano, *An invariant of systems in the ergodic theory*, Tôhoku Math. J. 30 (1978), 337–350.
[14] —, *The multiplicity of helices for a regularly increasing sequence of σ-fields*, ibid. 36 (1984), 141–148.
[15] —, *On helices for Kolmogorov system with two indices*, Math. J. Toyama Univ. 14 (1991), 213–226.
[16] P. Walters, *Some results on the classification of non-invertible measure preserving transformations*, in: Lecture Notes in Math. 318, Springer, 1973, 266–276.

Faculty of Mathematics and Informatics
Nicholas Copernicus University
Chopina 12/18
87-100 Toruń, Poland
E-mail: bkam@mat.uni.torun.pl

Institute of Mathematics
Wrocław Technical University
Wybrzeże Wyspiańskiego 27
50-370 Wrocław, Poland
E-mail: kowalski@banach.im.pwr.wroc.pl

Université de Provence
Centre de Mathématiques et d'Informatique
39, Rue Joliot Curie
F-13453 Marseille Cedex 13, France
E-mail: liardet@gyptis.univ-mrs.fr

Received August 27, 1996
Revised version December 6, 1996 (3726)

STUDIA MATHEMATICA 124 (2) (1997)

Approximation on the sphere by Besov analytic functions

by

EVGUENI DOUBTSOV (Barcelona)

Abstract. Boundary values of zero-smooth Besov analytic functions in the unit ball of $\mathbb{C}^n$ are investigated. Bounded Besov functions with prescribed lower semicontinuous modulus are constructed. Correction theorems for continuous Besov functions are proved. An approximation problem on great circles is studied.

1. Introduction. Let $\mathbb{C}^n$ be the n-dimensional complex space (usually $n \geq 2$) with the unit ball $B = B_n = \{|z| < 1\}$ and the unit sphere $S = S_n = \partial B$. By ν and σ we denote the normalized Lebesgue measures on B and S respectively. In dimension one we use the notation $\mathbb{D} = B_1$, $\mathbb{T} = \partial\mathbb{D}$. $C(S)$ is the space of all continuous functions on S, the symbol LSC stands for lower semicontinuous functions, and $H(B)$ is the space of all analytic functions $f : B \to \mathbb{C}$. Finally, $A(B) = H(B) \cap C(\overline{B})$ is the ball algebra and $H^\infty(B) := \{f \in H(B) : f \text{ is bounded}\}$.

In the present paper we investigate boundary values of some Besov analytic functions. More precisely, given $0 < p < \infty$ and $q > 0$, define

$$\|f\|^p_{A_{pq}(B)} = \int_B |f(z)|^p (1 - |z|)^{q-1}\, d\nu(z),$$

$$A^1_{pq}(B) = \Big\{ f \in H(B) : \|f\|_{A_{pq}(B)} + \sum_{j=1}^n \|\partial_j f\|_{A_{pq}(B)} < \infty \Big\},$$

where $\partial_j = \partial/\partial z_j$. (From a different point of view, $A^1_{pq}(B)$ is a weighted Sobolev space of analytic functions.) To avoid technicalities, we do not consider the spaces $A^m_{pq}(B)$, $m > 1$, defined in terms of higher derivatives.

To justify the term *Besov* space, consider the particular (unweighted) case $q = 1$. For $p > 1$, the Besov space $B^{1-1/p}_{pp}(S)$ on the sphere is defined

1991 *Mathematics Subject Classification*: Primary 32A40; Secondary 32A35, 32E25.
Supported by the Centre de Recerca Matemàtica, Institut d'Estudis Catalans (Barcelona) under a grant from DGICYT (Spain).

as

$$B_{pp}^{1-1/p}(S) = \left\{ f \in L^p(S) : \int_S\int_S \frac{|f(z)-f(w)|^p}{|z-w|^{2n+p-2}}\, d\sigma(z)\, d\sigma(w) < \infty \right\}.$$

By the corresponding classical trace theorem (this result holds in the general setting of Sobolev spaces when f is not assumed to be analytic, see for example [N]), if $f \in A_{p1}^1(B)$, then $\operatorname{trace}(f) \in B_{pp}^{1-1/p}(S)$.

Note also that the spaces under consideration have an equivalent description in terms of radial derivatives. Indeed, define

$$\mathcal{R} = \sum_{j=1}^{n} z_j \partial_j \quad \text{and} \quad \widetilde{A}_{pq}^1(B) = \{ f \in H(B) : \|f\|_{A_{pq}} + \|\mathcal{R} f\|_{A_{pq}} < \infty \}.$$

Then $A_{pq}^1(B) = \widetilde{A}_{pq}^1(B)$ for all $0 < p < \infty$ and $q > 0$ (see [BB], Theorem 5.3).

Our objects of investigation are the Besov spaces with zero smoothness $A_p^1(B) := A_{pp}^1(B)$. More precisely, we consider $H^\infty(B) \cap A_p^1(B)$ and $A(B) \cap A_p^1(B)$ when $0 < p < 2$. Note that $H^\infty(B) \cap A_p^1(B) \subset H^\infty(B) \cap A_q^1(B)$ if $0 < p < q$. Indeed, suppose that $f \in H^\infty(B)$. Then by the Cauchy inequality $|\partial_j f(z)|(1-|z|) \le C$, $z \in B$, $1 \le j \le n$, therefore

$$\int_B |\partial_j f(z)|^{(q-p)+p} (1-|z|)^{(q-p)+p-1}\, d\nu(z) \le C \int_B \frac{|\partial_j f(z)|^p}{(1-|z|)^{1-p}}\, d\nu(z).$$

Note also that the intersections under consideration are not trivial since $A(B) \setminus A_p^1(B) \ne \emptyset$ for all $0 < p < 2$. An explicit example of a function $f \in A(B) \setminus A_1^1(B)$ is given in [R3], Theorem 17.9 (moreover, f maps almost every radius into a curve of infinite length). This example has a generalization for all $p < 2$.

On the other hand, if $p \ge 2$, then $H^\infty(B) \cap A_p^1(B) = H^\infty(B)$ because in this case $A_p^1(B)$ contains even the Hardy space $H^p(B)$; moreover, $A_2^1(B) = H^2(B)$.

We are going to prove, in particular, the following results (g^* stands for the boundary values of g, as usual).

