STUDIA MATHEMATICA 124 (1) (1997)

On duals of Calderón–Lozanovskiĭ intermediate spaces

by

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Abstract. We give a description of the dual of a Calderón–Lozanovskiĭ intermediate space $\varphi(X, Y)$ of a couple of Banach Köthe function spaces as an intermediate space $\psi(X^*, Y^*)$ of the duals, associated with a "variable" function ψ.

Introduction. Given two Köthe function spaces over the same measure space, X_0 and X_1, the interpolation spaces $X_0^{1-\theta}X_1^{\theta}$, $0 < \theta < 1$, were defined by Calderón ([C]) as the order ideal generated by the functions $x_0^{1-\theta}x_1^{\theta}$ with $x_0 \in X_0$, $x_0 \geq 0$ and $x_1 \in X_1, x_1 \geq 0$. When X_0 or X_1 is reflexive, these spaces coincide (in the complex case) with the spaces $[X_0, X_1]_\theta$ obtained by the complex interpolation method. In this case the dual spaces can also be described by complex interpolation; more precisely, if $X_0 \cap X_1$ is dense in X_0 and X_1, then X_0^* and X_1^* embed naturally in $(X_0 \cap X_1)^*$, and $[X_0, X_1]_\theta^* = [X_0^*, X_1^*]_\theta$. The description of the dual of $X_0^{1-\theta}X_1^{\theta}$ without any restriction on the Banach lattices X_0 and X_1 (except their order completeness) was achieved by Lozanovskiĭ ([L1], [L2]). When $X_0 \cap X_1$ is dense in X_0 and X_1, then $(X_0^{1-\theta}X_1^{\theta})^* = X_0^{*1-\theta}X_1^{*\theta}$, the definition of this last space being unambiguous since X_0^* and X_1^* are order ideals of $(X_0 \cap X_1)^*$; in the general case Lozanovskiĭ shows how to realize X_0^* and X_1^* as order ideals of a common space of measurable functions and then identifies (isometrically and order isomorphically) $(X_0^{1-\theta}X_1^{\theta})^*$ with $X_0^{*1-\theta}X_1^{*\theta}$. A consequence of this fact is that the equality $(X_0^{1-\theta}X_1^{\theta})' = X_0'^{1-\theta}X_1'^{\theta}$ holds for the Köthe duals of the spaces X_0, X_1.

These results were (partially) extended to a more general class of interpolation spaces of Köthe function spaces, the so-called Calderón–Lozanovskiĭ spaces. Let us recall their definition. Consider a function $\varphi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ which is concave, positively homogeneous of degree one, continuous and not identically zero (we denote by $\mathcal{C}$ the set of such functions, which we call *Calderón–Lozanovskiĭ functions*). By rescaling if necessary, we may suppose that $\varphi(1, 1) = 1$ (we denote by $\mathcal{C}_1$ the subset of such normalized functions).

1991 *Mathematics Subject Classification*: Primary 46E30.

Then the space $\varphi(X_0, X_1)$ is the order ideal generated by the functions $\varphi(x_0, x_1)$ with $x_i \in X_i$, $x_i \geq 0$, $i = 0, 1$. This space is normed by the formula $\|z\| = \inf\{\|x_0\| \vee \|x_1\| : |z| \leq \varphi(x_0, x_1);\ x_i \in X_i,\ x_i \geq 0\}$.

The Calderón–Lozanovskiĭ function φ_* conjugate to φ is defined by $\varphi_*(s,t) = \inf\{(\alpha u + \beta v)/\varphi(\alpha, \beta) : \alpha, \beta > 0\}$. It is generally not normalized but $1 \leq \varphi_*(1,1) \leq 2$ if φ is normalized. Suppose that $X_0 \cap X_1$ is dense in X_0 and X_1; let Z_0 be the closure of $X_0 \cap X_1$ in $\varphi(X_0, X_1)$. Then Lozanovskiĭ proves ([L3]) that $Z_0^* = \varphi_*(X_0^*, X_1^*)$; this is an equality between subspaces of $(X_0 \cap X_1)^*$, but the norms are only equivalent up to a constant 2. However, following [R], one can obtain isometry by putting on $\varphi_*(X_0^*, X_1^*)$ the modified norm $\|z^*\| = \inf\{\|x_0^*\|_0 + \|x_1^*\|_1 : |z| \leq \varphi_*(x_0^*, x_1^*);\ x_i^* \in X_i^*,\ x_i^* \geq 0\}$. As a consequence one can deduce the equality $\varphi(X_0, X_1)' = \varphi_*(X_0', X_1')$ for the Köthe duals (without any density assumption). This last fact is reproved in [R], without considering the whole duals. When φ satisfies the two-sided "reverse Δ_2-condition"

$$\exists c > 0,\ \forall s, t > 0, \quad \varphi(s, ct) \leq \tfrac{1}{2}\varphi(s,t) \quad \text{and} \quad \varphi(cs, t) \leq \tfrac{1}{2}\varphi(s,t)$$

