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Received May 30, 1996
Revised version December 9, 1996 (3682)

STUDIA MATHEMATICA 124 (2) (1997)

Order functions of plurisubharmonic functions

by

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Abstract. We consider the following problem: find on $\mathbb{C}^2$ a plurisubharmonic function with a given order function. In particular, we prove that any positive ambiguous function on $\mathbb{CP}^1$ which is constant outside a polar set is the order function of a plurisubharmonic function.

1. The order function for plurisubharmonic functions. In this paper we study pointwise singularities of plurisubharmonic functions, i.e., the behavior of a plurisubharmonic function near isolated points where the function's value is $-\infty$. Singularities of plurisubharmonic functions on subsets of $\mathbb{C}^n$ have been studied by many authors (see [7] for references), in general using the notion of the Lelong number. Unfortunately, this number does not provide a detailed description of the singularity. For example, another important characteristic of singularities—the mass of the Monge–Ampère operator at these points—has little to do with their Lelong numbers (see Ex. 5.7 in [4]).

We concentrate on the notion of the order function which reflects more features of the function's behavior. Given a plurisubharmonic function u on the unit ball $B \subset \mathbb{C}^n$, centered at the origin, the *order function* o_u of u at 0 is defined as

$$o_u(z) = \lim_{r\to 0} \inf_{|\gamma|=r} \frac{u(\gamma z)}{\log|\gamma z|},$$

where z is in $\mathbb{C}^n \setminus \{0\}$ and $\gamma \in \mathbb{C}$. Since $o_u(z) = o_u(\gamma z)$, $\gamma \neq 0$, we may assume that the order function is defined on the unit sphere S in $\mathbb{C}^n$ or on the complex projective space $\mathbb{CP}^{n-1}$, which for $n = 2$ coincides with the Riemann sphere $\overline{\mathbb{C}}$.

1991 *Mathematics Subject Classification*: Primary 32F05; Secondary 31C10.
Key words and phrases: plurisubharmonic function, singularity, order function, G_δ-function.

This function is similar to the order function at infinity studied in [5], where the complete description of regularized order functions was given. (The function $\varphi_*(x) = \liminf_{y\to x,\, y\neq x} \varphi(x)$ is the regularization of a function $\varphi(x)$.) In the case of a pointwise singularity the regularization significantly simplifies the situation. It was proved in [1] that the order function is equal to a constant l on $\mathbb{CP}^{n-1}$ minus a pluripolar set and is greater than l on this set, so its regularization is a constant function. It can be proved (see [2]) that l is the Lelong number of u at 0. Therefore, regularized order functions only distinguish singularities with distinct Lelong numbers.

To understand deeper the nature of singularities we consider the inverse problem: Find a (continuous) plurisubharmonic function with a given order function. We solve this problem for almost all possible cases and show that the space of possible order functions has a quite complicated structure. In [3] we solve a similar problem for maximal plurisubharmonic functions. All results in this paper are obtained for plurisubharmonic functions on $\mathbb{C}^2$. The case of greater dimensions is more complicated due to the absence of a good description of complete pluripolar sets.

Let us find the order of a subharmonic function u defined on a neighborhood of the unit disk D in $\mathbb{C}$. We consider the Laplacian Δu as a positive Borel measure on the domain of u and denote by $\Delta u(\{0\})$ its mass at the origin.

THEOREM 1.1. *If u is a subharmonic function on $B\subset\mathbb{C}$, then the order function o_u at the origin is equal to $\frac{1}{2\pi}\Delta u(\{0\})$.*

Proof. It follows from [6] and [7] that

$$\lim_{r\to 0}\frac{\sup_{|\zeta|=r} u(\zeta)}{\log r} = \frac{1}{2\pi}\Delta u(\{0\}).$$

Since

$$o_u = \lim_{r\to 0}\frac{\sup_{|\zeta|=r} u(\zeta)}{\log r},$$

we get the theorem. ■

As a consequence we get

COROLLARY 1.2. *Let u be a plurisubharmonic function on B and $z\in\partial B$. Then $o_u(z) = \frac{1}{2\pi}\Delta u^z(\{0\})$, where $u^z(\zeta) = u(\zeta z)$.*

We need the following observation.

