STUDIA MATHEMATICA 124 (2) (1997)

On extremal and perfect σ-algebras for $\mathbb{Z}^d$-actions on a Lebesgue space

by

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Abstract. We show that for every positive integer d there exists a $\mathbb{Z}^d$-action and an extremal σ-algebra of it which is not perfect.

1. Introduction. Invariant σ-algebras form an important tool for solving a series of problems in ergodic theory. They have been used, among other things, in the spectral theory of dynamical systems ([6], [12]), in non-equilibrium mechanics ([1], [3]) and to investigate helices ([13]–[15]).

The theory of these σ-algebras has been developed by Rokhlin and Sinai for $\mathbb{Z}$-actions (cf. [12]) on a Lebesgue space and by the first author in the case of arbitrary $\mathbb{Z}^d$-actions, $d \geq 2$ (see [6]). A special role in this theory is played by extremal σ-algebras and perfect σ-algebras.

Recently, Jan Kwiatkowski has posed the following question. Do there exist, for any $d \geq 1$, a $\mathbb{Z}^d$-action and an extremal σ-algebra of it which is not perfect?

The purpose of this paper is to give an affirmative answer to this question. First we construct a non-invertible measure preserving transformation with finite entropy which is exact and has no generator with finite entropy. Its natural extension and the corresponding exhaustive σ-algebra give a solution of the Kwiatkowski question in the case $d = 1$.

Next, using actions defined by Conze ([2]), we define the desired $\mathbb{Z}^d$-action and an extremal σ-algebra of it for arbitrary $d \geq 1$.

The non-invertible transformation constructed in the paper has the following property which does not hold for invertible transformations. Namely, it is well known (cf. [10]) that every ergodic invertible measure-preserving

1991 *Mathematics Subject Classification*: Primary 28D15.
The first author is supported by KBN grant 2P 30103107, the second by KBN grant 2 P03A 07608 and the third by URA-CNRS No 225.

transformation with finite entropy has a finite generator. As we have announced above, our transformation does not have this property.

2. Preliminaries. Let $(X,\mathcal{B},\mu)$ be a Lebesgue probability space and let $\mathcal{N}$ be the trivial subalgebra of $\mathcal{B}$. Let $\mathbb{Z}^d$ denote the group of d-dimensional integers.

In order to avoid notational difficulties, we restrict our considerations to the case $d=2$. Our arguments may be easily extended to arbitrary $\mathbb{Z}^d$-actions.

Let $\prec$ denote the lexicographical order in $\mathbb{Z}^2$. An ordered pair (A,B) of subsets of $\mathbb{Z}^2$ is said to be a *cut* if $A\neq\emptyset$, $B\neq\emptyset$, $A\cup B=\mathbb{Z}^2$ and $g\prec h$ for every $g\in A$ and $h\in B$. A cut (A,B) is called a *gap* if A does not contain a greatest element and B does not contain a lowest element.

Let Φ be a $\mathbb{Z}^2$-action on $(X,\mathcal{B},\mu)$. The automorphism of $(X,\mathcal{B},\mu)$ corresponding to $g\in\mathbb{Z}^2$ is denoted by Φ^g. We denote by $h(\Phi)$ and $\pi(\Phi)$ the entropy and the Pinsker σ-algebra of Φ, respectively.

DEFINITION 1. A σ-algebra $\mathcal{A}\subset\mathcal{B}$ is said to be *extremal* if

(i) it is invariant, i.e. $\Phi^g\mathcal{A}\subset\mathcal{A}$ for $g\prec(0,0)$,

(ii) the family $(\Phi^g\mathcal{A})_{g\in\mathbb{Z}^2}$ is continuous, i.e. for every gap (A,B) in $\mathbb{Z}^2$,

$$\bigvee_{g\in A}\Phi^g\mathcal{A}=\bigcap_{g\in B}\Phi^g\mathcal{A},$$

(iii) $\bigvee_{g\in\mathbb{Z}^2}\Phi^g\mathcal{A}=\mathcal{B}$,

(iv) $\bigcap_{g\in\mathbb{Z}^2}\Phi^g\mathcal{A}=\pi(\Phi)$.

DEFINITION 2. $\mathcal{A}$ is called *perfect* if it is extremal and

(v) $h(\Phi)=H(\mathcal{A}\mid\mathcal{A}_\Phi^-)$

where $\mathcal{A}_\Phi^-=\bigvee_{g\prec(0,0)}\Phi^g\mathcal{A}$.

