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Almost multiplicative functionals

by

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Abstract. A linear functional F on a Banach algebra A is almost multiplicative if

$$|F(ab) - F(a)F(b)| \le \delta \|a\| \cdot \|b\| \quad \text{for } a, b \in A,$$

for a small constant δ. An algebra is called *functionally stable* or *f-stable* if any almost multiplicative functional is close to a multiplicative one. The question whether an algebra is f-stable can be interpreted as a question whether A lacks an *almost corona*, that is, a set of almost multiplicative functionals far from the set of multiplicative functionals.

In this paper we discuss f-stability for general uniform algebras; we prove that any uniform algebra with one generator as well as some algebras of the form $R(K)$, $K \subset \mathbb{C}$, and $A(\Omega)$, $\Omega \subset \mathbb{C}^n$, are f-stable. We show that, for a Blaschke product B, the quotient algebra H^∞/BH^∞ is f-stable if and only if B is a product of finitely many interpolating Blaschke products.

1. Introduction. Let G be a linear and multiplicative functional on a Banach algebra A and let Δ be a linear functional on A with $\|\Delta\| \le \varepsilon$. Put $F = G + \Delta$. We can easily check by direct computation that F is δ-*multiplicative*, that is,

$$|F(ab) - F(a)F(b)| \le \delta \|a\| \cdot \|b\| \quad \text{for } a, b \in A,$$

where $\delta = 3\varepsilon + \varepsilon^2$. The problem we want to discuss here is whether the converse is true; that is, whether an almost multiplicative functional must be near a multiplicative one. We are interested mostly in uniform algebras. We shall call a Banach algebra *functionally stable* or *f-stable* if

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall F \in \mathfrak{M}_\delta(A)\ \exists G \in \mathfrak{M}(A) \quad \|F - G\| \le \varepsilon,$$

where we denote by $\mathfrak{M}(A)$ the set of all linear multiplicative functionals on A, and by $\mathfrak{M}_\delta(A)$ the set of δ-multiplicative functionals on A. We shall

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call a family of Banach algebras *uniformly f-stable* if any algebra from the family is f-stable and for any $\varepsilon > 0$ we can choose the same $\delta > 0$ for all the members of the family.

2. History. The question whether an almost multiplicative map is close to a multiplicative one constitutes an interesting problem per se; nevertheless, it originated in the deformation theory of Banach algebras. There are two basic concepts of deformation of Banach algebras: metric and algebraic [14].

DEFINITION 1. We say that a Banach algebra B is a *metric δ-deformation* of a Banach algebra A if there is a linear (but not necessarily multiplicative) isomorphism $T : A \to B$ such that $\|T\| \cdot \|T^{-1}\| \le 1 + \delta$.

DEFINITION 2. For a Banach algebra $(A, \cdot)$ we say that a new multiplication $\times$ defined on the same Banach space A is an *algebraic δ-deformation* of $(A,\cdot)$ if $\| \times - \cdot \| \le \delta$; that is, if

$$\|a\cdot b - a\times b\| \le \delta\|a\|\cdot\|b\| \quad \text{for } a, b \in A.$$

While the two definitions lead to different theories for general Banach algebras, they are equivalent in a natural way for all uniform algebras [14]. In particular:

(i) two uniform algebras are isometric if and only if they are isomorphic as algebras,

(ii) a linear map $T : A \to B$ between uniform algebras almost preserves the distance if and only if it almost preserves the multiplication of the algebras [14].

By a *uniform algebra* we mean a closed subalgebra of an algebra $C(K)$ of all continuous functions on a compact set K, equipped with the sup norm. Equivalently, A is (isometrically isomorphic to) a uniform algebra if $\|a^2\| = \|a\|^2$ for any $a \in A$. We always assume that an algebra has a unit.

There are several important links between the deformation theory and other areas. For example, the theory provides a natural definition of deformation of an analytic manifold, or a domain Ω in $\mathbb{C}^n$. We may define the distance between two domains Ω and Ω' by

$$d(\Omega,\Omega') = \inf\{\|T\|\cdot\|T^{-1}\| : T : A(\Omega) \to A(\Omega')\},$$

where $A(\Omega)$ is the Banach space of analytic functions on Ω. It is an important and deep result due to R. Rochberg [24] that for one-dimensional Riemann surfaces the distance defined above is locally equivalent to the Teichmüller distance involving quasiconformal homeomorphisms. Still, almost nothing is known about domains in $\mathbb{C}^n$ for $n > 1$ [15].

If $\times$ is a small algebraic deformation of a Banach algebra $(A, \cdot)$, then any multiplicative functional on A is almost $\times$-multiplicative. Since the main objective of the deformation theory of Banach algebras is to compare structures of two close algebras we would like to know if an almost multiplicative functional must be close to a multiplicative one.

It is not difficult to show that the class of $C(K)$ algebras is uniformly f-stable. More precisely, we have the following result.

PROPOSITION 3 ([17]). *For any compact Hausdorff space K, for any $\delta < .1$, and for any $F \in \mathfrak{M}_\delta(C(K))$ there is a $G \in \mathfrak{M}(C(K)) \cong K$ such that $\|F - G\| \le 10\delta$.*

In 1986 B. E. Johnson [17] also proved that the disc algebra $A(\mathbb{D})$ and some related algebras are f-stable (Johnson uses the name *AMNM* algebras). It was then conjectured that all uniform algebras are f-stable. However, very recently S. J. Sidney provided an ingenious counterexample [27].

Later in this paper we show that any uniform algebra with one generator, as well as some algebras of the form $R(K)$, $K \subset \mathbb{C}$, and $A(\Omega)$, $\Omega \subset \mathbb{C}^n$, are f-stable. We show that, for a Blaschke product B, the f-stability of the quotient algebra H^∞/BH^∞ is related to the distribution of the zeros of B: H^∞/BH^∞ is f-stable if and only if B is a product of finitely many interpolating Blaschke products.

It is still an open problem if $H^\infty(\mathbb{D})$ is f-stable. In view of the importance of the Corona Theorem for $H^\infty(\mathbb{D})$ it is particularly interesting to know whether $H^\infty(\mathbb{D})$ does not have an *almost corona*, that is, a set of almost multiplicative functionals far from the set of multiplicative functionals.

3. Basic properties. Let $A \subseteq C(K)$ be a uniform algebra. A subset L of K is called a *weak peak set* if for any open neighborhood U of L there is an $f \in A$ with $\|f\| = 1 = f(k) > |f(k')|$ for any $k \in L$ and $k' \in K \setminus U$. Any intersection and any finite union of weak peak sets is a weak peak set [7]. We denote by $\mathrm{Ch}(A)$ the *Choquet boundary* of A, that is, the subset of K consisting of all k such that the functional δ_k of evaluation at the point k is an extreme point of the unit ball of the dual space A^*. Equivalently, $k \in \mathrm{Ch}(A)$ if and only if $\{k\}$ is a weak peak set [7]. Any continuous linear functional S on A can be represented by a regular measure ν on $\mathrm{Ch}(A)$ with $\|S\| = \mathrm{var}(\nu)$.

PROPOSITION 4. *If L is a weak peak set of a uniform algebra A then there is a net f_γ of elements of A such that*

(1) $$f_\gamma|_L \equiv 1 = \|f_\gamma\|,$$

(2) $$f_\gamma \to 0 \text{ uniformly on compact subsets of } \mathfrak{M}(A)\setminus L, \text{ and}$$

(3) $$\lim_\gamma \|1 - f_\gamma\| = 1.$$

Proof. The existence of such a net is well known even for more general function spaces A (see for example the proof of Lemma 1 in [13]). If A is a uniform algebra the construction is simple: Directly from the definition of a weak peak set there is a net f_γ of functions satisfying (1) and (2). By the Riemann Mapping Theorem there is an analytic homeomorphism χ of the unit disc $\mathbb{D}$ onto

$$\Omega_\varepsilon = \{z \in \mathbb{C} : |z| < 1,\ |\operatorname{Im} z| < \varepsilon,\ \operatorname{Re} z > -\varepsilon\};$$

any such homeomorphism can be extended to a homeomorphism of $\overline{\mathbb{D}}$ onto $\overline{\Omega}_\varepsilon$ ([5], p. 50). Composing χ with an appropriate automorphism of the unit disc we assume that $\chi(0) = 0$, $\chi(1) = 1$. If we replace f_γ with $\chi \circ f_\gamma$, where $\varepsilon \xrightarrow{\gamma} 0$, we get a net satisfying conditions (1)–(3). ■

Let F be a δ-multiplicative map on A and let μ_F be a measure on $\operatorname{Ch}(A)$ representing F and such that $\operatorname{var}(\mu_F) = \|F\|$. Any δ-multiplicative functional is continuous and $\|F\| \le 1 + \delta$ (see [14]). Since $|F(\mathbf{1}) - (F(\mathbf{1}))^2| \le \delta$, $F(\mathbf{1})$ is close to 1 or close to 0. In the latter case $F(a) = F(\mathbf{1}a) \approx F(\mathbf{1})F(a) \approx 0$, and F is close to the zero functional. If $F(\mathbf{1}) \approx 1$, then μ_F is close to a probability measure. By straightforward computation one can show the following.