LEMMA 1.3. *Let $u_j \le 0$ be plurisubharmonic functions on the unit ball. Suppose that $u = \sum_{j=1}^{\infty} u_j \not\equiv -\infty$ on each complex line passing through the origin. Then $o_u(v) = \sum_{j=1}^{\infty} o_{u_j}(v)$.*

Proof. Let $u^z(\zeta) = u(\zeta z)$. Since $u^z \not\equiv -\infty$,

$$\Delta u^z(\{0\}) = \sum_{j=1}^{\infty}\Delta u_j^z(\{0\}).$$

Since $o_u(z) = o_{u^z}(\zeta)$, the lemma follows from Corollary 1.2. ■

The following theorem is the starting point of our studies.

THEOREM 1.4. *The order function of a plurisubharmonic function u on B is the limit of a decreasing sequence of lower semicontinuous functions.*

Proof. It was proved in Lemma 2 of [1] that $o_u(z) = \lim_{r\to 0}\Psi_r(z)$, where the functions

$$\Psi_r(z) = \inf_{|\zeta|=r}\frac{u(\zeta z)}{\log r}$$

are plurisuperharmonic and increasing in r. Hence these functions are lower semicontinuous and o_u is the limit of a decreasing sequence of lower semicontinuous functions. ■

A plurisubharmonic function u is said to be *continuous* if the function e^u is continuous (we assume that $e^{-\infty}=0$). In this case, by the following theorem, the order function is upper semicontinuous.

THEOREM 1.5. *If u is a continuous plurisubharmonic function, then the order function o_u is upper semicontinuous.*

Proof. The function Ψ_r, introduced in the previous theorem, is the lower envelope of the family of continuous functions

$$\Psi_r(v,\gamma) = \frac{u(\gamma v)}{\log|\gamma|},$$

where $|\gamma| = r$. Thus, the functions $\Psi_r(z)$ are upper semicontinuous in z and increasing in r. Therefore, the order function

$$o_u(z) = \lim_{r\to 0}\Psi_r(z)$$

is also upper semicontinuous. ■

2. G_δ- and ambiguous functions. To describe the order functions of plurisubharmonic functions we need the classes of G_δ and ambiguous functions. A real-valued function φ on a topological space X is G_δ if for all $a\in\mathbb{R}$ the superlevel sets $E_a = \{x\in X : \varphi(x)\ge a\}$ are G_δ-sets. A function φ on X is *ambiguous* if for each $c\in\mathbb{R}$ the superlevel set E_c is ambiguous, i.e., it is both an F_σ- and a G_δ-set. The following theorem describes decreasing limits of sequences of lower semicontinuous functions as G_δ-functions.

THEOREM 2.1. *A nonnegative function φ on a topological space X is G_δ if and only if it is the pointwise limit of a decreasing sequence of lower semicontinuous functions.*

Proof. Note that every lower semicontinuous function ψ is G_δ, because for every $a \in \mathbb{R}$ the set $\{\psi \geq a\}$ is the intersection of open sets $E_j = \{\psi > a - 1/j\}$. If φ is the limit of a decreasing sequence of lower semicontinuous functions φ_j, then

$$\{\varphi(x) \geq a\} = \bigcap_{j=1}^{\infty} \{\varphi_j(x) \geq a\}$$

for any $a \in \mathbb{R}$. Therefore, the set $\{\varphi \geq a\}$ is a G_δ-set as the intersection of the G_δ-sets $\{\varphi_j(x) \geq a\}$. So φ is a G_δ-function.

If φ is a G_δ-function on X, then for $k = 0$ and for every nonnegative integer i we consider the G_δ-sets $A_i^0 = \{\varphi(x) \geq i\}$. Let $\widetilde{E}_{i,j}^0$ be open sets such that $\widetilde{E}_{i,j+1}^0 \subset \widetilde{E}_{i,j}^0$ and $A_i^0 = \bigcap_{j=0}^{\infty} \widetilde{E}_{i,j}^0$. Define open sets $E_{i,j}^0$ as

$$E_{i,j}^0 = \bigcap_{l=0}^{i} \widetilde{E}_{l,j}^0.$$

Then, clearly, $E_{i+1,j}^0 \subset E_{i,j}^0$ and $E_{i,j+1}^0 \subset E_{i,j}^0$. Moreover, $A_i^0 = \bigcap_{j=0}^{\infty} E_{i,j}^0$. In fact, $A_i^0 \subset \widetilde{E}_{l,j}^0$ for all $0 \leq l \leq i$. This implies that $A_i^0 \subset E_{i,j}^0$ and, therefore, $A_i^0 \subset \bigcap_{j=0}^{\infty} E_{i,j}^0$. But $E_{i,j}^0 \subset \widetilde{E}_{i,j}^0$, hence $A_i^0 = \bigcap_{j=0}^{\infty} E_{i,j}^0$.

