On duals of Calderón–Lozanovskii intermediate spaces

by

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Abstract. We give a description of the dual of a Calderón–Lozanovskii intermediate space \( \varphi(X, Y) \) of a couple of Banach Köthe function spaces as an intermediate space \( \varphi(X^*, Y^*) \) of the duals, associated with a "variable" function \( \varphi \).

Introduction. Given two Köthe function spaces over the same measure space, \( X_0 \) and \( X_1 \), the interpolation spaces \( X_0^{1-\theta}X_1^{\theta} \), \( 0 < \theta < 1 \), were defined by Calderón ([C]) as the order ideal generated by the functions \( x_0^{1-\theta}x_1^{\theta} \) with \( x_0 \in X_0 \), \( x_0 \geq 0 \) and \( x_1 \in X_1 \), \( x_1 \geq 0 \). When \( X_0 \) or \( X_1 \) is reflexive, these spaces coincide (in the complex case) with the spaces \([X_0, X_1]_\theta\) obtained by the complex interpolation method. In this case the dual space can also be described by complex interpolation; more precisely, if \( X_0 \cap X_1 \) is dense in \( X_0 \) and \( X_1 \), then \( X_0^{1-\theta}X_1^{\theta} \) embed naturally in \((X_0 \cap X_1)^*\), and \([X_0, X_1]_\theta = [X_0^*, X_1^*]_\theta\). The description of the dual of \( X_0^{1-\theta}X_1^{\theta} \) without any restriction on the Banach lattices \( X_0 \) and \( X_1 \) (except their order completeness) was achieved by Lozanovskii ([L1], [L2]). When \( X_0 \cap X_1 \) is dense in \( X_0 \) and \( X_1 \), then \((X_0^{1-\theta}X_1^{\theta})^* = X_0^{\theta-1}X_1^{1-\theta}\), the definition of this last space being unambiguous since \( X_0^\theta \) and \( X_1^{1-\theta} \) are order ideals of \((X_0 \cap X_1)^*\); in the general case Lozanovskii shows how to realize \( X_0^{\theta} \) and \( X_1^{1-\theta} \) as order ideals of a common space of measurable functions and then identifies (isometrically and order isomorphically) \((X_0^{1-\theta}X_1^{\theta})^*\) with \( X_0^{\theta-1}X_1^{1-\theta}\). A consequence of this fact is that the equality \((X_0^{1-\theta}X_1^{\theta})^* = X_0^{1-\theta}X_1^{\theta}\) holds for the Köthe duals of the spaces \( X_0, X_1 \).

These results were (partially) extended to a more general class of interpolation spaces of Köthe function spaces, the so-called Calderón–Lozanovskii spaces. Let us recall their definition. Consider a function \( \varphi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \) which is concave, positively homogeneous of degree one, continuous and not identically zero (we denote by \( C \) the set of such functions, which we call Calderón–Lozanovskii functions). By rescaling if necessary, we may suppose that \( \varphi(1, 1) = 1 \) (we denote by \( C_1 \) the subset of such normalized functions).

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Then the space $\varphi(X_0, X_1)$ is the order ideal generated by the functions $\varphi(x_0, x_1) = x_i$ with $x_i \in X_i, x_i \geq 0, i = 0, 1$. This space is normed by the formula $\|\varphi\| = \inf\{\|\varphi\| + \|\varphi\| : \|\varphi\| \leq \varphi(x_0, x_1); x_i \in X_i, x_i \geq 0\}$.

The Calderón–Lozanovskii function $\varphi_*$ conjugate to $\varphi$ is defined by $\varphi_*(s, t) = \inf\{\alpha s + \beta t : \alpha, \beta \geq 0\}$. It is generally not normalized but $1 \leq \varphi_*'(1, 1) \leq 2$ if $\varphi$ is normalized. Suppose that $X_0 \cap X_1$ is dense in $X_0$, and $X_1$; let $Z_0$ be the closure of $X_0 \cap X_1$ in $\varphi(X_0, X_1)$. Then Lozanovskii proves ([L3]) that $Z_0 = \varphi_*(X_0, X_1)$; this is an equality between subspaces of $(X_0 \cap X_1)^*$, but the norms are only equivalent up to a constant 2. However, following [R], one can obtain isometry by putting on $\varphi_*(X_0, X_1)$ the modified norm $\|\varphi_*\| = \inf\{\|\varphi_0\| + \|\varphi_1\| : \varphi_0 \in X_0^*, \varphi_1 \in X_1^*; \|\varphi_0\| \geq 0\}$. As a consequence one can deduce the equality $\varphi(X_0, X_1)^* = \varphi_*(X_0, X_1)$ for the Köthe duals (without any density assumption). This last fact is reproved in [R], without considering the whole duals. When $\varphi$ satisfies the two-sided “reverse $\Delta_2$-condition”

$$\exists c > 0, \forall s, t > 0, \varphi(s, ct) \leq \frac{1}{2}\varphi(s, t)$$

and $\varphi(cs, t) \leq \frac{1}{2}\varphi(s, t)$ (this is in particular the case for $\varphi(s, t) = s^{1-\alpha}t^\alpha$) then $X_0 \cap X_1$ is dense in $\varphi(X_0, X_1)$ and the preceding result gives a description of the whole dual $(\varphi(X_0, X_1))^*$ (under the density assumption).

A particular, well known class of Calderón–Lozanovskii spaces is that of Orlicz spaces: if we set $M^{-1}(t) = \varphi(t, 1)$, then $M$ is an Orlicz function, and the corresponding Orlicz space $L_M$ is simply $\varphi(L_1, L_\infty)$ (with equality of norms if $L_M$ is equipped with the so-called Luxemburg norm). Let $M_*$ be the Young conjugate of $M$; one has $M_*^{-1}(s) = \varphi_*(1, s)$. If $M$ satisfies the usual $\Delta_2$ condition, then $L_M = L_{M_*} = \varphi_*(L_\infty, L_1)$; if not, $L_M = L_M \oplus L$, where $L$ is an abstract (separable) $L_1$-space (Ando’s theorem [A]; see also [Z], [F]).

The purpose of this paper is to give a unified description of the dual of the space $\varphi(X_0, X_1)$ in the most general case. Let a generalized Calderón–Lozanovskii function, for short g.c.-L. function, defined on the measure space $(S, \Sigma, m)$, be a measurable map $\varphi: S \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that for a.e. $s \in S$, the partial function $\varphi_s = \varphi(s, \cdot)$ belongs to $C$ or is identically 0. If $Y_0, Y_1$ are Köthe function spaces over $(S, \Sigma, m)$, then the generalized Calderón–Lozanovskii space $\varphi(Y_0, Y_1)$ is the order ideal generated by the functions $\varphi(y_0, y_1) = \varphi_s(y_0(s), y_1(s))$, where $y_0 \in Y_0, y_1 \geq 0, s = 0, 1$. Then the dual of the space $\varphi(X_0, X_1)$ can be described as a g.c.-L. space $\varphi(X_0^*, X_1^*)$ (see Theorem 4.1), for a suitable realization of $X_0^*$ and $X_1^*$ as order ideals of a space $L_0(S, \Sigma, m)$, and a g.c.-L. function $\varphi$ over $(S, \Sigma, m)$. Moreover, for a.e. $s \in S$, the conjugate function $\varphi_*$ is a limit of "dilations of $\varphi$", i.e. functions $\varphi_{c, b}: (u, v) \mapsto \varphi(cu, bv)/c(d, b)$ (with the convention that $\varphi_s = c$ when $\varphi_s = 0$).

In Section 1 we recall some basic notions and facts firstly about Köthe function spaces, Köthe duality and the Vulikh–Lozanovskii representation of the duals, and secondly about the set $C$ of Calderón–Lozanovskii functions, remarkable subsets of $C$ associated with a given C.-L. function $\varphi$, and the conjugation operation on $C$. In Section 2, given two Köthe function spaces $X$, $Y$, we define a functional $\Psi$ over the product of the positive cones of the dual spaces, with values in $\varphi(X, Y)^*$, which provides a way to express the norm on $\varphi(X, Y)^*$ (Theorem 2.5). In Section 3 we consider a triple $(E, F, \Psi)$, consisting of a couple $(E, F)$ of Köthe function spaces and an abstract functional $\Psi$ defined on $E_F \times F_F$; with this triple is associated a Köthe space $\varphi(E, F)^*$, the Köthe dual of which can be expressed in terms of a dual functional $\Psi_*$. In Section 4 we give the above announced representation theorem for $\varphi_*(X, Y)^*$ as a space $\varphi(X^*, Y^*)$ (Theorem 4.1), and in Section 5 we refine this theorem, by decomposing the underlying measure space into disjoint parts over which $\psi$ takes its values in remarkable subsets of C.-L. functions asymptotically associated with $\varphi_*$ (Theorem 3.9). An example is given to show that this decomposition can be nontrivial (Section 6).

1. Preliminaries

(a) Köthe function spaces and their duals. A Köthe function space over the measure space $(\Omega, A, \mu)$ is an order dense order ideal (= solid subspace) of the space $L_0(\Omega, A, \mu)$ of all measurable functions over $(\Omega, A, \mu)$, equipped with a norm for which it is a Banach lattice for the natural order. By extension, we shall also consider function spaces whose elements have supports in a fixed subset $A \subseteq A$, and are Köthe function spaces over $(A, |A|_A, \mu|A)$ (generalized Köthe function spaces). We call $A$ the support of $X$.

In the case where $\mu$ is not $\sigma$-finite, we shall suppose that the measure space is decomposable (or strictly localizable), i.e. there exists a measurable partition $(A_i)_i$ of $\Omega$ into $\mu$-integrable sets such that a subset $E$ of $\Omega$ is $A$-measurable (resp. $\mu$-negligible) iff all the intersections $E \cap A_i$ are $A$-measurable (resp. $\mu$-negligible). In this case $L_0(\Omega, A, \mu)$ is Dedekind complete ([Fry]).

If $X$ is an abstract Banach lattice, its Nakano dual $X'$ is the subspace of $X''$ whose elements are order continuous, i.e. $x^* \in X'$ for all decreasing nets $(x_i)_{i \in I}$ with $\bigwedge x_i = 0$, has lim sup $(x_i, x) = 0$. The space $X'$ is a band in $X''$. When $X$ is a generalized Köthe space over $(\Omega, A, \mu)$, then $X'$ can be realized as a generalized Köthe space over $(\Omega, A, \mu)$ with the same support, the Köthe dual of $X$, consisting of the elements $f \in L_0(\Omega, A, \mu)$ such that $f x \in L_1$ for every $x \in X$, and living on the support of $X$; then $(X, \varphi) = \int f(x) \, d\mu$. The natural embedding $i: X \rightarrow X''$ takes values in $X''$ and is an isometric lattice isomorphism onto a sublattice of $X''$. Let $r$ be the restriction projection from $X''$ onto $X'$. Then $j = r \circ i$ is a lattice
homomorphism from $X$ into $X''$, which is injective in the case of Köthe function spaces. The equality $X = X''$, with equality of norms, is equivalent to the Fatou property of $X$, i.e. that every norm bounded increasing net of nonnegative elements has a supremum whose norm is the supremum of the norms of the elements. In particular, duals have the Fatou property, hence $X^{**} = X^*$.

Let $Y$ be an order complete Banach lattice. We can find in $Y$ a complete system of local units, i.e. a maximal system $(y_\alpha)_\alpha$ of disjoint non-zero, nonnegative elements. Then the order ideal $I_Y$ generated by the $y_\alpha$'s is order dense in $Y$. On the other hand, let $\mathcal{Z}(Y)$ be the center of $Y$, i.e. the closure in $L(Y)$, for the operator norm topology, of the space of operators of the type $\sum_{i=1}^m a_i p_i$, where the $a_i$ are scalars and the $p_i$ are disjoint band projections. Then every $y$ in $\mathcal{Z}(Y)$ can be formally written $y = \sum_\alpha \varphi_\alpha y_\alpha$, with $\varphi_\alpha \in \mathcal{Z}(Y)$; if $y \geq 0$, so are the $\varphi_\alpha$, and $\sum_\alpha \varphi_\alpha$ means simply the supremum.