PROPOSITION 5 ([17]). *If A is a uniform algebra and $F \in \mathfrak{M}_\delta(A)$ with $\delta \le 1/4$, then either $\|F\| \le 2\delta$ or there is a map $F' \in \mathfrak{M}_{5\delta}(A)$ such that $\|F - F'\| \le 2\delta$ and $\|F'\| = 1 = F'(\mathbf{1})$.*

Hence, considering the f-stability we may always assume that the functional F in question is represented by a probability measure μ_F. The next proposition shows we may also assume that μ_F has no atoms and has some other nice properties.

PROPOSITION 6. *Let A be a uniform algebra on K. Assume $F \in \mathfrak{M}_\delta(A)$ with $\delta \le 1/4$ is represented by a nonnegative measure μ_F on $\operatorname{Ch}(A)$. Then*

1. *If L is a weak peak set, or a complement of a weak peak set, then*

$$\mu_F(L) \le 2\delta \quad \text{or} \quad 1 - 2\delta \le \mu_F(L) \le 1 + 2\delta;$$

furthermore, if F_L is the functional on A represented by the restriction of μ_F to L then $F_L \in \mathfrak{M}_\delta(A)$.

2. *If $\mu_F = \sum \lambda_j \delta_{k_j} + \nu$ is the decomposition of μ_F into an atomic and a nonatomic part then either $\sum \lambda_j \le 2\delta$ or there is an atom k_{j_0} such that $\lambda_{j_0} \ge 1 - 2\delta$.*

3. *Either $\|F - G\| \le 2\delta$ for some $G \in \operatorname{Ch}(A) \subset K \subset \mathfrak{M}(A)$, or there is an $F' \in \mathfrak{M}_{2\delta}(A)$ such that $\|F - F'\| \le 6\delta$ and F' is represented by a nonatomic probability measure $\mu_{F'}$ on $\operatorname{Ch}(A)$.*

Proof. 1. Let L be a weak peak set and let $f_\gamma \in A$ be a net given by Proposition 4. Since μ_F is regular, $\int f_\gamma\, d\mu_F \to \mu_F(L)$ and we have

$$\begin{aligned} \delta &\ge |F(f_\gamma^2) - (F(f_\gamma))^2| \\ &= \left| \int f_\gamma^2\, d\mu_F - \left(\int f_\gamma\, d\mu_F\right)^2 \right| \to |\mu_F(L) - (\mu_F(L))^2| \ge 0. \end{aligned}$$

Hence $\mu_F(L) \le 2\delta$ or $1 + 2\delta \ge \mu_F(L) \ge 1 - 2\delta$.

Let F_L be the functional on A represented by the restriction of μ_F to L. Let f, g be norm one functions in A. If f_γ are as before then we have

$$\begin{aligned} |F_L(fg) - F_L(f)F_L(g)| &= \Big| \int_L fg\, d\mu_F - \int_L f\, d\mu_F \int_L g\, d\mu_F \Big| \\ &= \Big| \lim_\gamma \Big(\int f f_\gamma g f_\gamma\, d\mu_F - \int f f_\gamma\, d\mu_F \int g f_\gamma\, d\mu_F \Big) \Big| \\ &= |\lim_\gamma (F(f f_\gamma g f_\gamma) - F(f f_\gamma) F(g f_\gamma))| \\ &\le \lim_\gamma (\delta \|f f_\gamma\| \cdot \|g f_\gamma\|) = \delta \|f\|_L \|g\|_L \le \delta \|f\| \cdot \|g\|, \end{aligned}$$

so $F_L \in \mathfrak{M}_\delta(A)$.

Replacing above the net f_γ with the net $g_\gamma = 1 - f_\gamma$ we get the same conclusions for the complement of a weak peak set.

2. Assume $\{k_1, \ldots, k_s\} \subseteq \operatorname{Ch}(A)$ is a set of atoms of μ_F. By the first part of the Proposition, $\mu_F(\{k_1, \ldots, k_s\})$ is close to one or to zero. Since this holds for any subset of atoms, it follows that either μ_F has one atom of mass close to 1, or the sum of all the atoms of μ_F is small.

3. From the previous part, either $\|F - G\| \le 2\delta$ for some $G \in \operatorname{Ch}(A)$ or the sum of all the atoms of μ_F is smaller than or equal to 2δ. Assume the latter. Put $\widetilde{F} = F - \sum \lambda_j \delta_{k_j} = F|_{L'}$ where L' is the complement of the set of all atoms of μ_F. Put $F' = \widetilde{F}/\|\widetilde{F}\|$. It is clear that F' can be represented by a nonatomic probability measure.

Any finite set of atoms is a weak peak set, so by the first part of the proposition

$$\|\widetilde{F}\| \ge \mu_F(L') \ge 1 - 2\delta \quad \text{and} \quad \Big\| \sum \lambda_j \delta_{k_j} \Big\| \le 2\delta,$$

hence

$$\|F - F'\| \le \Big\| \sum \lambda_j \delta_{k_j} \Big\| + \|\widetilde{F} - F'\| \le 2\delta + |\|\widetilde{F}\| - 1| \le 4\delta.$$

From the first part of the proposition we also have $\widetilde{F} \in \mathfrak{M}_\delta(A)$, and simple computations give $F' \in \mathfrak{M}_{2\delta}(A)$. ■

THEOREM 7. *Let A be a uniform algebra and let $K \subseteq \mathfrak{M}(A)$ be a weak peak set for A. Then A is f-stable if and only if both $A|_K = \{f|_K : f \in A\}$ and $A_K = \{f \in A : f|_K = \text{const}\}$ are f-stable.*

Proof. We first assume that A is f-stable.

Let $F \in \mathfrak{M}_\delta(A|_K)$, and put $\widetilde{F}(f) = F(f|_K)$. For any $f_1, f_2 \in A$ we have

$$\begin{aligned}|\widetilde{F}(f_1 f_2) - \widetilde{F}(f_1)\widetilde{F}(f_2)| &= |F(f_1 f_2|_K) - F(f_1|_K)F(f_2|_K)|\\ &\le \delta\|f_1|_K\| \cdot \|f_2|_K\| \le \delta\|f_1\| \cdot \|f_2\|.\end{aligned}$$

So $\widetilde{F} \in \mathfrak{M}_\delta(A)$. Assume $\|\widetilde{F} - \delta_x\| \le \varepsilon < 1$ for some $x \in \mathfrak{M}(A)$. If x were not an element of K then, since K is a weak peak set, there would be a norm one element h of A such that $h|_K \equiv 1$ and $h(x) = 0$. Hence $(\widetilde{F} - \delta_x)(h) = 1$. The contradiction shows that $x \in K = \mathfrak{M}(A|_K)$.

For any $g \in A|_K$ there is a $\widetilde{g} \in A$ such that $\widetilde{g}|_K = g_1$ and $\|g\| = \|\widetilde{g}\|$ ([7]). So we get

$$|F(g) - g(x)| = |\widetilde{F}(\widetilde{g}) - \widetilde{g}(x)| \le \varepsilon\|\widetilde{g}\| = \varepsilon\|g\|,$$

and hence $\|F - \delta_x\| \le \varepsilon$. Thus $A|_K$ is f-stable.