Define $\varphi_0(x) = i + 1$ on $E_{i,0}^0 \setminus E_{i+1,0}^0$ and $\varphi_0(x) = \infty$ on $\bigcap_{i=0}^{\infty} E_{i,0}^0$. Then the function φ_0 is lower semicontinuous. Indeed, for any $a \in \mathbb{R}$, the set

$$\{\varphi_0(x) > a\} = \bigcup_{k=i_0}^{\infty} (E_{k,0}^0 \setminus E_{k+1,0}^0) = E_{i_0,0}^0,$$

where i_0 is the minimal integer such that $i_0 + 1 > a$, is open. Moreover, $\varphi_0(x) \geq \varphi(x)$ since $\varphi(x) < i + 1$ on $X \setminus E_{i+1,0}^0$.

Using induction and the argument above, for every $k \geq 1$ we may choose open sets $E_{i,j}^k$ such that:

(1) $A_i^k = \{\varphi(x) \geq i/2^k\} = \bigcap_{j=0}^{\infty} E_{i,j}^k$;

(2) $E_{i+1,j}^k \subset E_{i,j}^k$ and $E_{i,j+1}^k \subset E_{i,j}^k$;

(3) if $i = 2m$ is even, then $E_{i,j}^k \subset E_{m,j}^{k-1}$.

To satisfy condition (3) we note that $i2^{-k} = m2^{1-k}$. So $A_i^k = A_m^{k-1}$ and we choose $\widetilde{E}_{i,j}^k \subset \widetilde{E}_{m,j}^{k-1}$.

We define the function $\varphi_k(x) = (i + 1)2^{-k}$ if and only if $x \in E_{i,k}^k \setminus E_{i+1,k}^k$ and $\varphi_l(x) = \infty$ on $\bigcap_{i=0}^{\infty} E_{i,k}^k$. As before, one can see that φ_k is lower semicontinuous and $\varphi_k \geq \varphi$.

Let us show that $\varphi_k \leq \varphi_{k-1}$. If $x \in E_{i,k}^k \setminus E_{i+1,k}^k$ and $i = 2m$ is even, then

$$x \in E_{i,k}^k \subset E_{m,k}^{k-1} \subset E_{m,k-1}^{k-1}.$$

This implies that

$$\varphi_{k-1}(x) \geq \frac{m+1}{2^{k-1}} = \frac{i+2}{2^k} \geq \varphi_k(x).$$

If $i = 2m + 1$ then $E_{2m+1,k}^k \subset E_{2m,k}^k \subset E_{m,k-1}^{k-1}$ and

$$\varphi_k(x) = \frac{2m+2}{2^k} = \frac{m+1}{2^{k-1}} \leq \varphi_{k-1}(x).$$

Let $\widetilde{\varphi}(x) = \lim_{k\to\infty} \varphi_k(x)$. Then $\widetilde{\varphi} \geq \varphi$. Suppose $\varphi(x) < \widetilde{\varphi}(x)$ for some $x \in X$. We can find integers i and k such that $\varphi(x) < i2^{-k} < \widetilde{\varphi}(x)$. If $\varphi(x) < i2^{-k}$, then $x \notin A_i^k$. By (1), $x \notin E_{i,j}^k$ for all j greater than or equal to some j_0. Hence $x \notin E_{2^m i,j}^{k+m}$ when $j \geq j_0$. Therefore, if $k + m \geq j_0$ then $x \notin E_{2^m i,k+m}^{k+m}$ and this means that

$$\varphi_{k+m}(x) \leq \frac{2^m i + 1}{2^{k+m}} = \frac{i}{2^k} + \frac{1}{2^{k+m}}.$$

So $\widetilde{\varphi}(x) \leq i/2^k$ and this contradiction proves that $\widetilde{\varphi} \equiv \varphi$. ∎

This theorem and Theorem 1.4 immediately imply

COROLLARY 2.2. *If u is a plurisubharmonic function on the unit ball B in $\mathbb{C}^n$, then the order function o_u is a G_δ-function.*

It follows from the theorem above that if f and g are nonnegative G_δ-functions, then their sum and product are also G_δ-functions.

