Proof. Let $A \in [B]$ be a minimal element with respect to the ordering $\prec$. By Theorem 5 the pair $(A, -A)$ is minimal. According to Lemma 5.1 of [1] the function $d = \min(f_A, f_{-A})$ belongs to $\mathcal{B}$. Note that

$$d(a) = \lim_{\delta(P)\to 0} \sum_{k=0}^{n} \min(f_A(a_{k+1}) - f_A(a_k), f_{-A}(a_{k+1}) - f_{-A}(a_k)),$$

where $\delta(P)$ is the diameter of the partition $P = \{0 = a_0 < \ldots < a_{n+1} = a\}$. Assume that $d \neq 0$. If $a \in [0, 2\pi]$ is a point of discontinuity of d then $H_A(a), H_{-A}(a)$ are both non-trivial line segments in $\mathbb{R}^2$. For $b \in [0, 2\pi]$ such that $|a - b| = \pi$ we have $H_A(b) = -H_{-A}(a)$. In this case $H_A(a), H_A(b)$ and $H_{-A}(a), H_{-A}(b)$ are non-trivial, parallel line segments contained in the boundaries of A and $-A$ respectively. Let I be the shortest of them. Then I is a summand of both A and $-A$, thus the pair $(A, -A)$ is not minimal. Therefore $d = 0$. Now, observe that $(A, -A) = (B, -B - x)$ for some $x \in \mathbb{R}^2$. In this case we can use Lemma 8 and Proposition 4, which gives the desired result. ∎

Let us finally remark that a minimal element in a class $[B] \in \mathcal{K}(\mathbb{R}^2)/\approx$ is given by the formula $A_{f_B - \min(f_B, f_{-B})}$. Since all the functions necessary to evaluate this expression are given directly our approach to the minimality is constructive.

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Operators determining the complete norm topology of $C(K)$

by

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Abstract. For any uniformly closed subalgebra A of $C(K)$ for a compact Hausdorff space K without isolated points and $x_0 \in A$, we show that every complete norm on A which makes continuous the multiplication by x_0 is equivalent to $\|\cdot\|_\infty$ provided that $x_0^{-1}(\lambda)$ has no interior points whenever λ lies in $\mathbb{C}$. Actually, these assertions are equivalent if $A = C(K)$.

0. Introduction. It is well known that for any infinite-dimensional Banach space X there exists an uncountable set of pairwise nonequivalent complete norms. The situation becomes nicer if we restrict our attention to those complete norms on X which make continuous a sufficiently good bilinear map from $X \times X$ into X (see [1] and [2]).

Our purpose is to look for continuous linear operators T on the Banach space $C(K)$, of all continuous functions on a compact Hausdorff space K, endowed with its traditional supremum norm $\|\cdot\|_\infty$, for which every complete norm $\|\cdot\|$ on $C(K)$ making continuous the operator T from $(C(K), \|\cdot\|)$ into $(C(K), \|\cdot\|)$ is automatically equivalent to $\|\cdot\|_\infty$. Such an operator is said to *determine the complete norm topology* of $C(K)$. In the first section we prove that the complete norm topology of any uniformly closed subalgebra of $C(K)$ is determined by the multiplication operator by x_0 if $x_0^{-1}(\lambda)$ is nowhere dense whenever λ lies on $\mathbb{C}$. From this fact we deduce unexpected characterizations of multiplication operators determining the complete norm topology either of $C(K)$ or the disc algebra $A(\mathbb{D})$. In the second section we exhibit a class of continuous linear operators that determine the complete norm topology on a given Banach space. Further we show that this class is nonempty for the Banach space $C(K)$ whenever K is a compact metric space.

1. Multiplication operators. In order to measure the continuity of a linear operator T from a Banach space X into a Banach space Y we will

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use its *separating subspace*, which is defined as the subspace $\mathcal{S}(T)$ of those elements y in Y for which there exists a sequence $\{x_n\}$ in X with $\lim x_n = 0$ and $\lim Tx_n = y$. The closed graph theorem shows that T is continuous if, and only if, $\mathcal{S}(T) = 0$.