(this is in particular the case for $\varphi(s,t) = s^{1-\theta}t^{\theta}$) then $X_0 \cap X_1$ is dense in $\varphi(X_0, X_1)$ and the preceding result gives a description of the whole dual $(\varphi(X_0, X_1))^*$ (under the density assumption).

A particular, well known class of Calderón–Lozanovskiĭ spaces is that of Orlicz spaces: if we set $M^{-1}(t) = \varphi(t, 1)$, then M is an Orlicz function, and the corresponding Orlicz space L_M is simply $\varphi(L_1, L_\infty)$ (with equality of norms if L_M is equipped with the so-called Luxemburg norm). Let M_* be the Young conjugate of M; one has $M_*^{-1}(s) = \varphi_*(1, s)$. If M satisfies the usual Δ_2 condition, then $L_M^* = L_{M_*} = \varphi_*(L_\infty, L_1)$; if not, $L_M^* = L_{M_*} \oplus L$, where L is an abstract (nonseparable) L_1-space (Andô's theorem [A]; see also [Z], [F]).

The purpose of this paper is to give a unified description of the dual of the space $\varphi(X_0, X_1)$ in the most general case. Let a *generalized Calderón–Lozanovskiĭ function*, for short g.C.-L. function, defined on the measure space (S, Σ, m), be a measurable map $\psi : S \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that for a.e. $s \in S$, the partial function $\psi_s := \psi(s, \cdot, \cdot)$ either belongs to $\mathcal{C}$ or is identically 0. If Y_0, Y_1 are Köthe function spaces over (S, Σ, m), then the *generalized Calderón–Lozanovskiĭ space* $\psi(Y_0, Y_1)$ is the order ideal generated by the functions $\psi(y_0, y_1) = \psi(\cdot, y_0(\cdot), y_1(\cdot))$, where $y_i \in Y_i$, $y_i \geq 0$, $i = 0, 1$. Then the dual of the space $\varphi(X_0, X_1)$ can be described as a g.C.-L. space $\psi(X_0^*, X_1^*)$ (see Theorem 4.1), for a suitable realization of X_0^* and X_1^* as order ideals of a space $L_0(S, \Sigma, m)$, and a g.C.-L. function ψ over (S, Σ, m). Moreover, for a.e. $s \in S$, the conjugate function ψ_{s*} is a limit of "dilations of φ", i.e. functions $\varphi_{a,b} : (u, v) \mapsto \varphi(au, bv)/\varphi(a, b)$ (with the convention that $\psi_{s*} \equiv 0$ when $\psi_s \equiv 0$).

In Section 1 we recall some basic notions and facts firstly about Köthe function spaces, Köthe duality and the Vulikh–Lozanovskiĭ representation of the duals, and secondly about the set $\mathcal{C}$ of Calderón–Lozanovskiĭ functions, remarkable subsets of $\mathcal{C}$ associated with a given C.-L. function φ, and the conjugation operation on $\mathcal{C}$. In Section 2, given two Köthe function spaces X, Y we define a functional Ψ over the product of the positive cones of the dual spaces, with values in $\varphi(X, Y)^*$, which provides a way to express the norm on $\varphi(X, Y)^*$ (Theorem 2.5). In Section 3 we consider a triple (E, F, Ψ), consisting of a couple (E, F) of Köthe function spaces and an abstract functional Ψ defined on $E_+ \times F_+$; with this triple is associated a Köthe space $\Psi(E, F)$, the Köthe dual of which can be expressed in terms of a dual functional Ψ_*. In Section 4 we give the above announced representation theorem for $\varphi(X, Y)^*$ as a space $\psi(X^*, Y^*)$ (Theorem 4.1), and in Section 5 we refine this theorem, by decomposing the underlying measure space into disjoint parts over which ψ takes its values in remarkable subsets of C.-L. functions asymptotically associated with φ_* (Theorem 5.12). An example is given to show that this decomposition can be nontrivial (Section 6).