We need the following theorem.

THEOREM 2.3. *The class of bounded G_δ-functions is closed under uniform limits. In particular, the sum $\sum_{j=1}^{\infty} c_j \chi_{E_j}$ is a G_δ-function, where χ_{E_j} are the characteristic functions of G_δ-sets E_j and $c_j > 0$, $\sum c_j < \infty$.*

Proof. Let $(f_n)_{n\geq 1}$ be a sequence of bounded G_δ-functions converging uniformly to a function f. Let

$$\varepsilon_n = \sup |f - f_n|$$

for $n = 1, 2, \ldots$ Then $\varepsilon_n \to 0$ as $n \to \infty$. Thus $f_n - \varepsilon_n \leq f$ for all $n \geq 1$. For $a \in \mathbb{R}$,

$$\{f \geq a\} = \bigcap_{n=1}^{\infty} \{f_n \geq a - \varepsilon_n\}.$$

The right hand side is a G_δ-set, being a countable intersection of G_δ-sets. Therefore f is a G_δ-function. ∎

For applications to plurisubharmonic functions we need representations of G_δ- or upper semicontinuous functions as weighted sums of characteristic functions of G_δ- or closed sets. Moreover, we need series of weights of these representations to converge. We could neither prove nor disprove the existence of such representations in the general case. The following two theorems describe cases when such representations are possible.

THEOREM 2.4. *Let φ be a bounded ambiguous function on a metric space X which is equal to a constant $c \ge 0$ outside some set $E \subset X$. Let $\varphi(x) > c$ for all $x \in E$. Then φ can be represented as an infinite sum of functions $a_j + b_j\chi_{E_j}$, where the sets E_j are ambiguous, $a_j \ge 0$, $b_j > 0$, $\sum b_j$ and $\sum a_j$ converge.*

Proof. Let $c = 0$. Since φ is bounded there exists a positive number M such that $\varphi \le M$. Let $E_1 = \{x \in X : \varphi(x) \ge M/2\}$ and $\varphi_1 = (M/2)\chi_{E_1}$. For $n \ge 2$ set

$$E_n = \left\{x \in X : \varphi(x) - \sum_{j=1}^{n-1} \varphi_j \ge \frac{M}{2^n}\right\},$$

and define $\varphi_n = (M/2^n)\chi_{E_n}$. Since the difference of ambiguous functions is ambiguous, it follows by induction that the functions φ_n are ambiguous and $0 \le \varphi - s_n \le M2^{-n}$, where $s_n = \sum_{j=1}^n \varphi_j$. Thus

$$\varphi = \sum_{n=1}^\infty \frac{M}{2^n}\chi_{E_n}.$$

If $c > 0$ then we replace φ by $\varphi - c$ and represent φ as

$$\sum_{n=1}^\infty \left(\frac{c}{2^n} + \frac{M}{2^n}\chi_{E_n}\right). \ \blacksquare$$

THEOREM 2.5. *Let E be a closed 0-dimensional subset of a compact metric space X and φ be a bounded function on X which is continuous on E and equal to a constant $c \ge 0$ on $X \setminus E$. Let $\varphi(x) > c$ for all $x \in E$. Then φ can be represented as an infinite sum of functions $a_j + b_j\chi_{E_j}$, where the sets E_j are closed, $a_j \ge 0$, $b_j > 0$, $\sum b_j$ and $\sum a_j$ converge.*

Proof. We start with the case $c = 0$. Let $E_1' = \{x \in E : \varphi(x) \ge 2M/3\}$. The set E_1' is closed and, hence, compact. For every point $x \in E_1'$ we choose a relatively open-closed neighborhood $U_x \subset E_1'$ such that $\varphi \ge M/2$ on U_x. The neighborhoods U_x cover E and we choose a finite subcover $\{U_{x_j}\}$, $1 \le j \le k$, and let $E_1 = \bigcup_{j=1}^k U_{x_j}$. Then the set E_1 is open and closed in E. Since E is closed, so is E_1.

