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Convergence of conditional expectations for unbounded closed convex random sets

by

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Abstract. We discuss here several types of convergence of conditional expectations for unbounded closed convex random sets of the form $E^{\mathcal{B}_n}X_n$ where $(\mathcal{B}_n)$ is a decreasing sequence of sub-σ-algebras and (X_n) is a sequence of closed convex random sets in a separable Banach space.

1. Introduction. The Mosco convergence of sequences of sets or functions is known to be a useful tool in the approximation of optimization problems and variational inequalities (see e.g. [A, Mo, We]). Often these problems are considered in the presence of a parameter ω whose value depends on the outcome of a random experiment.

The present paper precisely concerns Mosco convergence in such a stochastic context. Indeed, our main contribution consists in the study of almost sure Mosco convergence for sequences of random sets of the form $E^{\mathcal{B}_n}X_n$, where $(\mathcal{B}_n)_{n\geq 1}$ is a decreasing sequence of sub-σ-algebras and (X_n) a sequence of Banach-valued closed convex random sets (recall that a *random set* is a random variable whose values are subsets of some given space).

It is worthwhile to observe that, even for real-valued random variables, results of such kind are not completely standard. That is why we provide a short and self-contained treatment of the problem in this special case (in Section 4.A). This is done in the same spirit as in the papers by Szynal and Zięba [SZ] and by Zięba [Zi].

On the other hand, we stress the fact that the values of the random sets we deal with are not assumed to be bounded. So, specific results borrowed from [He1, 2] are needed; they are recalled in Section 3 for convenience. Our main results and their proofs are presented in Sections 4.B and 4.C.

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We have mentioned above that our results are valid for random sets with unbounded values. This allows us to deduce, in Section 5, a property concerning Mosco epi-convergence of sequences of integrands.

Our results can be seen as extensions of previous results of Hiai [Hi2] and Hess [He2], where the sequence $(\mathcal{B}_n)$ was constant (i.e. $\mathcal{B}_n = \mathcal{B}$). They also provide variants of the results of Couvreux [Co] where the random sets X_n are equal to a fixed random set X.

2. Notations and definitions. Throughout this paper, $(\Omega, \mathcal{F}, P)$ denotes an abstract probability space, $(\mathcal{B}_n)_{n\ge1}$ is a decreasing sequence of sub-σ-algebras of $\mathcal{F}$ and $\mathcal{B}_\infty = \bigcap_{n\ge1}\mathcal{B}_n$.

Let E be a separable Banach space with dual space E^*. For each $C \subset E$, $\mathrm{cl}\, C$, w-$\mathrm{cl}\, C$ and $\overline{\mathrm{co}}\, C$ denote the *norm closure*, the *weak closure*, and the *closed convex hull* of C, respectively; the *distance function* $d(\cdot, C)$ and the *support function* $\delta^*(\cdot, C)$ of C are defined by

$$d(x,C) = \inf[\|x-y\| : y \in C], \quad x \in E,$$

and

$$\delta^*(x^*, C) = \sup[\langle x^*, x\rangle : x \in C], \quad x^* \in E^*.$$

Let $\mathrm{c}(E)$ (resp. $\mathrm{cc}(E)$, $\mathrm{cwk}(E)$) be the family of all nonempty closed (resp. closed convex, convex weakly compact) subsets of E. Let $\mathcal{L}\mathrm{wc}(E)$ be the family of nonempty closed convex weakly locally compact subsets of E which contain no line. Let $C \in \mathrm{cc}(E)$ and $x_0 \in C$. Recall that the *asymptotic cone* of C is the greatest convex cone Γ such that $x_0 + \Gamma \subset C$. This cone, which does not depend on x_0, is denoted by $\mathrm{As}(C)$. We also have

$$\mathrm{As}(C) = \bigcap_{t>0} t(C - x_0),$$

and $\mathrm{As}(C)$ is the polar cone of $\mathrm{dom}\,\delta^*(\cdot, C) = \{x^* \in E^* : \delta^*(x^*, C) < \infty\}$.

In the present paper, we shall use a notion of convergence for sequences of subsets which has been introduced by Mosco [Mo] and which is related to that of Kuratowski. Let t be a topology on E and $(C_n)_{n\ge1}$ a sequence in $\mathrm{c}(E)$. We put

$$t\text{-li}\, C_n = \{x \in E : x = t\text{-lim} x_n,\ x_n \in C_n,\ \forall n \ge 1\},$$
$$t\text{-ls}\, C_n = \{x \in E : x = t\text{-lim} x_k,\ x_k \in C_{n(k)},\ \forall k \ge 1\},$$

where $(C_{n(k)})_{k\ge1}$ is a subsequence of (C_n). The subsets t-li C_n and t-ls C_n are the *lower limit* and the *upper limit* of (C_n), relative to the topology t. We obviously have t-li $C_n \subset t$-ls C_n. A sequence (C_n) is said to converge to C_∞ *in the sense of Kuratowski* relative to the topology t if

$$t\text{-ls}\, C_n = t\text{-li}\, C_n = C_\infty.$$

In this case, we shall write $C_\infty = \text{M-lim}_n C_n$; this is true if and only if

$$t\text{-ls}\, C_n \subset C_\infty \subset t\text{-li}\, C_n.$$

Let us denote by s (resp. w) the strong (resp. weak) topology of E. A subset C_∞ is said to be the *Mosco limit* of the sequence $(C_n)_{n\ge1}$ (denoted by M-$\lim_n C_n$) if

$$C_\infty = \text{s-li}\, C_n = \text{w-ls}\, C_n,$$

which is true if and only if

$$\text{w-ls}\, C_n \subset C_\infty \subset \text{s-li}\, C_n.$$

The corresponding definitions of pointwise convergence and almost sure convergence for a sequence (X_n) of multifunctions defined on Ω are clear. In fact, in the above definitions, it suffices to replace C_n by $X_n(\omega)$ and C_∞ by $X_\infty(\omega)$ for all $\omega \in \Omega$ (or for almost every ω).

Concerning the Mosco convergence, we refer to Mosco [Mo], Wets [We], and Attouch [A]. In the present paper, $\mathbb{N}^*$ denotes the set of strictly positive integers, and $\mathbb{R}$ (resp. $\mathbb{R}^+$) the set of real numbers (resp. positive real numbers).