LEMMA 1 [4; Lemma 1.6]. *Let X and Y be Banach spaces and let $\{R_n\}$ and $\{S_n\}$ be sequences of continuous linear operators on X and Y, respectively. If T is a linear operator from X into Y satisfying $S_nT = TR_n$ for all $n \in \mathbb{N}$, then there is $N \in \mathbb{N}$ such that $\overline{(S_1 \ldots S_n)\mathcal{S}(T)} = \overline{(S_1 \ldots S_N)\mathcal{S}(T)}$ for all $n \geq N$.*

THEOREM 1. *Let A be a uniformly closed subalgebra of $C(K)$ for some compact Hausdorff space K without isolated points and let $x_0 \in A$. If $x_0^{-1}(\lambda)$ has no interior points for all $\lambda \in \mathbb{C}$, then the multiplication operator by x_0 determines the complete norm topology of A.*

Proof. Assume that x_0 has the required property and let $\|\cdot\|$ be a complete norm on A making continuous the multiplication operator by x_0. Denote by $\mathcal{S}$ the separating subspace for the identity map from the Banach space $(A, \|\cdot\|)$ into the Banach space $(A, \|\cdot\|_\infty)$ and consider the closed subset K_0 of K given by

$$K_0 = \{t \in K : x(t) = 0 \ \forall x \in \mathcal{S}\}.$$

If we prove that $K_0 = K$, then $\mathcal{S} = 0$, which shows that the identity map is continuous and consequently $\|\cdot\|$ is equivalent to $\|\cdot\|_\infty$. To obtain a contradiction, suppose that there is a t_0 in the open subset $G = K \setminus K_0$. Since $x_0^{-1}(x_0(t_0))$ has no interior points we have $G \not\subset x_0^{-1}(x_0(t_0))$. Accordingly we could find $t_1 \in G$ with $x_0(t_1) \neq x_0(t_0)$. Now we suppose inductively that elements $t_1, \ldots, t_n \in G$ have been chosen with $x_0(t_0), x_0(t_1), \ldots, x_0(t_n)$ pairwise different. The open subset $\{t \in G : x_0(t) \neq x_0(t_k),\ k = 1, \ldots, n\}$ contains t_0 and therefore it is not contained in $x_0^{-1}(x_0(t_0))$. Hence we could find $t_{n+1} \in G$ such that $x_0(t_{n+1}) \neq x_0(t_k)$ for $k = 0, 1, \ldots, n$. For every $n \in \mathbb{N}$ the map $x \mapsto (x_0 - x_0(t_n))x$ defines a continuous linear operator, say R_n, on $(A, \|\cdot\|)$ and also a continuous linear operator, say S_n, on $(A, \|\cdot\|_\infty)$. On account of Lemma 1, we have

$$\overline{(x_0 - x_0(t_1)) \ldots (x_0 - x_0(t_n))\mathcal{S}}^{\|\cdot\|_\infty} = \overline{(x_0 - x_0(t_1)) \ldots (x_0 - x_0(t_N))\mathcal{S}}^{\|\cdot\|_\infty}$$

for all $n \geq N$, for a suitable $N \in \mathbb{N}$. We thus get

$$(x_0 - x_0(t_1)) \ldots (x_0 - x_0(t_N))\mathcal{S} \subset \overline{(x_0 - x_0(t_1)) \ldots (x_0 - x_0(t_{N+1}))\mathcal{S}}^{\|\cdot\|_\infty},$$

which clearly forces

$$(x_0(t_{N+1}) - x_0(t_1)) \ldots (x_0(t_{N+1}) - x_0(t_N))x(t_{N+1}) = 0$$

and therefore $x(t_{N+1}) = 0$ for all $x \in \mathcal{S}$, which contradicts the choice of t_{N+1}. ■

It should be noted that the converse of the preceding theorem is false. We illustrate this fact in the following.

EXAMPLE 1. Let $A = \{f \in C([-1,1]) : f([-1,0]) = 0\}$ and consider the function in A defined by

$$x_0(t) = \begin{cases} 0 & \text{if } -1 \leq t \leq 0, \\ t & \text{if } 0 \leq t \leq 1. \end{cases}$$

Note that the restriction map $x \mapsto x_{|[0,1]}$ gives an isometric isomorphism from A onto $B = \{f \in C([0,1]) : f(0) = 0\}$. From Theorem 1 we deduce that $x_{0|[0,1]}$ determines the complete norm topology of B and from this it follows easily that x_0 determines the complete norm topology of A.

For $C(K)$ the converse of Theorem 1 is true.