THEOREM A. *Let $0 < p < 2$ and $\varphi \in \mathrm{LSC}(\overline{B}) \cap L^\infty(S)$, $\varphi > 0$. Then there exists a function $g \in H^\infty(B) \cap A_p^1(B)$ such that $|g| \le \varphi$ on B and $|g^*| = \varphi$ σ-a.e.*

THEOREM B. *Let $0 < p < 2$. Suppose that $\varphi \in C(\overline{B})$, $\varphi > 0$, and $\varepsilon > 0$. Then there exists a function $g \in A(B) \cap A_p^1(B)$ such that $|g| \le \varphi$ on $\overline{B}$ and $\sigma\{|g| = \varphi\} > 1 - \varepsilon$.*

In the final section of the paper we obtain a theorem of type B on approximation on the great circles $\mathbb{T}_\zeta = \{\lambda\zeta : \lambda \in \mathbb{T}\}$, $\zeta \in S$.

For the "pure" spaces $H^\infty(B)$ and $A(B)$ the above theorems were obtained in [A1] and [A2] respectively. If $p = 1$, then Theorem A (with $\varphi \in C(S)$, $\varphi > 0$) was proved in [Du1] (recall that the theorems are interesting for small $p > 0$). Theorem A in dimension one (again $\varphi \in C(\mathbb{T})$, $\varphi > 0$) was obtained in [Do]. Note that this result has an interpretation in terms of inner-outer factorization (in the sense of Beurling, see [Do] for details).

The base of the proofs, as in [Du1] and [Do], is the approximation construction of A. B. Aleksandrov in $L^p(S)$, $0 < p < 1$ (see [A1], and also [R2] for an exposition of this construction). The point is the possibility to keep estimates of the A_p^1-norm in the induction construction.

Comments. A1. It is necessary to explain why we do not consider $H^\infty(B) \cap A_{pq}^1(B)$ with $p \ne q$. First, let $q > p$. Then by the Cauchy inequality $H^\infty(B) \subset A_{pq}^1(B)$, so this case is degenerate.

On the other hand, if $q < p$, then the theorem is not valid for $A_{pq}^1(B)$. Indeed, suppose, without loss of generality, that $p > q > p - 1$ and $q \ge 1$. Let $f \in A_{pq}^1(B)$. Then the trace theorem yields

$$\int_S\int_S \frac{|f(z)-f(w)|^p}{|z-w|^{2n+p-q-1}}\, d\sigma(z)\, d\sigma(w) < \infty.$$

In other words, we have a restriction on the smoothness of f. In particular, there exists $\varphi \in C(\overline{B})$, $\varphi > 0$, such that the above integral diverges with φ in place of f.

A2. Theorem A is closely related to the following problem (see [R3], 19.16). Given $1 \le t \le 2$, is there an inner function f (i.e. $f \in H^\infty(B_n)$, $n \ge 2$, $|f^*| = 1$ σ-a.e. and f is not constant) such that $\operatorname{grad}(f) \in L^t(B)$? (Note that $\varphi \equiv 1$ is a very smooth function, so the argument from comment A1 is not applicable.)

The result by Y. Dupain [Du1] gives a positive answer for $t = 1$. Moreover, if $0 < p < 2$, then Theorem A shows that there exist inner functions f in B such that $\operatorname{grad}(f) \in A_p(B)$.

On the other hand, A. B. Aleksandrov observed that there are no inner functions with gradients in $L^{3/2}(B)$. To show this, we need the following result (here $f_r(\zeta) = f(r\zeta)$, $0 \le r < 1$, $\zeta \in S$).

THEOREM (M. Tamm [Ta], see also [A3], Chapter 5, 4.4). *Let $f \in H^\infty(B_n)$, $n \ge 2$, $0 < t < \infty$ and*

$$\|f_r - f^*\|_{L^t(S)}^t = o(1-r)^{1/2}, \qquad r \to 1-.$$

Then the essential range of f^* coincides with $\overline{f(B)}$. In particular, if $|f^*| = 1$ σ-a.e., then f is constant.

Now, let $\mathrm{grad}(f) \in L^t(B)$. Then the Hölder inequality gives

$$\|f_r - f^*\|^t_{L^t(S)} \le \int_S \Big(\int_r^1 |\mathrm{grad}(f)(x\zeta)|\,dx\Big)^t d\sigma(\zeta)$$

$$\le (1-r)^{t/t'} \int_S \int_r^1 |\mathrm{grad}(f)(x\zeta)|^t\,dx\,d\sigma(\zeta).$$

Therefore, if $t = 3/2$, then $\|f_r - f^*\|^t_{L^t(S)} = o(1-r)^{1/2}$, $r \to 1-$, so we apply the above theorem.

B1. Note that $\varphi \in C(\overline{B})$ is *arbitrary* in Theorem B. If the modulus φ is supposed to be smooth, then a stronger result holds.

THEOREM (B. Tomaszewski [To], Corollary 1). *Let $n \ge 2$. For every $\varepsilon > 0$ there exists $\alpha = \alpha(\varepsilon, n) > 0$ such that for every function $\varphi \in \mathrm{Lip}_1(S_n)$, $\varphi > 0$, there exist nonconstant functions $g \in A(B_n) \cap \mathrm{Lip}_\alpha$ such that $|g(\zeta)| \le \varphi(\zeta)$ for $\zeta \in S_n$, and*

$$\sigma\{\zeta \in S_n : |g(\zeta)| = \varphi(\zeta)\} > 1 - \varepsilon.$$

2. Auxiliary results. Given $\zeta, \eta \in S$, put $d(\zeta,\eta) = |1 - \langle \zeta, \eta\rangle|^{1/2}$ and $E_\eta(\delta) = \{\zeta \in S : d(\eta,\zeta) < \delta\}$, $0 < \delta \le \sqrt{2}$. Note that d defines a (nonisotropic) metric on S and the sets $E_\eta(\delta)$ are balls in this metric. Define also $V(\delta) = \sigma(E_\eta(\delta))$.