Let now $F \in \mathfrak{M}_\delta(A_K)$. We may assume that F is represented by a nonatomic probability measure μ_F on $\partial A_K \subseteq \mathfrak{M}(A_K)$. The maximal ideal space $\mathfrak{M}(A_K)$ of A_K is a quotient space of $\mathfrak{M}(A)$, where the set K has been collapsed to a single point which we denote by $\{K\}$. Since μ_F has no atoms we have $\mu_F(\{K\}) = 0$, so $\widetilde{F}(f) = \int_{\mathfrak{M}(A)} f\mu_F$ is a well defined linear functional on A. Let f_γ be a net of elements of A given by Proposition 4, that is, such that

$$f_\gamma|_K \equiv 1 = \|f_\gamma\|, \quad \lim_\gamma \|1 - f_\gamma\| = 1,$$

and $f_\gamma \to 0$ uniformly on compact subsets of $\mathfrak{M}(A) \setminus K$. Put $g_\gamma = 1 - f_\gamma$. For any $f, g \in A$ we have

$$\begin{aligned}|\widetilde{F}(fg) - \widetilde{F}(f)\widetilde{F}(g)| &= \lim_\gamma |\widetilde{F}(g_\gamma f g_\gamma g) - \widetilde{F}(g_\gamma f)\widetilde{F}(g_\gamma g)|\\ &= \lim_\gamma |F(g_\gamma f g_\gamma g) - F(g_\gamma f)F(g_\gamma g)|\\ &\le \lim_\gamma \delta\|g_\gamma f\| \cdot \|g_\gamma g\| = \delta\|f\| \cdot \|g\|.\end{aligned}$$

Thus $\widetilde{F} \in \mathfrak{M}_\delta(A)$. We assume that A is f-stable, so there is a $G \in \mathfrak{M}(A)$ close to $\widetilde{F}$, and the restriction of G to the subalgebra A_K is close to F. Hence A_K is f-stable, which proves the necessity part of the theorem.

To prove sufficiency assume now that both $A|_K$ and A_K are f-stable.

Let $F \in \mathfrak{M}_\delta(A)$. Since A_K is a subalgebra of A obviously $F \in \mathfrak{M}_\delta(A_K)$. We may assume that F is represented by a probability measure μ_F on $\partial A \subseteq \mathfrak{M}(A)$. By Proposition 6, μ_F is concentrated almost entirely on $\mathfrak{M}(A) \setminus K$ or on K. In the first case, since A_K is f-stable, there is an $x \in \mathfrak{M}(A) \setminus K$ such that

$$|F(f) - f(x)| \le \varepsilon\|f\| \quad \text{for } f \in A_K.$$

Using the same argument with the net g_γ we show that

$$|F(f) - f(x)| \le \varepsilon\|f\| \quad \text{for } f \in A,$$

so $\|F - \delta_x\| \le \varepsilon$. In the second case μ_F generates a δ-multiplicative functional $F|_K$ on $A|_K$, and since $A|_K$ is f-stable there is an $x \in \mathfrak{M}(A|_K) = K$ close to $F|_K$. Again, using the net g_γ, we show that x is close to F. Hence A is f-stable. ∎

The next result establishes an equivalence relation between f-stability of a uniform algebra and that of its antisymmetric components. We first need to recall some definitions and results.

Let $A \subseteq C(K)$ be a uniform algebra on K. By taking a quotient space we can always assume that A separates points of K. A set $L \subset K$ is called a *set of antisymmetry* if any $f \in A$ which is real-valued on L is constant on L, and L is a *maximal set of antisymmetry* if it is not contained in any bigger set of antisymmetry. Any maximal set of antisymmetry is closed and it is a weak peak set [29], consequently, $A|_L := \{f|_L \in C(L) : f \in A\}$ is a closed subalgebra of $C(L)$. Furthermore, the quotient norm on $A|_L$ coincides with the sup norm on L, so $A|_L$ is a uniform algebra on L, and if $\mathfrak{M}(A) = K$ then $\mathfrak{M}(A|_L) = L$. Two distinct maximal sets of antisymmetry are disjoint, hence K can be decomposed into the disjoint union of maximal sets of antisymmetry, called the *Bishop decomposition*. The Bishop Decomposition Theorem says:

THEOREM 8 ([1]). *Let A be a uniform algebra on K and let $f \in C(K)$. Then*

$$f \in A \quad \textit{iff} \quad f|_L \in A|_L \textit{ for any maximal set of antisymmetry } L.$$

For a uniform algebra A on K we denote by QA the largest C^* subalgebra of A; that is, $QA = A \cap \overline{A}$ where $\overline{A} = \{\overline{f} : f \in A\}$. For $x \in K$ we call

$$E_x = \{k \in K : \forall f \in QA,\ f(x) = f(k)\}$$

the *QA level set* corresponding to x. Two distinct QA level sets are disjoint, hence K can be decomposed into the disjoint union of QA level sets, called the *Shilov decomposition*. The Shilov Decomposition Theorem states:

THEOREM 9 ([26]). *Let A be a uniform algebra on K and let $f \in C(K)$. Then*

$$f \in A \quad \textit{iff} \quad f|_L \in A|_L \textit{ for any } QA \textit{ level set } L.$$

The first impression may be that the Bishop and Shilov decompositions must coincide. This is indeed the case for many uniform algebras. Furthermore, for any uniform algebra the Bishop decomposition is at least as fine as the Shilov decomposition; however, in general the Bishop decomposition may be strictly finer than the Shilov one. This means that for a QA level set L the algebra $A|_L$ may again have nontrivial QA level sets, which is in striking contrast with the Bishop decomposition: for a maximal set of antisymmetry L the algebra $A|_L$ has no nontrivial maximal sets of antisymmetry. $H^\infty(\mathbb{D}) + C(\partial\mathbb{D})$ is an example of an algebra where the two decompositions are different [9, 25].

We are now ready to state a decomposition theorem for f-stability.

THEOREM 10. *A uniform algebra A is f-stable if and only if the family*

$$\{A|_K : K \text{ is a maximal set of antisymmetry}\}$$

is uniformly f-stable.

Proof. Assume that the family $\{A|_K : K$ is a maximal set of antisymmetry$\}$ is not uniformly f-stable. Then there is an $\varepsilon > 0$ such that for any $\delta > 0$ there is a maximal set of antisymmetry K and an $F \in \mathfrak{M}_\delta(A|_K)$ such that $\|F - G\| \ge \varepsilon$ for any $G \in \mathfrak{M}(A|_K)$. The composition $A \xrightarrow{\pi} A|_K \xrightarrow{F} \mathbb{C}$ of F and the natural projection π is δ-multiplicative and at a distance at least ε from any multiplicative functional on A.

Assume now that A is not f-stable. Let $\varepsilon > 0$ be such that for any $\delta > 0$ there is an $F \in \mathfrak{M}_\delta(A)$ with $\|F - G\| \ge \varepsilon$ for any $G \in \mathfrak{M}(A)$. Without loss of generality we may assume that F is represented by a probability measure μ_F on $\mathrm{Ch}(A)$. Since F restricted to the subalgebra QA of A is also δ-multiplicative and QA is a $C(X)$ algebra, Proposition 3 tells us that there is a QA level set E such that $\mu_F(E) \ge 1 - 10\delta$. Any QA level set is a weak peak set, so if δ is small enough it follows from the first part of Proposition 6 that $\mu_F(E) \ge 1 - 2\delta$ and $F_E \in \mathfrak{M}_\delta(A)$. If the Shilov and Bishop decompositions coincide then E is a maximal set of antisymmetry and we are done since F_E gives a δ-multiplicative functional on $A|_E$ at a distance at least ε from any multiplicative one. In the general case we have to work some more and it will be crucial that in the first part of Proposition 6, $F_E \in \mathfrak{M}_\delta(A)$ with the same δ as the original functional F.

For each ordinal number ϖ we define a partition $\mathcal{P}_\varpi$ of $\mathfrak{M}(A)$ into weak peak sets: For $\varpi = 1$, $\mathcal{P}_\varpi$ is the Shilov decomposition. $\mathcal{P}_{\varpi+1}$ is the decomposition obtained from $\mathcal{P}_\varpi$ by applying the Shilov decomposition to each of the algebras $A|_E$, where $E \in \mathcal{P}_\varpi$. If ϖ is a limit ordinal then $x, y \in \mathfrak{M}(A)$ belong to the same $\mathcal{P}_\varpi$ set if and only if x, y belong to the same $\mathcal{P}_{\varpi'}$ for any $\varpi' < \varpi$.

By an obvious cardinality argument $\mathcal{P}_{\varpi'} = \mathcal{P}_\varpi$ for any ϖ', ϖ large enough. Also, $\mathcal{P}_{\varpi+1} = \mathcal{P}_\varpi$ if and only if all sets in $\mathcal{P}_\varpi$ are maximal sets of antisymmetry. Hence, there is an ϖ_0 such that $\mathcal{P}_{\varpi_0}$ is the Bishop decomposition. We already proved that there is exactly one $E \in \mathcal{P}_1$ such that $\mu_F(E) \ge 1 - 2\delta$ and $F_E \in \mathfrak{M}_\delta(A)$. By the same arguments as before, applied inductively, for any ϖ there is an $E_\varpi \in \mathcal{P}_\varpi$ such that $\mu_F(E_\varpi) \ge 1 - 2\delta$ and $F_{E_\varpi} \in \mathfrak{M}_\delta(A)$. Hence E_{ϖ_0} is a maximal set of antisymmetry such that $F_{E_{\varpi_0}}$ gives a δ-multiplicative functional on $A|_{E_{\varpi_0}}$ at a distance at least ε from any multiplicative one. ■

The next result provides a tool for constructing non-f-stable algebras; we will use it in Section 7. The proposition says that if there is a multiplicative functional on A close to an ideal J but not close to any particular element of the spectrum of J then it generates an almost multiplicative functional on the quotient algebra A/J which is not close to a multiplicative one.