Let us briefly recall now the realization of $X^*$ as a Köthe function space given in [VL] (for $X$ a Köthe function space over $(\mathcal{O}_1, A, \mu)$). If $x \in X$, we have an order continuous lattice homomorphism $\pi_x : X^* \to L_\infty(\mathcal{O}_1)^*$ defined by $(\pi_x x^*, h) = (x^*, h_x)$. These homomorphisms $\pi_x$ induce a bijection $\pi$ between the bands of $X^*$ and the subbands of a band $R_X$ of $L_\infty(\mathcal{O}_1)^*$ (by $\pi(V) =$ $\text{band}(\pi_x(x^*) : x^* \in V, x \in X)$). By identifying the bands with the associated band projections, $\pi$ is an isomorphism from the complete Boolean algebra $B(X^*)$ of projections of $X^*$ to that of $R_X$. This isomorphism induces naturally an isometric order isomorphism from $Z(X^*)$ onto $Z(R_X)$ (also denoted by $\pi$).

Conversely, we can define a homomorphism $\pi$ from the complete Boolean algebra $B(L_\infty(\mathcal{O}_1)^*)$ onto $B(X^*)$ by setting $(\pi(p)x, z) = (p \pi_x x, z)$ for all $x^* \in X^*$, $x \in X$ and $p \in B(L_\infty(\mathcal{O}_1)^*)$. Then $\pi$ is the natural restriction from $B(L_\infty(\mathcal{O}_1)^*)$ to $B(R_X)$. Note that $\pi$ induces a continuous homomorphism from $Z(L_\infty(\mathcal{O}_1)^*)$ onto $Z(X^*)$.

Note that $L_\infty(\mathcal{O}_1, A, \mu)^*$ is an abstract $L_1$ space, and thus identifies (isometrically and order isomorphically) with a space $L_1(\mathcal{E}, \Sigma, m)$ (see [LT]). Then $R_X$ is the band generated by a set $S_X \subset \Sigma$ in $L_1(S)$, $Z(L_\infty(\mathcal{O}_1)^*)$ identifies with $L_\infty(S, \Sigma, \Sigma)$, and $R_X$ with $L_\infty(S_X)$. Then $X^*$ appears as an $L_\infty(S)$-module for the action defined by $h.x^* = h(x^*) (x^* \in X^*, h \in L_\infty(S))$. If $x^* \in X^*$, and $p_x$ is the projection onto the band generated by $x^*$, we set the set $S_{x^*} \subset \Sigma$ whose indicator function is identified with $\pi(p_x x^*)$ the support of $x^*$.

We can choose then a complete system $(x^*_\alpha)_\alpha$ of local units in $X^*$ and another one $(\nu_\alpha)_\alpha$ in $R_X$, such that $\pi(\text{band } y_\alpha) = \text{band } \nu_\alpha$ for every $\alpha$. We can suppose that $\nu_\alpha$ is an indicator function $1_{S_{\alpha}}$, with $S_{\alpha} \subset \Sigma$, and $m(S_{\alpha}) < \infty$. Then with every element $x^* = \sum_\alpha \varphi_\alpha x^*_\alpha$ in the order ideal generated by the $x^*_\alpha$'s we associate $\pi(x^*) = \sum_\alpha \pi(\varphi_\alpha) 1_{S_{\alpha}}$ and we extend $\pi$ by order density to an order isomorphism from $X^*$ onto an ideal of $L_\infty(S, \Sigma, m)$ (supported by $S_{\alpha})$: then $\pi(X^*)$ is the desired realization of $X^*$ as a Köthe function space over $(\mathcal{O}_1, \Sigma, m)$.

We call such a realization of $X^*$ in $L_\infty(S, \Sigma, m)$ a standard realization (associated with the complete systems $(x^*_\alpha)$ and $(\nu_\alpha)$ of local units). If $w \in L_\infty(S)_+$ with $w > 0$ a.e., then $\pi(w)$ defined by $\pi(w) = \pi(w x^*)$ gives another standard realization of $X^*$: for, we may assume that $w$ is bounded from above and below on the support $S_{\alpha}$ of each $\pi(x^*_\alpha)$, and set $x^*_\alpha = (1_{S_{\alpha}} w)^{-1} x^*_\alpha$, thus obtaining a new complete system of local units of $X^*$ for which $\pi(w x^*_\alpha) = \pi(x^*_\alpha) 1_{S_{\alpha}}$. We say that the new standard realization of $X^*$ is obtained from the old one by a change of density.

A standard realization of $X^*$ induces in turn a realization of $X^*$ in $L_\infty(S, \Sigma, m)$. The embedding $i_X$ of $X$ into $X^*$ is then characterized by the relations $i_X(x) \cdot \pi(x^*) = \pi(x^*)$ for every $x^* \in X^*$ (or equivalently $1_{S_{\alpha}} i_X(x) = \pi(\nu_\alpha) x^*$ for every $\alpha$). The order ideal $I_X$ generated by $X$ in $X^*$ consists of the elements $h.i_X(x)$, $x \in X$, $h \in L_\infty(S)$, and one has $X^* = I_X$ (since clearly $X^* = I_X$). Hence nonnegative elements of $X^*$ are supreme of norm-bounded directed families of nonnegative elements of $I_X$ (see [Z]).

We can find a maximal system $(S_{\alpha})_\alpha$ of disjoint subsets of $S_X$ whose indicator functions $1_{S_{\alpha}}$ are simultaneously in $[1]$ (the realization of $X^*$ and that of $X^*$). By a change of density, we can obtain a standard realization of $X^*$ for which $1_{S_{\alpha}} = 1_{S_{\alpha}} i_X(x^*_\alpha)$ for a certain complete system $(x^*_\alpha)_\alpha$ of local units of $X$.

(b) The set of Calderón–Lozanovskii functions. Now let us say a few words about the set $C_\varphi$ of normalized Calderón–Lozanovskii functions. We equip $C_\varphi$ with the topology of simple convergence on the open quadrant $P = \{x, y : u > 0, v > 0\}$. Using Ascoli’s theorem, it is easy to see that this topology coincides with the topology of uniform convergence on compact subsets of $P$ (or of the open segment $A = \text{int}(P \cap \{(u, v) : u + v = 1\})$). This topology is metrizable; in fact, one obtains a compatible metric setting $d(\varphi, \psi) = \sum_{n=1}^\infty 2^{-n} |\varphi(u_n, v_n) - \psi(u_n, v_n)|$, where $(u_n, v_n)_{n=\infty}^\infty$ is, say, the set of rational couples in $A$. Note that the balls relative to this metric are convex. Moreover, $C_\varphi$ is compact for this topology. The same is true of course for the set $C_{\varphi, b} = \{\varphi \in C : 0 < \varphi(1, b) \leq \varphi(1, 1) \leq b\}$ for all positive numbers $a, b$. Given a $\varphi \in C_\varphi$, we shall denote by $I_{\varphi}^y$ the subset of $C_\varphi$ consisting of all $\varphi$-dilations $\varphi_{a, b}$ (defined by $\varphi_{a, b}(u, v) = \varphi(a u, b v) / \varphi(a, b)$) where $a, b > 0$; and by $\Gamma_{\varphi}$ the closure of $I_{\varphi}^y$ in $C_\varphi$. Denote also by $\Gamma_{\varphi}^M$ the closure of $\{\varphi_{a, b} : a > M > 0\}$ and by $\Gamma_{\varphi}^{\infty, M}$ that of $\{\varphi_{a, b} : b > M a > 0\}$, and finally let $\Gamma_{\varphi}^{\infty} = \bigcap_M \Gamma_{\varphi}^{M}$, resp. $\Gamma_{\varphi}^{\infty, M} = \bigcap_M \Gamma_{\varphi}^{\infty, M}$.

Let us show that the conjugates of the elements of $\Gamma_{\varphi}$ appear after normalization as elements of the set $C_{\varphi}^*$ associated with the conjugate of $\varphi$. 


LEMMA 1.1. The conjugation map \( \varphi \mapsto \varphi_* \) is continuous from \( C_1 \) into \( C \).

Proof. We have to prove that if \( \varphi_n \to \varphi \) uniformly on compact sets then \( \varphi_n \to \varphi \) pointwise.

From the inequalities
\[
\forall s, t > 0, \quad \varphi_n(s, t) \leq us + vt
\]
we deduce
\[
\forall s, t > 0, \quad \varphi(s, t) \leq us + vt
\]
where \( \psi(u, v) = \lim sup_{n \to \infty} \varphi_n(u, v) \), which means that \( \psi \leq \varphi \).

Conversely, fix \( u, v > 0 \), and let \( 0 < a < 1 \). Set
\[
\varphi_n^a(u, v) = \psi_n^a(u, v) = \varphi_n(u, v) \overline{\varphi_n(1, a)}
\]
and define \( \varphi_n^{(a)} \) similarly. From \( \varphi_n^{(a)} \geq \varphi_n \), we deduce \( \varphi_n^{(a)} \leq \varphi_n \). Now compute \( \varphi_n^{(a)}(u, v) \). We have
\[
\varphi_n^{(a)}(u, v) = \inf_{s, t > 0} \frac{us + vt}{\varphi_n^{(a)}(s, t)} = \inf_{s, t > 0} \frac{us + vt}{\varphi_n^{(a)}(s, t)} = \inf_{at \leq s \leq a^{-1}t} \varphi_n(s, t) \leq \frac{us + vt}{\varphi_n(1, a)}
\]
and
\[
\psi(u, v) = \lim sup_{n \to \infty} \varphi_n(u, v) = \psi_n(u, v).
\]

Since
\[
\frac{\varphi_n(s, t)}{s + t} \to \frac{\varphi(s, t)}{s + t}
\]
uniformly on the set \( \{ s, t > 0 : at \leq s \leq a^{-1}t \} \), we have
\[
\inf_{at \leq s \leq a^{-1}t} \varphi_n(s, t) \to \inf_{at \leq s \leq a^{-1}t} \varphi(s, t),
\]
whence \( \varphi_n^{(a)}(u, v) \to \varphi^{(a)}(u, v) \), hence \( \lim \inf_{n \to \infty} \varphi_n^{*} \geq \varphi^{(a)} \), for every \( 0 < a < 1 \). But we have
\[
\frac{u}{\varphi(a, 1)} \geq \frac{u}{\varphi^{(a)}(u, v)} \quad \text{and} \quad \frac{u}{\varphi(1, a)} \geq \frac{u}{\varphi^{(a)}(u, v)},
\]
whence
\[
\varphi^{(a)}(u, v) \geq \left( \frac{u}{\varphi(a, 1)} \right) \varphi^{(a)}(u, v) \to \varphi^{*}(u, v).
\]

COROLLARY 1.2. The conjugates of the elements of \( \Gamma_\varphi \) (resp. \( \Gamma_\varphi^{1, \infty}, \Gamma_\varphi^{1, \infty} \)) are proportional to elements of \( \Gamma_{\varphi^*} \) (resp. \( \Gamma_{\varphi^*}^{1, \infty}, \Gamma_{\varphi^*}^{1, \infty} \)).

Proof. If \( \psi \in \Gamma_\varphi \), i.e. \( \psi(u, v) = \varphi(au, bv)/\varphi(a, b) \) for \( u, v > 0 \), we have
\[
\psi(u, v) = \inf_{s, t > 0} \frac{us + vt}{\varphi(a, b)} \varphi(a, b)
\]
and
\[
\varphi = \inf_{s, t > 0} \frac{us + vt}{\varphi(a, b)} \varphi(a, b).
\]

In particular, \( \psi(1, 1) = \varphi(1/a, 1/b) \varphi(a, b) \), hence
\[
\psi(u, v) = \frac{\varphi(a, b)}{\varphi(1/a, 1/b)} \varphi(a, b).
\]

Now if \( \psi \in \Gamma_\varphi \), \( \psi(u, v) = \lim_{n \to \infty} \varphi(a_n u, b_n v)/\varphi(a_n, b_n) \), then by Lemma 1.1 we have
\[
\psi(u, v) = \lim_{n \to \infty} \varphi(a_n, b_n) \psi(1, 1).
\]

2. THE \( \Psi \)-FUNCTIONAL AND THE NORM ON \( \varphi(X, Y)^* \). Let \( X \) and \( Y \) be two Köthe function spaces over the same measure space \( (\Omega, A, \mu) \), and \( \varphi \) a normalized Calderón-Lozanovskii function. We set once for all \( Z = \varphi(X, Y) \).