THEOREM 2. *Let K be a compact Hausdorff space without isolated points and let $x_0 \in C(K)$. Then the multiplication operator by x_0 determines the complete norm topology of $C(K)$ if, and only if, $x_0^{-1}(\lambda)$ has no interior points for all $\lambda \in \mathbb{C}$.*

Proof. Suppose that there is $\lambda \in \mathbb{C}$ for which the set $x_0^{-1}(\lambda)$ has a nonempty interior, say G. Since K has no isolated points, it may be concluded that G is infinite. Let $t_0 \in G$ and let u be a continuous function on K that takes the value 1 at t_0 and the value 0 at all points of $K \setminus G$. Since G is infinite, it follows that $\dim C(\overline{G}) = \infty$. Consequently, there exists a discontinuous linear functional g on $C(\overline{G})$ with $g(u_{|\overline{G}}) = 1$. We get a discontinuous linear functional f on $C(K)$ by defining $f(x) = g(x_{|\overline{G}})$ which satisfies $f(u) = 1$. As $x_0 - \lambda$ vanishes on $\overline{G}$ we have $0 = f((x_0 - \lambda)x) = f(x_0x) - \lambda f(x)$ for all $x \in C(K)$. The map $x \mapsto 2x - f(x)u$ defines a discontinuous linear bijection from $C(K)$ onto itself and therefore we may define a complete norm $\|\cdot\|$ on $C(K)$ which is not equivalent to $\|\cdot\|_\infty$ by

$$\|x\| = \|2x - f(x)u\|_\infty.$$

It remains to prove that this norm makes continuous the multiplication by x_0. Let $\{x_n\}$ be a sequence in $C(K)$ with $\lim \|x_n\| = 0$. For every $n \in \mathbb{N}$ we have

$$\begin{aligned} 2x_0x_n - f(x_0x_n)u &= 2x_0x_n - \lambda f(x_n)u \\ &= x_0(2x_n - f(x_n)u) + f(x_n)(x_0 - \lambda)u \\ &= x_0(2x_n - f(x_n)u). \end{aligned}$$

From this we conclude that $\|x_0 x_n\| \le \|x_0\|_\infty \|x_n\|$ and therefore $\lim \|x_0 x_n\| = 0$, which concludes the proof. ∎

In the following we exhibit examples of continuous functions on the interval $[0,1]$ whose multiplication operators determine the complete norm topology of $C([0,1])$, although each of the subalgebras generated by them is far from being $C([0,1])$.

EXAMPLES 2. 1. The function x_0 defined on $[0,1]$ by $x_0(0) = 0$ and $x_0(t) = t\sin(1/t)$ otherwise satisfies the requirement of the preceding theorem.

2. If $0 \le \varrho < 1$, then there exists a Cantor set C with Lebesgue measure ϱ. Define the continuous function x_0 on $[0,1]$ to be 0 on C and if $]a,b[$ is one of the open intervals forming $[0,1]\setminus C$, then define x_0 on $]a,b[$ by

$$x_0((1-t)a+tb) = \begin{cases} t & \text{if } 0 \le t \le 1/2, \\ 1-t & \text{if } 1/2 \le t \le 1. \end{cases}$$

Since no Cantor set contains an open interval we deduce that $x_0^{-1}(\lambda)$ is nowhere dense, although the set $x_0^{-1}(0)$ could have a nonzero Lebesgue measure.

Theorem 1 and the uniqueness theorem for holomorphic functions gives a surprising property of the disc algebra.

COROLLARY 1. *The multiplication operator by $x_0 \in A(\mathbb{D})$ determines the complete norm topology of $A(\mathbb{D})$ if, and only if, x_0 is not constant.*

Two questions still unanswered are the following.

Q1. Is Theorem 2 true for any uniform algebra?

Q2. Does every uniform algebra have a function x_0 satisfying the requirements of Theorem 1?

2. Shift operators. By a *shift operator* on a Banach space X we mean a continuous linear operator T on X for which $\ker T = 0$ and $\bigcup_{n=1}^{\infty} \ker(T^*)^n$ separates the points of X, where T^* stands for the dual operator of T.

EXAMPLES 3. 1. If a Banach space X has a shift operator T and Φ is an isomorphism from X onto another Banach space Y, then $\Phi T \Phi^{-1}$ is a shift operator on Y.

2. The traditional shift operators on the Banach spaces c_0 and l_p ($1 \le p \le \infty$) defined by $(Tx)(1) = 0$ and $(Tx)(n+1) = x(n)$ satisfy our preceding requirements. In fact, for every Banach space X, the Banach sequence spaces $c_0(X)$ and $l_p(X)$ are endowed with a shift operator defined in an obvious way. In particular, every infinite-dimensional Hilbert space H may be equipped with a shift operator, since in such a case H is isomorphic to $l_2(H)$.

3. The operator T defined on $L_p([0,1])$ by

$$(Tx)(t) = \begin{cases} 0 & \text{if } 0 \le t \le 1/2, \\ x(2t-1) & \text{if } 1/2 \le t \le 1, \end{cases}$$

is easily seen to be a shift operator.