PROPOSITION 11. *Let A be a commutative Banach algebra and let J be a closed ideal in A. Put $X = \mathfrak{M}(A)$ and $K = \{H \in X : H|_J = 0\}$. Assume that there is an $F_0 \in X \setminus K$ such that for $F_0|_J : J \to \mathbb{C}$ we have $\|F_0|_J\| \le \eta < 1$ and $\|F_0 - H\| > \beta$ for any $H \in K$. Then there is a $G_0 \in \mathfrak{M}_{4\eta}(A/J)$ such that $\|G_0 - H\| > \beta - \eta$ for any $H \in \mathfrak{M}(A/J) = K$.*

Proof. Let $S : A \to \mathbb{C}$ be a linear map such that $\|S\| = \|F_0|_J\| \le \eta$ and $S|_J = F_0|_J$. Put $G = F_0 - S$ and let $G_0 : A/J \to \mathbb{C}$ be defined by $G_0(f + J) = G(f)$. By direct computation, $G \in \mathfrak{M}_{4\eta}(A)$. Let $f + J$ and $g + J$ be arbitrary elements of A/J with norms less than one. Let $f' \in f + J$ and $g' \in g + J$ be elements of A such that $\|f'\| < 1$ and $\|g'\| < 1$. We have

$$|G_0(f + J)G_0(g + J) - G_0(fg + J)| = |G(f')G(g') - G(f'g')| \le 4\eta,$$

hence $G_0 \in \mathfrak{M}_{4\eta}(A/J)$.

Let $H \in K$. Since $\|F_0 - H\| > \beta$ there is an $f_0 \in A$ with $\|f_0\| < 1$ such that $|F_0(f_0) - H(f_0)| > \beta$. Hence $\|f_0 + J\| \le \|f_0\| < 1$ and

$$\|G_0 - H\| \ge |G_0(f_0 + J) - H(f_0)| = |F_0(f_0) - H(f_0) - S(f_0)| > \beta - \eta. \ ■$$

4. The ball algebras. Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. By $A(\Omega)$ we denote the uniform algebra of all functions holomorphic on Ω and continuous on $\bar{\Omega}$. If Ω is equal to the n-dimensional unit ball

$$B_n = \Big\{\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n : \|\mathbf{z}\| = \Big(\sum_{k=1}^n |z_k|^2\Big)^{1/2} < 1\Big\},$$

we call $A(B_n)$ the *n-ball algebra*. For $n = 1$, $B_1 = \mathbb{D}$ so $A(B_1) = A(\mathbb{D})$ is the disc algebra.

THEOREM 12. *The ball algebras are f-stable.*

Proof. Let F be a δ-multiplicative functional on $A(B_n)$. We need to prove that F is close to a multiplicative functional. By the results of the previous section we may assume that F is represented by a nonatomic probability measure μ_F on ∂B_n. For $\mathbf{w} = (w_1, \ldots, w_n) \in B_n$ we define a function $\Phi_{\mathbf{w}} : \overline{B}_n \to \mathbb{C}^n$ by

$$\Phi_{\mathbf{w}}(\mathbf{z}) = \frac{\mathbf{w} - P_{\mathbf{z}} - \sqrt{1 - \|\mathbf{w}\|^2}(\mathbf{z} - P_{\mathbf{z}})}{1 - \langle \mathbf{z}, \mathbf{w} \rangle},$$

where

$$\langle \mathbf{z}, \mathbf{w} \rangle = \sum_{k=1}^{n} z_k \overline{w}_k \quad \text{and} \quad P_{\mathbf{z}} = \frac{\langle \mathbf{z}, \mathbf{w} \rangle}{\|\mathbf{w}\|^2} \mathbf{w}.$$

It is easy to check ([20], p. 391) that $\Phi_{\mathbf{w}}$ are automorphisms of B_n. We define a function $\varphi : B_n \to \mathbb{C}^n$ by

$$\varphi(\mathbf{w}) = \int \Phi_{\mathbf{w}} \, d\mu_F.$$

If $\mathbf{w}_0 \in \partial B_n$ and $\mathbf{w}_j$ is a sequence of points in B_n that converges to $\mathbf{w}_0$ then straightforward computations show that $\Phi_{\mathbf{w}_j}(z) \to \mathbf{w}_0$ pointwise on $\overline{B}_n \setminus \{\mathbf{w}_0\}$. So, since μ_F has no atoms, φ extends to a continuous function on $\overline{B}_n$ such that $\varphi(\mathbf{w}) = \mathbf{w}$ for $\mathbf{w} \in \partial B_n$. Hence, there is a $\mathbf{w}_0 \in B_n$ such that $\varphi(\mathbf{w}_0) = \mathbf{0}$. Since $f \mapsto f \circ \Phi_{\mathbf{w}_0}$ is an isometric isomorphism of $A(B_n)$ onto itself, F_0 defined by

$$F_0(f) = F(f \circ \Phi_{\mathbf{w}_0}) \quad \text{for } f \in A(B_n)$$

is a δ-multiplicative functional on $A(B_n)$, and

$$F_0(z_k) = 0 \quad \text{for } k = 1, \ldots, n. \tag{4}$$

By [2] (see also [28], pp. 151–153), the linear map T from the product of n copies of $A(B_n)$ into

$$A_0(B_n) := \{f \in A(B_n) : f(\mathbf{0}) = 0\}$$

defined by

$$T(f_1, \ldots, f_n) = \sum_{k=1}^{n} z_k f_k$$

is surjective. Hence, by the Open Mapping Theorem there is a constant C such that for any $f \in A_0(B_n)$ there are $S_k f = f_k \in A(B_n)$ with $\|f_k\| \le C\|f\|$ such that $T(f_1, \ldots, f_n) = f$.

For any norm one function f in $A(B_n)$ we have

$$F_0(f) = F_0\Big(f(\mathbf{0}) + \sum_{k=1}^{n} z_k S_k(f - f(\mathbf{0}))\Big) = f(\mathbf{0}) + \sum_{k=1}^{n} F_0(z_k S_k(f - f(\mathbf{0}))).$$

Since F_0 is δ-multiplicative, and $\|S_k(f - f(\mathbf{0}))\| \le 2C$, the last expression is at a distance $2nC\delta$ from

$$f(\mathbf{0}) + \sum_{k=1}^{n} F_0(z_k) F_0(S_k(f - f(\mathbf{0}))),$$

which by (4) is equal to $f(\mathbf{0})$. Hence $\|F_0 - \delta_{\mathbf{0}}\| \le 2nC\delta$, so $\|F - \delta_{\mathbf{w}_0}\| \le 2nC\delta$. ∎

As a special case, for $n = 1$ we get Johnson's theorem.

COROLLARY 13 ([17]). *The disc algebra $A(\mathbb{D})$ is f-stable.*

In Section 7 we show that the finite-dimensional quotients of the disc algebra are not uniformly f-stable.

B. E. Johnson also proved that the polydisc algebras $A(\mathbb{D}^n)$ are f-stable [17]. The author does not know if the same is true for the $A(\Omega)$ algebras in general. It may be interesting to notice that the non-f-stable uniform algebra constructed by Sidney [27] contains $A(\Omega)$, for some disconnected set $\Omega \subset \mathbb{C}^2$.

5. Algebras with one generator

THEOREM 14. *The family of all uniform algebras with one generator is uniformly f-stable.*

Proof. For a compact subset K of the complex plane $\mathbb{C}$ we denote by $P(K)$ the closure of the algebra of all polynomials in the topology of uniform convergence on K. Any uniform algebra with one generator is isometrically isomorphic to an algebra of the form $P(K)$ for a simply connected K equal to the spectrum of a generator. If K is disconnected then any component of K is a peak set, hence by Proposition 6 we can restrict our attention to algebras $P(K)$ with K connected and simply connected.