We identify \( L_\infty(\Omega)^* \) with \( L_1(S, \Sigma, m) \).

DEFINITION 2.1. For every \( x^* \in X_+^* \), \( y^* \in Y_+^* \) and \( z \in Z_+ \), set
\[
\Psi(x^*, y^*) = \inf \{ (u^*, z) + (y^*, y) : z \in X_+, y \in Y_+, z \leq \varphi(x, y) \}.
\]

PROPOSITION 2.2. The map \( \Psi(x^*, y^*) \) extends to a bounded positive linear form over \( Z \).

Proof. We have to prove that \( \Psi(x^*, y^*) \) is positively linear over \( Z_+ \). Let \( z = x_1 + z_2 \), with \( x_1, y \in Z_+ \).

Let \( \epsilon > 0 \), and let \( x_1 \in X_+ \) and \( y_1 \in Y_+ \) be such that \( z_1 \leq \varphi(x_1, y_1) \) and \( \Psi(x^*, y^*)(z_1) \geq \langle x^*, x_1 \rangle + \langle y^*, y_1 \rangle - \epsilon \). Then (by concavity and homogeneity of \( \varphi \))
\[
z_1 + z_2 \leq \varphi(x_1, y_1) + \varphi(x_2, y_2) \leq \varphi(x_1 + x_2, y_1 + y_2),
\]
whence
\[
\Psi(x^*, y^*)(z_1 + z_2) \leq \langle x^*, x_1 + x_2 \rangle + \langle y^*, y_1 + y_2 \rangle \leq \Psi(x^*, y^*)(z_1) + 2\epsilon
\]

Conversely, let \( x \in X_+ \) and \( y \in Y_+ \) be such that \( z \leq \varphi(x, y) \) and \( \Psi(x^*, y^*) \geq \langle x^*, x \rangle + \langle y^*, y \rangle - \epsilon \). We may write \( x_1 = \mu_1 x + h^1 \in L_\infty(\Omega) \) and \( h^1 + h^2 = 1_{\Omega} \). Set \( x_1 = h^1 x \) and \( y_1 = h^2 y \). We have
\[
z_1 \leq h^1 \varphi(x, y) = \varphi(h^1 x, h^2 y) = \varphi(x_1, y_1),
\]

thus
\[
\Psi(x^*, y^*)(z_1) \leq \langle x^*, x_1 \rangle + \langle y^*, y_1 \rangle
\]
and
\[ \Psi(x^*, y^*)(x_1) + \Psi(x^*, y^*)(x_2) \leq (x^*, x_1 + x_2) + (y^*, y_1 + y_2) = (x^*, x) + (y^*, y) \leq \Psi(x^*, y^*)(x) + \varepsilon. \]

To prove the boundedness of \( \Psi(x^*, y^*) \), choose for any \( z \in Z_+ \) two elements \( x \in X_+ \) and \( y \in Y_+ \) with \( z \leq \varphi(x, y) \) and \( \|x\| \vee \|y\| \leq (1 + \varepsilon) \|z\| \). Then
\[ \Psi(x^*, y^*)(z) \leq (x^*, x) + (y^*, y) \leq (1 + \varepsilon) \|z\| (\|x^*\| + \|y^*\|). \]

Thus \( \Psi(x^*, y^*) \in Z_+^* \) and \( \|\Psi(x^*, y^*)\| \leq \|x^*\| + \|y^*\| \).

**Proposition 2.3.** For every \( x_0 \in X_+ \), \( y_0 \in Y_+ \), and \( z \in Z_+^* \), we have
\[ \inf \{ (x^*, x_0) + (y^*, y_0) : \Psi(x^*, y^*) \geq z \} = \limsup_{x \geq 0, y \geq 0} (x^*, \varphi(x, y)). \]

(Note that we set \( \inf \emptyset = +\infty \).

**Proof.** For all \( x \in X_+ \), \( y \in Y_+ \), and \( z \in Z_+^* \), we have for all \( x^* \in X^* \) and \( y^* \in Y^* \) such that \( \Psi(x^*, y^*) \geq z \),
\[ (z, \varphi(x, y)) \leq (\Psi(x^*, y^*), \varphi(x, y)) \leq (x^*, x) + (y^*, y) \]
by the very definition of \( \Psi(x^*, y^*) \). Hence
\[ \limsup_{x \geq 0, y \geq 0} (z, \varphi(x, y)) = \limsup_{x \geq 0, y \geq 0} (x^*, x) + (y^*, y). \]

For the proof of the converse inequality, set
\[ h(x_0, y_0) := \limsup_{x \geq 0, y \geq 0} (z, \varphi(x, y)). \]

Note that we allow \( z = x_0 \) or \( y = y_0 \) in this limsup, thus \( h(x_0, y_0) \geq (z, \varphi(x_0, y_0)) \). The function \( h \) is clearly upper semicontinuous (it is the u.s.c. envelope of the function \( (x, y) \mapsto (z, \varphi(x, y)) \)). It is also straightforward to verify that \( h \) is positively homogeneous and concave over \( X_+ \times Y_+ \). As a consequence of the Hahn–Banach Theorem, for all \( (x_0, y_0) \in X_+ \times Y_+ \) and \( \varepsilon > 0 \), there exists a \( F \in (X \times Y)_+^* \) such that
\[ \begin{cases} F(x, y) \geq h(x, y), & \forall x \in X_+, \forall y \in Y_+, \\ F(x_0, y_0) \leq h(x_0, y_0) + \varepsilon. \end{cases} \]

We may write \( F(x, y) = (x^*, x) + (y^*, y) \) for certain \( x^* \in X_+^* \) and \( y^* \in Y_+^* \). We then have, for every \( x \in X_+ \) and \( y \in Y_+ \),
\[ (z, \varphi(x, y)) \leq h(x, y) \leq F(x, y), \]

hence for every \( z \in Z_+^* \),
\[ (z, z) \leq \inf \{ F(x, y) : x \leq \varphi(x, y) \} = \Psi(x^*, y^*), \]
which means that \( z \leq \Psi(x^*, y^*) \). On the other hand,
\[ (x^*, x_0) + (y^*, y_0) = F(x_0, y_0) \leq h(x_0, y_0) + \varepsilon. \]

**Remark.** If \( \varphi \) satisfies a two-sided reverse \( \Delta_2 \)-condition (see the introduction), then in fact
\[ \inf \{ (x^*, x_0) + (y^*, y_0) : \Psi(x^*, y^*) \geq z \} = (z, \varphi(x_0, y_0)). \]

For, in this case, the map \( X_+ \times Y_+ \rightarrow Z_+^*, (x, y) \mapsto \varphi(x, y) \), is continuous.

**Corollary 2.4.** For every \( z^* \in Z_+^* \), there exist \( x^* \in X_+^* \) and \( y^* \in Y_+^* \) such that \( z^* \leq \Psi(x^*, y^*) \).

**Proof.** We have
\[ \limsup_{x \geq 0, y \geq 0} (z^*, \varphi(x, y)) \leq \limsup_{x \geq 0, y \geq 0} \|z^*\| (\|x\| \vee \|y\|) \]
\[ = \|z^*\| (\|x_0\| \vee \|y_0\|) < \infty. \]

**Theorem 2.5.** The norm of any element \( z^* \in Z^* \) is given by
\[ \|z^*\| = \inf \{ \|x^*\| + \|y^*\| : z^* \leq \Psi(x^*, y^*) \}. \]

**Proof.** Let \( \|z^*\| \) be the right hand side in this relation. The inequality \( \|z^*\| \leq \|z^*\| \) results from the fact that if \( 0 \leq |z| \leq \varphi(x, y) \) with \( \|x\| \vee \|y\| \leq (1 + \varepsilon) \|z\| \), and if \( |z| \leq \Psi(x^*, y^*) \), then
\[ |(z, x)| \leq (\Psi(x^*, y^*), \varphi(x, y)) \leq (x^*, x) + (y^*, y) \]
\[ \leq (\|x^*\| + \|y^*\|) (\|x\| \vee \|y\|) \leq (1 + \varepsilon) \|z\| (\|x^*\| + \|y^*\|). \]

Conversely, let \( a < \|z^*\| \). Set \( H_a = \{ (x^*, y^*) \in X_+^* \times Y_+^* : \Psi(x^*, y^*) \geq z^* \} \). This set is nonempty (by Corollary 2.4), convex and \( w^* \)-closed: for every \( x^*, y^* \in H_a \), \( \varphi(x, y) \in H_a \), where \( H_{x, y} \) is the \( w^* \)-closed hyperplane \( \{ (x^*, y^*) \in X_+^* \times Y_+^* : (x^*, x) + (y^*, y) \geq (z, \varphi(x, y)) \} \).

Set \( B_a = \{ (x^*, y^*) \in X_+^* \times Y_+^* : \|x^*\| + \|y^*\| \leq a \} \). By the Hahn–Banach Theorem, we can separate \( H_a \) from the nonempty \( w^* \)-compact set \( B_a \) by a \( w^* \)-closed hyperplane, i.e. there exists a nonzero couple \( (x_0, y_0) \in X \times Y \) such that
\[ \inf \{ (x^*, x_0) + (y^*, y_0) : \Psi(x^*, y^*) \geq z^* \} \]
\[ \geq \sup \{ (x^*, x_0) + (y^*, y_0) : \|x^*\| + \|y^*\| \leq a \} = a \|x_0\| \vee \|y_0\|. \]

We may suppose that \( x_0, y_0 \geq 0 \) (replacing these elements by their absolute values). By Proposition 2.3, we deduce that
\[ \limsup_{x \geq 0, y \geq 0} (z^*, \varphi(x, y)) \geq a \|x_0\| \vee \|y_0\|. \]
But the left hand side is less than
\[ \|z\| \lim_{x \to y} \sup_{y \to y_0} |x| \vee |y| = \|z\| (\|z_0\| \vee \|y_0\|). \]

Hence \(\|z\| \geq a. \]

In the following proposition we list some important properties of the \(\Psi\)-functional.

**Proposition 2.6.** (a) The function \(\Psi\) is concave, positively homogeneous on \(X_+ \times Y_+\) and \((w^*, w^*)\)-upper semicontinuous.

(b) The function \(\Psi\) is nondecreasing, i.e., \(x^* \geq x_1^*, \ y^* \geq y_2^*\) implies \(\Psi(x^*, y^*) \geq \Psi(x_1^*, y_2^*).\)

(c) The function \(\Psi\) is \(L^\infty(S)\)-homogeneous, i.e., for every \(h \in L^\infty(S)\), we have \(\Psi(hx_1, hy_1) = h \Psi(x_1, y_1).\)

(d) The function \(\Psi\) is order continuous, in the sense that for all increasing nets \(x^*_a \uparrow x^*\) and \(y^*_a \uparrow y^*\), we have \(\Psi(x^*_a, y^*_a) \uparrow \Psi(x^*, y^*),\) and similarly for decreasing nets.

(By \((w^*, w^*)\)-upper semicontinuity of \(\Psi\) we mean that for every \(z \in Z_+\), the map \((x^*, y^*) \mapsto \langle \Psi(x^*, y^*), z \rangle\) is \(w^*-\)upper semicontinuous).

**Proof.** (a) For every \(z \in Z_+,\) the map \((x^*, y^*) \mapsto \langle \Psi(x^*, y^*), z \rangle\) is defined as a g.l.b. of \(w^*-\)continuous linear forms. The positive homogeneity of \(\Psi\) is straightforward.