We may express Definitions 1 and 2 in a simpler form using the automorphisms $T=\Phi^{(1,0)}$ and $S=\Phi^{(0,1)}$. Namely, $\mathcal{A}$ is extremal iff

(i′) $S^{-1}\mathcal{A}\subset\mathcal{A}$, $T^{-1}\mathcal{A}_S\subset\mathcal{A}$ where $\mathcal{A}_S=\bigvee_{n=-\infty}^{\infty}S^n\mathcal{A}$, and

(ii′) $\bigcap_{n=0}^{\infty}S^{-n}\mathcal{A}=T^{-1}\mathcal{A}_S$,

(iii′) $\bigvee_{n=-\infty}^{\infty}T^n\mathcal{A}_S=\mathcal{B}$,

(iv′) $\bigcap_{n=-\infty}^{\infty}T^n\mathcal{A}_S=\pi(\Phi)$.

$\mathcal{A}$ is perfect iff it is extremal and

(v′) $h(\Phi)=H(\mathcal{A}\mid S^{-1}\mathcal{A})$.

It is known (cf. [7], p. 122) that in the case $h(\Phi)<\infty$ the properties (i)–(iii) and (v) imply (iv). Therefore, it was natural to ask whether the properties (i)–(iv) imply (v). Now, we show that this question has a negative answer.

THEOREM. *There exists a $\mathbb{Z}^2$-action on a Lebesgue space with an extremal σ-algebra which is not perfect.*

Proof. Let Z and $\overline{Z}$ denote the spaces of one-sided and two-sided sequences, respectively, with values in $\{-1,1\}$. Let P and $\overline{P}$ be the zero-time partitions in Z and $\overline{Z}$, respectively. We denote by $\sigma:Z\to Z$ the one-sided shift and by $\overline{\sigma}:\overline{Z}\to\overline{Z}$ the two-sided shift. We assume that Z and $\overline{Z}$ are equipped with the product σ-algebras and Bernoulli measures p and $\overline{p}$, respectively, determined by the vector $(1/2,1/2)$.

Let the space $\Omega=Z\times\overline{Z}$ be equipped with the product σ-algebra $\mathcal{F}$ and let $\nu=p\times\overline{p}$. We denote by T the endomorphism of Ω defined by the formula

$$T(x,y)=(\sigma x,\overline{\sigma}^{x(0)}y),\qquad x\in Z,\ y\in\overline{Z}.$$

First, note that the partition $P\times\overline{P}$ is a generator for T. This follows from the fact that P and $\overline{P}$ are generators for σ and $\overline{\sigma}$, respectively, and the sequence (χ_n) of random variables defined by $\chi_n(x)=\sum_{k=0}^{n-1}x(k)$, $x\in Z$, is a random walk (cf. [4]).

Now, observe that T is exact. The automorphism $\overline{T}:\overline{Z}\times\overline{Z}\to\overline{Z}\times\overline{Z}$ which is the natural extension of T takes the form

$$\overline{T}(\overline{x},y)=(\overline{\sigma}(\overline{x}),\overline{\sigma}^{\overline{x}(0)}y),\qquad \overline{x},y\in\overline{Z}.$$

It follows from [11] that $\overline{T}$ is a K-automorphism. Hence and from the fact that $P\times\overline{P}$ is a generator for T it follows that T is exact.

Let $\tau:\Omega\to\Omega$ be the endomorphism defined by $\tau=T_1\circ T$ $(=T\circ T_1)$ where $T_1(x,y)=(x,\overline{\sigma}(y))$, i.e.

$$\tau(x,y)=(\sigma(x),\overline{\sigma}^{x(0)+1}y),\qquad x\in Z,\ y\in\overline{Z}.$$

Since T is exact, the transformation τ is also exact.

Now we shall check that

$$H(\mathcal{F}\mid\tau^{-1}\mathcal{F})<h(\tau),\tag{1}$$

i.e. τ has no generator with finite entropy. Let $S=\{S_x\}$ where

$$S_xy=\overline{\sigma}^{x(0)+1}y,\qquad x\in Z,\ y\in\overline{Z}.$$

Let $h_\sigma(S)$ denote the mixed entropy of fibers of τ, i.e. $h_\sigma(S)=\sup h_\sigma(S,Q)$ where the supremum is taken over all finite partitions Q of $\overline{Z}$ and

$$h_\sigma(S,Q)=\lim_{n\to\infty}\frac{1}{n}\int_{\overline{Z}}H(Q_{\overline{x}}^n)\,\overline{p}(d\overline{x}),$$

$$Q_{\overline{x}}^n=Q\vee S_x^{-1}Q\vee\ldots\vee S_x^{-1}S_{\sigma x}^{-1}\ldots S_{\sigma^{n-1}x}^{-1}Q.$$