1. Preliminaries

(a) *Köthe function spaces and their duals.* A *Köthe function space* over the measure space $(\Omega, \mathcal{A}, \mu)$ is an order dense order ideal (= solid subspace) of the space $L_0(\Omega, \mathcal{A}, \mu)$ of all measurable functions over $(\Omega, \mathcal{A}, \mu)$, equipped with a norm for which it is a Banach lattice for the natural order. By extension, we shall also consider function spaces whose elements have supports in a fixed subset $A \in \mathcal{A}$, and are Köthe function spaces over $(A, \mathcal{A}|_A, \mu|_A)$ (*generalized Köthe function spaces*). We call A the *support* of X.

In the case where μ is not σ-finite, we shall suppose that the measure space is *decomposable* (or *strictly localizable*), i.e. there exists a measurable partition $(A_\alpha)_\alpha$ of Ω into μ-integrable sets such that a subset E of Ω is $\mathcal{A}$-measurable (resp. μ-negligible) iff all the intersections $E \cap A_\alpha$ are $\mathcal{A}$-measurable (resp. μ-negligible). In this case $L_0(\Omega, \mathcal{A}, \mu)$ is Dedekind complete ([Fr]).

If X is an abstract Banach lattice, its *Nakano dual* X' is the subspace of X^* whose elements are order continuous, i.e. $x^* \in X'$ iff for all decreasing nets $(x_i)_{i\in I}$ with $\bigwedge_i x_i = 0$, one has $\lim_i \langle x^*, x_i\rangle = 0$. The space X' is a band in X^*. When X is a generalized Köthe space over $(\Omega, \mathcal{A}, \mu)$, then X' can be realized as a generalized Köthe space over $(\Omega, \mathcal{A}, \mu)$ with the same support, the Köthe dual of X, consisting of the elements $f \in L_0(\Omega, \mathcal{A}, \mu)$ such that $fx \in L_1$ for every $x \in X$, and living on the support of X; then $\langle f, x\rangle = \int fx\,d\mu$. The natural embedding $i : X \to X^{**}$ takes values in $X^{*\prime}$ and is an isometric lattice isomorphism (onto a sublattice of $X^{*\prime}$). Let r be the restriction projection from $X^{*\prime}$ onto X''. Then $j = r \circ i$ is a lattice homomorphism from X into X'', which is injective in the case of Köthe function spaces. The equality $X = X''$, with equality of norms, is equivalent to the Fatou property of X, i.e. that every norm bounded increasing net of nonnegative elements has a supremum whose norm is the supremum of the norms of the elements. In particular, duals have the Fatou property, hence $X^{*\prime\prime} = X^*$.

Let Y be an order complete Banach lattice. We can find in Y a *complete system of local units*, i.e. a maximal system $(y_\alpha)_\alpha$ of disjoint nonzero, nonnegative elements. Then the order ideal $\mathcal{I}$ generated by the y_α's is order dense in Y. On the other hand, let $\mathcal{Z}(Y)$ be the center of Y, i.e. the closure in $\mathcal{L}(Y)$, for the operator norm topology, of the space of operators of the type $\sum_{i=1}^n a_i p_i$, where the a_i are scalars and the p_i are disjoint band projections. Then every y in $\mathcal{I}$ can be formally written $y = \sum_\alpha \varphi_\alpha y_\alpha$, with $\varphi_\alpha \in \mathcal{Z}(Y)$; if $y \geq 0$, so are the φ_α, and $\sum_\alpha$ means simply the supremum.

Let us briefly recall now the realization of X^* as a Köthe function space given in [VL] (for X a Köthe function space over $(\Omega, \mathcal{A}, \mu)$). If $x \in X$, we have an order continuous lattice homomorphism $\pi_x : X^* \to L_\infty(\Omega)^*$ defined by $\langle \pi_x x^*, h\rangle = \langle x^*, hx\rangle$. These homomorphisms π_x induce a bijection $\widetilde{\pi}$ between the bands of X^* and the subbands of a band R_X of $L_\infty(\Omega)^*$ (by $\widetilde{\pi}(V) = \mathrm{band}\{\pi_x(x^*) : x^* \in V,\ x \in X\}$). By identifying the bands with the associated band projections, $\widetilde{\pi}$ is an isomorphism from the complete Boolean algebra $\mathcal{B}(X^*)$ of projections of X^* to that of R_X. This isomorphism induces naturally an isometric order isomorphism from $\mathcal{Z}(X^*)$ onto $\mathcal{Z}(R_X)$ (also denoted by $\widetilde{\pi}$).