Assume $X \subset \mathbb{C}$ is homeomorphic to a closed unit disc $\overline{\mathbb{D}}$. By the Riemann Mapping Theorem $\operatorname{int} X$ and $\mathbb{D}$ are holomorphically diffeomorphic, and by ([5], p. 50) any such diffeomorphism can be extended to a homeomorphism of X onto $\overline{\mathbb{D}}$. Since, by Mergelyan's theorem [7], $P(X)$ consists of all continuous complex-valued functions on X that are holomorphic on $\operatorname{int} X$, it follows that $P(X)$ is isometrically isomorphic to the disc algebra. Consequently, the family of all $P(X)$ algebras with X homeomorphic to $\overline{\mathbb{D}}$ is uniformly f-stable.

Fix an $\varepsilon > 0$. Let $\delta > 0$ be such that any δ-multiplicative functional on the disc algebra is within ε from a multiplicative functional. Let K be a compact, connected and simply connected subset of the complex plane. Let F be a δ-multiplicative functional on $P(K)$ represented by a probability measure μ on K. Let $(X_n)_{n=1}^{\infty}$ be a decreasing sequence of subsets of $\mathbb{C}$ such that, for any n,

$$X_n \text{ is homeomorphic to } \overline{\mathbb{D}}, \quad K \subset \operatorname{int} X_n, \quad \text{and} \quad \bigcap_{n=1}^{\infty} X_n = K.$$

For any n we have $P(X_n) \subset P(K)$ and for any $f \in P(X_n)$ the $P(X_n)$-norm of f is at least as large as the $P(K)$-norm of f. So F is a δ-multiplicative functional on $P(X_n)$. Hence, there is a multiplicative functional G_n on $P(X_n)$, represented by a probability measure μ_n on X_n, and such that $\|F - G_n\|_{P(X_n)} \le \varepsilon$. Without loss of generality we may assume that the sequence μ_n is convergent in the weak* topology of $(C(X_1))^*$ to a probability measure μ on K. We denote by G the functional on $P(K)$ represented by μ. For any polynomials p, q we have

$$G(p)G(q) = \lim_n G_n(p) \lim_n G_n(q) = \lim_n G_n(pq) = G(pq),$$

and since polynomials are dense in $P(K)$, the functional G is multiplicative. Assume now that p, is a polynomial with the $P(K)$-norm less than one. Since p is uniformly continuous on bounded sets the $P(X_n)$-norms of p are less than one for all n sufficiently large. Hence

$$\begin{aligned} |F(p) - G(p)| &= \Big|F(p) - \int p\, d\mu\Big| = \Big|\lim_n \Big(F(p) - \int p\, d\mu_n\Big)\Big| \\ &\le \lim_n \|F - G_n\| \cdot \|p\|_{P(X_n)} \le \varepsilon, \end{aligned}$$

so $\|F - G\|_{P(K)} \le \varepsilon$. ∎

6. $R(K)$ **algebras.** For a compact subset K of the complex plane we denote by $R(K)$ the uniform algebra on K generated by all rational functions with poles off K. If K is simply connected then $P(K) = R(K)$. While any uniform algebra generated by polynomials of a single element is isomorphic to some $P(K)$, any uniform algebra generated by rational functions of a single element is isomorphic to $R(K)$. We do not know if all $R(K)$ algebras are f-stable; we can prove this if K is sufficiently regular.

THEOREM 15. *If $K \subset \mathbb{C}$ is such that $\mathbb{C} \setminus K$ has finitely many components and the closures of the components are disjoint then $R(K)$ is f-stable.*

Proof. Without loss of generality we may assume that K is a subset of the unit disc and that $\mathbb{C} \setminus K$ is not connected. Let $\overline{\mathbb{C}}$ denote the one-point compactification of the complex plane, let $\mathbf{Z}$ be the identity function on $\overline{\mathbb{C}}$, let U_0 be the component of $\overline{\mathbb{C}} \setminus K$ that contains the point at infinity, let $U_1, \ldots, U_n$ be the other components of $\overline{\mathbb{C}} \setminus K$, and let $r > 0$ be such that the distance between any two components of $\mathbb{C} \setminus K$ is at least r. For any $k = 0, 1, \ldots, n$ we fix a point $w_k \in U_k$ (with w_0 not the point at infinity) and set

$$a := \inf\{\operatorname{dist}(w_k, K) : k = 0, 1, \ldots, n\} > 0.$$

For $k = 0, 1, \ldots, n$ let A_k be the (uniformly closed) algebra of continuous functions on $\overline{\mathbb{C}} \setminus U_k$ that are holomorphic on $\overline{\mathbb{C}} \setminus \overline{U}_k$. By Mergelyan's theorem, A_k is the closed subalgebra of $R(K)$ generated by $(\mathbf{Z} - w_k)^{-1}$ if $k = 1, \ldots, n$,

and by $\mathbf{Z}$ if $k = 0$; the maximal ideal space $\mathfrak{M}(A_k)$ of A_k can be identified with $\overline{\mathbb{C}} \setminus U_k$ ([4]). Furthermore, any $f \in R(K)$ can be decomposed in a unique way into a sum

$$f = f_0 + f_1 + \ldots + f_n$$

such that $f_k \in A_k$ and $f_1(\infty) = \ldots = f_n(\infty) = 0$; the maps

$$R(K) \ni f \overset{T_k}{\longmapsto} f_k \in A_k$$

are continuous and linear. Put $C = \sup\{\|T_k\| : k = 0, 1, \ldots, n\}$.

We define hyperbolic metrics $\varrho_k(z, w)$ on $\mathfrak{M}(A_k)$ by

$$\varrho_k(z, w) = \|\delta_z - \delta_w\| = \sup\{|f(z) - f(w)| : f \in A_k,\ \|f\| \le 1\}.$$

A hyperbolic metric is locally equivalent to the Euclidean metric. So, there is a constant $c > 0$ such that for any $k = 0, 1, \ldots, n$ and any $z, w \in \mathfrak{M}(A_k)$, if the Euclidean distance between $\overline{\mathbb{C}} \setminus \mathfrak{M}(A_k) = U_k$ and at least one of z and w is larger than $r/3$, then

$$\varrho_k(z, w) \le c|z - w|. \tag{5}$$

Let $\varepsilon > 0$. Let $\eta > 0$ be such that

$$\frac{2a}{a - 4\eta} < 3, \quad \frac{12\eta}{a} < \frac{r}{3} \quad \text{and} \quad (n+1)C\eta\Big(1 + \frac{12c}{a}\Big) < \varepsilon. \tag{6}$$

By Theorem 14 there is a $\delta > 0$ such that any δ-multiplicative functional on an algebra A with one generator is within η from the set of multiplicative functionals on A; we may also assume $\delta < \eta$.

Let F be a δ-multiplicative functional on $R(K)$. We may assume that $F(1) = 1 = \|F\|$. We need to show that F is within distance ε of $\mathfrak{M}(R(K))$. Let F_k be the restriction of F to A_k; obviously $F_k \in \mathfrak{M}_\delta(A_k)$. By the definition of δ there are functionals $G_k \in \mathfrak{M}(A_k)$ such that $\|F_k - G_k\| < \eta$. Let z_k be the point in $\overline{\mathbb{C}}$ corresponding to G_k. Let $F(\mathbf{Z}) = F_0(\mathbf{Z}) = \widehat{z}$. Since K is contained in the unit disc we have $\|\mathbf{Z}\| \le 1$. For $k = 1, \ldots, n$ we have $|w_k| < 1$ and, by the definition of a, $\|(\mathbf{Z} - w_k)^{-1}\| \le 1/a$, hence

$$\begin{aligned} |z_k - w_k|^{-1}|z_k - \widehat{z}| &= |1 - (z_k - w_k)^{-1}(\widehat{z} - w_k)| \\ &= |1 - G_k((\mathbf{Z} - w_k)^{-1})F(\mathbf{Z} - w_k)| \\ &\le |1 - F_k((\mathbf{Z} - w_k)^{-1})F(\mathbf{Z} - w_k)| \\ &\quad + \|F_k - G_k\| \cdot \|(\mathbf{Z} - w_k)^{-1}\| \cdot \|F\| \cdot \|\mathbf{Z} - w_k\| \\ &\le \delta\|(\mathbf{Z} - w_k)^{-1}\| \cdot \|\mathbf{Z} - w_k\| + \eta\|(\mathbf{Z} - w_k)^{-1}\| \cdot 2 \\ &\le \frac{2\delta}{a} + \frac{2\eta}{a} \le \frac{4\eta}{a}. \end{aligned}$$

Therefore $z_k \ne \infty$ and

$$|z_k - \widehat{z}| \le \frac{4\eta}{a}|z_k - w_k| \quad \text{for } k = 1, \ldots, n. \tag{7}$$