LEMMA 2.1 ([R1], 5.1.4). *Let $\eta \in S = S_n$ and $0 < \delta \le \Delta < 1$. Then*

$$V(\Delta)/V(\delta) \le (\Delta/\delta)^{2n} \quad \text{and} \quad 2^{-n}\delta^{2n} \le V(\delta) \le 2^n \delta^{2n}.$$

LEMMA 2.2 (see, for instance, [R1], 1.4.10). *Let $w \in B$, $a > 0$, $b > -1$. Then*

$$\int_B \frac{(1-|z|)^b\,d\nu(z)}{|1 - \langle z, w\rangle|^{n+1+a+b}} \le \frac{\mathrm{const}(a,b)}{(1-|w|)^a}.$$

LEMMA 2.3. *Let $n \ge 2$, $0 < p < 1$, $q \in \mathbb{N}$, $pq \ge n+1$, and $\eta \in S$. Define*

$$E(\Delta) = E_\eta(\Delta), \quad \Delta \in (0,1),$$

$$h(t,z) = \frac{i}{(2 + t - t\langle z, \eta\rangle)^q}, \quad t \ge 2,\ z \in \overline{B}.$$

Then there exist $M_0 = M_0(p,q) \ge 8$ and $\alpha = \alpha(p,q,M_0) \in (0,1)$ such that

$$\|\mathbf{1}_{E(\Delta)}(\cdot) - \mathrm{Re}\, h(M_0\Delta^{-2}, \cdot)\|^p_{L^p(S)} < \alpha V(\Delta), \tag{2.1}$$

$$\Big\|\frac{\partial h(M_0\Delta^{-2}, z)}{\partial z_j}\Big\|^p_{A_p(B)} < V(\Delta), \quad 1 \le j \le n, \tag{2.2}$$

for all $\Delta \in (0,1)$.

Proof. 1. By Lemma 2 of [A1] (see also [R2], Lemma 3.7), there exists $M_1 \ge 2$ such that (2.1) holds if $M_0 \ge M_1$.

2. Let $M \ge 2$ and $\Delta \in (0,1)$. Then put $t := M\Delta^{-2} \ge 2$. Since

$$\Big|\frac{\partial h(t,z)}{\partial z_j}\Big| \le \frac{tq}{|2 + t - t\langle \zeta, \eta\rangle|^{q+1}},$$

Lemma 2.2, with $a = pq - n > 0$ and $b = p - 1 > -1$, and Lemma 2.1 provide

$$\Big\|\frac{\partial h(t,z)}{\partial z_j}\Big\|^p_{A_p(B)} = \int_B \frac{t^p q^p (t+2)^{-pq-p}(1-|z|)^{p-1}}{|1 - t(t+2)^{-1}\langle z, \eta\rangle|^{pq+p}}\,d\nu(z)$$

$$\le C(p,q)\frac{(t+2)^{-pq}}{(1 - t(t+2)^{-1})^{pq-n}} \le C(p,q)\,t^{-n} \le C_2 V(\Delta) M^{-n}.$$

Now, if $M_2^{-n} C_2 < 1$ and $M_0 \ge M_2$, then (2.2) holds. To finish the argument, put $M_0 = \max\{8, M_1, M_2\}$. ■

3. Approximation in L^p, $0 < p < 1$. First, we approximate the characteristic functions of the sets $E_\zeta(r) \subset S$.

Let $E \subset S$, $R \in [0,1]$, and define $\Lambda(E,R) = \{r\zeta : \zeta \in E,\ R \le r \le 1\}$ (the truncated E-cone).

LEMMA 3.1. *For $p \in (0,1)$, there exists a constant $\beta = \beta(p) \in (0,1)$ with the following property: Suppose that $E = E_\zeta(r) \subset S$ ($\zeta \in S$, $r \in (0,1)$), $\varkappa \in (0,1)$ and $R \in [0,1)$. Then there is an $f \in A(B)$ such that*

$$|f| < 1 \text{ on } \overline{B} \quad \text{and} \quad |f| < \varkappa \text{ on } \overline{B} \setminus \Lambda(E,R), \tag{3.1}$$

$$\|\mathbf{1}_E - \mathrm{Re}\, f\|^p_{L^p(S)} < \beta\sigma(E), \tag{3.2}$$

$$\|\partial_j f\|^p_{A_p(B)} < \sigma(E), \quad 1 \le j \le n. \tag{3.3}$$

Remark. It is useful to imagine that $\sigma(E)$ and $\varkappa$ are small and R is close to 1.

Proof. Put $\delta = \varkappa \min\{r, (1-R)\}$ and fix $q = q(p) \in \mathbb{N}$ such that $pq \ge 2n+1$ (in particular, $q \ge 2n+1$).

Let $\{\zeta_k\}_{k=1}^N \subset E_\zeta(r/2)$ be maximal with respect to having the sets $E_{\zeta_k}(\delta)$ pairwise disjoint. Note that $NV(\delta) \le V(r)$, so $N\delta^{2n} \le r^{2n}$ by Lemma 2.1.

Let $M_0 \ge 8$ be the constant provided by Lemma 2.3. We are going to check that the function

$$f(z) = \sum_{k=1}^N h_k(z) := \sum_{k=1}^N \frac{i}{(2 + M_0\delta^{-2}(1 - \langle z, \zeta_k\rangle))^q}$$

satisfies the conditions (3.1)–(3.3).

1. Let $\xi \in E$. First, there exists at most one point ζ_k such that $d(\xi,\zeta_k) < \delta$. Second, given $m \in \mathbb{N}$, put $H_m = \{\zeta_k : 2^{m-1}\delta \le d(\xi,\zeta_k) < 2^m\delta\}$. If $\zeta_k \in H_m$, then $E_{\zeta_k}(\delta) \subset E_\xi(4^m\delta)$; therefore, the cardinality of H_m is estimated by $V(4^m\delta)/V(\delta) \le 4^{2mn}$ (see Lemma 2.1).