Applying the ergodicity of $\bar{\sigma}$ we obtain, for a.e. $x \in Z$,

$$h_{\sigma}(S, \overline{P} \vee \bar{\sigma}^{-1}\overline{P}) = \lim_{n\to\infty} \frac{1}{n} H\Big(\bigvee_{k=0}^{l_n(x)} \bar{\sigma}^{-2k}(\overline{P} \vee \bar{\sigma}^{-1}\overline{P})\Big)$$
$$= \lim_{n\to\infty} \frac{2l_n(x)}{n} \cdot \ln 2 = \ln 2$$

where $l_n(x)$ denotes the number of ones in the sequence $(x(0), \ldots, x(n-1))$. Therefore, by the well known Abramov–Rokhlin formula we get

$$h(\tau) = h(\sigma) + h_{\sigma}(S) \geq \ln 4.$$

Let J_{τ}, J_{σ} and $J_{\bar{\sigma}}$ denote the Jacobians of τ, σ and $\bar{\sigma}$, respectively. By Lemma 3 of [9] we have $J_{\tau}(x,y) = J_{\sigma}(x) \cdot J_{\bar{\sigma}}^{x(0)+1}(y)$ and so $J_{\tau}(x,y) = J_{\sigma}(x)$. Therefore, applying the entropy formula from [16], we have

$$H(\mathcal{F} \mid \tau^{-1}\mathcal{F}) = \int_{\Omega} \ln J_{\tau}(x,y)\, \nu(dxdy) = \int_{Z} \ln J_{\sigma}(x)\, p(dx) = \ln 2.$$

Thus, we obtain the desired inequality.

Let $(Y, \mathcal{G}, \lambda, \varphi)$ be the natural extension of $(\Omega, \mathcal{F}, \nu, \tau)$, i.e. there exists a σ-algebra $\mathcal{D} \subset \mathcal{G}$ such that $\varphi^{-1}\mathcal{D} \subset \mathcal{D}$, $\bigvee_{n=-\infty}^{\infty} \varphi^n \mathcal{D} = \mathcal{G}$ and the factor endomorphism $\varphi_{\mathcal{D}}$ is conjugate to τ. Since τ is exact, we have

$$\bigcap_{n=0}^{\infty} \varphi^{-n}\mathcal{D} = \mathcal{N}, \tag{2}$$

i.e. $\mathcal{D}$ is a Kolmogorov σ-algebra for φ. Moreover, we have $H(\mathcal{F} \mid \tau^{-1}\mathcal{F}) = H(\mathcal{D} \mid \varphi^{-1}\mathcal{D})$ and so, using (1), we get

$$H(\mathcal{D} \mid \varphi^{-1}\mathcal{D}) < h(\varphi). \tag{3}$$

Let now $\Phi = \Phi_{\varphi}$ be the $\mathbb{Z}^2$-action defined as follows (cf. [2], Example (3)). We consider the product probability space

$$(X, \mathcal{B}, \mu) = \prod_{n=-\infty}^{\infty} (Y_n, \mathcal{G}_n, \lambda_n)$$

where $Y_n = Y$, $\mathcal{G}_n = \mathcal{G}$, $\lambda_n = \lambda$, $n \in \mathbb{Z}$. We equip the space $(X, \mathcal{B}, \mu)$ with the transformations T and S defined by

$$(Tx)(n) = x(n+1), \quad (Sx)(n) = \varphi x(n), \quad n \in \mathbb{Z}.$$

It is clear that T and S are commuting automorphisms of $(X, \mathcal{B}, \mu)$. Let Φ be the $\mathbb{Z}^2$-action generated by T and S, i.e.

$$\Phi^g = T^m \circ S^n, \quad g = (m,n) \in \mathbb{Z}^2.$$

We denote by π_0 the projection of X on the zero coordinate and we put

$$\mathcal{C} = \pi_0^{-1}(\mathcal{D}), \qquad \mathcal{A} = \mathcal{C} \vee \bigvee_{g \prec (0,0)} \Phi^g \mathcal{C} = \mathcal{C} \vee \mathcal{C}_S^- \vee (\mathcal{C}_S)_T^-.$$

It is clear that $\mathcal{A}$ satisfies the condition (i′). The equality (ii′) is equivalent to

$$\bigcap_{n=0}^{\infty} (S^{-n}\mathcal{C}_S^- \vee (\mathcal{C}_S)_T^-) = (\mathcal{C}_S)_T^-. \tag{4}$$

Since μ is the product measure, the σ-algebras $\mathcal{C}_S^-$ and $(\mathcal{C}_S)_T^-$ are independent. Therefore, it follows from Theorem 1 of [8] that (4) is equivalent to

$$\bigcap_{n=0}^{\infty} S^{-n}\mathcal{C}_S^- \vee (\mathcal{C}_S)_T^- = (\mathcal{C}_S)_T^-. \tag{5}$$

This equality is valid because, in view of (2), $\bigcap_{n=0}^{\infty} S^{-n}\mathcal{C}_S^- = \mathcal{N}$.