Conversely, we can define a homomorphism ϱ from the complete Boolean algebra $\mathcal{B}(L_\infty(\Omega)^*)$ onto $\mathcal{B}(X^*)$ by setting $\langle \varrho(p)x^*, x\rangle = \langle p\pi_x x^*, \mathbf{1}_\Omega\rangle$ for all $x^* \in X^*$, $x \in X$ and $p \in \mathcal{B}(L_\infty(\Omega)^*)$. Then $\varrho\widetilde{\pi} = \mathrm{id}_{\mathcal{B}(X^*)}$, while $\widetilde{\pi}\varrho$ is the natural restriction from $\mathcal{B}(L_\infty(\Omega)^*)$ to $\mathcal{B}(R_X)$. Note that ϱ induces a continuous homomorphism from $\mathcal{Z}(L_\infty(\Omega)^*)$ onto $\mathcal{Z}(X^*)$.

Note that $L_\infty(\Omega, \mathcal{A}, \mu)^*$ is an abstract L_1 space, and thus identifies (isometrically and order isomorphically) with a space $L_1(S, \Sigma, m)$ (see [LT]). Then R_X is the band generated by a set $S_X \in \Sigma$ in $L_1(S)$; $\mathcal{Z}(L_\infty(\Omega)^*)$ identifies with $L_\infty(S, \Sigma, m)$, and $\mathcal{Z}(R_X)$ with $L_\infty(S_X)$. Then X^* appears as an $L_\infty(S)$-module for the action defined by $h.x^* = \varrho(h)(x^*)$ ($x^* \in X^*$, $h \in L_\infty(S)$). If $x^* \in X^*$, and p_{x^*} is the projection onto the band generated by x^*, we call the set $S_{x^*} \in \Sigma$ whose indicator function is identified with $\widetilde{\pi}(p_{x^*})$ the *support* of x^*.

We can choose then a complete system $(x^*_\alpha)_\alpha$ of local units in X^* and another one $(\nu_\alpha)_\alpha$ in R_X, such that $\widetilde{\pi}(\mathrm{band}\, y^*_\alpha) = \mathrm{band}\, \nu_\alpha$ for every α. We can suppose that ν_α is an indicator function $\mathbf{1}_{S_\alpha}$, with $S_\alpha \in \Sigma$, and $m(S_\alpha) < \infty$. Then with every element $x^* = \sum_\alpha \varphi_\alpha x^*_\alpha$ in the order ideal generated by the x^*_α's we associate $\widetilde{\pi}(x^*) = \sum_\alpha \widetilde{\pi}(\varphi_\alpha)\mathbf{1}_{S_\alpha}$ and we extend $\widetilde{\pi}$ by order density

to an order isomorphism from X^* onto an ideal of $L_0(S, \Sigma, m)$ (supported by S_X): then $\widetilde{\pi}(X^*)$ is the desired realization of X^* as a Köthe function space over (S, Σ, m).

We call such a realization of X^* in $L_0(S, \Sigma, m)$ a *standard realization* (associated with the complete systems (x^*_α) and (ν_α) of local units). If $w \in L_0(S)_+$ with $w > 0$ a.e., then $\widetilde{\pi}^{(1)}$ defined by $\widetilde{\pi}^{(1)}(x^*) = w.\widetilde{\pi}(x^*)$ gives another standard realization of X^*: for, we may assume that w is bounded from above and below on the support S_α of each $\widetilde{\pi}(x^*_\alpha)$, and set $x^{*(1)}_\alpha = (\mathbf{1}_{S_\alpha} w^{-1}).x^*_\alpha$, thus obtaining a new complete system of local units of X^* for which $\widetilde{\pi}^{(1)}(x^{*(1)}_\alpha) = \widetilde{\pi}(x^*_\alpha) = \mathbf{1}_{S_\alpha}$. We say that the new standard realization of X^* is obtained from the old one *by a change of density*.

A standard realization of X^* induces in turn a realization of $X^{*\prime}$ in $L_0(S, \Sigma, m)$. The embedding i_X of X into $X^{*\prime}$ is then characterized by the relations $i_X(x).\widetilde{\pi}(x^*) = \pi_x(x^*)$ for every $x^* \in X^*$ (or equivalently $\mathbf{1}_{S_\alpha} i_X(x) = \pi_x(x^*_\alpha)$ for every α). The order ideal $\mathcal{I}_X$ generated by X in $X^{*\prime}$ consists of the elements $h.i_X(x)$, $x \in X$, $h \in L_\infty(S)$, and one has $X^{*\prime} = \mathcal{I}''_X$ (since clearly $X^* = \mathcal{I}'_X$). Hence nonnegative elements of $X^{*\prime}$ are suprema of norm-bounded directed families of nonnegative elements of $\mathcal{I}_X$ (see [Z]).