A closed-valued *multifunction* X, i.e., a map from Ω to $\mathrm{c}(E)$, is said to be *measurable* if, for every open subset U of E, the subset

$$X^-U = \{\omega \in \Omega : X(\omega) \cap U \ne \emptyset\}$$

belongs to $\mathcal{F}$. A measurable multifunction is also called a *random set*. A function f from Ω to E is said to be a *selection* of X if, for any ω in Ω, $f(\omega) \in X(\omega)$. A *Castaing representation* of X is a sequence $(f_n)_{n\ge1}$ of measurable selections of X such that for all ω, $X(\omega) = \mathrm{cl}\{f_n(\omega) : n \ge 1\}$. It is known ([CV], Theorem III.9) that a closed-valued multifunction X is measurable if and only if it has a Castaing representation, or if and only if the real function $d(x, X(\cdot))$ is measurable for all x in E. Further, the *Effros σ-field* $\mathcal{E}$ on $\mathrm{c}(E)$ is generated by the subsets

$$U^{-} := \{F \in \mathrm{c}(E) : F \cap U \ne \emptyset\}$$

where U is an open subset in E. Then a multifunction $\Gamma : \Omega \to \mathrm{c}(E)$ is measurable if and only if, for any $B \in \mathcal{E}$, one has $\Gamma^{-1}(B) \in \mathcal{F}$. Let $L^1(\Omega, \mathcal{F}, P, E) = L^1(\Omega, E)$ denote the Banach space of (equivalence classes of) measurable functions f from Ω to E such that

$$\|f\|_1 = E\|f\| = \int_\Omega \|f(\omega)\|\, P(d\omega)$$

is finite. For any $\mathcal{F}$-measurable random set X, we put

$$S^1_X(\mathcal{F}) = \{f \in L^1(\Omega, E) : f(\omega) \in X(\omega) \text{ a.s.}\},$$

which is a closed subset of $L^1(\Omega, E)$ and is nonempty if and only if the real function $d(0, X(\cdot))$ is in $L^1(\Omega, \mathbb{R})$. In this case, we shall say that the random set X is *integrable*. On the other hand, a random set X is said to be *strongly integrable* or *integrably bounded* if the function $|X(\cdot)|$ is in L^1 where $|X(\cdot)|$ is defined for all ω in Ω by $|X(\omega)| = \sup\{\|x\| : x \in X(\omega)\}$. Given a sub-$\sigma$-field $\mathcal{B}$ of $\mathcal{F}$, and an $\mathcal{F}$-measurable integrable random set X, Hiai and Umegaki ([HU]) showed the existence of a $\mathcal{B}$-measurable random set G such that

$$S^1_G(\mathcal{B}) = \mathrm{cl}\{E^{\mathcal{B}} f : f \in S^1_X(\mathcal{F})\},$$

the closure being taken in $L^1(\Omega, E)$. Such a random set G is the *multivalued conditional expectation* of X relative to $\mathcal{B}$ and is denoted by $E^{\mathcal{B}}X$.

3. Preliminaries. Throughout this section, E^* is assumed to be endowed with the Mackey topology and D^* denotes a countable dense subset of E^*. We begin with the following proposition concerning the basic algebraic properties of the multivalued conditional expectation which are borrowed from [Hi1] (see also [HU] for the integrably bounded case).

PROPOSITION 3.1. *If F and G are two integrable random sets with closed values in E, and $\mathcal{B}$ is a sub-σ-algebra of $\mathcal{F}$, then*

(a) $E^{\mathcal{B}}(F+G) = \mathrm{cl}[E^{\mathcal{B}}F + E^{\mathcal{B}}G]$ *a.s.*

(b) *If r is a real $\mathcal{B}$-measurable function such that rF is integrable, then*

$$E^{\mathcal{B}}(rF) = rE^{\mathcal{B}}F \quad a.s.$$

(c) *If g is a bounded scalarly $\mathcal{B}$-measurable function from Ω to E^*, then*

$$\delta^*(g, E^{\mathcal{B}}F) = E^{\mathcal{B}}\delta^*(g, F) \quad a.s.$$

In particular, $\delta^*(x^*, E^{\mathcal{B}}F) = E^{\mathcal{B}}\delta^*(x^*, F)$ *a.s. for every* $x^* \in E^*$.

(d) $E^{\mathcal{B}}\,\overline{\mathrm{co}}\,F = \overline{\mathrm{co}}\,E^{\mathcal{B}}F$ *a.s.*

(e) *Let F be $\mathcal{B}$-measurable, with values in* $\mathrm{cc}(E)$, *and r an $\mathcal{F}$-measurable positive function such that rF is integrable; then*

$$E^{\mathcal{B}}(rF) = E^{\mathcal{B}}(r)F \quad a.s.$$

In particular, $E^{\mathcal{B}}F = F$ *a.s.*

As mentioned in the introduction we also need some specific results allowing us to deal with random sets whose values are closed convex, but not necessarily bounded. They are due to Hess [He1, 2] and are briefly recalled for convenience. The next result will enable us to state our multivalued version of the dominated convergence theorem in a more general situation: only the majorization by $\mathcal{B}_\infty$-measurable multifunctions will be required, instead of the majorization by constant ones. It can be seen as a measurable parametrization of Lemma 3.3 below and it provides a tool which may be useful in other circumstances.

PROPOSITION 3.2. *Let F be an $\mathcal{F}$-measurable and integrable random set with values in* $\mathcal{L}\mathrm{wc}(E)$. *Then there exists a sequence $(g^*_k)_{k\ge1}$ of measurable functions from Ω into D^* satisfying*:

(i) *For every $\omega \in \Omega$, the sequence $(g^*_k(\omega))_{k\ge1}$ is a dense subset of* $\mathrm{int}\,\mathrm{dom}\,\delta^*(\cdot, F(\omega))$.

(ii) *For every $k \ge 1$, the real-valued measurable function $\delta^*(g^*_k(\cdot), F(\cdot))$ is integrable and $\sup\{\|g^*_k(\omega)\| : \omega \in \Omega\}$ is finite.*

Proof. [He2], Proposition 3.4.

LEMMA 3.3. *Let $C \in \mathrm{c}(E)$, $L \in \mathcal{L}\mathrm{wc}(E)$, and $M^* = \mathrm{dom}\,\delta^*(\cdot, L)$. Then the following two statements are equivalent*:

(a) *C is contained in L.*

(b) *$\delta^*(x^*, C) \le \delta^*(x^*, L)$ for every $x^* \in D^* \cap \mathrm{int}\,M^*$.*

Proof. [He2], Lemma 3.1.