Also for $k = 1, \ldots, n$,

$$\begin{aligned}2|F((\mathbf{Z}-w_k)^{-1})| &\geq |F(\mathbf{Z}-w_k)|\cdot|F((\mathbf{Z}-w_k)^{-1})|\\ &\geq 1-|1-F(\mathbf{Z}-w_k)F((\mathbf{Z}-w_k)^{-1})|\\ &\geq 1-\delta\|(\mathbf{Z}-w_k)^{-1}\|\cdot\|\mathbf{Z}-w_k\|\\ &\geq 1-\frac{2\delta}{a}\geq 1-\frac{2\eta}{a}>0,\end{aligned}$$

which implies

$$|F((\mathbf{Z}-w_k)^{-1})| \geq \frac{a-2\eta}{2a} > 0,$$

and then

$$\begin{aligned}|z_k-w_k|^{-1} &\geq |F((\mathbf{Z}-w_k)^{-1})| - |F((\mathbf{Z}-w_k)^{-1})-(z_k-w_k)^{-1}|\\ &\geq \frac{a-2\eta}{2a} - |(F_k-G_k)((\mathbf{Z}-w_k)^{-1})|\\ &\geq \frac{a-2\eta}{2a}-\frac{\eta}{a}=\frac{a-4\eta}{2a}>0.\end{aligned}$$

Hence, by (6) and (7) for $k = 1, \ldots, n$ we have $|z_k - \widehat{z}| < 12\eta/a$. Since also $|z_0 - \widehat{z}| = |G_0(\mathbf{Z}) - F_0(\mathbf{Z})| \leq \|G_0 - F_0\| < \eta$, we get

$$|z_k - z_j| < 12\eta/a < r/3 \quad \text{for } k = 0, 1, \ldots, n. \tag{8}$$

Let $\mathrm{dist}(z_{k_0}, U_{k_0}) = \min\{\mathrm{dist}(z_k, U_k) : k = 0, 1, \ldots, n\}$. We show that $z_{k_0} \in K$ and that F is within ε of $\delta_{z_{k_0}}$. By the definition of r, for all $k = 0, 1, \ldots, n$ except possibly $k = k_0$, $\mathrm{dist}(z_k, U_k) \leq r/3$, so in view of (8), $z_{k_0} \in \mathbb{C} \setminus U_k$, for $k = 0, 1, \ldots, n$. Hence $z_{k_0} \in K$ and by (8) we have $\varrho_k(z_{k_0}, z_k) \leq c|z_{k_0} - z_k|$. Let f be a norm one element of $R(K)$. Then

$$\begin{aligned}|F(f)-f(z_{k_0})| &\leq \sum_{k=0}^{n}|F(T_kf)-(T_kf)(z_{k_0})|\\ &\leq \sum_{k=0}^{n}|F(T_kf)-G_k(T_kf)|+\sum_{k=0}^{n}|(T_kf)(z_k)-(T_kf)(z_{k_0})|\\ &\leq \sum_{k=0}^{n}\eta\|T_kf\|+\sum_{k=0}^{n}\|\delta_{z_{k_0}}-\delta_k\|\cdot\|T_kf\|\\ &\leq (n+1)\eta C+nc\frac{12\eta}{a}C\leq (n+1)C\eta\left(1+\frac{12\eta}{a}\right)<\varepsilon,\end{aligned}$$

where we have used (8) and (6). Hence $\|F - \delta_{z_{k_0}}\| \leq \varepsilon$ as promised. ■

The author does not know if the family of algebras described in the last theorem is uniformly f-stable. If it is, then using arguments very similar to those in the proof of Theorem 14 one can show that all algebras of the form $R(K)$ are f-stable. At least some of the algebras not covered by the last theorem are f-stable. For example, if $K = \{z \in \mathbb{C} : |z| \leq 1, |z - 1/2| \geq 1/2\}$ then $R(K)$ is isometrically isomorphic to a subalgebra $\{f \in A(\mathbb{D}) : f(-1) = f(1)\}$ of the disc algebra. Hence, by Theorem 7 and Corollary 13, $R(K)$ is f-stable.

7. Quotient algebras H^∞/BH^∞. We prove that the quotient algebra H^∞/BH^∞ is f-stable if and only if B is a product of finitely many interpolating Blaschke products; equivalently, if the measure defined by B is a Carleson measure. This type of Blaschke product has been investigated in a number of papers; see for example [3, 6, 10, 18, 19, 21, 22, 23, 30].

We first need to recall some basic properties of $H^\infty = H^\infty(\mathbb{D})$ and the Blaschke products as can be found in [8]. First, there are several natural and equivalent ways we will interpret an element f in H^∞: it can be seen as a bounded analytic function on $\mathbb{D}$, or as an element of $L^\infty = L^\infty(\partial\mathbb{D})$, or as a continuous function on $\mathfrak{M}(H^\infty)$, or as a function on $\mathfrak{M}(L^\infty) \subset \mathfrak{M}(H^\infty)$.

Suppose $\{\alpha_n\}$ is a sequence in $\mathbb{D}$ such that

$$\sum_{n=1}^{\infty}(1-|\alpha_n|)<\infty.$$

This is the necessary and sufficient condition for the sequence $\{\alpha_n\}$ to be the zero sequence of a bounded analytic function on $\mathbb{D}$. When the condition is satisfied, we have the associated Blaschke product

$$B(z,\{\alpha_n\})=\prod_{n=1}^{\infty}\frac{\overline{\alpha}_n}{|\alpha_n|}\cdot\frac{\alpha_n-z}{1-\overline{\alpha}_nz}.$$

This product converges uniformly on compact subsets of $\mathbb{D}$, and defines a function in $H^\infty(\mathbb{D})$. Furthermore, a function f in $H^\infty(\mathbb{D})$ vanishes on $\{\alpha_n\}$ if and only if $f = B(\cdot, \{\alpha_n\})g$, where $g \in H^\infty(\mathbb{D})$ and $\|f\| = \|g\|$. Hence, for any Blaschke product B,

$$BH^\infty := \{Bg : g \in H^\infty\}$$

is a closed ideal; we denote by π_B the natural projection from H^∞ onto the quotient algebra H^∞/BH^∞. The algebra H^∞/BH^∞ can be seen as a subalgebra of l^∞:

$$H^\infty/BH^\infty \ni [f] \overset{\iota_B}{\longmapsto} \{f(\alpha_n)\} \in l^\infty.$$

We call a Blaschke product B *interpolating* if the map $\iota_B \circ \pi_B : H^\infty \to l^\infty$ is surjective; in that case ι_B^{-1} is automatically continuous, and we denote by M_B the norm of ι_B^{-1}. Recall that the hyperbolic distance on $\mathbb{D}$ is defined by

$$\varrho(z,w)=\|\delta_z-\delta_w\|=\sup\{|f(z)-f(w)| : f\in H^\infty,\ \|f\|\leq 1\},$$

and that

$$\frac{\varrho(z,w)}{2} \le \left|\frac{z-w}{1-\overline{z}w}\right| \le \varrho(z,w), \qquad z,w\in\mathbb{D}.$$

The following is the most fundamental result in the theory of Blaschke products.

THEOREM 16. *If $\{\alpha_n\}$ is a sequence in $\mathbb{D}$ and B is the corresponding Blaschke product, then the following are equivalent*:

1. *$\{\alpha_n\}$ is an interpolating sequence.*
2. *We have the inequality*

$$\inf_k \prod_{j\ne k} \varrho(\alpha_j,\alpha_k) =: \delta_B > 0.$$

3. *We have*

$$\inf_{j\ne k} \varrho(\alpha_j,\alpha_k) =: a_B > 0 \tag{9}$$

and

$$\sup_{\substack{0<r<1\\ 0\le\theta<\pi}} \sum_{\substack{1-r<|\alpha_n|\\ |\arg(\alpha_n)-\theta|<r}} (1-|\alpha_n|)/r =: A_B < \infty. \tag{10}$$

A sequence $\{\alpha_n\}$ in $\mathbb{D}$ which satisfies (10) is called a *Carleson sequence*. The next result is a combination of results from [18, 19, 21, 22] (see also [3], pp. 68–69). We will need only the implication (2)⇒(3) (see [19], p. 534).