On the other hand, if $d(\xi,\zeta_k) \ge 2^{m-1}\delta$, then

$$|h_k(\xi)| \le (M_0 2^{2m-2})^{-q} \le 4^{-mq}.$$

Thus

$$|f(\xi)| \le 2^{-q} + \sum_{m=1}^{\infty} (4^{2n-q})^m < 1.$$

Now, let $\xi \in S \setminus E$. Then $d(\xi,\zeta_k) \ge r/2$ for all $1 \le k \le N$. Note that $|1-\varrho\lambda| \ge \varrho|1-\lambda|$ if $\varrho \in [0,1]$ and $\lambda \in \mathbb{D}$. Therefore $|1-\langle \varrho\xi,\zeta_k\rangle| \ge r^2/8$ for all $\varrho \in [0,1]$ and $\xi \in S \setminus E$. Since $M_0 \ge 8$ and $N\delta^{2n} \le r^{2n}$, we obtain

$$|f(\varrho\xi)| \le \sum_{k=1}^{N} |h_k(\varrho\xi)| \le N(\delta/r)^{2q} \le \varkappa^{2q-2n} < \varkappa.$$

To finish the proof of (3.1), it is sufficient to estimate $|f(z)|$ for $|z| < R$. In this case $|1-\langle z,\zeta_k\rangle| > 1-R$ for all $1 \le k \le N$. Thus, as above,

$$|f(z)| \le N\delta^{2q}(1-R)^{-q} \le N\delta^{2n}\varkappa^q < \varkappa.$$

2. Since the union of the $E_{\zeta_k}(2\delta)$ covers $E_\zeta(r/2)$, we have $NV(2\delta) \ge V(r/2)$, so Lemma 2.1 provides $2^{4n}NV(\delta) \ge V(r) = \sigma(E)$. The triangle inequality for p-norms, $0<p<1$, and the estimate (2.1), with $\Delta = \delta$, give

$$\begin{aligned}\|\mathbf{1}_E - \operatorname{Re} f\|^p_{L^p(S)} &\le \sigma(E) - NV(\delta) + \sum_{k=1}^{N} \|\mathbf{1}_{E_{\zeta_k}(\delta)} - \operatorname{Re} h_k\|^p_{L^p(S)}\\ &< \sigma(E) - NV(\delta)(1-\alpha(p)) \le \beta(p)\sigma(E),\end{aligned}$$

where $\beta(p) = 1 - 2^{-4n}(1-\alpha(p)) < 1$.

3. By (2.2), we have

$$\|\partial_j f\|^p_{A_p(B)} \le \sum_{k=1}^{N} \|\partial_j h_k\|^p_{A_p(B)} < NV(\delta) < \sigma(E).$$

The proof is complete. ■

Remark. We suppose in Lemma 2.3 that $n \ge 2$. On the other hand, an analogue of this result in dimension one does hold (a similar statement is Lemma 2.2 of [Do]), so Lemma 3.1 (together with all results below) holds for all dimensions n.

Now we are able to approximate LSC functions.

LEMMA 3.2. *For $0<p<1$, there exists a constant $\gamma = \gamma(p) \in (0,1)$ with the following property: Suppose that $\psi \in \mathrm{LSC}(\overline{B}) \cap L^p(S)$, $\psi > 0$, and $\varepsilon > 0$. Then there is a function $F \in A(B)$ such that*

$$|F| < \psi \quad \text{on } \overline{B}, \tag{3.4}$$

$$\|\psi - \operatorname{Re} F\|^p_{L^p(S)} < \gamma\|\psi\|^p_{L^p(S)}, \tag{3.5}$$

$$\|\partial_j F\|^p_{A_p(B)} < \|\psi\|^p_{L^p(S)}, \quad 1 \le j \le n. \tag{3.6}$$

Proof. Since ψ is lower semicontinuous, there is a $\psi_1 \in C(\overline{B})$ such that $\psi \ge \psi_1 > 0$ and ψ_1 approximates ψ in $L^p(S)$. Therefore, we assume that $\psi \in C(\overline{B})$.

Construct a finite set of (nonisotropic) disjoint balls $E_m \subset S$ such that a linear combination of their characteristic functions $\sum_{m=1}^{M} c_m \mathbf{1}_{E_m}$, $c_m > 0$, approximates ψ from below in $L^p(S)$. More precisely, define $\Lambda_m = \Lambda(E_m, R)$ (the truncated E_m-cones) and $h = \sum_{m=1}^{M} c_m \mathbf{1}_{\Lambda_m}$ such that $\psi - h \ge 2\delta > 0$ on S (for some δ) and

$$2\|\psi - h\|^p_{L^p(S)} < (1-\beta(p))\|\psi\|^p_{L^p(S)}. \tag{3.7}$$

Now take (and fix) $R<1$ so close to 1 that

$$\psi - h \ge \delta \quad \text{on } \overline{B}. \tag{3.8}$$

Put $c_0 = \max\{1, c_m : 1 \le m \le M\}$ and $\varkappa = \delta/(2c_0M)$. Given E_m, $\varkappa > 0$, and $R \in (0,1)$, Lemma 3.1 provides f_m.

Define $\gamma = (1+\beta)/2$. We claim that the function $F := \sum_{m=1}^{M} c_m f_m$ is as required.

Since the sets Λ_m are mutually disjoint and $c_0 M\varkappa = \delta/2$, the properties (3.1) and (3.8) provide $\psi \ge |F| + \delta/2$ on $\overline{B}$, so we have (3.4).

Recall that $0<p<1$, so (3.2) and (3.7) imply (3.5). Indeed,

$$\begin{aligned}\|\psi - \operatorname{Re} F\|^p_{L^p(S)} &\le \|\psi - h\|^p_{L^p(S)} + \sum_{m=1}^{M} c_m^p \|\mathbf{1}_{E_m} - \operatorname{Re} f_m\|^p_{L^p(S)}\\ &< \tfrac{1}{2}(1-\beta)\|\psi\|^p_{L^p(S)} + \beta\|h\|^p_{L^p(S)} \le \gamma\|\psi\|^p_{L^p(S)}.\end{aligned}$$

Finally, (3.3) yields

$$\|\partial_j F\|^p_{A_p(B)} \le \sum_{m=1}^{M} c_m^p \|\partial_j f_m\|^p_{A_p(B)} \le \sum_{m=1}^{M} c_m^p \sigma(E_m) < \|\psi\|^p_{L^p(S)},$$

so (3.6) holds. ■

4. Bounded Besov analytic functions. In this section we use Lemma 3.2 to construct a function from $H^\infty(B) \cap A^1_p(B)$ with a prescribed strictly positive LSC modulus.

THEOREM 4.1. *Let $0 < p < 2$ and $\varphi \in \mathrm{LSC}(\overline{B}) \cap L^\infty(S)$, $\varphi > 0$. Then there exists a function $g \in H^\infty(B) \cap A_p^1(B)$ such that $|g| \le \varphi$ on B and $|g^*| = \varphi$ σ-a.e.*

Proof. Define $\psi_0 = \log \varphi$. Without loss of generality, we suppose that $\psi_0 > 0$ and $0 < p < 1$.