LEMMA 3.4. (a) *If (C_n) is a sequence in $\mathrm{c}(E)$, then*

$$\forall x^* \in E^*, \qquad \delta^*(x^*, \text{w-ls}\,C_n) \le \limsup_{n\to\infty} \delta^*(x^*, C_n).$$

(b) *Moreover, if L is an element of $\mathcal{L}\mathrm{wc}(E)$ which contains all the C_n, then*

$$\forall x^* \in \mathrm{int}\,\mathrm{dom}\,\delta^*(\cdot, L), \qquad \limsup_{n\to\infty} \delta^*(x^*, C_n) \le \delta^*(x^*, \overline{\mathrm{co}}(\text{w-ls}\,C_n)).$$

Proof. [He2], Lemma 3.2.

We close this section with a useful lemma concerning the asymptotic cones of closed convex sets, which will help us to take into account the unboundedness of the random set in the proof of our main result on convergence of conditional expectations.

PROPOSITION 3.5. *Let (C_n) be a sequence in $\mathrm{cc}(E)$ such that* $\text{w-li}\,C_n$ *is nonempty. Then*

$$\text{w-ls}\,\mathrm{As}(C_n) \subset \mathrm{As}(\overline{\mathrm{co}}(\text{w-ls}\,C_n)).$$

Proof. [He2], Lemma 3.6.

4.A. Convergence of conditional expectations for sequences of real-valued random variables. We present convergence results for special sequences of real-valued random variables (r.v. for short). These sequences are of the type $E^{\mathcal{B}_n}X_n$, where $(\mathcal{B}_n)$ is a decreasing sequence of sub-σ-algebras of $\mathcal{F}$ and (X_n) a sequence of r.v. We shall provide extensions of Fatou's lemma and Lebesgue's dominated convergence theorem for such sequences. For this purpose, three preparatory results will be needed. We begin by a known simple lemma whose proof is recalled for convenience (see e.g. Lemma III-4 in [Co]).

LEMMA 4.1. *Let Y be a positive r.v., that is, with values in $[0,\infty]$, and $\mathcal{B}$ a sub-σ-algebra of $\mathcal{F}$. Then*

$$(4.1.1)\qquad \{\omega\in\Omega:(E^{\mathcal{B}}Y)(\omega)<\infty\}\subset\{\omega\in\Omega:Y(\omega)<\infty\}\quad a.s.$$

More generally, if $\mathcal{C}$ and $\mathcal{D}$ are two sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{C}\subset\mathcal{D}$, then

$$(4.1.2)\qquad \{\omega\in\Omega:(E^{\mathcal{C}}Y)(\omega)<\infty\}\subset\{\omega\in\Omega:(E^{\mathcal{D}}Y)(\omega)<\infty\}\quad a.s.$$

Proof. Denote by A the left-hand side of (4.1.1) and define, for each integer $k\ge 1$, the $\mathcal{B}$-measurable subset A_k by setting

$$A_k:=\{\omega\in\Omega:(E^{\mathcal{B}}Y)(\omega)<k\}.$$

Clearly A is the union of the sequence (A_k) and we have

$$\int_{A_k} Y\,dP=\int_{A_k}E^{\mathcal{B}}Y\,dP\le kP(A_k).$$

Consequently, for every $k\ge 1$, there exists a P-negligible subset N_k contained in A_k and such that $Y(\omega)<\infty$ for each $\omega\in A_k\setminus N_k$. Clearly, we have $Y(\omega)<\infty$ a.s. on $\bigcup_{k\ge1}(A_k\setminus N_k)$. Set $N=\bigcup_{k\ge1}N_k$. So, N is a negligible set and

$$A\setminus N=\bigcup_{k\ge1}(A_k\setminus N)\subset\bigcup_{k\ge1}(A_k\setminus N_k).$$

The proof of (4.1.2) follows easily. ■

The following lemma is a slight extension of Lebesgue's monotone convergence theorem for the conditional expectation.

LEMMA 4.2. *Let $\mathcal{B}$ be a fixed sub-σ-field of $\mathcal{F}$ and $(X_n)_{n\ge1}$ a decreasing sequence of extended real-valued random variables. Also define $X:=\inf_{n\ge1}X_n$ and assume that*

$$(4.2.1)\qquad E^{\mathcal{B}}X_1^+ \text{ is a.s. finite } (^1).$$

Then

$$(4.2.2)\qquad E^{\mathcal{B}}X=\inf_{n\ge1}E^{\mathcal{B}}X_n=\lim_{n\to\infty}E^{\mathcal{B}}X_n.$$

Proof. The inequality

$$(4.2.3)\qquad E^{\mathcal{B}}X\le\inf_{n\ge1}E^{\mathcal{B}}X_n$$

is clear. In order to show the opposite inequality consider, for any fixed integer $k\ge1$, the following member of $\mathcal{B}$:

$$B_k:=\{\omega\in\Omega:(E^{\mathcal{B}}X_1)(\omega)\le k\},$$

(1) In the classical version of this result $E^{\mathcal{B}}X_1^+$ is assumed to be integrable.

and note that integration of X_1 over B_k leads to

$$\int_{B_k}E^{\mathcal{B}}X_1\,dP\le kP(B_k).$$

This shows that, provided the integration is restricted to B_k, the classical monotone convergence theorem for conditional expectations applies and yields the equality

$$\inf_{n\ge1}(E^{\mathcal{B}}X_n)(\omega)=(E^{\mathcal{B}}\inf_{n\ge1}X_n)(\omega)$$

almost surely for $\omega\in B_k$. Thus, by (4.2.3) we have,

$$(E^{\mathcal{B}}X)(\omega)=\inf_{n\ge1}(E^{\mathcal{B}}X_n)(\omega)$$

almost surely for $\omega\in B_k$. The proof is completed by noting that, due to (4.2.1), Ω is a.s. equal to the union over k of the B_k. ■

The following lemma extends Lemma 4.2 to the case where the sub-σ-field $\mathcal{B}$ depends on n. It can be observed that the claim at the beginning of the proof is an easy improvement of Corollary V.3.12 of [Ne].