THEOREM 17. *If $\{\alpha_n\}$ is a sequence in $\mathbb{D}$ then the following are equivalent*:

1. *$\{\alpha_n\}$ is a Carleson sequence.*
2. *For any $\varepsilon_0>0$ there is a $\delta>0$ such that for any $z\in\mathbb{D}$ we have*

$$\text{if}\quad |B(z,\{\alpha_n\})|<\delta \quad\text{then}\quad \inf\{\varrho(z,\alpha_n): n=1,2,\ldots\}<\varepsilon_0.$$

3. *$\{\alpha_n\}$ is a union of finitely many interpolating sequences.*

THEOREM 18. *Let B be a Blaschke product. Then H^∞/BH^∞ is f-stable if and only if B is a product of finitely many interpolating Blaschke products.*

Proof. We first observe that if B is an interpolating Blaschke product then the quotient algebra H^∞/BH^∞ is isomorphic to the algebra $l^\infty = C(\beta N)$ of all bounded sequences, so by Proposition 3 it is f-stable. One should, however, notice that the isomorphism between H^∞/BH^∞ and l^∞ may have a large norm, so while it follows that each of the algebras H^∞/BH^∞, with B an interpolating Blaschke product, is f-stable, it does not follow and is not true (see Proposition 19) that the family of all algebras of the form H^∞/BH^∞ with B an interpolating Blaschke product is uniformly f-stable.

Assume now that $B=\prod_{j=1}^p B_j$, where B_j are interpolating Blaschke products and $F\in\mathfrak{M}_\delta(H^\infty/BH^\infty)$. Without loss of generality we may assume that $\|F\|=1$ (see [17]).

In the algebra H^∞/BH^∞ we have $\prod_{j=1}^p(B_j+BH^\infty)=B+BH^\infty=0$, so, since F is almost multiplicative, we have

$$\prod_{j=1}^p F(B_j+BH^\infty)\approx F\Big(\prod_{j=1}^p(B_j+BH^\infty)\Big)=F(0)=0.$$

It follows that one of the numbers $F(B_j+BH^\infty)$ is small. By a direct computation we can verify that

$$|F(B_{j_0}+BH^\infty)|\le \sqrt[p]{(p-1)\delta}=:\delta_1, \tag{11}$$

for at least one index j_0.

Put $I=B_{j_0}H^\infty/BH^\infty$. Since $F\in\mathfrak{M}_\delta(H^\infty/BH^\infty)$, by (11) for any $f+BH^\infty\in H^\infty/BH^\infty$ we have

$$\begin{aligned}|F((f+BH^\infty)\cdot(B_{j_0}+BH^\infty))|&\\ &\le \delta\|f+BH^\infty\|+|F(f+BH^\infty)\cdot F(B_{j_0}+BH^\infty)|\\ &\le \delta\|f+BH^\infty\|+\delta_1|F(f+BH^\infty)|\\ &\le(\delta+\delta_1)\|f+BH^\infty\|,\end{aligned}$$

so $\|F|_I\|\le\delta+\delta_1$. Let $\Delta\in(H^\infty/BH^\infty)^*$ be such that $\|\Delta\|=\|F|_I\|$ and $\Delta|_I=F|_I$. Put $F_1=F-\Delta$. Since $I\subset\ker F_1$, F_1 induces a linear functional $\widetilde F$ on the quotient algebra $(H^\infty/BH^\infty)/I\cong H^\infty/B_{j_0}H^\infty$. By a direct computation, $F_1\in\mathfrak{M}_{5\delta_1}(H^\infty/BH^\infty)$, so

$$\widetilde F\in\mathfrak{M}_{5\delta_1}(H^\infty/B_{j_0}H^\infty).$$

Since B_{j_0} is an interpolating Blaschke product, the algebra $H^\infty/B_{j_0}H^\infty$ is isomorphic to l^∞. By Proposition 3 there is a $\widetilde G\in\mathfrak{M}(l^\infty)$ such that $\|\widetilde F-\widetilde G\|\le\delta_2$, where δ_2 depends on $\delta+\delta_1$ and the norm of the isomorphism between l^∞ and $H^\infty/B_{j_0}H^\infty$, and tends to zero as $\delta\to0$. Let π be the natural projection from H^∞/BH^∞ onto $H^\infty/B_{j_0}H^\infty$ and put $G=\widetilde G\circ\pi$. We have

$$G\in\mathfrak{M}(H^\infty/BH^\infty)\quad\text{and}\quad \|F-G\|\le\|\Delta\|+\|\widetilde F-\widetilde G\|\le\delta+\delta_1+\delta_2.$$

Now assume that $B=B(\cdot,\{\alpha_n\})$ is not a product of finitely many interpolating Blaschke products, and let $K=\{H\in\mathfrak{M}(H^\infty): H|_{BH^\infty}=0\}$ be the spectrum of BH^∞. By Theorem 17 there is an $\varepsilon_0>0$ such that for any $\delta>0$ there is a $z_\delta\in\mathbb{D}$ with

$$|B(z_\delta,\{\alpha_n\})|<\delta\quad\text{and}\quad \inf\{\varrho(z_\delta,\alpha_n): n=1,2,\ldots\}\ge\varepsilon_0. \tag{12}$$

Notice that K is the union of the set $\{\alpha_n : n = 1, 2, \ldots\}$ and a subset of $\mathfrak{M}(H^\infty) \setminus \mathbb{D}$. Since $\mathbb{D}$ is a Gleason part of H^∞ ([12]), the norm distance between z_δ and any point from $\mathfrak{M}(H^\infty) \setminus \mathbb{D}$ is equal to 2. Hence, and by (12), any point of K is far from z_δ, so by Proposition 11, H^∞/BH^∞ is not f-stable. ∎

PROPOSITION 19. *Let $\mathcal{BF}$ be the set of finite Blaschke products. Then the families*

$$(13) \qquad \{A(\mathbb{D})/BA(\mathbb{D}) : B \in \mathcal{BF}\} \quad \text{and} \quad \{H^\infty/BH^\infty : B \in \mathcal{BF}\}$$

of finite-dimensional algebras are not uniformly f-stable.

Note that all of the algebras in the families (13) are finite-dimensional so they are all f-stable.

Proof of Proposition 19. Let A be equal to $A(\mathbb{D})$ or to H^∞, let $\delta > 0$, and let α_j, $j = 1, \ldots, n$, be points in a disc of radius $1/2$ around the origin such that

$$\prod_{j=1}^{n} \varrho(3/4, \alpha_j) < \delta.$$

Let B be a finite Blaschke product with zeros at $\{\alpha_j\}$. Let

$$J = BA = \{f : f|_{\{\alpha_j\}_{j=1}^n} = 0\}.$$

For any $j = 1, \ldots, n$ and for any $f \in J$ of norm 1 we have

$$\varrho(3/4, \alpha_j) \ge 1/4 \quad \text{and} \quad |f(3/4)| < \delta.$$

Hence, by Proposition 11, the families A/J of (13) are not uniformly f-stable. ∎

8. The algebra $H^\infty(\mathbb{D})$. The question whether a Banach algebra A is f-stable can be interpreted as a question whether A has an *almost corona*, that is, a set of almost multiplicative functionals far from the set of multiplicative functionals. In view of the importance of the Corona Theorem for $H^\infty(\mathbb{D})$ it would be particularly interesting to know whether $H^\infty(\mathbb{D})$ has an almost corona. We have been unable to answer this question; we prove, however, some results linking f-stability with the approximation properties of interpolating Blaschke products.

We need to use repeatedly the Douglas–Rudin Theorem ([8], p. 428):

THEOREM 20. *Suppose $u \in L^\infty$ and $|u| = 1$ almost everywhere. Let $\varepsilon > 0$. Then there exist interpolating Blaschke products B_1 and B_2 such that*

$$\|u - B_1/B_2\|_\infty < \varepsilon.$$

PROPOSITION 21. *For an arbitrary regular, nonatomic probability measure μ on $\partial H^\infty(\mathbb{D}) = \mathfrak{M}(L^\infty)$ there is an interpolating Blaschke product B with $|\int B\,d\mu| < 3/4$.*

Notice that we do not assume that μ is absolutely continuous with respect to the Lebesgue measure, nor even that μ is a measure on the unit circle; it is a measure on a much bigger set $\mathfrak{M}(L^\infty)$. In this setting B is a continuous function on $\mathfrak{M}(L^\infty)$.

