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Département de Mathématiques
Case 051
Université Montpellier II
F-34095 Montpellier Cedex 5, France
E-mail: castaing@darboux.math.univ-montp2.fr

Département de Mathématiques Mohammed V
Faculté des Sciences Agdal
Rabat, Maroc

CEREMADE
URA CNRS No 749
Université Paris-Dauphine
75775 Paris Cedex 16, France
E-mail: hess@ceremade.dauphine.fr

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Minimality in asymmetry classes

by

MICHAŁ WIERNOWOLSKI (Poznań)

Abstract. We examine minimality in asymmetry classes of convex compact sets with respect to inclusion. We prove that each class has a minimal element. Moreover, we show there is a connection between asymmetry classes and the Rådström–Hörmander lattice. This is used to present an alternative solution to the problem of minimality posed by G. Ewald and G. C. Shephard in [4].

1. Introduction. We will denote by $\mathcal{K}(X)$ the space of non-empty convex compact subsets of a topological vector space X. The space $\mathcal{K}(X)$ has been widely investigated, especially in connection with the Minkowski sum defined by $A+B := \{a+b : a \in A,\ b \in B\}$. Although $\mathcal{K}(X)$ with the Minkowski sum forms a commutative semigroup with the cancellation law (see R. Urbański [5]), it is not a vector space. In [4] G. Ewald and G. C. Shephard introduced some normed vector spaces consisting of classes of convex compact sets. Let us recall one of those concepts, the so called asymmetry classes. As in [4] we will restrict our examination to the finite-dimensional case ($X = \mathbb{R}^n$, $n \in \mathbb{N}$). We define the relation of asymmetry: $A \approx B$ iff there exist centrally symmetric ([1]) sets S, T such that $A+S$ is a translation of $B+T$ (we can require S and T to be symmetric instead of centrally symmetric). It has been proved in [4] that $\approx$ is an equivalence relation and $\mathcal{K}(\mathbb{R}^n)/\approx$ is a normed vector space.

In [4] the authors posed the question whether each class of asymmetry can be expressed in the form $\{M+S : S \text{ centrally symmetric}\}$, where $M \in \mathcal{K}(X)$ is a certain "minimal" element. This problem has been solved by R. Schneider [3]. It is proved there that for $n = 2$ every asymmetry class contains a minimal member (Theorem I). In the proof, measure theory as well as surface area functions are employed. We will present a different approach.

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([1]) Centrally symmetric sets are translations of certain symmetric sets.

First of all, notice that if M is "minimal" then it must be minimal with respect to inclusion. This suggests examining the ordering

$$A \prec B \quad \text{iff} \quad A \approx B,\ A \subseteq B.$$

We will show that in each asymmetry class there are minimal elements with respect to this ordering.

Secondly, we will show that asymmetry classes have a connection with the Rådström–Hörmander lattice. Recall that this is the quotient space $\mathcal{K}^2(X)/\sim$ where $(A,B) \sim (C,D)$ iff $A+D = B+C$. We will also use a special ordering in $\mathcal{K}^2(X)$, namely

$$(A,B) \le (C,D) \quad \text{iff} \quad A \subseteq B,\ C \subseteq D,\ (A,B) \sim (C,D).$$

It has been proved by D. Pallaschke, S. Scholtes and R. Urbański [2] that in each class of $\sim$ there exists a minimal element with respect to $\le$.

2. Auxilliary lemmas. In this section we assume that X is a normed vector space. We use the Hausdorff metric which is defined for closed bounded subsets of X by the formula

$$d_{\mathrm{H}}(A,B) := \inf\{\lambda > 0 : A \subseteq B + \lambda U,\ B \subseteq A + \lambda U\},$$

where U denotes the closed unit ball.