(b) is straightforward.

(c) Suppose first the case where \(h\) is an element of \(L^\infty(O, A, \mu).\) Let \(z \in Z_+\) and let \(x \in X_+\) and \(y \in Y_+\) satisfy \(z \leq \varphi(x, y).\) Then clearly \(hz \leq \varphi(hx, hy),\)
\[ \langle h\Psi(x^*, y^*), z \rangle = \langle \Psi(x^*, y^*), hx \rangle \leq \langle x^*, hx \rangle + \langle y^*, hy \rangle = \langle hx^*, x \rangle + \langle hy^*, y \rangle. \]

Passing to the infimum with respect to \(x, y\) in the last expression, we obtain
\[ h\Psi(x^*, y^*) \leq \Psi(hx^*, hy^*). \]

Suppose now that \(h = 1,\) and set \(g = 1 - h.\) We also have
\[ g\Psi(x^*, y^*) \leq \Psi(gx^*, gy^*). \]

Adding these two inequalities, we obtain
\[ \Psi(x^*, y^*) \leq \Psi(hx^*, hy^*) + \Psi(gx^*, gy^*) \leq \Psi(x^*, y^*) \]

where the last inequality is a consequence of the concavity and the positive homogeneity of \(\Psi\) (and the fact that \(g + h = 1).\) Hence in this relation, the inequality in the middle is an equality, and the same is true for the two preceding relations we added.

If now \(h \in L^\infty(S, \Sigma, m)_+,\) then there exists, by the lattice version of Helly’s theorem (see [K]), a net \((h_a)\) in \((L^\infty(O))_+\) which converges to \(h\) for the \(w^*-\)topology of \(L^\infty(O)_+.\) Then \(h_a \to x^*, h_a y^* \to h_y^*\) and \(h_a \varphi(x^*, y^*) \to h \varphi(x^*, y^*)\) for the appropriate \(w^*-\)topology. Using the upper semicontinuity of \(\Psi,\) we obtain the inequality
\[ h \Psi(x^*, y^*) \leq \Psi(hx^*, hy^*) \]

and we derive equality as before.

(d) The case of decreasing nets is a consequence of the \(w^*-\)upper semicontinuity of \(\Psi\) (see (a)). Consider increasing nets \(x^*_a \uparrow x^*\) and \(y^*_a \uparrow y^*.\) We have \(x^*_a = h_a x^*\) and \(y^*_a = k_a y^*,\) with \(h_a, k_a \in L^\infty(S)\) and \(0 \leq h_a \downarrow 1, 0 \leq k_a \uparrow 1.\) Then
\[ \Psi(h_a x^*, k_a y^*) \geq \Psi(h_a \wedge h_a x^*, h_a \wedge k_a y^*) = h_a \wedge k_a \Psi(x^*, y^*) \uparrow \Psi(x^*, y^*). \]

**Remark 2.7.** Suppose that the two Köthe spaces are identical: \(X = \Psi = \Delta.\) Then the \(\Psi\)-functional is simply given by
\[ \forall t^*_1, t^*_2 \in \Delta, \quad \Psi(t^*_1, t^*_2) = \varphi(t^*_1, t^*_2). \]

**Proof.** Set \(z^* = t^*_1 + t^*_2;\) using Proposition 2.6, we can reduce to the case where \(t^*_1 = a^*_1 x^*\) for some nonnegative reals \(a_1, a_2.\) Then \(\varphi(x^*, y^*) = \varphi(a_1, a_2) z^*\).

For every \((t_1, t_2) \in \Delta^2\) such that \(t = \varphi(t_1, t_2),\) we have
\[ \langle t^*_1, t_1 \rangle + \langle t^*_2, t_2 \rangle = \langle x^*, a_1 t_1 + a_2 t_2 \rangle \geq \langle z^*, \varphi(a_1, a_2) \varphi(t_1, t_2) \rangle = \langle \varphi(a_1, a_2) x^*, t \rangle, \]

and conversely: if \(\varepsilon \geq 0\) choose positive reals \(u_1, u_2\) such that \(\varphi(u_1, u_2) = 1\) and \(u_1 a_1 + a_2 u_2 \leq \varphi(a_1, a_2) + \varepsilon.\) Set \(t_1 = u_1 t\) and \(t_2 = u_2 t.\) Then \(\varphi(t_1, t_2) = t,\) and
\[ \langle z^*, a_1 t_1 + a_2 t_2 \rangle \leq \langle z^*, (\varphi(a_1, a_2) + \varepsilon) t \rangle = \langle \varphi(a_1, a_2) x^*, t \rangle + \varepsilon \langle z^*, t \rangle. \]

Then let \(\varepsilon \to 0.\]

**Remark 2.8.** The \(\Psi\)-functional is also characterized by the following formula:
\[ \langle \Psi(x^*, y^*), z \rangle = \lim_{x, y \to z} \langle \varphi(x', y'), z \rangle \]

for every \(x^* \in X^*_+\) and \(y^* \in Y^*_+\) and \(z \in Z.\)

This formula is analogous to the formula given in Proposition 2.3 (whose left hand side is related to the dual functional \(\Psi : X^*_+ \times Y^*_+ \to Z^*_+\), see §3), except that here the \(w^*-\)convergence on \(X^*\) and \(Y^*\) is involved, not the norm convergence. The better properties of the norm convergence justify that we prefer to consider \(\Psi,\) rather than \(\Psi\) in the subsequent sections.
Proof. Consider the map \( h : X_+ \times Y_+ \to \mathbb{R}_+ \) defined by
\[
  h(x^*, y^*) = \limsup_{z \to (x^*, y^*)} \langle \varphi_*(x, y), z \rangle.
\]
The map \( h \) is \( \omega^* \)-u.s.c. and concave. The inequality \( h(x^*, y^*) \leq \langle \Psi(x^*, y^*), z \rangle \) is easy. Conversely, using the Hahn–Banach theorem, for every couple \((x_0^*, y_0^*)\) in \( X_+ \times Y_+ \) and \( \varepsilon > 0 \), we can find a couple \((x, y)\) in \( X_+ \times Y_+ \) with
\[
  \begin{cases}
  (x, y) + (x_0^*, y_0^*) \geq h(x^*, y^*) & \text{for every } (x^*, y^*) \in X_+ \times Y_+; \\
  (x_0^*, y_0^*) \geq h(x^*, y^*) + \varepsilon.
  \end{cases}
\]

We apply the first inequality to a couple \((x', y')\) in \( X_+ \times Y_+ \); we obtain
\[
  (x', z) + (y', y) \geq (x, z) + (y_0^*, y_0^*) \geq h(x_0^*, y_0^*) + \varepsilon.
\]

Let \( x' \in Z_+ \). We see that
\[
  \inf \{ (x', z) + (y', y) : x' \in X_+, \ y' \in Y_+, \ \varphi_*(x, y) \geq z \} \geq (x', z).
\]

From Lozanovskiǐ’s paper [L3] (Lemma 17), or from Proposition 3.5 below, we know that the left hand side in this last inequality is nothing but \( (x', \varphi(x, y)) \). Hence \( (x', \varphi(x, y)) - z \geq 0 \) for every \( x' \in Z_+ \), which suffices to show that \( \varphi(x, y) \geq z \). Thus, by the definition of \( \Psi \),
\[
  \langle \Psi(x_0^*, y_0^*), z \rangle \leq (x_0^*, y_0^*) \leq (x_0^*, y_0^*) + \varepsilon.
\]

Then let \( \varepsilon \to 0 \).


Let \( E, F \) and \( G \) be three (generalized) Köthe function spaces over the measure space \( (S, \Sigma, m) \), and let \( \Psi : E_+ \times F_+ \to G_+ \) be a map which is concave, nondecreasing, \( L_\infty(S)_+ \)-homogeneous, order continuous and onto, and suppose that the norm of \( G \) is given by the relation
\[
  \| g \| = \inf \{ \| e \| + \| f \| : e \in E_+, \ f \in F_+, \ \| g \| \leq \Psi(e, f) \}.
\]

Let us call \( G \) an abstract Calderón–Lozanovskiǐ space. Our main candidates for \( G \) and \( \Psi \) will be of course \( \varphi(X, Y) \)* and the \( \Psi \)-functional of Section 2.

**Lemma 3.1**. For every \((e^*, f^*) \in E^*_+ \times F^*_+ \) and \( g \in G \), set
\[
  \Psi_*(e^*, f^*) := \inf \{ (e^*, e) + (f^*, f) : \Psi(e, f) \geq g \}.
\]

Then \( \Psi_* \) extends to an element of \( G^*_+ \). The functional \( \Psi_* : E^*_+ \times F^*_+ \to G^*_+ \) is concave, \( (\omega^*, \omega^*) \)-upper semicontinuous, nondecreasing, \( L_\infty(S)_+ \)-homogeneous, and order continuous. Moreover, the restriction of \( \Psi_* \) to \( E^*_+ \times F^*_+ \) takes values in \( G^*_+ \).

**Proof.** It is clear that \( \Psi_*(e^*, f^*) \) extends to an element of \( G^*_+ \) (same proof as for Proposition 2.2). The properties of \( \Psi_* \) are proved like those of \( \Psi \) in Proposition 2.6. Let now \( e^* \in E^*_+ \) and \( f^* \in F^*_+ \). If \( (g_0) \) is a decreasing net in \( G, g_0 \downarrow 0 \), then we can write \( g_0 = g_0 \downarrow g_0 \) with \( g_0 \in L_\infty(S)_+ \), \( g_0 \downarrow 0 \). Let \( (e_0, f_0) \) be such that \( \Psi(e_0, f_0) \geq g_0 \) and set \( e = e_0, f = f_0 \). Then \( e_0 \downarrow 0, f_0 \downarrow 0 \) and \( \Psi(e_0, f_0) \geq g_0 \) (hence \( \Psi_*(e^*, f^*) \geq g^* \)) order continuous.

Let us denote by \( \pi \) the natural band projection \( E^* \to E^* \) (resp. \( F^* \to F^* \), \( G^* \to G^* \)) associating with a linear form its absolutely continuous part.

**Lemma 3.2.** For every \((e^*, f^*) \in E^*_+ \times F^*_+ \), we have \( \pi \Psi_*(e^*, f^*) = \Psi_*(\pi e^*, \pi f^*) \).

**Proof.** Represent \( L_\infty(S)_+ \) as a space \( L_1(T, T, \tau) \). Since \( L_1(T) \) is a band in \( L_1(T) \), \( S \) can be considered as a subset of \( T \). It is easy to see that the natural band projection \( \pi \) coincides with the action of the indicator function \( 1_S \) on all duals of Köthe function spaces over \( (S, \Sigma, m) \). The assertion is then a consequence of the \( L_\infty(T)_+ \)-homogeneity of \( \Psi_* \), whose proof is analogous to that of Proposition 2.6(c).

**Proposition 3.3.** The map \( \Psi_* : E^*_+ \times F^*_+ \to G^*_+ \) is onto, and the norm of every \( g^* \in G^*_+ \) is given by
\[
  \| g^* \| = \inf \{ \| e^* \| + \| f^* \| : e^* \in E^*_+, \ f^* \in F^*_+, \ g^* \leq \Psi_*(e^*, f^*) \}.
\]

**Proof.** By the reasoning of Corollary 2.4, the map \( \Psi_* : E^*_+ \times F^*_+ \to G^*_+ \) is onto; so is the map \( \pi \Psi_* : E^*_+ \times F^*_+ \to G^*_+ \). Upon using Lemma 3.2, it becomes clear that the map \( \Psi_* : E^*_+ \times F^*_+ \to G^*_+ \) is onto. Similarly the reasoning of Theorem 2.5 gives the formula
\[
  \| g^* \| = \inf \{ \| e^* \| + \| f^* \| : e^* \in E^*_+, \ f^* \in F^*_+, \ g^* \leq \Psi_*(e^*, f^*) \},
\]
and an appeal to Lemma 3.2 allows us to replace \( e^*, f^* \) by \( e' = \pi e^*, f' = \pi f^* \) in this formula.