Let $\psi_1 = (M/3)\chi_{E_1}$. The function $\varphi_1 = \varphi - \psi_1$ is continuous on E because the set E_1 is open and closed. It is equal to 0 on $X \setminus E$ and is greater than 0 on E. Moreover, $\varphi_1 < 2M/3$ on X.

Repeating this procedure inductively for the function φ_1 and so on, we get sequences of closed sets $E_j \subset E$ and functions

$$\psi_j = \frac{1}{3}\left(\frac{2}{3}\right)^j M\chi_{E_j}$$

such that the functions

$$\varphi_k = \varphi - \sum_{j=1}^k \psi_j$$

satisfy all conditions of the theorem and $\varphi_k \le (2/3)^k M$. Therefore, $\varphi = \sum \psi_j$.

If $c > 0$ then we represent $\varphi - c$ as a sum of $b_j\chi_{E_j}$, where the sets E_j are closed. Then

$$\varphi = \sum_{j=1}^\infty \left(\frac{c}{2^j} + b_j\chi_{E_j}\right). \ \blacksquare$$

3. Construction of plurisubharmonic functions in $\mathbb{C}^2$. In this section we show that functions on $\overline{\mathbb{C}}$ which admit representations as weighted sums of characteristic functions with convergent series of coefficients can be realized as order functions of plurisubharmonic functions. We start with the case of characteristic functions.

A plurisubharmonic function u on $\mathbb{C}^n$ is called *logarithmically homogeneous* with the coefficient $c \ge 0$ if $u(\lambda z) = c\log|\lambda| + u(z)$ whenever $\lambda \in \mathbb{C}$ and $z \in \mathbb{C}^n$.

LEMMA 3.1. *Let v be a subharmonic function on the complex plane $\mathbb{C}$ satisfying the growth condition $v(\zeta) \le \log^+|\zeta| = \max(\log|\zeta|, 0)$ and $0 < a < b$. Then the function u defined on $\mathbb{C}^2$ as*

$$u(z) = av(z_1/z_2) + b\log|z_2|$$

if $z_2 \ne 0$ and as $-\infty$ if $z_2 = 0$ is plurisubharmonic on $\mathbb{C}^2$. Moreover, u is logarithmically homogeneous with coefficient b and, if v is continuous, then so is u.

Proof. Clearly, u is plurisubharmonic when $z_2 \ne 0$. Hence, to prove that u is plurisubharmonic on $\mathbb{C}^2$, it suffices to check its upper semicontinuity at points of the plane $\{(z_1, z_2) : z_2 = 0\}$. Fix w_1 and let $z_2 \ne 0$ tend to 0 and z_1 to w_1. Then

$$\begin{aligned}\limsup_{z_1\to w_1,\, z_2\to 0} & u(z_1, z_2) \\ &\le \limsup_{z_1\to w_1,\, z_2\to 0} (\max(a\log|z_1| - a\log|z_2|, 0) + b\log|z_2|) \\ &= \limsup_{z_1\to w_1,\, z_2\to 0} \max(a\log|z_1| + (b-a)\log|z_2|, b\log|z_2|) \\ &= -\infty = u(w_1, 0).\end{aligned}$$

Therefore, u is usc on $\mathbb{C}^2$ and hence plurisubharmonic on $\mathbb{C}^2$.

Since $u(\zeta z) = b\log|\zeta| + u(z)$ for all $z \in \mathbb{C}^2$ and $\zeta \in \mathbb{C}$, u is logarithmically homogeneous with coefficient b. ∎

LEMMA 3.2. *Let E be a polar G_δ-subset of $\overline{\mathbb{C}}$ and $\varphi = a + b\chi_E$, $a, b > 0$, be a function on $\overline{\mathbb{C}}$. Then there exists a plurisubharmonic function u on $\mathbb{C}^2$, negative on the unit ball, such that $o_u(z) = \varphi(z)$ for all $z \neq 0$.*