LEMMA 1. *If a family $\{A_\alpha\}$ of compact sets is a chain then there exists a countable subfamily $\{A_{\alpha_n}\}$ such that*

$$\bigcap_\alpha A_\alpha = \bigcap_n A_{\alpha_n} \quad \textit{and} \quad \lim_{n\to\infty} d_{\mathrm{H}}\Big(A_{\alpha_n}, \bigcap_\alpha A_\alpha\Big) = 0.$$

Proof. Assume that $A_\alpha \subseteq A_\beta$ for $\alpha \le \beta$. Let $B := \bigcap_\alpha A_\alpha$. The set B is non-empty as the intersection of a chain of compact sets. Since for $\alpha \le \beta$ we have $0 \le d_{\mathrm{H}}(A_\alpha, B) \le d_{\mathrm{H}}(A_\beta, B)$, the limit $\lim_{\alpha\downarrow} d_{\mathrm{H}}(A_\alpha, B)$ exists. We show that it is 0. If this is not the case then $\exists_{\varepsilon>0}\forall_\alpha d_{\mathrm{H}}(A_\alpha,B) > \varepsilon$. This is equivalent to $\forall_\alpha \exists_{x_\alpha\in A_\alpha} \varrho(x_\alpha, B) > \varepsilon$, where ϱ is the standard distance between a point and a set. Let C_α be the closure of the set $\{x_\beta : \beta \le \alpha\}$. The family $\{C_\alpha\}$ is a chain of compact sets, therefore $C := \bigcap_\alpha C_\alpha$ is not empty. Plainly $C \subseteq B$, while on the other hand $\forall_{x\in C}\varrho(x,B) \ge \varepsilon$. This contradiction proves that A_α converges to B. Now, it is enough to choose α_n satisfying $d_{\mathrm{H}}(A_{\alpha_n}, B) \le 1/n$ and $\alpha_{n+1} \le \alpha_n$. ■

LEMMA 2. *If $\{B_\alpha\}$ is a chain of compact sets then $A + \bigcap_\alpha B_\alpha = \bigcap_\alpha (A + B_\alpha)$ for every compact set A.*

Proof. $\subseteq$ always holds.

$\supseteq$ In view of the previous lemma we can assume to have a countable family $\{B_n\}$. Moreover, we assume $B_{n+1} \subseteq B_n$. Let $B := \bigcap_n B_n$. We have to show that for a given $z \in \bigcap_n (A + B_n)$ there exist $a \in A$ and $b \in B$ such that $z = a + b$. Obviously $z = a_n + b_n$ for some $a_n \in A$ and $b_n \in B_n$. The sequences $\{a_n\}$ and $\{b_n\}$ are contained in compact sets, thus there are subsequences $\{a_{n_k}\}$ and $\{b_{n_k}\}$ converging to some a and b. Plainly $a \in A$, $b \in B$ and $a + b = \lim a_{n_k} + \lim b_{n_k} = \lim(a_{n_k} + b_{n_k}) = \lim z = z$. ■

3. Minimality with respect to inclusion. Now we present several properties of the relations $\approx$ and $\prec$.

PROPOSITION 3. *$A \approx B$ iff $A - B$ is centrally symmetric.*

Proof. $\Rightarrow$ If $A \approx B$ then there exist symmetric sets S, T and $x \in X$ such that $A + S = B + T + x$. This implies $-A + S = -B + T - x$. Let us add the above two equations crosswise: $A + S - B + T - x = -A + S + B + T + x$. Using the cancellation law we get $A - B = B - A + 2x$.

$\Leftarrow$ If $A - B$ is centrally symmetric then $A - B = B - A + x$ for some $x \in X$. By adding B we get $A + (B - B) = B + (B - A) + x$. Observe that $B - B$ and $B - A$ are centrally symmetric, hence $A \approx B$. ■

PROPOSITION 4. *If $A \approx B$ and $B = A + P$ then P is centrally symmetric.*

Proof. By Proposition 3 the set $A - B$ is centrally symmetric, so there exists $x \in X$ such that $A - B = B - A + x$. Replacing B with $A + P$ gives $A - A - P = A + P - A + x$. This implies that $-P = P + x$, thus P is centrally symmetric. ■

THEOREM 5. *Each asymmetry class has a minimal element with respect to the ordering $\prec$.*

Proof. We use the Kuratowski–Zorn Lemma. Let $\{A_\alpha\}$ be a chain contained in an asymmetry class $[B]$. We will show that $A := \bigcap_\alpha A_\alpha$ is a lower bound of this chain. First observe A is non-empty and convex, since A_α are compact and convex. In view of Lemma 1 we can skip to a countable chain $\{A_n\}$. Without loss of generality we can assume $0 \in A \cap B$ and $A_{n+1} \subseteq A_n$.