**Lemma 3.4** (Reciprocity formula). For every \((e_0, f_0) \in E_+ \times F_+ \) and every \( g' \in G_+ \), the following relation holds:
\[
  \langle g', \Psi(e_0, f_0) \rangle = \inf \{ \langle e^*, e_0 \rangle + \langle f^*, f_0 \rangle : e^* \in E^*_+ \}, \ f^* \in F^*_+, \ \Psi_*(e^*, f^*) \geq g' \}.
\]

**Proof.** By the proof of Proposition 2.3, we have
\[
  \inf \{ \langle e^*, e_0 \rangle + \langle f^*, f_0 \rangle : e^* \in E^*_+ \}, \ f^* \in F^*_+, \ \Psi_*(e^*, f^*) \geq g' \} \leq \limsup_{\substack{e \to e_0 \\ f \to f_0}} \langle g', \Psi(e, f) \rangle,
\]
but by Lemma 3.2, \( \Psi_*(e^*, f^*) \geq g' \) implies \( \Psi_*(\pi e^*, \pi f^*) \geq g' \), hence in (**) the left hand side equals \( \inf \{ \langle e^*, e_0 \rangle + \langle f^*, f_0 \rangle : e^* \in E^*_+, \ f^* \in F^*_+, \ \Psi_*(e^*, f^*) \geq g' \} \). Let \( (e_n) \) and \( (f_n) \) be such that the right hand side in (**) equals
Find a family \((h_\alpha)_{\alpha \in \mathcal{J}}\) of normalized g.C.L. functions by the preceding construction applied to the sets \((A_\alpha)_{\alpha \in \mathcal{J}}\); set
\[
\psi = \sum_{\alpha \in \mathcal{J}} \Psi(1_A,1_A)h_\alpha + \sum_{\alpha \in \mathcal{J}} \Psi(1_{A_0},0)p_1 + \sum_{\alpha \in \mathcal{J}} \Psi(0,1_A)p_2
\]
where \(p_1, p_2\) are the (constant) C.L. functions \(p_1(u,v) = u\) and \(p_2(u,v) = v\).

The equality \(\Psi(e,f)(s) = \psi(s,e(s),f(s))\) is then verified first for step functions; then for arbitrary \(e, f\), by using the order continuity of \(\Psi\).

Define now \(\varphi\) by \(\varphi(s,a,b) = (\varphi_s)_*(a,b)\), for every \(s \in S_G\). It is in fact measurable, because in the definition of the conjugate functions \((\varphi_s)_*\) one can restrict the infimum to the positive rationals (or by Lemma 1.1). For \(e' \in E_+\) and \(f' \in F_+\), define \(\Phi(e',f')(s) = \varphi(s,e'(s),f'(s))\) if \(s \in S_G\), and 0 if \(s \notin S_G\). Since for all \(e \in E_+\) and \(f \in F_+\), we have \(\varphi(s,e'(s),f'(s))\psi(s,e(s),f(s)) \leq \varphi(s,e'(s),f(s))\psi(s,e(s),f'(s))\) for a.e. \(s\), it is clear that \(\Phi(e',f') \leq \Phi_s(e',f')\). Conversely, using a suitable version of the von Neumann measurable selection theorem (as in [Au]), we can find, for every \(e' > 0\), two measurable maps \(h, k : S_G \to \mathbb{R}_+\) such that
\[
h(s)e'(s) + k(s)f'(s) \leq (1 + e)\varphi(s,e'(s),f'(s))\]
for a.e. \(s \in S_G\). For all \(A \in \Sigma, \ A \subset S_G\), with \(m(A) > 0\), there exists \(B \in \Sigma, B \subset A\), with \(m(B) > 0\), such that \(1_Bh \in E \) and \(1_Bk \in F\). Then \(\Psi(1_Bh,1_Bk) = 1_B\), and
\[
\int_B \Psi_s(e',f') \, dm \leq \int_B he' \, dm + \int_B kf' \, dm \leq (1 + e) \int_B \Phi(e',f') \, dm.
\]
This shows \(\Psi_s(e',f') \leq \Phi(e',f')\) (since \(\Psi_s(e',f')\) is supported by \(S_G\)).

Note that by the reciprocity formula of Lemma 3.4, we would obtain the same result by constructing first \(\varphi\) from \(\psi_s\), and then setting \(\psi_s = (\varphi_s)_*\).

4. The representation theorem. In this section, we prove the following representation theorem:

\[\text{Theorem 4.1. Let } X \text{ and } Y \text{ be two Köthe function spaces over the same measure space } (\Omega, \mathcal{A}, \mu), \varphi \text{ a normalized Calderón–Lozanovskii function and } \varphi(X,Y) \text{ the corresponding Calderón–Lozanovskii space. Given two standard realizations of the duals } X^*, Y^* \text{ as (generalized) Köthe function spaces over the measure space } (S, \Sigma, m), \text{ there is a standard realization of } \varphi(X,Y)^* \text{ and a generalized Calderón–Lozanovskii function } \varphi \text{ over } (S, \Sigma, m) \text{ such that } m\text{-almost all nonzero partial functions } \psi_s \text{ have their conjugate functions in the set } I_{\varphi} \text{ and } \varphi(X,Y)^* = \psi(X^*, Y^*).\]

Note. By Corollary 1.2, the normalized functions \(\psi_s/\psi_s(1,1)\) belong in fact to \(I_{\varphi}\).
We postpone the proof of Theorem 4.1 after that of the following Proposition 4.2. Let $\Phi = \varphi_s$ be the conjugate functional $X^*_s \times Y^*_s \rightarrow Z^*_s$ (as defined in Section 3) of the $\varphi$-functional $X_s \times Y_s \rightarrow Z_s$ defined in Section 2.

**Proposition 4.2.** Let $\xi \in X^*_s$ and $\eta \in Y^*_s$ be such that $\varphi(\xi, \eta) \neq 0$. There exists a normalized $g.c.l.$-function $\tilde{\varphi} = \varphi_{\xi, \eta}$ defined over the support $S_{\xi, \eta}$ of $\varphi(\xi, \eta)$, with partial functions $\tilde{\varphi}_s$ belonging to $\Gamma_{\varphi}$ for a.e. $s \in S_{\xi, \eta}$, such that for every $a, b \in \mathbb{R}$, we have $\tilde{\varphi}(a \xi, b \eta) = \varphi(a, b)\varphi(\xi, \eta)$. For two such functions $\tilde{\varphi}, \tilde{\varphi}'$, one has $\tilde{\varphi}_s = \tilde{\varphi}'_s$ for a.e. $s \in S_{\xi, \eta}$. Proof. The existence and unicity of $\tilde{\varphi}_{\xi, \eta}$ are clear (see the proof of Proposition 3.5), the point is to prove $\tilde{\varphi}_{\xi, \eta} \in \Gamma_{\varphi}$.

We first reduce to the case where $\xi \in \mathcal{I}_{X}(X_s)$ and $\eta \in \mathcal{I}_{Y}(Y_s)$ (where $i_X, i_Y$ are the natural injections $X \rightarrow X^s, Y \rightarrow Y^s$). For, if the lemma is true in this case, it is then trivially true when $\xi = \sum_i a_i X_i$ and $\eta = \sum_i i_i Y_i$, where $(A_i) \subset \Sigma$ is a system of disjoint sets. In the general case, we can find directed nets $\eta \uparrow \xi$ and $\eta \uparrow \eta$, where $\xi \in X^s$ and $\eta \in Y^s$ have the preceding form. Then $\Phi(a \xi, a \eta) \Rightarrow \Phi(a \xi, a \eta)$ for every $a, b \geq 0$, whence $\varphi_{\xi, \eta}(a, b) \Rightarrow \varphi_{\xi, \eta}(a, b)$ for every $a, b \geq 0$. Hence for a.e. $s \in S_{\xi, \eta}$ we have $\varphi_{s, \eta}(a, b) \Rightarrow \varphi_{\xi, \eta}(a, b)$, a priori for all rationals, but in fact for all nonnegative reals, a by a continuity argument. Hence $\tilde{\varphi}_{s, \eta} \in \Gamma_{\varphi}$ for a.e. $s$.

Fix $x_0 \in X_s$ and $y_0 \in Y_s$ (we shall identify $x_0, y_0$ with their images $i_X(x_0), i_Y(y_0)$). We shall prove the following claim:

**Claim.** For every $x^* \in Z^*_s$ such that $\langle \Phi(x_0, y_0), z^* \rangle > 0$, and every $\varepsilon > 0$, there exists $A \in A$ such that $\langle \Phi(x_0, y_0), z^* \rangle > 0$ and that the map

$$H^*_A : (u, v) \mapsto \frac{\langle \Phi(u, x_0, y_0), z^* \rangle}{\langle \Phi(x_0, y_0), z^* \rangle}$$

lies in $C_1$ at a distance from $\Gamma_{\varphi}$ less than $\varepsilon$.

**Suppose** that the claim is proved. Assume that the function $\varphi := \varphi_{x_0, y_0}$ has partial functions $\varphi_s$ not belonging to $\Gamma_{\varphi}$ for $s$ in a nonmeasurable set. Since the function $S_{x_0, y_0} : R_+ \rightarrow \mathbb{R}_+$, $s \mapsto d(\varphi_s, \Gamma_{\varphi})$, is measurable (see (31) for the definition of the distance $d$), there exist $\varepsilon > 0$ and a subset $S_1 \subset S$ such that $d(\varphi_s, \Gamma_{\varphi}) > \varepsilon$ for every $s \in S_1$. Since $C_1$ can be covered by a finite number of $d$-balls of diameter less than $\varepsilon/3$, we can find $s \in S_1$ with $d(\varphi_s, \Gamma_{\varphi}) > 2\varepsilon/3$ and a subset $S_2 \subset S_1$ such that $d(\varphi_s, \theta) < \varepsilon/3$ for every $s \in S_2$. For every $z^* \in Z^*_s$ with support in $S_2$, the map $H^*_A : (u, v) \mapsto \langle \Phi(u, x_0, y_0), z^* \rangle/\langle \Phi(x_0, y_0), z^* \rangle$ satisfies $d(H^*_A, \theta) \leq \varepsilon/3$, since

$$H^*_A : (u, v) = \frac{\langle \varphi(s, u, v), z^*(s) \rangle}{z^*(s)} \leq d(s)$$

and the $d$-balls are convex. Fixing such a $z^*$, and considering a set $A \in A$ given by the Claim (with $\varepsilon/3$ in place of $\varepsilon$), we obtain a contradiction for $z^*_s$.