Suppose, as induction hypothesis, that $m \in \mathbb{N}$, $\{F_k\}_{k=1}^m \subset A(B)$ and

$$\mathrm{Re}\Big(\sum_{k=1}^m F_k\Big) < \psi_0 \quad \text{on } \overline{B}, \tag{4.1}$$

$$\|F_m\|^p_{L^p(S)} < \gamma^{m-1}\|\psi_0\|^p_{L^p(S)}, \tag{4.2}$$

$$\Big\|\psi_0 - \mathrm{Re}\Big(\sum_{k=1}^m F_k\Big)\Big\|^p_{L^p(S)} < \gamma^m \|\psi_0\|^p_{L^p(S)}, \tag{4.3}$$

$$\|\partial_j F_m\|^p_{A_p(B)} < \gamma^{m-1}\|\psi_0\|^p_{L^p(S)}, \quad 1 \le j \le n. \tag{4.4}$$

Base of induction. Put $\psi = \psi_0$. Then Lemma 3.2 yields an $F \in A(B)$. Define $F_1 := F$.

Step $m+1$. Lemma 3.2, with $\psi = \psi_0 - \mathrm{Re}\,(\sum_{k=1}^m F_k) > 0$, yields the function F_{m+1}. Clearly (3.4)–(3.6) provide (4.1)–(4.4) for $m+1$. So the induction construction works.

Define $h = \sum_{k=1}^\infty F_k$ and $g = \exp h$. Notice that $\sum_{k=1}^\infty \gamma^m < \infty$, therefore $h \in H^p(B)$ (see (4.2)), so $g \in H^\infty(B)$; moreover, (4.1) gives $|g| \le \varphi$ on $\overline{B}$.

The standard identification $h^* \leftrightarrow h$ and (4.3) provide $\mathrm{Re}\, h^* = \psi_0$ σ-a.e., hence $|g^*| = \varphi$ σ-a.e.

Finally, we claim that $g \in A_p^1(B)$. Indeed, (4.4) implies $h \in A_p^1(B)$; on the other hand, $|\partial_j g| = |g| \cdot |\partial_j h| \le \mathrm{const}\, |\partial_j h|$. This completes the proof. ∎

Remark. We use the same $p \in (0,1)$ in the estimates (3.5) and (3.6). This fact has an explanation. Indeed, if we use a smaller $p > 0$ in (3.5), then we obtain a weaker type of convergence for $\sum_{k=1}^\infty F_k$, but H^p-convergence is sufficient with any $p > 0$. On the other hand, we gain in (3.6) because the A_p^1-norm of F_m (which is of interest for small $p > 0$) can be estimated by the L^p-norm of $\psi_0 - \mathrm{Re}(\sum_{k=1}^m F_k)$ and the latter norm rapidly decreases.

It is natural to ask what happens when the modulus φ is not LSC and has zeros (at least, on the sphere). The situation is complicated already for the space $H^\infty(B_n)$, $n > 1$.

If $n > 1$, then some LSC hypothesis is not unreasonable. Indeed, let $g \in H^\infty(B)$. Then

$$\zeta \to M_g(\zeta) = \operatorname*{ess\,sup}_{\lambda \in \mathbb{T}} |g^*(\lambda\zeta)|$$

is an LSC function on S (for further LSC information see, for example, [R3], Chapter 12).

If we consider a modulus φ with zeros on S, then we have to be careful. Let $n \ge 1$ and $g \in H^\infty(B)$. Then $\int_S \log|g^*|\, d\sigma > -\infty$. In dimension one this property characterizes the set $\{|g^*| : g \in H^\infty(B)\}$, but this is not true if $n > 1$. The following statement is a good example.

THEOREM ([KM], Theorem 2). *Let $n \ge 2$. There exists a nonnegative function $\varphi \in C(S)$, $\int_S \log\varphi\, d\sigma > -\infty$, which vanishes at one point only, but such that the estimate $|g^*| \le \varphi$ σ-a.e. for $g \in H^\infty(B)$ implies $g \equiv 0$.*

After the above discussion we give an immediate corollary of Theorem 4.1.

COROLLARY 4.2. *Assume that $0 < p < 2$, $\varphi \in L^\infty(S)$, $f \in H^\infty(B) \cap A_p^1(B)$, $K|f^*| \ge \varphi \ge k|f^*|$ σ-a.e. for some constants $K, k > 0$, and $\varphi/|f^*|$ is LSC. Then there is a function $g \in H^\infty(B) \cap A_p^1(B)$ such that $|g^*| = \varphi$ σ-a.e., and g and f have the same zeros in B.*

Proof. Put $\varphi_1 = \varphi/|f^*|$. Then Theorem 4.1 yields $g_1 \in H^\infty(B) \cap A_p^1(B)$ such that $|g_1^*| = \varphi_1$ σ-a.e. and g_1 has no zeros in B. So define $g = fg_1 \in H^\infty(B) \cap A_p^1(B)$. ∎

5. A correction theorem for $A(B) \cap A_p^1(B)$.

The correction theorem for the ball algebra (see [A2], see also [R3], Chapter 15, for a presentation of this result) says that any function $\psi \in C(S)$ can be modified on a set of arbitrarily small measure in such a way that the new function ψ_1 satisfies the estimate $\psi_1 \le \psi$ and $\psi_1 = \mathrm{Re}\, g^*$ for some $g \in A(B)$. In the present section we prove the latter result with $g \in A(B) \cap A_p^1(B)$.

First, iteration of Lemma 3.2 yields the following approximation lemma.

LEMMA 5.1. *Let $0 < p < 1$ and $M \in \mathbb{N}$. Suppose that $\psi \in C(\overline{B})$, $\psi > 0$. Then there exists a function $\Sigma_M \in A(B)$ such that*

$$|\Sigma_M| < 2^M \psi \quad \text{on } \overline{B}, \tag{5.1}$$

$$\mathrm{Re}\, \Sigma_M < \psi \quad \text{on } \overline{B}, \tag{5.2}$$

$$\|\psi - \mathrm{Re}\, \Sigma_M\|^p_{L^p(S)} < \gamma^M \|\psi\|^p_{L^p(S)}, \tag{5.3}$$

$$\|\partial_j \Sigma_M\|^p_{A_p(B)} < \frac{1}{1-\gamma}\|\psi\|^p_{L^p(S)}, \quad 1 \le j \le n, \tag{5.4}$$

where $\gamma = \gamma(p) \in (0,1)$ is the constant from Lemma 3.2. ∎

Now we can apply an abstract approximation scheme from [A2] with control of the A_p^1-norm.