Proof. We introduce on $\overline{\mathbb{C}}$ the homogeneous coordinates of $\mathbb{CP}^1$. We may assume E is non-empty and $(1:0)$ is in E. We introduce the coordinate $\zeta = z_1/z_2$ on $\mathbb{C} = \overline{\mathbb{C}} \setminus \{(1:0)\}$, where z_1 and z_2 are coordinates on $\mathbb{C}^2$. Let $E' = \zeta(E)$ be a set in $\mathbb{C}$. Clearly, for every integer $j \geq 2$ the sets $E_j = \{\zeta \in E' : j \leq |\zeta| \leq j+1\}$ are polar and G_δ. Thus there are subharmonic functions v_j on $\mathbb{C}$ such that $v_j(\zeta) = -\infty$ if and only if $\zeta \in E_j$. We may assume that for every j the measure $\mu_j = \Delta v_j$ is supported by the ring $\{j-1 \leq |\zeta| \leq j+2\}$ and $\mu_j(\mathbb{C}) = 2^{-j}$. The function

$$v_j'(\zeta) = \int_{\mathbb{C}} \log|\zeta - \xi|\,\mu_j(d\xi)$$

differs from v_j by a harmonic function and, therefore, is equal to $-\infty$ at the same points as v_j.

Let

$$v = \sum_{j=2}^{\infty} v_j'.$$

Since $|v_j'(\zeta)| \leq 2^{-j}\log(|\zeta| + j + 2)$, the series converges uniformly on compacta and v is subharmonic. Clearly, $v = -\infty$ on E. If $\zeta \notin E$, then $v_j'(\zeta) \neq -\infty$ for all j and $v_j'(\zeta) > 0$ when $j > |\zeta| + 2$. Thus $v(\zeta) \neq -\infty$.

If $|\zeta| > 1$, then

$$v(\zeta) \leq \sum_{j=2}^{\infty} 2^{-j}\log(|\zeta| + j + 2)$$
$$\leq \frac{1}{2}\log|\zeta| + \sum_{j=2}^{\infty} 2^{-j}\log(j+3) \leq \frac{1}{2}\log|\zeta| + c.$$

Let $u^1 = v - c$ and

$$u_1(z_1, z_2) = \begin{cases} u^1(z_1/z_2) + \log|z_2| & \text{if } z_2 \neq 0, \\ -\infty & \text{if } z_2 = 0. \end{cases}$$

Then by Lemma 3.1, u_1 is a logarithmically homogeneous plurisubharmonic function on $\mathbb{C}^2$ and is negative on B because $u^1(\zeta) \leq 0$ if $|\zeta| \leq 1$. Define $u(z) = \max(au_1(z), (a+b)\log\|z\|)$. The function $u(z)$ is plurisubharmonic on $\mathbb{C}^2$. Furthermore, if $v \in E$, then $u(\lambda v) = (a+b)\log\|\lambda v\|$. Hence $o_u(v) = a + b$. If $v \in \overline{\mathbb{C}} \setminus E$, then $o_u(v) = a$. Therefore, $o_u = \varphi$. ∎

If the set E is closed, then we have the following result.

LEMMA 3.3. *Let φ be a function of the form $\varphi = a + b\chi_E$ on $\overline{\mathbb{C}}$, where E is a closed polar subset of $\overline{\mathbb{C}}$ and $a, b > 0$. Then there exists a continuous plurisubharmonic function u such that $o_u = \varphi$.*

Proof. The proof repeats that of the previous lemma. One only takes into account that the sets E_j are compact. Therefore, by a result on page 181 of [8], the functions v_j and v_j' can be chosen to be continuous on $\mathbb{C}$. Now it is easy to check the continuity of u. ∎

The following two theorems show that realization of a function as an order function is possible for functions admitting representations as sums of characteristic functions.

THEOREM 3.4. *Let $\varphi = \sum_{j=1}^{\infty} \varphi_j$, where the $\varphi_j \geq 0$ are simple G_δ-functions on $\overline{\mathbb{C}}$ of the form $\varphi_j = a_j + b_j\chi_{E_j}$ with the following properties:*

(1) *$a_j, b_j > 0$ for all $j = 1, 2, \ldots$;*
(2) *E_j are polar G_δ-sets for $j = 1, 2, \ldots$;*
(3) *$\sum_{j=1}^{\infty} a_j + \sum_{j=1}^{\infty} b_j = a + b < \infty$.*

Then there exists a plurisubharmonic function $0 \geq u \in PSH(B)$ such that $o_u(v) = \varphi(v)$ for all $v \in \overline{\mathbb{C}}$.