LEMMA 4.3. *Let $(\mathcal{B}_n)_{n\ge1}$ be a nonincreasing sequence of sub-σ-fields of $\mathcal{F}$ and (X_n) a nonincreasing sequence of extended real-valued random variables. Also define the sub-σ-field $\mathcal{B}_\infty$ and the random variable X_∞ by*

$$\mathcal{B}_\infty=\bigcap_{n\ge1}\mathcal{B}_n\quad\text{and}\quad X_\infty(\omega):=\inf_{n\ge1}X_n(\omega)$$

and assume that

$$(4.3.1)\qquad E^{\mathcal{B}_\infty}X_1^+ \text{ is almost surely finite.}$$

Then

$$(4.3.2)\qquad E^{\mathcal{B}_\infty}X_\infty=\lim_{n\to\infty}E^{\mathcal{B}_n}X_n\quad\text{almost surely.}$$

Proof. First, we claim that (4.3.2) holds in the special case where the sequence (X_n) is constant, that is, $X_n=X$ for every $n\ge1$. Applying Corollary V.3.12 of [Ne] to X^- and X^+ we obtain

$$(4.3.3)\qquad E^{\mathcal{B}_\infty}X^-=\lim_{n\to\infty}E^{\mathcal{B}_n}X^-,\quad E^{\mathcal{B}_\infty}X^+=\lim_{n\to\infty}E^{\mathcal{B}_n}X^+\quad\text{a.s.}$$

Further, we observe that, due to (4.3.1), the r.v.

$$E^{\mathcal{B}_\infty}X=E^{\mathcal{B}_\infty}X^+-E^{\mathcal{B}_\infty}X^-$$

is well defined (but it can take the value $-\infty$) and that Lemma 4.1 implies, for every $n\ge1$, the a.s. finiteness of $E^{\mathcal{B}_n}X^+$. In turn this implies that $E^{\mathcal{B}_n}X=E^{\mathcal{B}_n}X^+-E^{\mathcal{B}_n}X^-$ is well defined and allows us to deduce the claim from (4.3.3).

Now, let us prove (4.3.2) in full generality. First, using the claim and the monotonicity hypothesis on (X_n) we get

$$\liminf_{n\to\infty} E^{\mathcal{B}_n} X_n \ge \liminf_{n\to\infty} E^{\mathcal{B}_n} X_\infty = E^{\mathcal{B}_\infty} X_\infty .$$

Similarly, for every fixed integer $k \ge 1$, we have

$$\limsup_{n\to\infty} E^{\mathcal{B}_n} X_n \le \limsup_{n\to\infty} E^{\mathcal{B}_n} X_k = E^{\mathcal{B}_\infty} X_k . \tag{4.3.4}$$

The proof is easily finished by letting k go to infinity on the right-hand side of (4.3.4) and applying Lemma 4.2, which is possible by condition (4.3.1). ∎

As already mentioned, the following result and its corollary are the main results of this section. They are extensions of the classical Fatou lemma and the Lebesgue dominated convergence theorem for conditional expectations to the case where both the r.v. X_n and the σ-algebras $\mathcal{B}_n$ depend on n.

THEOREM 4.4. *If Y is a positive r.v. such that*

$$E^{\mathcal{B}_\infty} Y < \infty \quad a.s. \tag{4.4.1}$$

and (X_n) is a sequence of r.v. satisfying

$$X_n \le Y \quad a.s. \quad \forall n \ge 1, \tag{4.4.2}$$

then

$$\limsup_{n\to\infty} E^{\mathcal{B}_n} X_n \le E^{\mathcal{B}_\infty} \limsup_{n\to\infty} X_n \quad a.s.$$

Proof. Apply Lemma 4.3 to the r.v.

$$Z_n := \sup_{k\ge n} X_k \quad \text{and} \quad Z := \limsup_{n\to\infty} Z_n = \inf_{n\ge 1} Z_n .$$

This yields

$$E^{\mathcal{B}_\infty} Z = \inf_{n\ge 1} E^{\mathcal{B}_n} Z_n \ge \limsup_{n\to\infty} E^{\mathcal{B}_n} X_n . \ ∎$$

COROLLARY 4.5. *If Y is a positive r.v. satisfying* (4.4.1) *and (X_n) a sequence of r.v. satisfying*

$$\forall n \in \mathbb{N}^*, \quad |X_n| \le Y \quad a.s., \tag{*}$$

$$\lim_{n\to\infty} X_n = X_\infty \quad a.s., \tag{**}$$

then

$$\lim_{n\to\infty} E^{\mathcal{B}_n} X_n = E^{\mathcal{B}_\infty} X_\infty \quad a.s.$$

Remark. (1) Corollary 4.5 holds true if (X_n) is a sequence of r.v. in $L^1(\Omega, E)$ satisfying

(i) $\forall n \in \mathbb{N}^*$, $\|X_n(\omega)\| \le Y(\omega)$ a.s.,

(ii) X_n converges almost surely to an integrable function X_∞.

This follows easily from Corollary 4.5 and a result due to Choukairi ([Ch], Lemme 3.1, p. 52).

(2) Similar results were obtained by the second author ([Ez1], [Ez2]) in the case when the sequence of sub-σ-algebras $\mathcal{B}_n$ is increasing and where $\mathcal{B}_\infty$ is defined by $\mathcal{B}_\infty := \sigma(\bigcup_{n\ge 1} \mathcal{B}_n)$. In this situation (4.4.1) is replaced by $E^{\mathcal{B}_1} Y < \infty$ and the proofs are based on martingale techniques in [Ne].

4.B. Fatou's lemma for random sets. We present a Fatou lemma for the conditional expectation of the strong lower limit of a sequence of unbounded random sets which is a generalization of a result due to Hiai ([Hi2], Theorem 2.3).