THEOREM 5.2. *Let $0 < p < 2$. Suppose that $\psi \in C(\overline{B})$ and $\varepsilon > 0$. Then there exists a function $g \in A(B) \cap A_p^1(B)$ such that*

$$\mathrm{Re}\, g \le \psi \quad \text{on } \overline{B}, \quad \sigma\{\mathrm{Re}\, g < \psi\} < \varepsilon.$$

Proof. Put $\psi_0 = \psi$. We suppose that $1 \geq \psi_0 > 0$ and $0 < p < 1$, as usual. Fix $N \in \mathbb{N}$ such that

$$\gamma^N < 4^{-p}(1-\gamma)\varepsilon.$$

Suppose, as induction hypothesis, that $m \in \mathbb{Z}_+$, $\{\psi_k\}_{k=0}^m \subset C(\overline{B})$, $0 < \psi_k \leq 4^{-k}$ on $\overline{B}$, and $\{g_k\}_{k=0}^m \subset A(B)$. Assume also that

$$|g_k| \leq 2^{N+k}\psi_k \leq 2^{N-k} \quad \text{on } \overline{B}, \tag{5.5}$$

$$\operatorname{Re} g_k < \psi_k \quad \text{on } \overline{B}, \tag{5.6}$$

$$\|\psi_k - \operatorname{Re} g_k\|^p_{L^p(S)} < \gamma^{N+k}\|\psi_k\|^p_{L^p(S)} \leq \gamma^{N+k}4^{-pk}, \tag{5.7}$$

$$\|\partial_j g_k\|^p_{A_p(B)} < \frac{\|\psi_k\|^p_{L^p(S)}}{1-\gamma} \leq \frac{4^{-pk}}{1-\gamma}, \quad 1 \leq j \leq n, \tag{5.8}$$

for all $0 \leq k \leq m$.

Base of induction. Lemma 5.1, with $M = N$ and $\psi = \psi_0$, yields a function $\Sigma_M \in A(B)$. Define $g_0 := \Sigma_M$.

Step $m+1$. Put $\psi_{m+1} = \min(4^{-m-1}, \psi_m - \operatorname{Re} g_m)$. Note that $\psi_{m+1} > 0$ (see (5.6)). Again, given $M = N + m + 1$, $\psi = \psi_{m+1}$, Lemma 5.1 provides Σ_M. Define $g_{m+1} = \Sigma_M$. Since (5.1)–(5.4) $\Rightarrow$ (5.5)–(5.8), the induction construction works.

Define

$$g = \sum_{k=1}^{\infty} g_k.$$

We claim that g satisfies the conditions of the theorem. First, the estimate (5.5) yields $g \in A(B)$.

Now define auxiliary functions $\omega_m = (\psi_m - \operatorname{Re} g_m) - \psi_{m+1} \geq 0$. Then

$$\psi = \sum_{k=0}^{m} \operatorname{Re} g_k + \sum_{k=0}^{m} \omega_k + \psi_{m+1}$$

for all $m \in \mathbb{N}$. Recall that $|\psi_{m+1}| \leq 4^{-m-1}$; therefore,

$$\psi = \operatorname{Re} g + \sum_{k=0}^{\infty} \omega_k,$$

in particular, $\operatorname{Re} g \leq \psi$. On the other hand, the definition of ψ_{k+1} and (5.7) yield

$$\begin{aligned}\sigma\{\operatorname{Re} g < \psi\} &\leq \sum_{k=0}^{\infty} \sigma\{\omega_k > 0\} = \sum_{k=0}^{\infty} \sigma\{\psi_k - \operatorname{Re} g_k > 4^{-k-1}\} \\ &\leq \sum_{k=0}^{\infty} 4^{pk+p}\|\psi_k - \operatorname{Re} g_k\|^p_p \leq 4^p \sum_{k=0}^{\infty} \gamma^{N+k} = \frac{4^p\gamma^N}{1-\gamma} < \varepsilon.\end{aligned}$$

The last property $g \in A^1_p(B)$ follows immediately from (5.8). ∎

COROLLARY 5.3. *Let $0 < p < 2$. Suppose that $\varphi \in C(\overline{B})$, $\varphi > 0$, and $\varepsilon > 0$. Then there exists a function $g \in A(B) \cap A^1_p(B)$ such that*

$$|g| \leq \varphi \quad \text{on } \overline{B}, \quad \sigma\{|g| < \varphi\} < \varepsilon.$$

Proof. Note that $g \in A(B) \cap A^1_p(B)$ implies $\exp g \in A(B) \cap A^1_p(B)$. So it is sufficient to apply Theorem 5.2 with $\psi = \log \varphi$. ∎

6. Approximation on great circles. In §5 we considered approximation with respect to one measure. In the present section we prove a theorem of type B on approximation with respect to a *family* of measures. The Besov version of this result is extremely technical, so we restrict our attention to the "pure" ball algebra.

Recall that the function

$$M_f(\zeta) = \operatorname*{ess\,sup}_{\lambda \in \mathbb{T}} |f^*(\lambda\zeta)| \quad (f \in H^\infty(B),\ \zeta \in S)$$

is an important tool in the investigation of the boundary values of $H^\infty(B)$. W. Ramey asked (see [R3], 19.22) whether there is an $f \in H^\infty(B)$ (or even $f \in A(B)$), with $|f^*|$ not constant a.e., for which M_f is constant. The following result yields such an $f \in H^\infty(B)$.

THEOREM (Y. Dupain [Du2]). *Let $\varphi \in C(S)$, $\varphi > 0$. Then there exists a nonconstant $f \in H^\infty(B)$ such that, for every $\zeta \in S$,*

$$\lim_{r \to 1-} |f(r\lambda\zeta)| = \varphi(\lambda\zeta) \quad \textit{for almost all } \lambda \in \mathbb{T}.$$

The main result of this section provides, in particular, a nonconstant $f \in A(B)$ with constant M_f (note that $|f^*|$ is not constant a.e. automatically). To present the corresponding statement, we will use an abstract approach.