Proof of Proposition 21. Because μ is regular and nonatomic, there is a compact set $K \subset \mathfrak{M}(L^\infty)$ such that $\mu(K) = 1/2$. For any $n \in \mathbb{N}$ let $f_n \in C(\mathfrak{M}(L^\infty)) \cong L^\infty$ be such that $f_n : \mathfrak{M}(L^\infty) \to [0,1]$,

$$f_n = 1 \text{ on } K \quad \text{and} \quad \Big|\int_{\mathfrak{M}(L^\infty)} f_n\,d\mu\Big| < \frac{1}{2} + \frac{1}{n}.$$

Put $g_n = -\exp(\pi i f_n)$. The function g_n is a unimodular function equal to 1 on K and is μ-close to -1 on the complement of K. By Theorem 20 there are interpolating Blaschke products $B_{n,1}, B_{n,2}$ with $\|g_n - B_{n,1}/B_{n,2}\|_\infty < 1/n$. Hence

$$\|g_n B_{n,2} - B_{n,1}\|_\infty < 1/n$$

so $B_{n,1}, B_{n,2}$ are almost identical on K and almost opposite off K. Put

$$\lambda_{n,i} = \int_K B_{n,i}\,d\mu, \qquad \int_{\mathfrak{M}(L^\infty)\setminus K} B_{n,i}\,d\mu = \lambda'_{n,i}, \quad \text{for } i = 1, 2.$$

As $n \to \infty$ we have

$$\Big|\int_{\mathfrak{M}(L^\infty)} B_{n,2} g_n\,d\mu - (\lambda_{n,1} + \lambda'_{n,1})\Big| \to 0,$$

$$\Big|\int_{\mathfrak{M}(L^\infty)} B_{n,2} g_n\,d\mu - (\lambda_{n,2} - \lambda'_{n,2})\Big| \to 0,$$

so

$$(14) \qquad |(\lambda_{n,1} + \lambda'_{n,1}) - (\lambda_{n,2} - \lambda'_{n,2})| \to 0.$$

Since the absolute values of all the λ's are smaller than or equal to $1/2$, for any n we have

$$|\lambda_{n,2} - \lambda'_{n,2}|^2 + |\lambda_{n,2} + \lambda'_{n,2}|^2 = 2(|\lambda_{n,2}|^2 + |\lambda'_{n,2}|^2) \le 1,$$

so

$$\liminf \Big|\int_{\mathfrak{M}(L^\infty)} B_{n,2}\,d\mu\Big| = \liminf |\lambda_{n,2} + \lambda'_{n,2}| \le \frac{\sqrt{2}}{2},$$

or, by (14),

$$\liminf \Big| \int_{\mathfrak{M}(L^\infty)} B_{n,1}\, d\mu \Big| = \liminf |\lambda_{n,2} - \lambda'_{n,2}| \le \frac{\sqrt{2}}{2}.$$

Hence, some of the numbers $|\int_{\mathfrak{M}(L^\infty)} B_{n,i}\, d\mu|$, $i = 1, 2$, $n \in \mathbb{N}$, are smaller than $3/4$. ∎

PROPOSITION 22. *For any $\varepsilon > 0$ there is a $\delta > 0$ such that for any $F \in \mathfrak{M}_\delta(H^\infty)$, either there is an $x \in \partial H^\infty$ with $\|F - \delta_x\| < 2\delta$, or there is an interpolating Blaschke product B with $|F(B)| < \varepsilon$.*

Proof. Let k be such that $(3/4)^k < \varepsilon/4$ and δ such that $k\delta < \varepsilon/4$. By Propositions 5 and 6 we may assume that $F \in \mathfrak{M}_\delta(H^\infty)$ is represented by a nonatomic probability measure on $\mathfrak{M}(L^\infty)$ so by Proposition 21 there is a Blaschke product B_0 such that $|F(B_0)| < 3/4$. Since F is δ-multiplicative, by a simple induction, we have

$$|F(B_0^k) - (F(B_0))^k| \le (k-1)\delta < \varepsilon/4 - \delta.$$

Put $I = B_0^k$. We have $|F(I)| < \varepsilon/2 - \delta$. By Theorem 20 there are interpolating Blaschke products $B, \widetilde{B}$ with $\|I - B/\widetilde{B}\|_\infty < \varepsilon/2$. Hence $\|I\widetilde{B} - B\|_\infty < \varepsilon/2$, so since F is δ-multiplicative, we get

$$\begin{aligned} |F(B)| &\le |F(B) - F(I\widetilde{B})| + |F(I\widetilde{B}) - F(I)F(\widetilde{B})| + |F(I)F(\widetilde{B})| \\ &\le \varepsilon/2 + \delta + (\varepsilon/2 - \delta) = \varepsilon. \ ∎ \end{aligned}$$

Remark 1. Assume that for a given $F \in \mathfrak{M}_\delta(H^\infty)$ we could replace, in the last proposition, $\varepsilon > 0$ with $\varepsilon = 0$ and that we could control the interpolating constant M_B. Then F induces a δ-multiplicative functional on an f-stable quotient algebra H^∞/BH^∞. Hence F has to be close to a multiplicative functional δ_x for some $x \in \mathfrak{M}(H^\infty/BH^\infty) \subset \mathfrak{M}(H^\infty)$.

The maximal ideal space $\mathfrak{M}(H^\infty)$ of H^∞ is the union of four disjoint sets: the unit disc $\mathbb{D}$, the Shilov boundary ∂H^∞, the set $\mathcal{P}$ of all trivial Gleason parts not in ∂H^∞, and $\mathcal{G}$, the union of all nontrivial Gleason parts other than $\mathbb{D}$ ([12]). If $x \in \mathcal{P}$ then $\|\delta_x - \delta_y\| = 2$ for any point $y \in \partial H^\infty$, so by Proposition 22, there is an interpolating Blaschke product B such that $|B(x)| < \varepsilon$. Combining this with classical results by Hoffman [12] we get the following (known) proposition.

PROPOSITION 23. *Let $x \in \mathfrak{M}(H^\infty) \setminus \mathbb{D}$. Then*

(i) *$x \in \partial H^\infty$ if and only if $B(x) \ne 0$ for any Blaschke product B; furthermore, if $x \in \partial H^\infty$ then $|B(x)| = 1$ for any Blaschke product B.*

(ii) *$x \in \partial H^\infty \cup \mathcal{P}$ if and only if $B(x) \ne 0$ for any interpolating Blaschke product B; furthermore, if $x \in \mathcal{P}$ then for any $\varepsilon > 0$ there is an interpolating Blaschke product B such that $|B(x)| < \varepsilon$.*

9. Open problems. We list here some open problems concerning f-stability of uniform algebras.

PROBLEM 1. Is H^∞ f-stable?

PROBLEM 2. Are Douglas algebras f-stable?

PROBLEM 3. Let K be a compact subset of the complex plane. Is $R(K)$ f-stable? Is $H^\infty(K)$ f-stable?

PROBLEM 4. Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. Are $A(\Omega)$ and $H^\infty(\Omega)$ f-stable?

PROBLEM 5. Is any uniform algebra with two generators f-stable?

PROBLEM 6. Let A be an f-stable uniform algebra. Is the algebra

$$l^\infty(A) = \{(f_n)_{n=1}^\infty : \forall n,\ f_n \in A, \text{ and } \|(f_n)\| = \sup_n \|f_n\| < \infty\}$$

f-stable?

PROBLEM 7. Let A be an f-stable uniform algebra. Is an ultrapower of A f-stable? (See [11, 16] for basic properties of ultraproducts.)

PROBLEM 8. Let A be a uniform algebra such that the family of all quotient algebras A/I is uniformly f-stable, where I is a closed ideal in A. Is $A = C(K)$ for some compact set K?

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Two-sided estimates of the approximation numbers of certain Volterra integral operators

by

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Abstract. We consider the Volterra integral operator $T : L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+)$ defined by

$$(Tf)(x) = v(x)\int_0^x u(t)f(t)\,dt.$$

Under suitable conditions on u and v, upper and lower estimates for the approximation numbers $a_n(T)$ of T are established when $1 < p < \infty$. When $p = 2$, these yield

$$\lim_{n\to\infty} n a_n(T) = \pi^{-1}\int_0^\infty |u(t)v(t)|\,dt.$$

We also provide upper and lower estimates for the ℓ^α and weak ℓ^α norms of $(a_n(T))$ when $1 < \alpha < \infty$.

1. Introduction. In this paper we study the approximation numbers of the Volterra integral operator T given by

$$(Tf)(x) = v(x)\int_0^x u(t)f(t)\,dt \tag{1.1}$$

for $x \in \mathbb{R}^+ := [0,\infty)$ and $f \in L^p(\mathbb{R}^+)$. Here $1 < p < \infty$, and u, v are real-valued functions, with $u \in L^{p'}_{\mathrm{loc}}(\mathbb{R}^+)$ and $v \in L^p(\mathbb{R}^+)$; as usual, $p' = p/(p-1)$. The paper is a continuation of our earlier work [4], in which we gave a necessary and sufficient condition for $T : L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+)$ to be compact and also provided a scheme for obtaining upper and lower estimates for the approximation numbers of T. As an illustrative example we showed that when $u(x) = e^{Ax}$ and $v(x) = e^{-Bx}$, where $0 < A < B$, then the nth approximation number $a_n(T)$ of T is bounded above and below by positive multiples of n^{-1}. However, the general scheme mentioned above

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