4. It is a simple matter to show that

$$(Tx)(t) = \begin{cases} 0 & \text{if } 0 \le t \le 1/2, \\ (t-1/2)x(2t-1) & \text{if } 1/2 \le t \le 1. \end{cases}$$

defines a shift operator on $C([0,1])$.

THEOREM 3. *Every shift operator on a Banach space $(X, \|\cdot\|)$ determines the complete norm topology of X.*

Proof. Let $|\cdot|$ be a complete norm on X and let $\mathcal{S}$ denote the separating subspace for the identity map from the Banach space $(X,|\cdot|)$ into the Banach space $(X,\|\cdot\|)$. Lemma 1 now yields $N \in \mathbb{N}$ such that $\overline{T^n\mathcal{S}} = \overline{T^N\mathcal{S}}$ for all $n \ge N$. Accordingly, for every $f \in \bigcup_{n=N+1}^{\infty} \ker(T^*)^n$ we have $f(T^N\mathcal{S}) = 0$ and therefore $T^N(\mathcal{S}) = 0$, since $\bigcup_{n=N+1}^{\infty} \ker(T^*)^n$ separates the points of X. Since $\ker T = 0$, it may be concluded that $\mathcal{S} = 0$, which shows that $|\cdot|$ is equivalent to $\|\cdot\|$. ∎

THEOREM 4. *Let K be an infinite compact metric space. Then $C(K)$ is equipped with a shift operator.*

Proof. If K is uncountable, then the Milyutin theorem [3; Theorem 21.5.10] shows that $C(K)$ is isomorphic to $C([0,1])$. On account of Examples 3.1 and 3.4 there is a shift operator on $C(K)$.

If K is countable, then by the Mazurkiewicz–Sierpiński theorem [3; Theorem 8.6.1], there are a countable ordinal τ and a finite ordinal n such that K is homeomorphic to $\omega^\tau n + 1$. Consequently, $C(K)$ is isomorphic to $C(\omega^\tau n+1)$. We only need to show that a shift operator may be defined on $C(\omega^\tau n+1)$. For every $x \in C(\omega^\tau n+1)$ we define Tx to be zero at every limit ordinal in $\omega^\tau n+1$, at 0, and at every nonzero nonlimit ordinal in $\omega^\tau n+1$ whose predecessor is a limit ordinal, and otherwise we define $(Tx)(\alpha) = x(\text{predecessor of } \alpha)d(\alpha, \min\{\text{limit ordinals greater than } \alpha\})$. It is easy to check that T is a well-defined shift operator on $C(\omega^\tau n+1)$. ∎

It would be desirable to answer the following question.

Q3. Is the preceding theorem true for nonmetrizable compact Hausdorff spaces?

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Order functions of plurisubharmonic functions

by

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Abstract. We consider the following problem: find on $\mathbb{C}^2$ a plurisubharmonic function with a given order function. In particular, we prove that any positive ambiguous function on $\mathbb{CP}^1$ which is constant outside a polar set is the order function of a plurisubharmonic function.

1. The order function for plurisubharmonic functions. In this paper we study pointwise singularities of plurisubharmonic functions, i.e., the behavior of a plurisubharmonic function near isolated points where the function's value is $-\infty$. Singularities of plurisubharmonic functions on subsets of $\mathbb{C}^n$ have been studied by many authors (see [7] for references), in general using the notion of the Lelong number. Unfortunately, this number does not provide a detailed description of the singularity. For example, another important characteristic of singularities—the mass of the Monge–Ampère operator at these points—has little to do with their Lelong numbers (see Ex. 5.7 in [4]).

We concentrate on the notion of the order function which reflects more features of the function's behavior. Given a plurisubharmonic function u on the unit ball $B \subset \mathbb{C}^n$, centered at the origin, the *order function* o_u of u at 0 is defined as

$$o_u(z) = \lim_{r\to 0} \inf_{|\gamma|=r} \frac{u(\gamma z)}{\log|\gamma z|},$$

where z is in $\mathbb{C}^n \setminus \{0\}$ and $\gamma \in \mathbb{C}$. Since $o_u(z) = o_u(\gamma z)$, $\gamma \neq 0$, we may assume that the order function is defined on the unit sphere S in $\mathbb{C}^n$ or on the complex projective space $\mathbb{CP}^{n-1}$, which for $n = 2$ coincides with the Riemann sphere $\overline{\mathbb{C}}$.

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