From Proposition 3 it follows that $A_n - B = B - A_n + x_n$ for some x_n. Using Lemma 2 we can derive

$$A - B = \bigcap_n A_n - B = \bigcap_n (A_n - B) = \bigcap_n (B - A_n + x_n)$$
$$= B + \bigcap_n (x_n - A_n).$$

Since $0 \in B \cap A_n$ we have $x_n \in A_n - B \subseteq A_1 - B$. The last set is compact, so (x_n) must have a cluster point. Since we can always skip to a subsequence, we will assume even more, namely $x_n \to x$. We will show that $\bigcap_n (x_n - A_n) = x - A$ (notice that $\{x_n - A_n\}$ is a chain). Using the properties of the Hausdorff metric we get

$$
\begin{aligned}
d_{\mathrm{H}}\Big(x - A, \bigcap_n (x_n - A_n)\Big) &\le d_{\mathrm{H}}(x - A, x_n - A) + d_{\mathrm{H}}(x_n - A, x_n - A_n) \\
&\quad + d_{\mathrm{H}}\Big(x_n - A_n, \bigcap_n (x_n - A_n)\Big) \\
&= d(x, x_n) + d_{\mathrm{H}}(A, A_n) + d_{\mathrm{H}}\Big(x_n - A_n, \bigcap_n (x_n - A_n)\Big).
\end{aligned}
$$

Lemma 1 guarantees the above distances to be arbitrarily small. This yields $A - B = B + (x - A) = B - A + x$. Again from Proposition 3 we get $A \approx B$. Finally, $\forall_n A \prec A_n$. Since every chain has a lower bound there exists a minimal element in the class $[B]$. ∎

THEOREM 6. *The set A is minimal (with respect to $\prec$) iff the pair $(A, -A)$ is minimal (with respect to $\le$).*

Proof. $\Leftarrow$ Assume that the pair $(A, -A)$ is minimal, while the set A is not. In this case there exists a set B such that $B \prec A$, $B \ne A$. Proposition 3 gives $A - B = B - A + x$ for some $x \in X$. Using this equality and the cancellation law we get $(B, -B - x) \le (A, -A)$. This contradicts the minimality of $(A, -A)$.

$\Rightarrow$ Suppose that A is minimal but $(A, -A)$ is not. Then there exists a pair $(B, C) \le (A, -A)$, $(B, C) \ne (A, -A)$. From the definition of $\sim$ it follows that $A + C = B - A$. Multiplying this equation by -1 gives $-A - C = A - B$. By adding the above equations crosswise we get $2A + C - B = -2A + B - C$. This leads to the equation

$$A - \frac{B - C}{2} = \frac{B - C}{2} - A.$$

Finally, $\frac{B-C}{2} \prec A$, which contradicts the minimality of A. ∎

4. Applications to the minimality problem of G. Ewald and G. C. Shephard. In order to apply our observations from the previous section we need some more definitions.

In [1] J. Grzybowski proved that for $X = \mathbb{R}^2$ minimal pairs are unique up to translation. We will use J. Grzybowski's approach using arc-length functions.