DEFINITION. Suppose that K is a compact Hausdorff space, $C(K)$ is the space of all (complex) continuous functions on K, $X \subset C(K)$ is a closed subspace, $\mathcal{P}(K)$ is the set of all probability measures on K, and $\mathcal{M} \subset \mathcal{P}(K)$. Let $0 < p < 1$. Then the triple $(X, K, \mathcal{M})$ is said to be *p-regular* if there exists a $\gamma \in (0,1)$ with the following property: For every $\psi \in C(K)$, $\psi > 0$, there is an $f \in X$ such that

$$|f| < \psi,$$
$$\|\psi - \operatorname{Re} f\|^p_{L^p(\mu)} < \gamma\|\psi\|^p_{L^p(\mu)} \quad \text{for all } \mu \in \mathcal{M}.$$

If $\mathcal{M} = \{\mu\}$ for a probability measure μ, then we obtain one of the equivalent definitions of the regular triple in the sense of [A2].

It is important in the above definition that we use the *same* function f for all $\mu \in \mathcal{M}$. To illustrate this remark, recall that $(A(B), S, \mu)$ is regular for any $\mu \in \mathcal{P}(S)$ (see [A2]), but $(A(B), S, \mathcal{P}(S))$ is not regular.

The following statements are analogues of Lemma 5.1 and Theorem 5.2 (the details of the proofs are even simpler because we can forget about Besov norms).

LEMMA 6.1. *Let $(X, K, \mathcal{M})$ be p-regular, $0 < p < 1$, and $M \in \mathbb{N}$. Suppose that $\psi \in C(K)$, $\psi > 0$. Then there exists a $\Sigma_M \in X$ such that*

$$|\Sigma_M| < 2^M\psi, \qquad \operatorname{Re}\Sigma_M < \psi,$$
$$\|\psi - \operatorname{Re}\Sigma_M\|^p_{L^p(\mu)} < \gamma^M \|\psi\|^p_{L^p(\mu)} \quad \text{for all } \mu \in \mathcal{M}. \ \blacksquare$$

THEOREM 6.2. *Suppose that $(X, K, \mathcal{M})$ is p-regular for some $p \in (0,1)$, $\psi \in C(K)$, $\psi > 0$, and $\varepsilon > 0$. Then there exists a function $g \in X$ such that $\operatorname{Re} g \le \psi$ and*

$$\mu\{\operatorname{Re} g = \psi\} \ge 1 - \varepsilon \quad \text{for all } \mu \in \mathcal{M}. \ \blacksquare$$

Remark. If the space X contains constants, then the restriction $\psi > 0$ is obviously superfluous.

The above results are of no interest without explicit examples of regular triples with sufficiently rich $\mathcal{M}$. So we move to "great circles". Given $\zeta \in S$, put $\mathbb{T}_\zeta = \{\lambda\zeta : \lambda \in \mathbb{T}\} \subset S$. Let m_ζ be the normalized Lebesgue measure on $\mathbb{T}_\zeta$ (the symbol m corresponds to $\mathbb{T}$). Then define $\mathcal{M}_1 = \{m_\zeta : \zeta \in S\}$. We consider the triple $(A(B), S, \mathcal{M}_1)$.

First, recall a notion from [Du2]. Let $\{k_j\}_{j=1}^n, \{m_j\}_{j=1}^n \subset \mathbb{Z}$ and $N \in \mathbb{N}$. The corresponding *fundamental set* is defined by the equality

$$E(\{k_j\}, \{m_j\}, N) = \Big\{\zeta \in S : \operatorname{Re} z_j \in \Big[\frac{2k_j - 1}{N}, \frac{2k_j+1}{N}\Big), \\ \operatorname{Im} z_j \in \Big[\frac{2m_j - 1}{N}, \frac{2m_j + 1}{N}\Big),\ 1 \le j \le n\Big\}.$$

Note that often $E(\{k_j\}, \{m_j\}, N) = \emptyset$, so we consider only nonempty fundamental sets. A specific geometry of these sets permits establishing the following lemma on "multi-approximation" in L^p, $0 < p < 1$.

LEMMA 6.3 (Y. Dupain [Du2]). *For $0 < p < 1$, there exists a constant $\beta = \beta(p) \in (0,1)$ with the following property: Let E be a fundamental set and $\varkappa > 0$. Then there is an $f \in A(B)$ such that*

$$(6.1) \qquad |f(z)| < 1 \quad \text{if } z \in S, \qquad |f(z)| < \varkappa \quad \text{if } z \in S \setminus E,$$

$$(6.2) \qquad \int_{\mathbb{T}} |\mathbf{1}_E(\lambda\zeta) - \operatorname{Re} f(\lambda\zeta)|^p\, dm(\lambda) < \beta m_\zeta(\mathbb{T}_\zeta \cap E) + \varkappa \quad \text{for all } \zeta \in S.$$

Remark. The geometry of the fundamental sets plays a very important role only in the proof of the above lemma. In applications it is essential that fundamental sets in the Nth generation are small provided N is large.

PROPOSITION 6.4. *The triple $(A(B), S, \mathcal{M}_1)$ is p-regular for all $0<p<1$.*

Proof. Let $\psi \in C(S)$, $\psi > 0$, $\|\psi\|_{C(S)} = 1$. Put

$$\varepsilon = \frac{1-\beta}{3} \min_{\zeta \in S} \int_{\mathbb{T}} |\psi(\lambda\zeta)|^p\, dm(\lambda) > 0.$$

Take N sufficiently large and represent the sphere as the union of disjoint fundamental sets $\{E_j\}_{j=1}^m$ such that, for some $\delta = \delta(\varepsilon) > 0$,

$$\psi - \varepsilon^{1/p} < \sum_{j=1}^m a_j \mathbf{1}_{E_j} < \psi - \delta, \qquad a_j > 0.$$

Put $A = \max_{1\le j\le m}\{a_j, 1\}$. Given the sets E_j and $\varkappa := (mA)^{-1}\min\{\varepsilon, \delta\}$, Lemma 6.3 yields functions f_j. Define

$$f = \sum_{j=1}^m a_j f_j.$$

1. Let $\zeta \in E_j$. Then (6.1) provides $|f(\zeta)| \le a_j + mA\varkappa \le a_j + \delta < \psi(\zeta)$, therefore $|f| < \psi$.