The equality (iii′) is an easy consequence of the equality $\bigvee_{n=-\infty}^{\infty} \varphi^n \mathcal{D} = \mathcal{G}$.

Since the σ-algebras $T^n\mathcal{C}_S, n \in \mathbb{Z}$, are independent the Kolmogorov zero-one law implies

$$\bigcap_{n=-\infty}^{\infty} T^n \mathcal{A}_S = \bigcap_{n=-\infty}^{\infty} T^n \mathcal{C}_S = \mathcal{N}.$$

On the other hand, the fact that φ is a K-automorphism and Theorem 3 of [5] imply that Φ is a K-action of $\mathbb{Z}^2$, i.e. $\pi(\Phi) = \mathcal{N}$. This means that (iv′) is satisfied and so $\mathcal{A}$ is extremal.

It remains to show that $H(\mathcal{A} \mid S^{-1}\mathcal{A}) < h(\Phi)$. We know from [2] that

$$h(\Phi) = h(\varphi). \tag{6}$$

On the other hand, the independence of the σ-algebras $\mathcal{C} \vee \mathcal{C}_S^-$ and $(\mathcal{C}_S)_T^-$ implies

$$H(\mathcal{A} \mid S^{-1}\mathcal{A}) = H(\mathcal{C} \mid \mathcal{C}_S^- \vee (\mathcal{C}_S)_T^-) = H(\mathcal{C} \mid \mathcal{C}_S^-). \tag{7}$$

It is easy to check that

$$H(\mathcal{C} \mid \mathcal{C}_S^-) = H(\mathcal{D} \mid \varphi^{-1}\mathcal{D}). \tag{8}$$

Combining (3) and (6)–(8), we obtain $H(\mathcal{A} \mid S^{-1}\mathcal{A}) < h(\Phi)$, i.e. $\mathcal{A}$ is not perfect. ∎

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Received August 27, 1996
Revised version December 6, 1996 (3726)

STUDIA MATHEMATICA 124 (2) (1997)

Approximation on the sphere by Besov analytic functions

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Abstract. Boundary values of zero-smooth Besov analytic functions in the unit ball of $\mathbb{C}^n$ are investigated. Bounded Besov functions with prescribed lower semicontinuous modulus are constructed. Correction theorems for continuous Besov functions are proved. An approximation problem on great circles is studied.

1. Introduction. Let $\mathbb{C}^n$ be the n-dimensional complex space (usually $n \geq 2$) with the unit ball $B = B_n = \{|z| < 1\}$ and the unit sphere $S = S_n = \partial B$. By ν and σ we denote the normalized Lebesgue measures on B and S respectively. In dimension one we use the notation $\mathbb{D} = B_1$, $\mathbb{T} = \partial\mathbb{D}$. $C(S)$ is the space of all continuous functions on S, the symbol LSC stands for lower semicontinuous functions, and $H(B)$ is the space of all analytic functions $f : B \to \mathbb{C}$. Finally, $A(B) = H(B) \cap C(\overline{B})$ is the ball algebra and $H^\infty(B) := \{f \in H(B) : f \text{ is bounded}\}$.

In the present paper we investigate boundary values of some Besov analytic functions. More precisely, given $0 < p < \infty$ and $q > 0$, define

$$\|f\|^p_{A_{pq}(B)} = \int_B |f(z)|^p (1 - |z|)^{q-1}\, d\nu(z),$$

$$A^1_{pq}(B) = \Big\{ f \in H(B) : \|f\|_{A_{pq}(B)} + \sum_{j=1}^n \|\partial_j f\|_{A_{pq}(B)} < \infty \Big\},$$

where $\partial_j = \partial/\partial z_j$. (From a different point of view, $A^1_{pq}(B)$ is a weighted Sobolev space of analytic functions.) To avoid technicalities, we do not consider the spaces $A^m_{pq}(B)$, $m > 1$, defined in terms of higher derivatives.

To justify the term *Besov* space, consider the particular (unweighted) case $q = 1$. For $p > 1$, the Besov space $B^{1-1/p}_{pp}(S)$ on the sphere is defined

1991 *Mathematics Subject Classification*: Primary 32A40; Secondary 32A35, 32E25.

Supported by the Centre de Recerca Matemàtica, Institut d'Estudis Catalans (Barcelona) under a grant from DGICYT (Spain).