Let $\mathcal{A}$ be the set of all functions $f : [0, 2\pi] \to \mathbb{R}_+$ such that

1. f is non-decreasing, $f(0) = 0$,
2. $2f(a) = f(a+) + f(a-)$ for $a \in (0, 2\pi)$,
3. $f(2\pi) = f(0+) + f(2\pi-)$,

where $f(x+)$, $f(x-)$ denote the right and left limits of f at x. We define an ordering in $\mathcal{A}$ in the following manner: $f \angle g$ iff $g - f \in \mathcal{A}$. For any $f, g \in \mathcal{A}$ there exists $h \in \mathcal{A}$ such that $h = \min(f, g)$ with respect to $\angle$. We will denote by $\mathcal{B}$ the subset of $\mathcal{A}$ consisting of all functions f such that $f([0, 2\pi])$ has no more than 3 elements.

Let $e^{ia} = (\cos a, \sin a)$ and $A \in \mathcal{K}(X)$. For $a \in [0, 2\pi]$ we denote by $h_A(a)$ the center of the set $H_A(a) = \{x \in A : \langle e^{ia}, x\rangle = \max_{y \in A}\langle e^{ia}, y\rangle\}$, where $\langle \cdot, \cdot \rangle$ is the scalar multiplication in $\mathbb{R}^2$. The set $H_A(a)$ is either a point or a line segment in $\mathbb{R}^2$. For any $A \in \mathcal{K}(X)$ we define $f_A \in \mathcal{A}$ by

$$f_A(a) = \lim_{\delta(P) \to 0} \sum_{k=0}^{n} \|h_A(a_{k+1}) - h_A(a_k)\|,$$

where $\delta(P)$ is the diameter of the partition $P = \{0 = a_0 < \ldots < a_{n+1} = a\}$. Note that $f_A(a)$ is the length of the boundary arc of A connecting $h_A(0)$ and $h_A(a)$. For $f \in \mathcal{A}$ we define a function $h_f : [0, 2\pi] \to \mathbb{R}^2$ by

$$h_f(a) = \lim_{\delta(P) \to 0} \sum_{k=0}^{n} (f(a_{k+1}) - f(a_k)) e^{i(a_k + \pi/2)}.$$

If $h_f(2\pi) = 0$ for $f \in \mathcal{A}$ then we denote by A_f the smallest convex compact set containing $h_f([0, 2\pi])$.

LEMMA 7. *A is a summand* [2] *of B iff $f_A \angle f_B$.*

Proof. $\Rightarrow$ See Proposition 3.4 of [1].

$\Leftarrow$ If $f_A \angle f_B$ then $f_B = f_A + (f_B - f_A)$. Using Proposition 4.8 of [1] we deduce that $f_B - f_A + g = f_C$ for some $g \in \mathcal{B}$ and $C \in \mathcal{K}(X)$. This gives $f_B + g = f_A + f_C = f_{A+C}$. Observe that

$$
\begin{aligned}
h_g(2\pi) &= h_{f_{A+C}}(2\pi) - h_{f_B}(2\pi) \\
&= h_{A+C}(2\pi) - h_{A+C}(0) - (h_B(2\pi) - h_B(0)) = 0
\end{aligned}
$$

(see [1], Proposition 4.2). Since $g \in \mathcal{B}$ we have $\|h_g(2\pi)\| = g(2\pi) = 0$, thus $g = 0$. This implies $f_B = f_{A+C}$. By Proposition 4.5 of [1], $B = A + C + x$ for some $x \in X$. ∎

LEMMA 8. *If $(A, B) \sim (C, D)$ and $\min(f_A, f_B) = 0$ then A is a summand of C.*

Proof. Using Theorem 2.5 of [1] for the functions $f_A, f_C - \min(f_C, f_D)$, $f_B, f_D - \min(f_C, f_D)$ we deduce that $f_A = f_C - \min(f_C, f_D)$. This gives $f_A \angle f_C$. From Lemma 7 it follows that A is a summand of C. ∎

THEOREM 9. *Each asymmetry class $[B] \in \mathcal{K}(\mathbb{R}^2)/\approx$ contains a "minimal" element M such that $[B] = \{M + S : S$ centrally symmetric$\}$. Moreover, the set M is minimal with respect to the ordering $\prec$.*

---

(2) A is called a *summand* of B if there exists a set C such that $A + C = B$.