Now we prove the claim. It suffices to prove that for every $s > 0$, $u_1, \ldots, u_m > 0$, and $v_1, \ldots, v_m > 0$ there exist $\theta \in \Gamma_{\varphi}$ such that for all $i, j = 1, \ldots, m$,

$$\langle H(u_i, v_j, \theta) \rangle - \langle H(u_i, v_j, \theta) \rangle < \varepsilon.$$  

By Proposition 2.3, for every $u, v > 0$, we can find sequences $(x_n) \subset X_s$ and $(y_n) \subset Y_s$ such that $x_n \rightarrow x_0, y_n \rightarrow y_0$, and

$$\langle z^*, \varphi(u x_n, v y_n) \rangle \rightarrow \langle \Phi(u x_0, v y_0), z^* \rangle.$$  

In fact, we may assume that $x_n \geq x_0$ and $y_n \geq y_0$ for all $n$ (since this limit is a lim sup). We can find sequences $(x_n) $ and $(y_n) $ which give rise to this limit ($\ast$) simultaneously for the $(m + 1)^2$ couples $(u_i, v_j)$, $i = 0, \ldots, m$, $(\text{where we set } u_0 = 1, v_0 = 1)$, in place of $(u, v)$: for, we choose for each $(u, v)$

sequence $(x(n(u,v)))_n, (y(n(u,v)))_n$ greater than $x_0, y_0$, converging to $x_0, y_0$, respectively, satisfying ($\ast$), and then set

$$x_n = \bigvee_{i,j=0}^m x_i(y_i), y_n = \bigvee_{i,j=0}^m y_i(y_i).$$

The point now is that in fact we have

$$\langle t^*, \varphi(u x_i, v y_i) \rangle \rightarrow \langle \Phi(u x_0, v y_0), t^* \rangle$$

uniformly for all $t^* \in Z^*_s$ with $t^* \leq z^*$. For, we have

$$\langle \Phi(u x_i, v y_0), t^* \rangle - \langle \Phi(u x_i, v y_i), t^* \rangle = \limsup_{n \rightarrow \infty} \langle t^*, \varphi(u x_i, v y_i) - \varphi(u x_i, v y_i) \rangle$$

$$\leq \limsup_{n \rightarrow \infty} \langle t^*, \varphi(u x_i, v y_i) - \varphi(u x_i, v y_i) \rangle$$

$$\leq \limsup_{n \rightarrow \infty} \langle z^*, \varphi(u x_i, v y_i) - \varphi(u x_i, v y_i) \rangle$$

$$\rightarrow 0$$

since

$$\limsup_{n \rightarrow \infty} \langle z^*, \varphi(u x_i, v y_i) - \varphi(u x_i, v y_i) \rangle = \langle \Phi(u x_0, v y_0), z^* \rangle$$

$$= \lim_{n \rightarrow \infty} \langle \Phi(u x_0, v y_0), z^* \rangle.$$

Hence

$$\langle \Phi(u x_0, v y_0), t^* \rangle - \langle \Phi(u x_i, v y_i), t^* \rangle \rightarrow 0$$

where the sequence $(\varepsilon_n)_n$, converging to zero, can be chosen independent of $t^* \leq z^*$. By applying this also to $z^* - t^*$ in place of $z^*$, we obtain the desired uniform convergence.
Define a pseudometric on $C_1$ by
\[ \forall \theta_1, \theta_2 \in C_1, \quad \delta(\theta_1, \theta_2) = \sup_{i,j=0, \ldots, m} |\theta_1(u_i, v_j) - \theta_2(u_i, v_j)|. \]

Consider now a finite covering $\{\Gamma_1, \ldots, \Gamma_N\}$ of $\Gamma_\rho$ by Borel subsets of $\delta$-diameter less than $\varepsilon/2$. For every $(x, y) \in X_+ \times Y_+$, define $\varphi_{x,y} : \Omega \to \Gamma_\rho$ by
\[ \varphi_{x,y}(\omega)(u, v) = \frac{\varphi(u x(w), v y(w))}{\varphi(u x, v y)} \]
when $\varphi(x(u), y(v)) \neq 0$ and $= \varphi(u, v)$ if not. Set $A_\rho^N = \{ \omega \in \Omega : \varphi_{x_n, y_n}(\omega) \in \Gamma_p \}$ for $p = 1, \ldots, N$. Note that $\bigcup_{p=1}^N A_\rho^N = \Omega$. Then we have
\[ \forall p = 1, \ldots, N, \forall \varepsilon, j = 0, \ldots, m, \]
\[ \lim_{n \to \infty} \langle z^*, 1_{A_\rho^N} \varphi_{u_j x_n, v_j y_n} \rangle = \lim_{n \to \infty} \langle \Phi(u_j x_0, v_j y_0), 1_{A_\rho^N} z^* \rangle \]
by the uniform convergence result proved above (up to taking a subsequence we may suppose that all these limits do exist). There is a $p_0$ such that this limit $\lim_{n \to \infty} \langle \Phi(u_0 x_0, v_0 y_0), 1_{A_\rho^N} z^* \rangle > 0$. We have
\[ \forall j, p = 1, \ldots, m, \]
\[ \lim_{n \to \infty} \langle z^*, 1_{A_{\rho_0}^N} \varphi_{u_j x_n, v_j y_n} \rangle = \lim_{n \to \infty} \langle \Phi(u_j x_0, v_j y_0), 1_{A_{\rho_0}^N} z^* \rangle. \]

In other words, for sufficiently large $n$, the map $H = H_{\rho_0}^*$ has the property that $\delta(H - H) \leq \varepsilon/2$ for some $\theta$ in the closed convex hull (in $C_1$) of $\Gamma_\rho$ (set $\theta(u, v) = (z^*, 1_{A_\rho^N} \varphi_{u x_n, v y_n})/\langle z^*, 1_{A_\rho^N} \varphi_{x_n, y_n} \rangle$). By convexity of $\delta$, the function $\delta(\theta - H)$ lies at a $\delta$-distance of $\Gamma_\rho$ less than or equal to $\varepsilon/2$. Hence there is a $\theta_0 \in \Gamma_\rho$ with $\delta(H, \theta_0) \leq \varepsilon$, which finishes the proof of the Claim, and of Proposition 4.2.

Remark. When $\varphi$ satisfies a two-sided reverse $A_\Delta$-condition, we have $\Phi(u x_0, y_0) = \varphi(u x_0, y_0)$ (see the Remark following Proposition 2.3). In this case the ratio $\Phi(u x_0, y_0) / \Phi(u x, y)$ defines an element $h_{u x, y} \in L_0(\Omega) \subset L_0(\Omega)$ (the embedding here comes from the embedding $L_0(\Omega) \subset L_0(\Omega)$, which is not the conjugate of the dual projection $L_0(\Omega)^* \to L_1(\Omega)$). Viewed in $L_0(\Omega)$, these $h_{u x, y}$ define an element of $L_0(\Omega)$; $\Gamma_{\rho_0}$); but viewed in $L_0(\Omega)$, they define an element of $L_0(\Omega; \Gamma_{\rho_0})$ (not $L_0(\Omega; \Gamma_\rho)$ in general).

Proof of Theorem 4.1. An appeal to Proposition 3.5 shows that $Z^*$ and $Z^\ast$ are identified with generalized Calderón–Lozanovskiǐ spaces $\psi(X^*, Y^*)$ and $\psi_*(X^*, Y^*)$ for some conjugate g.C.L. functions $\psi$ and $\psi_*$; the functionals $\Phi$ and $\Phi_*$ are then related to $\psi$ and $\psi_*$ by the formulas $\Phi(x(s), y(s)) = \psi(s, x(s), y(s))$ and $\Phi_*(s, \eta(s)) = \psi_*(s, \xi(s), \eta(s))$ (for a.e. $s \in S$). It remains to show that $\psi_*(s) \in \Gamma_\rho$ (when nonzero).

Given a standard realization of $X^*$ and $Y^*$ as (generalized) Köthe function spaces over $(\Sigma, \Sigma, m)$, we can realize $Z^*$ in such a way that for all indicator functions $1_A \in X^*$ and $1_B \in Y^*$, the element $\Phi(1_A, 1_B)$ of $Z^*$ is an indicator function. For, let $S_{X^*}$ and $S_{Y^*}$ be the supports of $X^*$ and $Y^*$. It suffices to show that $\Phi(1_A, 1_B)$, $\Phi(1_A, 0)$ and $\Phi(0, 1_B)$ are realized as indicator functions, for every $A \subset S_{X^*} \cap S_{Y^*}$, $B \subset S_{X^*} \cap S_{Y^*}$, and $C \subset S_{Y^*} \cap S_{X^*}$, and this can be obtained by a simple change of density.

We now perform the construction of the proof of Proposition 3.5, but starting from the $\Phi$-functional (and considering $\Psi$ as the conjugate $\Phi_*$, by Lemma 3.4). Proposition 4.2 shows that the resulting g.C.L. function has partial functions a.e. in $\Gamma_\rho$ (when nonzero).

5. Refinement of the representation theorem. In this section we make more precise the set of partial functions $\psi_\ast$ of the g.C.L. function $\psi$ which describes $\phi(X, Y)^*$, according to the position of $s$ in $S_{X^*}$ (Theorem 5.12 at the end of the section).

We can already treat the case where $s \in S_{X^*} \setminus S_{Y^*}$, resp. $s \in S_{Y^*} \setminus S_{X^*}$: in this case it follows from the proof of Proposition 3.5 that necessarily $\psi_\ast$ is linear, and depends only on the first, resp. second variable; that is, $\psi_\ast(u, v) = u$, resp. $\psi_\ast(u, v) = v$. Hence the band generated in $Z^*$ by $S_{X^*} \cap S_{Y^*}$, resp. $S_{Y^*} \cap S_{X^*}$, coincides with that generated by the same set in $X^*$, resp. $Y^*$.

Denote by $\Delta$ the intersection $X \cap Y$ (equipped with its natural norm). Let $X_{\Delta}$ be the band in $X^*$ whose elements are normal extensions of their restrictions to $\Delta$, in the sense that
\[ \forall s \in X_+, \quad \langle \text{ext}(s), e \rangle = \sup \{ \langle \text{ext}(s), t \rangle : t \in \Delta, 0 \leq t \leq e \}, \]
and let $(X_{\Delta})^\perp$ be the complementary band in $X^*$. The latter band is simply the band of elements $z^* \in X^*$ having zero restriction to $\Delta$. Let $X_0$ be the closure of $\Delta$ in $X$; then there is a canonical isometric order isomorphism $z^* \mapsto \text{ext}(z^*)$ from $X_0^\perp$ onto $X_{\Delta}$, defined for nonnegative $z^* \in X_0^\perp$ by
\[ \forall s \in X_+, \quad \langle \text{ext}(z^*), e \rangle = \sup \{ \langle z^*, t \rangle : t \in \Delta, 0 \leq t \leq e \}. \]
If $r$ is the restriction map $X^* \to X_{\Delta}^\perp$, then $\text{ext}(r)$ is the band projection from $X^*$ onto $X_{\Delta}$ (see [VL]).

The relation between the respective restrictions of $X^*$, $Y^*$ and $Z^*$ to $\Delta$ (which we denote indifferently by $\pi_{\Delta}$) is given by the following proposition.

PROPOSITION 5.1. For every $x^* \in X_{\Delta}^\perp$ and $y* \in Y_{\Delta}^\perp$, we have
\[ \pi_{\Delta} \psi(x^*, y^*) = \psi_\ast(x^*, \pi_{\Delta} y^*) = \psi_\ast(x^*, \pi_{\Delta} y^*). \]

Note that the last member of these equalities is well defined, since $\pi_{\Delta} x^*$ and $\pi_{\Delta} y^*$ belong to the same Köthe space $\Delta^\ast$. 

Proof. Let \( x^* \in X^*_+ \), \( y^* \in Y^*_+ \) and \( t \in \Delta^*_+ \). We have
\[
\langle \pi \Delta \psi(x^*, y^*), t \rangle = \inf \{ \langle x^*, x \rangle + \langle y^*, y \rangle : x \in X_+, y \in Y_+, t \leq \varphi(x, y) \}
\leq \inf \{ \langle x^*, t_1 \rangle + \langle y^*, t_2 \rangle : t_1, t_2 \in \Delta_+, t \leq \varphi(t_1, t_2) \}
= \langle \pi \Delta \psi(x^*, y^*), t \rangle.
\]
So it remains to prove the reverse inequality. Let \( \varepsilon > 0 \), \( x \in X_+ \) and \( y \in Y_+ \) be such that \( t \leq \varphi(x, y) \) and
\[
\langle \pi \Delta \psi(x^*, y^*), t \rangle \geq \langle x^*, x \rangle + \langle y^*, y \rangle - \varepsilon.
\]
Suppose first that \( \lim_{M \to \infty} \varphi(M, 1) = \infty \) and then for every \( \varepsilon > 0 \), we can find \( M \) such that \( \varphi(M, \delta) \geq 1 + \varphi(1, M) \). Then, we have
\[
t \leq \varphi((x \wedge Mt) \vee t, (y \wedge Mt) \vee t) \leq t_1 \leq \varphi(t_1, t_2) \leq \varphi(t_1, t_2) + \varepsilon,
\]
whence
\[
\langle x^*, t_1 \rangle + \langle y^*, t_2 \rangle \leq \langle x^*, x \rangle + \langle y^*, y \rangle + \varepsilon(\|x^*\| \cdot \|x\| + \|y^*\| \cdot \|y\|)
\leq \langle \pi \Delta \psi(x^*, y^*), t \rangle + \varepsilon.
\]
for sufficiently small \( \delta \). Since \( t \leq \varphi(t_1, t_2) \), we obtain the desired inequality.