THEOREM 4.6. *Let Z be a positive random variable such that $E^{\mathcal{B}_\infty} Z$ is finite almost surely. Let E be a separable Banach space and $(X_n)_{n\in\mathbb{N}^*}$ a sequence of integrable closed convex random sets such that, for all $n \in \mathbb{N}^*$, $d(0, X_n) \le Z$ a.s. Set $X_\infty = \text{s-li}\, X_n$ and assume that X_∞ is integrable. Then*

$$E^{\mathcal{B}_\infty} X_\infty \subset \text{s-li}\, E^{\mathcal{B}_n} X_n \quad a.s.$$

Proof. Set $Y = \max\{Z, d(0, X_\infty)\}$. We still have $E^{\mathcal{B}_\infty} Y < \infty$ a.s. Let $f \in \mathcal{S}^1_{X_\infty}(\mathcal{F})$. For each $n \in \mathbb{N}^*$ and each $\omega \in \Omega$, set

$$G_n(\omega) = \{x \in X_n(\omega) : \|f(\omega) - x\| \le d(f(\omega), X_n(\omega)) + 1/n\}.$$

It is clear that G_n is a multifunction from Ω to $\mathrm{cc}(E)$ such that $\mathrm{Gr}(G_n) \in \mathcal{F} \otimes \mathcal{B}(E)$. We have

$$\begin{aligned} d(0, G_n(\omega)) &\le d(0, f(\omega)) + d(f(\omega), G_n(\omega)) \\ &\le \|f(\omega)\| + d(f(\omega), X_n(\omega)) + 1/n \\ &\le 2\|f(\omega)\| + d(0, X_n(\omega)) + 1/n. \end{aligned}$$

Since X_n is an integrable multifunction, $d(0, X_n(\cdot))$ is an integrable positive function. Hence the above inequalities show that $d(0, G_n(\cdot))$ is integrable, which implies $S^1_{G_n}(\mathcal{F}) \ne \emptyset$. Let f_n be an integrable selection of G_n. Then

$$\begin{aligned} \|f(\omega) - f_n(\omega)\| \le d(f(\omega), X_n(\omega)) + 1/n &\le \|f(\omega)\| + d(0, X_n(\omega)) + 1/n \\ &\le \|f(\omega)\| + Y + 1. \end{aligned}$$

Hence we have

$$\|f_n(\omega)\| \le 2\|f(\omega)\| + Y + 1 \quad \text{a.s.} \tag{4.6.1}$$

Since $X_\infty(\cdot) = \text{s-li}\, X_n(\cdot)$ a.s. and $\|f(\omega) - f_n(\omega)\| \le d(f(\omega), X_n(\omega)) + 1/n$, we get

$$\lim_{n\to\infty} \|f(\omega) - f_n(\omega)\| \le \lim_{n\to\infty} (d(f(\omega), X_n(\omega)) + 1/n) = 0. \tag{4.6.2}$$

From (4.6.1), (4.6.2) and Corollary 4.5, it follows that

$$\lim_{n\to\infty} E^{\mathcal{B}_n} f_n(\omega) = E^{\mathcal{B}_\infty} f(\omega) \quad \text{a.s.}$$

Since $E^{\mathcal{B}_n} f_n(\omega) \in (E^{\mathcal{B}_n} X_n)(\omega)$ a.s., we obtain

$$E^{\mathcal{B}_\infty} f(\omega) \in \text{s-li}\,(E^{\mathcal{B}_n} X_n)(\omega) \quad \text{a.s.}$$

Let $g \in \mathcal{S}^1_{E^{\mathcal{B}_\infty} X_\infty}(\mathcal{B}_\infty)$. There exists a sequence $(g_i)_{i\ge 1}$ in $\mathcal{S}^1_{X_\infty}(\mathcal{B}_\infty)$ such that $E^{\mathcal{B}_\infty} g_i \to g$ a.s. Set $F(\omega) = \text{s-li}\,(E^{\mathcal{B}_n} X_n)(\omega)$. From what has already been proved, we deduce that for each $i \ge 1$, $E^{\mathcal{B}_\infty} g_i(\omega) \in F(\omega)$ a.s. Since $F(\omega)$ is closed in E, we get $g(\omega) \in F(\omega)$ a.s. Let $(h_n)_{n\ge1}$ be a Castaing representation of $E^{\mathcal{B}_\infty} X_\infty$ in $\mathcal{S}^1_{E^{\mathcal{B}_\infty} X_\infty}(\mathcal{B}_\infty)$. Then $h_n(\omega) \in F(\omega)$ a.s. for all $n \ge 1$. Therefore we have

$$(E^{\mathcal{B}_\infty} X_\infty)(\omega) \subset \text{s-li}\,(E^{\mathcal{B}_n} X_n)(\omega) \quad \text{a.s.} \ \blacksquare$$

4.C. Dominated convergence theorem for unbounded random sets. We are now in a position to state a version of Lebesgue's dominated convergence theorem for a sequence of random sets whose values are not assumed to be bounded. Here, the domination condition is expressed by the inclusion of the random sets X_n in a suitable fixed random set.

THEOREM 4.7. *Let E be a separable Banach space. Let H be a $\mathcal{B}_\infty$-measurable and integrable random set with values in $\mathcal{L}\mathrm{wc}(E)$. Let Y_1 and Y_2 be positive random variables satisfying* (4.4.1). *Let X_∞ be a random variable in* $\mathrm{cc}(E)$ *and $(X_n)_{n\in\mathbb{N}^*}$ be a sequence of integrable random sets in* $\mathrm{cc}(E)$. *Assume the following conditions are satisfied*:

(i) $\forall n \in \mathbb{N}^*$, $d(0, X_n) \le Y_1$ *a.s.*,
(ii) $\forall n \in \mathbb{N}^*$, $X_n \subset G + Y_2 \cdot H$ *a.s. where G is an integrably bounded random set with values in* $\mathrm{cwk}(E)$.

Also assume the following hypotheses:

(iii) $\text{M-lim}_{n\to\infty} X_n = X_\infty$ *a.s.*,
(iv) $\mathrm{As}(H) \subset \text{w-ls}\,\mathrm{As}(X_n)$ *a.s.*,
(v) *the random set $Y_2 \cdot H$ is integrable*,
(vi) *the random set X_∞ is integrable.*

Under the foregoing conditions we have:

(a) $d(0, X_\infty) \le Y_1$ *a.s.*,
(b) $X_\infty \subset G + Y_2 \cdot H$ *a.s.*,
(c) $E^{\mathcal{B}_\infty} X_\infty = \text{M-lim}_{n\to\infty} E^{\mathcal{B}_n} X_n$ *a.s.*

Proof. According to Proposition 6.4.8 of [He3], condition (iii) implies that for every $x \in E$,

$$(4.7.1) \qquad d(x, X_\infty) = \lim_{n\to\infty} d(x, X_n) \quad \text{a.s.}$$

Thus, by (i) we get (a). Inclusion (b) is an easy consequence of (ii) and (4.7.1). Now let us prove equality (c). This will be done in four steps.