2. By (6.2), we have

$$\int_{\mathbb{T}} |\psi(\lambda\zeta) - \operatorname{Re} f(\lambda\zeta)|^p\, dm(\lambda) \le \varepsilon + \sum_{j=1}^m a_j^p \int_{\mathbb{T}} |\mathbf{1}_{E_j}(\lambda\zeta) - \operatorname{Re} f(\lambda\zeta)|^p\, dm(\lambda)$$
$$\le \varepsilon + mA\varkappa + \beta \int_{\mathbb{T}} |\psi(\lambda\zeta)|^p\, dm(\lambda)$$
$$< \frac{\beta+2}{3} \int_{\mathbb{T}} |\psi(\lambda\zeta)|^p\, dm(\lambda) \quad \text{for all } \zeta \in S.$$

So $(A(B), S, \mathcal{M}_1)$ is p-regular with $\gamma = (\beta+2)/3$. $\blacksquare$

COROLLARY 6.5. *Let $\varphi \in C(S)$, $\varphi > 0$, and $\varepsilon > 0$. Then there exists a function $f \in A(B)$ such that $|f| \le \varphi$ and $m_\zeta\{|f| = \varphi\} \ge 1 - \varepsilon$ for all $\zeta \in S$. In particular, there is an $f \in A(B)$ such that $|f|$ is not constant on S but M_f is constant.*

Proof. Let $\varphi > 1$. Then we apply Theorem 6.2 for $\psi = \log\varphi$ and take the exponent. $\blacksquare$

Remark. If $(X, K, \mathcal{M})$ is p-regular, then an abstract analogue of the above corollary holds. To prove this, we have to use the technique of [A2], Theorem 37.

Acknowledgements. The author is grateful to P. Ahern and A. B. Aleksandrov for valuable discussions.

References

[A1] A. B. Aleksandrov, *The existence of inner functions in the ball*, Mat. Sb. 118 (160) (1982), 147–163 (in Russian); English transl.: Math. USSR-Sb. 46 (1983), 143–159.

[A2] —, *Inner functions on compact spaces*, Funktsional. Anal. i Prilozhen. 18 (2) (1984), 1–13 (in Russian); English transl.: Funct. Anal. Appl. 18 (2) (1984), 87–98.

[A3] —, *Function theory in the ball*, in: Itogi Nauki i Tekhniki 8, VINITI, Moscow, 1985, 115–190 (in Russian); English transl.: G. M. Khenkin and A. G. Vitushkin (eds.), Encyclopaedia Math. Sci. 8 (Several Complex Variables II), Springer, Berlin, 1994, 107–178.

[BB] F. Beatrous and J. Burbea, *Sobolev spaces of holomorphic functions in the ball*, Dissertationes Math. 276 (1989).

[Do] E. Doubtsov, *Corrected outer functions*, Proc. Amer. Math. Soc., to appear.

[Du1] Y. Dupain, *Gradients des fonctions intérieures dans la boule unité de* $\mathbb{C}^n$, Math. Z. 193 (1986), 85–94.

[Du2] —, *Fonctions intérieures dans la boule unité de* $\mathbb{C}^n$ *dont les fonctions traces sont aussi intérieures*, ibid. 198 (1988), 191–206.

[KM] B. Korenblum and J. E. McCarthy, *The range of Toeplitz operators on the ball*, Rev. Mat. Iberoamericana 12 (1996), 47–61.

[N] J. Nečas, *Les méthodes directes en théorie des équations elliptiques*, Academia, Prague, 1967.

[R1] W. Rudin, *Function Theory in the Unit Ball of* $\mathbb{C}^n$, Grundlehren Math. Wiss. 241, Springer, Berlin, 1980.

[R2] —, *Inner functions in the unit ball of* $\mathbb{C}^n$, J. Funct. Anal. 50 (1983), 100–126.

[R3] —, *New Constructions of Functions Holomorphic in the Unit Ball of* $\mathbb{C}^n$, CBMS Regional Conf. Ser. in Math. 63, Amer. Math. Soc., Providence, R.I., 1986.

[Ta] M. Tamm, *Sur l'image par une fonction holomorphe bornée du bord d'un domaine pseudoconvexe*, C. R. Acad. Sci. Paris 294 (1982), 537–540.

[To] B. Tomaszewski, *Interpolation by Lipschitz holomorphic functions*, Ark. Mat. 23 (1985), 327–338.

Centre de Recerca Matemàtica
Institut d'Estudis Catalans
Apartat 50
E-08193 Bellaterra (Barcelona), Spain
E-mail: doubtsov@crm.es

Received August 30, 1996
Revised version December 23, 1996 (3728)

STUDIA MATHEMATICA 124 (2) (1997)

Multiplicative functionals and entire functions, II

by

KRZYSZTOF JAROSZ (Edwardsville, Ill.)

Abstract. Let $\mathcal{A}$ be a complex Banach algebra with a unit e, let F be a nonconstant entire function, and let T be a linear functional with $T(e)=1$ and such that $T\circ F:\mathcal{A}\to\mathbb{C}$ is nonsurjective. Then T is multiplicative.

Introduction. Let T be a nonzero multiplicative functional on a complex Banach algebra $\mathcal{A}$ with a unit e, and let $\mathcal{A}^{-1}$ denote the set of all invertible elements of $\mathcal{A}$. Then $T(e)=1$, and $T(x)\neq 0$ for any $x\in\mathcal{A}^{-1}$. A. M. Gleason [5] and, independently, J. P. Kahane & W. Żelazko [8], [9] proved that the converse implication also holds.

THEOREM 1 [G-K-Ż]. *If T is a linear functional on a complex unital Banach algebra $\mathcal{A}$ such that $T(e)=1$ and*

$$T(x)\neq 0 \quad \textit{for } x\in\mathcal{A}^{-1},$$

then T is multiplicative.

In fact, they proved even a stronger result.

THEOREM 2 [G-K-Ż]. *If T is a linear functional on a complex unital Banach algebra $\mathcal{A}$ such that $T(e)=1$ and*

$$T(x)\neq 0 \quad \textit{for } x\in\exp\mathcal{A}, \tag{1}$$

then T is multiplicative.

Here $\exp\mathcal{A}=\{\exp y : y\in\mathcal{A}\}$. In 1987 R. Arens asked if the exponential function above can be replaced by an arbitrary nonconstant entire function F, that is, whether

$$T(x)\neq 0 \quad \text{for } x\in F(\mathcal{A}):=\{F(y):y\in\mathcal{A}\}$$

1991 *Mathematics Subject Classification*: Primary 46J05; Secondary 46H05, 46H30, 46J15, 46J20.

Research was supported in part by a grant from the International Research & Exchanges Board, with funds provided by the National Endowment for the Humanities and the U.S. Department of State.