Suppose now that \( \lim_{M \to \infty} \varphi(M, 1) = \infty > \lim_{M \to \infty} \varphi(1, M) \). By the same trick as before, we find \( t_0 \in \Delta_+ \) and \( y_1 \in Y^*_+ \) with \( t \leq \varphi(t_0, y_1) \) and
\[
\langle x^*, t_0 \rangle + \langle y^*, y_1 \rangle \leq \langle \pi \Delta \psi(x^*, y^*), t \rangle + 2\varepsilon.
\]
For all \( \varepsilon > 0 \), there exist \( M_0 \) such that \( \sup M \varphi(1, M) \leq (1 + \varepsilon) \varphi(1, M_0) \). Then
\[
\varphi(t_0, y_1) \leq (1 + \varepsilon) \varphi(t_0, y_1 \wedge M_0) \leq \varphi(t_0, y_1 \wedge M_0).
\]
Set \( t_1 = (1 + \varepsilon) t_0 \) and \( t_2 = (1 + \varepsilon) y_1 \wedge M_0 t_0 \). We obtain \( t \leq \varphi(t_1, t_2) \) and finally, if \( \lim_{M \to \infty} \varphi(M, 1) < \infty \) and \( \lim_{M \to \infty} \varphi(1, M) < \infty \) we apply the second trick above simultaneously to \( x \) and \( y \).

For the second equality in Proposition 5.1, see Remark 2.7..
Proof. Consider the restriction maps \( \pi_{X,\Delta} : \Delta^* \to \Delta^* \) and \( \pi_{Y,\Delta} : \Delta^* \to \Delta^* \); these order continuous order homomorphisms have injective restrictions to \( \Delta^* \), resp. \( \Delta^* \).

Suppose first that we are in the case (a) of Corollary 5.4. Consider a complete system \( (x^*_k) \) of local units of the band \( V \) of \( \Delta^* \) generated by \( \pi_{\Delta}(x^* \cap \Delta^* \cap \Delta^*) \). We may assume that for all \( \alpha \), \( x^*_\alpha \in \pi_{\Delta}(x^* \cap \Delta^* \cap \Delta^*) \); then \( x^*_\alpha \in \pi_{X,\Delta}(x^*_\alpha) \) for uniquely determined elements \( x^*_\alpha \in \Delta^* \), \( y^*_\alpha \in \Delta^* \), which have the same supports as the \( t_\alpha \) (Lemma 5.3); then \( (x^*_\alpha)_\alpha \) and \( (y^*_\alpha)_\alpha \) are complete systems of local units of the band \( \Delta^* \); \( \pi_{\Delta}(x^* \cap \Delta^* \cap \Delta^*) \) and their supports are both \( \pi_{\Delta}(x^* \cap \Delta^* \cap \Delta^*) \). Finally, set \( x^*_\alpha \in \Delta^* \), \( y^*_\alpha \in \Delta^* \). Then \( x^*_\alpha \in \pi_{\Delta}(x^* \cap \Delta^* \cap \Delta^*) \), \( y^*_\alpha \in \pi_{\Delta}(y^* \cap \Delta^* \cap \Delta^*) \). By Proposition 5.1, we have \( \pi_{\Delta}(x^*_\alpha \cap \Delta^* \cap \Delta^*) \). For the same reason, we obtain \( \pi_{\Delta}(x^*_\alpha \cap \Delta^* \cap \Delta^*) \). But \( (x^*_\alpha, y^*_\alpha) \in \pi_{\Delta}(x^* \cap \Delta^* \cap \Delta^*) \), \( (y^*_\alpha, y^*_\alpha) \in \pi_{\Delta}(y^* \cap \Delta^* \cap \Delta^*) \). Hence \( (x^*_\alpha, y^*_\alpha) \in \pi_{\Delta}(x^* \cap \Delta^* \cap \Delta^*) \), \( (y^*_\alpha, y^*_\alpha) \in \pi_{\Delta}(y^* \cap \Delta^* \cap \Delta^*) \).

Using the continuity of \( \Psi \), we conclude that \( \Psi(x^*, y^*) \) is realized as \( \varphi(x^*, y^*) \) for every \( x^*, y^* \) with support included in \( \pi_{\Delta}(x^* \cap \Delta^* \cap \Delta^*) \).

In the case (b) of Corollary 5.4, we complete the system \( (x^*_\alpha) \) by some system \( (y^*_\alpha) \), to obtain a complete system of local units in \( \pi_{\Delta}(x^* \cap \Delta^* \cap \Delta^*) \); then consider \( x^*_\alpha \in \pi_{\Delta}(x^* \cap \Delta^* \cap \Delta^*) \), \( y^*_\alpha \in \pi_{\Delta}(y^* \cap \Delta^* \cap \Delta^*) \), and set \( x^*_\alpha \in \pi_{\Delta}(x^* \cap \Delta^* \cap \Delta^*) \). We proceed analogously in the case (c) of Corollary 5.4.

Remark 5.6. (a) The preceding realization of \( X^*, Y^*, \Delta^* \) induces a realization of \( X^*, Y^*, Z^* \) for which \( Z^* = \pi_{\Delta}(X^*, Y^*) \) and \( \pi_{\Delta}(X^*, Y^*) \in \Delta^* \).

(b) If we start from arbitrary standard realizations of \( X^*, Y^*, \Delta^* \), and apply the procedure of the proof of Theorem 4.5, we only find that \( \varphi_{\Delta} \in \pi_{\Delta}(X^*, Y^*) \) (see §2 for the definition of this set) for a.e. \( s \in \pi_{\Delta}(x^* \cap \Delta^* \cap \Delta^*) \).

For (b), note that by different changes of density on \( X^*, Y^* \), the space \( \pi_{\Delta}(X^*, Y^*) \) becomes \( \pi_{\Delta}(Y, \Delta^*) \) with \( \pi_{\Delta}(Y, \Delta^*) \in \Delta^* \) for a.e. \( s \).

The following corollary is a slight improvement of Theorem 1 of [L3].

Corollary 5.7. Suppose that \( \Delta^* \) is dense in \( X^* \) and \( Y^* \). Then the dual \( Z^* = \varphi(X, Y) \) can be identified with \( V \otimes \varphi(X^*, Y^*) \otimes W \), where \( V, W \), resp. \( V, W \), is a band in \( X^* \), resp. \( Y^* \).
Conversely, let $x_n \to x_0$ and $y_n \to t_0$ be such that
\[ \langle x^*, \varphi(x_n, y_n) \rangle \to \langle \Phi(x_0, t_0), x^* \rangle. \]
We may assume that $y_n \geq t_0$. Set
\[ y'_n = (y_n - t_0) \wedge \frac{1}{M} x_n + t_0 \quad \text{and} \quad x'_n = x_n + M t_0. \]
We have clearly $x'_n \to x_0 + M t_0$, $y'_n \geq M y_n$, and $t_0 \leq y'_n \leq y_n$, hence $y_n \to t_0$. We now check that replacing $(x_n, y_n)$ by $(x'_n, y'_n)$ can only increase (hence does not change) the preceding limit. We have (by right subadditivity of $\varphi$)
\[ 0 \leq \varphi(x_n, y_n) - \varphi(x_n, y'_n) \leq 1_{\{y_n < y'_n\}} \varphi(x_n, y_n - y'_n) \]
\[ \leq 1_{\{x_n \leq M(y_n - t_0)\}} \varphi(x_n, (y_n - t_0)) \leq \varphi(x_n \wedge M(y_n - t_0), y_n - t_0). \]
Observe that $t_n := x_n \wedge M(y_n - t_0) \in \Delta$; hence writing $z^* \leq T(x^*, y^*)$ with $z^* \in X^*_\Delta$, we obtain
\[ \langle x^*, \varphi(t_n, y_n - t_0) \rangle \leq \langle y^*, y_n - t_0 \rangle \to 0 \quad \text{as} \quad n \to \infty, \]
hence
\[ \langle z^*, \varphi(x_n, y_n) \rangle \to \langle \Phi(x_0, t_0), z^* \rangle. \]
Then we have a fortiori
\[ \liminf_{n \to \infty} \langle z^*, \varphi(x'_n, y'_n) \rangle \geq \langle \Phi(x_0, t_0), z^* \rangle. \]

**Proof of Proposition 5.10.** It is sufficient to prove the assertion when $\xi = 1_{A, x_0}$ and $\eta = 1_{A, y_0}$, where $x_0 \in X_+$, $y_0 \in Y_+$, and $A \subseteq X_\Delta$ is such that $1_{A, y} \in Y_\Delta$. In fact, we may assume that $y \in Y_0$ (since $Y_0 = Y_\Delta$), and even that $y_0 \in \Delta$ by density: more precisely, there exists a nondecreasing sequence $(t_n)$ in $\Delta$ with $t_n \to y$ in $Y$-norm; then $\xi(t_n)$ is nondecreasing, and $1_{A, t_n} \to 1_{A, y}$ in $Y^*$-norm; then $1_{A, y_0} = \sqrt{\lambda} \xi(t_n)$, whence
\[ \langle \Phi(v_1 A, x_0, v_1 A, y_0) \rangle = \langle \Phi(v_1 A, x_0, v_1 A, y_0) \rangle. \]

By Lemma 5.11, from $y_0 \in \Delta$ we deduce that for every $z^* \in Z_+$, and $u, v > 0$, we have
\[ \langle \Phi(v_1 A, x_0, v_1 A, y_0) \rangle = \limsup_{z^* \to M t_0, u \to M y} \langle 1_{A, z^*}, \varphi(u z, v y) \rangle. \]

Now the reasoning of the proof of Proposition 4.2 shows that for $s \in A$, the partial function $\varphi_s$ belongs to $I_{\psi}^M$. Since this is true for all $M \geq 0$, we conclude that $\varphi_s \in I_{\psi}^M$ for a.e. $s \in A$. \[ \Box \]

Applying Corollary 1.2, we can sum up the main results of this section in the following theorem:

**Theorem 5.12.** The partial functions of the normalization $\psi/\psi(1, 1)$ of the g.C.L. function $\psi$ of Theorem 4.1 belong to the set $I_{\psi}^M$, for a.e. $s \in \text{Supp } Z_\Delta$, to $I_{\psi}^\infty$, for a.e. $s \in \text{Supp } Z^* \cap \text{Supp } Y_\Delta \cap \text{Supp } X_\Delta$, and to $I_{\psi}^\infty$, for a.e. $s \in \text{Supp } Z^* \cap \text{Supp } X_\Delta \cap \text{Supp } Y_\Delta$. Finally, $\psi_s(u, v) = u$ for a.e. $s \in \text{Supp } Z^* \cap \text{Supp } Y$ and $\psi_s(u, v) = v$ for a.e. $s \in \text{Supp } Z^* \cap \text{Supp } X^*$. Moreover, on $\text{Supp } Z_\Delta$, we can obtain $\psi = \varphi$, by choosing appropriate realizations of $X^*$ and $Y^*$.