Step 1. Let E^* be the dual space of E endowed with the Mackey topology, and D^* a countable dense subset of E^*. By Proposition 3.2 there exists a sequence (g_k^*) of $\mathcal{B}_\infty$-measurable functions from Ω into D^* satisfying:

(*) for every $\omega \in \Omega$, $\{g_k^*(\omega) : k \ge 1\}$ is dense in $\mathrm{int}\,\mathrm{dom}\,\delta^*(\cdot, H(\omega))$,
(**) for every $k \ge 1$, the real-valued $\mathcal{B}_\infty$-measurable function $\delta^*(g_k^*, H)$ is integrable and $\sup\{g_k^*(\omega) : \omega \in \Omega\} < \infty$.

By (ii) for every $k \ge 1$ and $n \in \mathbb{N}^*$, we have

$$(4.7.2) \qquad \delta^*(g_k^*, X_n) \le \delta^*(g_k^*, G) + Y_2 \cdot \delta^*(g_k^*, H) \quad \text{a.s.}$$

Set $Z_k := |\delta^*(g_k^*, G)| + Y_2|\delta^*(g_k^*, H)|$. Then by (4.7.2), we have $\delta^*(g_k^*, X_n) \le Z_k$ a.s. It is clear that the Z_k so defined is a positive random variable. Moreover, since the random set H and the functions g_k^* are $\mathcal{B}_\infty$-measurable, it is not difficult to check that

$$(4.7.3) \qquad E^{\mathcal{B}_\infty} Z_k < \infty \quad \text{a.s.} \quad \forall k \ge 1.$$

Step 2. Now we are going to prove that for every $k \in \mathbb{N}^*$, we have

$$(4.7.4) \qquad \delta^*(g_k^*, \text{w-ls}\, E^{\mathcal{B}_n} X_n) \le \delta^*(g_k^*, E^{\mathcal{B}_\infty} X_\infty) \quad \text{a.s.}$$

Let k be fixed in $\mathbb{N}^*$. By Lemma 3.4(b) and Proposition 3.1(c), we can write

$$\begin{aligned} \delta^*(g_k^*, \text{w-ls}\, E^{\mathcal{B}_n} X_n) &\le \limsup_{n\to\infty} \delta^*(g_k^*, E^{\mathcal{B}_n} X_n) \\ &= \limsup_{n\to\infty} E^{\mathcal{B}_n} \delta^*(g_k^*, X_n) \quad \text{a.s.} \end{aligned}$$

By inequality (4.7.4) it is possible to apply Theorem 4.4 to the sequence $(\delta^*(g_k^*, X_n))_{n\ge1}$, which gives

$$(4.7.5) \qquad \limsup_{n\to\infty} E^{\mathcal{B}_n} \delta^*(g_k^*, X_n) \le E^{\mathcal{B}_\infty} \limsup_{n\to\infty} \delta^*(g_k^*, X_n) \quad \text{a.s.}$$

So we have

$$(4.7.6) \qquad \delta^*(g_k^*, \text{w-ls}\, E^{\mathcal{B}_n} X_n) \le E^{\mathcal{B}_\infty} \limsup_{n\to\infty} \delta^*(g_k^*, X_n) \quad \text{a.s.}$$

Also observe that inclusion (ii) allows us to apply Lemma 3.4(b), for almost all $\omega \in \Omega$, which by (iii) entails

$$(4.7.7) \qquad \limsup_{n\to\infty} \delta^*(g_k^*, X_n) \le \delta^*(g_k^*, \text{w-ls}\, X_n) = \delta^*(g_k^*, X_\infty).$$

Then taking the conditional expectation in (4.7.7) with respect to $\mathcal{B}_\infty$, and invoking Proposition 3.1(c) we get

$$E^{\mathcal{B}_\infty} \limsup_{n\to\infty} \delta^*(g_k^*, X_n) \le E^{\mathcal{B}_\infty} \delta^*(g_k^*, X_\infty) = \delta^*(g_k^*, E^{\mathcal{B}_\infty} X_\infty) \quad \text{a.s.}$$

Returning to (4.7.5) yields the desired conclusion.

Step 3. In this step we shall show that (4.7.4) implies w-ls $E^{\mathcal{B}_n}X_n \subset E^{\mathcal{B}_\infty}X_\infty$ a.s. By hypothesis (v) the random set $Y_2 \cdot H$ is integrable. Moreover, from (b) we know that

$$X_\infty \subset G + Y_2 \cdot H \quad \text{a.s.}$$

Hence, by Proposition 3.1((a), (e)),

$$E^{\mathcal{B}_\infty}X_\infty \subset E^{\mathcal{B}_\infty}(G + Y_2 \cdot H) = E^{\mathcal{B}_\infty}G + E^{\mathcal{B}_\infty}Y_2 \cdot H \quad \text{a.s.}$$

Since G is integrably bounded and convex weakly compact valued, so is $E^{\mathcal{B}_\infty}X_\infty$. Consequently, it is not difficult to see that the values of $E^{\mathcal{B}_\infty}X_\infty$ are members of $\mathcal{L}\mathrm{wc}(E)$. Further, by property $(*)$ of Step 1 we know that almost surely $\{g_k^* : k \geq 1\}$ is a dense subset of $\operatorname{int}\operatorname{dom}\delta^*(\cdot, H(\omega))$. Moreover, it is not difficult to see that $\operatorname{int}\operatorname{dom}\delta^*(\cdot, H(\omega)) = \operatorname{int}\operatorname{dom}\delta^*(\cdot, G(\omega) + Y_2(\omega)H(\omega))$. So, in order to show that (4.7.4) implies

$$\text{w-ls}\, E^{\mathcal{B}_n}X_n \subset E^{\mathcal{B}_\infty}X_\infty \quad \text{a.s.}$$

it suffices, in view of Lemma 3.3 applied with $L = E^{\mathcal{B}_\infty}X_\infty$ and $C = \text{w-ls}\, E^{\mathcal{B}_n}X_n$, to show the following inclusion:

$$\operatorname{int}\operatorname{dom}\delta^*(\cdot, E^{\mathcal{B}_\infty}X_\infty) \subset \operatorname{int}\operatorname{dom}\delta^*(\cdot, H) \quad \text{a.s.},$$

which, by polarity, is equivalent to

$$\text{(4.7.8)} \qquad \operatorname{As}(H) \subset \operatorname{As}(E^{\mathcal{B}_\infty}X_\infty) \quad \text{a.s.}$$