Proof. Let $A \in [B]$ be a minimal element with respect to the ordering $\prec$. By Theorem 5 the pair $(A, -A)$ is minimal. According to Lemma 5.1 of [1] the function $d = \min(f_A, f_{-A})$ belongs to $\mathcal{B}$. Note that

$$d(a) = \lim_{\delta(P)\to 0} \sum_{k=0}^{n} \min(f_A(a_{k+1}) - f_A(a_k), f_{-A}(a_{k+1}) - f_{-A}(a_k)),$$

where $\delta(P)$ is the diameter of the partition $P = \{0 = a_0 < \ldots < a_{n+1} = a\}$. Assume that $d \neq 0$. If $a \in [0, 2\pi]$ is a point of discontinuity of d then $H_A(a), H_{-A}(a)$ are both non-trivial line segments in $\mathbb{R}^2$. For $b \in [0, 2\pi]$ such that $|a - b| = \pi$ we have $H_A(b) = -H_{-A}(a)$. In this case $H_A(a), H_A(b)$ and $H_{-A}(a), H_{-A}(b)$ are non-trivial, parallel line segments contained in the boundaries of A and $-A$ respectively. Let I be the shortest of them. Then I is a summand of both A and $-A$, thus the pair $(A, -A)$ is not minimal. Therefore $d = 0$. Now, observe that $(A, -A) = (B, -B - x)$ for some $x \in \mathbb{R}^2$. In this case we can use Lemma 8 and Proposition 4, which gives the desired result. ∎

Let us finally remark that a minimal element in a class $[B] \in \mathcal{K}(\mathbb{R}^2)/\approx$ is given by the formula $A_{f_B - \min(f_B, f_{-B})}$. Since all the functions necessary to evaluate this expression are given directly our approach to the minimality is constructive.

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Faculty of Mathematics and Computer Science
Adam Mickiewicz University
Matejki 48/49
60-769 Poznań, Poland
E-mail: mike@math.amu.edu.pl

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Operators determining the complete norm topology of $C(K)$

by

A. R. VILLENA (Granada)

Abstract. For any uniformly closed subalgebra A of $C(K)$ for a compact Hausdorff space K without isolated points and $x_0 \in A$, we show that every complete norm on A which makes continuous the multiplication by x_0 is equivalent to $\|\cdot\|_\infty$ provided that $x_0^{-1}(\lambda)$ has no interior points whenever λ lies in $\mathbb{C}$. Actually, these assertions are equivalent if $A = C(K)$.

0. Introduction. It is well known that for any infinite-dimensional Banach space X there exists an uncountable set of pairwise nonequivalent complete norms. The situation becomes nicer if we restrict our attention to those complete norms on X which make continuous a sufficiently good bilinear map from $X \times X$ into X (see [1] and [2]).

Our purpose is to look for continuous linear operators T on the Banach space $C(K)$, of all continuous functions on a compact Hausdorff space K, endowed with its traditional supremum norm $\|\cdot\|_\infty$, for which every complete norm $\|\cdot\|$ on $C(K)$ making continuous the operator T from $(C(K), \|\cdot\|)$ into $(C(K), \|\cdot\|)$ is automatically equivalent to $\|\cdot\|_\infty$. Such an operator is said to *determine the complete norm topology* of $C(K)$. In the first section we prove that the complete norm topology of any uniformly closed subalgebra of $C(K)$ is determined by the multiplication operator by x_0 if $x_0^{-1}(\lambda)$ is nowhere dense whenever λ lies on $\mathbb{C}$. From this fact we deduce unexpected characterizations of multiplication operators determining the complete norm topology either of $C(K)$ or the disc algebra $A(\mathbb{D})$. In the second section we exhibit a class of continuous linear operators that determine the complete norm topology on a given Banach space. Further we show that this class is nonempty for the Banach space $C(K)$ whenever K is a compact metric space.

1. Multiplication operators. In order to measure the continuity of a linear operator T from a Banach space X into a Banach space Y we will

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