Proof. By Lemma 3.2, for each j, there is a plurisubharmonic function u_j on $\mathbb{C}^2$ which is negative on B and satisfies $o_{u_j} = \varphi_j$. This function can be represented in the form

$$u_j(z) = \max\left(a_j u_j^1\left(\frac{z_1}{z_2}\right) + a_j\log|z_2|, a_j u_j^2\left(\frac{z_2}{z_1}\right) + a_j\log|z_1|, (a_j + b_j)\log|z|\right).$$

We let

$$u = \sum_{j=1}^{\infty} u_j.$$

Then

$$u(z) = \sum_{j=1}^{\infty} u_j(z) \geq \sum_{j=1}^{\infty} (a_j + b_j)\log|z| = (a+b)\log|z|.$$

Hence u is locally bounded on $B^* = B \setminus \{0\}$ and, consequently, plurisubharmonic on B. Clearly, $u < 0$ on B. By Lemma 1.3,

$$o_u(z) = \sum_{j=1}^{\infty} o_{u_j}(z) = \sum_{j=1}^{\infty} \varphi_j(z) = \varphi(z)$$

for all $z \in B \setminus \{0\}$. ∎

THEOREM 3.5. *Let $\varphi = \sum_{j=1}^{\infty} \varphi_j$, where the $\varphi_j = a_j + b_j \chi_{E_j}$ are functions on $\overline{\mathbb{C}}$ with the following properties*:

(1) $a_j, b_j > 0$ *for all* $j = 1, 2, \ldots$;
(2) E_j *are closed polar sets for* $j = 1, 2, \ldots$;
(3) $\sum_{j=1}^{\infty} a_j + \sum_{j=1}^{\infty} b_j = a + b < \infty$.

Then there exists a continuous plurisubharmonic function $0 \geq u \in PSH(B)$ *such that* $o_u(z) = \varphi(z)$ *for all* $z \in \overline{\mathbb{C}}$.

Proof. We note that condition (3) implies that the series for φ converges uniformly, which means that φ is an upper semicontinuous function.

To prove this theorem one simply repeats the argument for Theorem 3.4 and takes into account that by Lemma 3.3 for each j the functions u_j can be chosen to be continuous. Moreover, $|u_j(z)| \leq (a_j + b_j)|\log|z||$. Therefore, the series $\sum_{j=1}^{\infty} u_j$ converges uniformly on $B \setminus \{0\}$. ■

Theorems 2.4 and 2.5 allow us to describe some solutions of our problem more geometrically.

COROLLARY 3.6. *Let φ be a nonnegative bounded ambiguous function on $\overline{\mathbb{C}}$ which is equal to a constant $c > 0$ outside a polar set $E \subset \overline{\mathbb{C}}$ and greater than c on E. Then there exists a plurisubharmonic function u, negative on the unit ball B of $\mathbb{C}^2$ and such that $o_u = \varphi$.*

Proof. By Theorem 2.4 we have

$$\varphi = \sum_{j=1}^{\infty} (a_j + b_j \chi_{E_j}),$$

where the sets E_j are ambiguous and the series of coefficients converge. Now the theorem follows from Lemma 1.3 and Theorem 3.4. ■

COROLLARY 3.7. *Let E be a closed polar subset of $\overline{\mathbb{C}}$ and φ be a nonnegative bounded function on $\overline{\mathbb{C}}$ which is equal to a constant $c > 0$ outside E, continuous on E and greater than c on E. Then there exists a continuous plurisubharmonic function u, negative on the unit ball B of $\mathbb{C}^2$ and such that $o_u = \varphi$.*

Proof. By Theorem 2.5 we can represent φ as the sum of functions $a_j + b_j \chi_{E_j}$, where the sets E_j are closed and the series of coefficients converge. By Theorem 3.5 there is a continuous plurisubharmonic function u on B such that $o_u = \varphi$. ■

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Received July 17, 1996
Revised version December 23, 1996 (3716)