But, by hypothesis (iv) and Lemma 3.5 applied for almost every $\omega \in \Omega$ to the sequence $(X_n(\omega))_{n\geq 1}$ we have

$$\operatorname{As}(H(\omega)) \subset \text{w-ls}\operatorname{As}(X_n(\omega)) \subset \operatorname{As}(\text{w-ls}\, X_n(\omega)),$$

which implies

$$\text{(4.7.9)} \qquad \operatorname{As}(H(\omega)) \subset \operatorname{As}(X_\infty(\omega)) \quad \text{a.s.}$$

Now, consider the multifunction $\operatorname{As}(H)$. Like H, it is $\mathcal{B}_\infty$-measurable. This can be proved by using the equality

$$\operatorname{As}(H(\omega)) = \bigcap_j (H(\omega) - h(\omega))/j$$

where $j \in \mathbb{N}^*$ and h is a fixed but arbitrary element of $\mathcal{S}_H^1(\mathcal{B}_\infty)$, and by invoking Theorem 4.2 of [He4]. Therefore, taking conditional expectation relative to $\mathcal{B}_\infty$ of both sides of (4.7.9) and noticing that this operation is monotonic, we get

$$E^{\mathcal{B}_\infty}\operatorname{As}(H)(\omega) = \operatorname{As}(H(\omega)) \subset E^{\mathcal{B}_\infty}\operatorname{As}(X_\infty(\omega)) \subset \operatorname{As}(E^{\mathcal{B}_\infty}X_\infty(\omega)) \text{ a.s.},$$

which gives the desired inclusion.

Step 4. By (i) and Theorem 4.6 we have $E^{\mathcal{B}_\infty}X_\infty \subset \text{s-li}\, E^{\mathcal{B}_n}X_n$ a.s., which ends the proof of (c). ∎

Remark. (vi) is satisfied if $(d(0, X_n))_{n\geq 1}$ is bounded in L^1 or, more generally, if $\liminf_{n\to\infty} d(0, X_n) \in L^1$. This is an easy consequence of Fatou's lemma and of (4.7.1).

COUNTEREXAMPLE. The following counterexample taken from [He1, 2] (see e.g. Remark 4.4 in [He2]) is useful to show that hypothesis (iv) of Theorem 4.7, although restrictive, cannot be removed. Let $\Omega := [0,1]$, $\mathcal{F} :=$ the Borel σ-field of Ω, $P :=$ Lebesgue measure, $E = \mathbb{R}$, $\mathcal{B}_n = \mathcal{B}_\infty := \{\Omega, \emptyset\}$, $G := \{0\}$ and $H := \mathbb{R}_+$ (so G and H are constant multifunctions). Moreover, for every $n \geq 1$, define the interval $A_n := [0, 1/n]$ and for every $\omega \in \Omega$, the random set X_n by

$$X_n(\omega) = \begin{cases} \mathbb{R}_+ & \text{if } \omega \in A_n, \\ \{0\} & \text{if } \omega \notin A_n. \end{cases}$$

For each $n \geq 1$, we have $\operatorname{As}(X_n) = X_n$ and we can see that $EX_n = H = \operatorname{As}(H) = \mathbb{R}_+$, whence $\lim_{n\to\infty} EX_n = H$. All the hypotheses of Theorem 4.7 are satisfied except (iv); in particular, we have

$$X_\infty = \operatorname{As}(X_\infty) = \lim_{n\to\infty} X_n(\omega) = \{0\} \quad \text{a.s.} \quad \text{and} \quad EX_\infty = \{0\}.$$

This shows that (iv) is not satisfied and contradicts conclusion (c) of Theorem 4.7. Also observe that, due to hypothesis (ii), the converse inclusion of (iv) always holds.

5. Application: Convergence of conditional expectations of a random lower semicontinuous integrand. Our main aim in this section is to reformulate Theorem 4.7 in terms of integrands. We shall present a result concerning the Mosco convergence of a sequence $(E^{\mathcal{B}_n}X_n)$ where, for every $n \geq 1$, X_n is the epigraphic multifunction associated with a random lower semicontinuous convex integrand F_n defined on $\Omega \times E$. Let us recall some definitions and known facts.

If f is a *numerical function*, defined on E (i.e. with values in the extended reals), its *epigraph*, denoted by epi$f$, is the subset of $E \times \mathbb{R}$ defined as

$$\operatorname{epi} f = \{(x, \lambda) \in E \times \mathbb{R} : f(x) \leq \lambda\}.$$

If f is convex, lower semicontinuous, and proper, the *asymptotic function* of f, denoted by $\operatorname{As}(f)$, is defined by the equality

$$\operatorname{epi}\operatorname{As}(f) = \operatorname{As}(\operatorname{epi} f).$$

The function f is said to be *inf-weakly compact for a certain slope* if there exists $x^* \in E^*$ such that the function $x \mapsto f(x) - \langle x^*, x\rangle$ is inf-weakly compact ([CV], Theorem 1.14). If f is convex lower semicontinuous, and proper, this is equivalent to

$$\operatorname{epi} f \in \mathcal{L}\mathrm{wc}(E \times \mathbb{R}).$$

A sequence (f_n) of numerical functions defined on E is *Mosco-convergent* to a numerical function f if epi f is the Mosco-limit of the sequence (epi f_n) in $E \times \mathbb{R}$. This convergence will be denoted by

$$f = \operatorname*{M-lim}_{n\to\infty} f_n.$$

A map R defined on $\Omega \times E$ with values in $\overline{\mathbb{R}}$ will be called a *normal convex integrand* if it has the following two properties: (i) The function $R(\omega, \cdot)$ is convex and lower semicontinuous a.s. (ii) The epigraphical multifunction $\omega \mapsto \operatorname{epi} R(\omega, \cdot)$, with closed convex values in $E \times \mathbb{R}$, is measurable.

The normal convex integrand R is said to be *integrable* if its epigraphical multifunction is integrable. This is equivalent to the existence of u in $L^1(\Omega, E)$ such that $R(\cdot, u(\cdot))^+$ is integrable.

The following theorem is a reformulation of Theorem 4.7 in terms of integrands.