Let us remark that when the linear functions $(u, v) \mapsto u$ and $(u, v) \mapsto v$ do effectively appear as possible values of $(\psi_s)_s$, they belong in fact to $I_{\psi}^\infty$ or $I_{\psi}^\infty$. For instance, when $s \in \text{Supp } Z^* \cap \text{Supp } X_\Delta \cap \text{Supp } Y^*$, we have $\{\psi_s\}_s \in I_{\psi}^\infty$, by Proposition 5.10. When $s \in \text{Supp } Z^* \cap \text{Supp } Y$, we have, on the contrary, $(\psi_s)_s \in I_{\psi}^\infty$. To see that, it is sufficient to note that if $t_0 \in \Delta_+$ and $\text{Supp } z^* \subseteq \text{Supp } X_\Delta \cap \text{Supp } Y^*$, then for every $M \geq 0$, we have
\[ \langle \Phi(t_0, 0), z^* \rangle = \langle \Phi(t_0, M t_0), z^* \rangle = \limsup_{y \to t_0, y \geq t_0} \langle z^*, \psi(t_0, M y) \rangle, \]
the first equality because $z^* \leq \psi(x^*, y^*)$ then in fact $z^* \leq \psi(x^*, 0)$, hence
\[ \langle \Phi(t_0, M t_0), z^* \rangle = \sup \{ \langle z^*, t_0 \rangle : z^* \leq \psi(x^*, 0) \} = \langle \Phi(t_0, 0), z^* \rangle, \]
and the second one because if $x_n \to t_0$ and $y_n \to M t_0$ with $x_n \geq t_0$, $y_n \geq M t_0$, and $(z^*, \varphi(x_n, y_n) \to \langle \Phi(t_0, M t_0), z^* \rangle$, then, writing again $z^* \leq \psi(x^*, 0)$,
\[ \langle z^*, \varphi(x_n, y_n) \rangle \leq \langle z^*, \varphi(x_n - t_0, y_n) \rangle \to (x^*, x_n - t_0) \to 0 \quad \text{as} \quad n \to \infty. \]

**6. Examples**

(a) Spaces $E_M$ where $E$ is order continuous. Denote by $M$ the Orlicz function associated with the C.L. function $\varphi$ by $M^{-1}(t) = \varphi(t, 1)$. If $E$ is a Köthe function space over $(\Omega, A, \mu)$, we denote by $E_M$ the space $\varphi(E, L_\infty)$. We have $E_M = \{ f \in \text{L}_0(\Omega, A, \mu) : \lambda > 0, M(\| f \| / \lambda) \in E \text{ and } \| f \| E_M = \inf \{ \lambda > 0 : \| f \| / \lambda \leq 1 \} \}$. If $E$ is order continuous, we have $E' = E^*_E = E_A^*$ (the second equality is because $E$ is dense in $E$), hence $\text{Supp } E' = \Omega$ (considering $\Omega$ as embedded in $S = \text{Supp } L_{\infty}$) is contained in the support of the dual of every Köthe space, in particular in that of $L_{\infty}^\infty = L_{\infty, 0}$ (where $L_{\infty, 0}$ is the closure of $\Delta = E \cap L_{\infty}$ in $L_{\infty}$). We then obtain, by Theorem 5.12,
\[ E_M = T \{ E' \in (E', L_1) \} \otimes \mathcal{L} \]
where $\mathcal{L}$ is (isometrically order isomorphic to) a band in $L_{\infty}$, i.e. an abstract $L_1$ space. In particular, if $E = L_1$, we have $E_M = L_{\infty, 0}$, and we recover the case of Orlicz spaces. If $E$ is the Lorentz space $L_{w, 1}$ associated with the
weight $w$, then $E_M$ is a Lorentz–Orlicz space $L_{w,M}$. The description of the dual $L_{w,M}^*$ reduces thus to that of the Köthe dual.

(b) Regularly varying C.-L. functions. We say that the Calderón–Lozanovskii function $\varphi$ is regularly varying if the limits

$$
\lim_{a \to \infty} \frac{\varphi(au,v)}{\varphi(a,1)} =: \varphi_1(u,v) \quad \text{and} \quad \lim_{a \to 0} \frac{\varphi(au,v)}{\varphi(a,1)} =: \varphi_\infty(u,v)
$$

exist (for every $u,v > 0$). Then $\varphi_1$ and $\varphi_\infty$ are necessarily Calderón interpolation functions: $\varphi_1(u,v) = u^{1-\theta_1}v^\theta_1$, $\varphi_\infty(u,v) = u^{1-\theta_\infty}v^\theta_\infty$ (for some $0 \leq \theta_1, \theta_\infty \leq 1$). The conjugate functions $\varphi_1^*, \varphi_\infty^*$ are respectively identical (up to a constant factor) to $\varphi_1, \varphi_\infty$. Hence we have

$$
\varphi(X,Y)^* = \varphi_1(X_0^*, Y_0) \cup U_1^{1-\theta_1}V_1^\theta_1 \cup U_\infty^{1-\theta_\infty}V_\infty^\theta_\infty,
$$

where $X_0, Y_0$ are the closures of $\Delta = X \cap Y$ in $X$, resp. $Y$; $U_1, V_1$ are the bands of $X^*$, resp. $Y^*$ with common support $S_1 = \text{Supp} \varphi(X,Y)^* \cap \text{Supp} X_0^* \cap \text{Supp} Y_0^*$, and $U_\infty, V_\infty$ are the bands of $X^*$, resp. $Y^*$ supported by $S_\infty = \text{Supp} \varphi(X,Y)^* \cap \text{Supp} X_\infty^* \cap \text{Supp} Y_\infty^*$.

A simple example of a regularly varying C.-L. function is

$$
\varphi(u,v) = u^{1-\alpha}v^\alpha \wedge u^{1-\beta}v^\beta.
$$

(c) Couples $(X,Y)$ with nontrivial sets $S_1, S_\infty$. We now give an example of a couple $(X,Y)$ such that $\text{Supp} Y_0^* \cap \text{Supp} X_0^* \neq \emptyset$, and moreover this set intersects $\text{Supp} \varphi(X,Y)^*$ for every C.-L. function $\varphi$.

We take $X = v_{\infty}(l_2)$, $Y = v_{\infty}(l_2)$, with, say, $2 < p < \infty$. In this case $\Delta = X$, hence $X_\infty^* = X^*$. Define sequences $(f_n) \subset l_2$ and $(g_n) \subset l_\infty$ by

$$
f_n = e_1 + \ldots + e_n \quad \text{and} \quad g_n = e_1 + \ldots + e_n \quad \text{for} \quad n = n_1^{1/p_1},
$$

where $\{e_i\}$ denote orthogonally the $l_p$ basis (for all $r$) and $1/p + 1/p_1 = 1$. Define $F^* \in X^*$ and $G^* \in Y^*$ by

$$
\langle F^*, (f_n) \rangle = \lim_{n \to \infty} \langle f_n, f_n \rangle, \quad \langle G^*, (g_n) \rangle = \lim_{n \to \infty} \langle g_n, g_n \rangle
$$

for every $(f_n) \in l_\infty(l_2)$ and $(g_n) \in l_\infty(l_2)$, where $U$ is some nontrivial ultrafilter over $\mathbb{N}$. Then $G^*|X = 0$, i.e. $G^* \in Y_0^*$, since for all $F = (f_n) \in X$,

$$
\langle F^*, F \rangle \leq \|F\| \cdot \lim_{n \to \infty} \|g_n\| \|g_n\| = 0.
$$

On the other hand, it is easy to verify that $\text{Supp} G^* \subset \text{Supp} F^*$. For, let $G \geq 0$ in $Y$ with $(G^*, G) \neq 0$. We may suppose that $G = (g_n)$, for every $n, g_n$ is supported by $\{e_1, \ldots, e_n\}$. Then $\|g_n\| \leq \|g_n\|_{p} \leq n_1^{1/2-1/p} \|g_n\|_{p} \leq n_1^{1/2-1/p} \|G\|$; so if we set $f_n = n_1^{1/p_1-1/2}g_n$, we have $F := (f_n) \in X$ and

$$
\langle f_n, f_n \rangle = n_1^{1/p_1-1/2} \|g_n\|_{p} \leq n_1^{1/2-1/p} \|g_n\|_{p} = \langle g_n, g_n \rangle.
$$

hence $\langle F^*, F \rangle = \langle G^*, G \rangle$. More generally, if $h \in l_\infty(N \times N)$, it is easy to see that $(F^*, hF) = \langle G^*, hG \rangle$, which means that $\pi_{G^*} = \pi_{F^*}$. Thus $\text{Supp} \pi_{G^*} \subset \text{Supp} F^*$ for all $G \in Y$, which means that $\text{Supp} G^* \subset \text{Supp} F^*$.

Let $\varphi$ be an arbitrary element of $C_1$. We have

$$
\varphi(\ell_\infty(2), \ell_\infty(2)) = \ell_\infty(\varphi(\ell_2, \ell_2))
$$

isometrically. Let $r$ be such that $1/2 = 1/p + 1/r$. We have $\ell_2 = \ell_p, \ell_r$, so that

$$
\varphi(\ell_2, \ell_p) = \varphi(\ell_p, \ell_p, \ell_\infty) = \ell_p \varphi(\ell_r, \ell_\infty) \quad (2\text{-isomorphically}) = \ell_p \ell_M \text{ with } M^{-1}(s) = \varphi(s^{1/r}, 1).
$$

For every $n \in \mathbb{N}$, denote by $\lambda(n), \lambda_1(n)$ and $\lambda_M(n)$ the norms of $e_1 + \ldots + e_n$ in the spaces $\varphi(\ell_2, \ell_p)$, $(\varphi(\ell_2, \ell_p))^*$ and $\ell_M$ respectively. Then $\lambda(n) \sim n^{1/2} \lambda_M(n)$, and $\lambda_1(n) = n/\lambda(n) \sim n^{1/2} \lambda_M(n)$. Define $H^* \in Z^*$ by

$$
\langle H^*, H \rangle = \lim_{n \to \infty} \|h_n\|_p \|h_n\|_p.
$$

With each $G = (g_n) \in Y$, we associate $H = (h_n) \in \varphi(X,Y)$, where $h_n = \lambda_M(n)^{-1} g_n$. Then $\langle H^*, H \rangle \sim \langle G^*, G \rangle$, and by the same reasoning as for $F^*$, we obtain $\text{Supp} G^* \subset \text{Supp} H^*$.

References


Almost multiplicative functionals

by

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Abstract. A linear functional $F$ on a Banach algebra $A$ is almost multiplicative if
\[ |F(ab) - F(a)F(b)| \leq \delta ||a|| \cdot ||b|| \]
for a small constant $\delta$. An algebra is called functionally stable or f-stable if any almost
multiplicative functional is close to a multiplicative one. The question whether an algebra
is f-stable can be interpreted as a question whether $A$ lacks an almost corona, that is, a
set of almost multiplicative functionals far from the set of multiplicative functionals.

In this paper we discuss f-stability for general uniform algebras; we prove that any
uniform algebra with one generator as well as some algebras of the form $R(K)$, $K \subset \mathbb{C}$,
and $A(D)$, $D \subset \mathbb{C}$, are f-stable. We show that, for a Blaschke product $B$, the quotient
algebra $H^\infty / BH^\infty$ is f-stable if and only if $B$ is a product of finitely many interpolating
Blaschke products.

1. Introduction. Let $G$ be a linear and multiplicative functional on a
Banach algebra $A$ and let $\Delta$ be a linear functional on $A$ with $||\Delta|| \leq \varepsilon$.
Put $F = G + \Delta$. We can easily check by direct computation that $F$ is $\delta$-multiplicative,
that is,
\[ |F(ab) - F(a)F(b)| \leq \delta ||a|| \cdot ||b|| \]
for $a, b \in A$,
where $\delta = 3\varepsilon + \varepsilon^2$. The problem we want to discuss here is whether the
converse is true; that is, whether an almost multiplicative functional must
be near a multiplicative one. We are interested mostly in uniform algebras.
We shall call a Banach algebra functionally stable or f-stable if
\[ \forall \varepsilon > 0 \exists \delta > 0 \forall F \in \mathcal{M}_G(A) \exists G \in \mathcal{M}(A) \quad ||F - G|| \leq \varepsilon, \]
where we denote by $\mathcal{M}(A)$ the set of all linear multiplicative functionals
on $A$, and by $\mathcal{M}_G(A)$ the set of $\delta$-multiplicative functionals on $A$. We shall

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