THEOREM 5.1. *Let Y be a positive random variable such that $E^{\mathcal{B}_\infty} Y < \infty$ almost surely. Let F_∞ and $(F_n)_n$ be convex normal integrands defined on $\Omega \times E$ and satisfying the following conditions:*

(i) $\text{M-lim}_n F_n(\omega, \cdot) = F_\infty(\omega, \cdot)$ *a.s.,*

(ii) *there is a sequence $(f_n)_{n\ge1}$ in $L^1_E(\mathcal{F})$ such that $\|f_n(\cdot)\| + F_n(\cdot, f_n(\cdot))^+$ and $\liminf_n(\|f_n(\cdot)\| + F_n(\cdot, f_n(\cdot))^+)$ are integrable,*

(iii) $\forall n \ge 1$, $(\|f_n(\cdot)\| + F_n(\cdot, f_n(\cdot))^+) \le Y(\cdot)$ *a.s.,*

(iv) *there exists an integrand Z defined on $\Omega \times E$ such that for every $\omega \in \Omega$, $Z(\omega, \cdot)$ is convex lower semicontinuous, proper and for every $x \in E$, $Z(\cdot, x)$ is $\mathcal{B}_\infty$-measurable and integrable and such that a.s. for $\omega \in \Omega$,*

(a) $Z(\omega, \cdot)$ *is inf-weakly compact for a certain slope,*
(b) $\forall n \ge 1$, $F_n(\omega, \cdot) \ge Z(\omega, \cdot)$ *a.s.,*
(c) $\operatorname{As}(Z(\omega, \cdot)) = \text{w-li} \operatorname{As}(F_n(\omega, \cdot))$.

Then

(a′) F_∞ *is an integrable convex lower semicontinuous integrand and*
(b′) $\text{M-lim}_n E^{\mathcal{B}_n} F_n(\omega, \cdot) = E^{\mathcal{B}_\infty} F_\infty(\omega, \cdot)$ *a.s.*

Proof. We define the random sets X_∞ and X_n, for $n \ge 1$, as follows:

$$\forall \omega \in \Omega, \quad X_\infty(\omega) = \operatorname{epi} F_\infty(\omega, \cdot) \quad \text{and} \quad X_n(\omega) = \operatorname{epi} F_n(\omega, \cdot).$$

Since F_n is a convex lower semicontinuous integrand, by assumption (ii) we deduce that X_n is an $\mathcal{F}$-measurable and integrable multifunction with closed convex values in $E \times \mathbb{R}$ and by (iii) for all $n \ge 1$, $d(0, X_n(\cdot)) \le Y(\cdot)$ a.s. Analogously, X_∞ is an $\mathcal{F}$-measurable multifunction with closed convex values, and by assumption (i), (a), (b) and ([He4], Proposition 6.4.8), we have

$$d(0, \operatorname{epi} F_\infty(\omega, \cdot)) = \lim_n d(0, \operatorname{epi} F_n(\omega, \cdot)) \quad \text{a.s.}$$

Since the function $\liminf_{n\to\infty}(\|f_n(\cdot)\| + F_n(\cdot, f_n(\cdot))^+)$ is integrable and $d(0, \operatorname{epi} F_\infty(\omega, \cdot)) \le \liminf_{n\to\infty}(\|f_n(\omega)\| + F_n(\omega, f_n(\omega))^+)$ almost surely, X_∞ is integrable. For all $\omega \in \Omega$, set

$$Y(\omega) = Y_1(\omega), \qquad H(\omega) = \operatorname{epi}(Z(\omega, \cdot),$$
$$G(\omega) = \text{singleton}\,\{(0,0)\} \text{ of } X \times \mathbb{R}, \qquad Y_2(\omega) = 1.$$

By the previous results and assumption (iv), we conclude that H takes its values in $\mathcal{L}\mathrm{wc}(E \times \mathbb{R})$, and for all $n \ge 1$ and $\omega \in \Omega$, $X_n(\omega) \subset H(\omega)$ and $\operatorname{As}(H(\omega)) \subset \text{w-ls} \operatorname{As}(X_n(\omega))$. Then by Theorem 4.7 we have

$$\operatorname*{M-lim}_{n\to\infty} E^{\mathcal{B}_n} X_n = E^{\mathcal{B}_\infty} X_\infty \quad \text{a.s.}$$

Therefore

$$\operatorname*{M-lim}_{n\to\infty} E^{\mathcal{B}_n} F_n(\omega, \cdot) = E^{\mathcal{B}_\infty} F_\infty(\omega, \cdot) \quad \text{a.s.},$$

which proves the theorem. ■

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Minimality in asymmetry classes

by

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Abstract. We examine minimality in asymmetry classes of convex compact sets with respect to inclusion. We prove that each class has a minimal element. Moreover, we show there is a connection between asymmetry classes and the Rådström–Hörmander lattice. This is used to present an alternative solution to the problem of minimality posed by G. Ewald and G. C. Shephard in [4].

1. Introduction. We will denote by $\mathcal{K}(X)$ the space of non-empty convex compact subsets of a topological vector space X. The space $\mathcal{K}(X)$ has been widely investigated, especially in connection with the Minkowski sum defined by $A+B := \{a+b : a \in A,\ b \in B\}$. Although $\mathcal{K}(X)$ with the Minkowski sum forms a commutative semigroup with the cancellation law (see R. Urbański [5]), it is not a vector space. In [4] G. Ewald and G. C. Shephard introduced some normed vector spaces consisting of classes of convex compact sets. Let us recall one of those concepts, the so called asymmetry classes. As in [4] we will restrict our examination to the finite-dimensional case ($X = \mathbb{R}^n$, $n \in \mathbb{N}$). We define the relation of asymmetry: $A \approx B$ iff there exist centrally symmetric ([1]) sets S, T such that $A+S$ is a translation of $B+T$ (we can require S and T to be symmetric instead of centrally symmetric). It has been proved in [4] that $\approx$ is an equivalence relation and $\mathcal{K}(\mathbb{R}^n)/\approx$ is a normed vector space.

In [4] the authors posed the question whether each class of asymmetry can be expressed in the form $\{M+S : S \text{ centrally symmetric}\}$, where $M \in \mathcal{K}(X)$ is a certain "minimal" element. This problem has been solved by R. Schneider [3]. It is proved there that for $n = 2$ every asymmetry class contains a minimal member (Theorem I). In the proof, measure theory as well as surface area functions are employed. We will present a different approach.

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([1]) Centrally symmetric sets are translations of certain symmetric sets.