We can find a maximal system $(S_\alpha)_\alpha$ of disjoint subsets of S_X whose indicator functions $\mathbf{1}_{S_\alpha}$ are simultaneously in (the realization of) X^* and (that of) $X^{*\prime}$. By a change of density, we can obtain a standard realization of X^* for which $\mathbf{1}_{S_\alpha} = \mathbf{1}_{S_\alpha} i_X(x_\alpha)$ for a certain complete system $(x_\alpha)_\alpha$ of local units of X.

(b) *The set of Calderón–Lozanovskiĭ functions*. Now let us say a few words about the set $\mathcal{C}_1$ of normalized Calderón–Lozanovskiĭ functions. We equip $\mathcal{C}_1$ with the topology of simple convergence on the open quadrant $\mathcal{P} = \{(u, v) : u > 0,\ v > 0\}$. Using Ascoli's theorem, it is easy to see that this topology coincides with the topology of uniform convergence on compact subsets of $\mathcal{P}$ (or of the open segment $\Lambda = \mathcal{P} \cap \{(u, v) : u + v = 1\}$). Thus this topology is metrizable; in fact, one obtains a compatible metric setting $d(\varphi, \psi) = \sum_{i=1}^\infty 2^{-i}|\varphi(u_i, v_i) - \psi(u_i, v_i)|$, where $(u_i, v_i)_{i=1}^\infty$ is, say, the set of rational couples in Λ. Note that the balls relative to this metric are convex. Moreover, $\mathcal{C}_1$ is compact for this topology. The same is true of course for the set $\mathcal{C}_{a,b} = \{\varphi \in \mathcal{C} : a \leq \varphi(1, 1) \leq b\}$ for all positive numbers a, b. Given a $\varphi \in \mathcal{C}_1$, we shall denote by Γ^f_φ the subset of $\mathcal{C}_1$ consisting of all *φ-dilations* $\varphi_{a,b}$ (defined by $\varphi_{a,b}(u, v) = \varphi(au, bv)/\varphi(a, b)$) where $a, b > 0$; and by Γ_φ the closure of Γ^f_φ in $\mathcal{C}_1$. Denote also by $\Gamma^{l,M}_\varphi$ the closure of $\{\varphi_{a,b} : a > Mb > 0\}$ and by $\Gamma^{r,M}_\varphi$ that of $\{\varphi_{a,b} : b > Ma > 0\}$, and finally let $\Gamma^{l,\infty}_\varphi = \bigcap_M \Gamma^{l,M}_\varphi$, resp. $\Gamma^{r,\infty}_\varphi = \bigcap_M \Gamma^{r,M}_\varphi$.

Let us show that the conjugates of the elements of Γ_φ appear after normalization as elements of the set Γ_{φ_*} associated with the conjugate of φ.

LEMMA 1.1. *The conjugation map $\varphi \mapsto \varphi_*$ is continuous from $\mathcal{C}_1$ into $\mathcal{C}$.*

Proof. We have to prove that if $\varphi_n \to \varphi$ uniformly on compact sets then $\varphi_{n*} \to \varphi_*$ pointwise.

From the inequalities

$$\forall n,\ \forall u,v,s,t>0,\qquad \varphi_n(s,t)\varphi_{n*}(u,v)\le us+vt$$

we deduce

$$\forall u,v,s,t>0,\qquad \varphi(s,t)\psi(u,v)\le us+vt$$

where $\psi(u,v)=\limsup_{n\to\infty}\varphi_{n*}(u,v)$, which means that $\psi\le\varphi_*$.

Conversely, fix $u,v>0$, and let $0<a<1$. Set

$$\varphi_n^{(a)}(u,v)=\varphi_n(u,v)\vee(v\varphi_n(a,1))\vee(u\varphi_n(1,a))$$

and define $\varphi^{(a)}$ similarly. From $\varphi_n^{(a)}\ge\varphi_n$, we deduce $\varphi_{n*}^{(a)}\le\varphi_{n*}$. Now compute $\varphi_{n*}^{(a)}$. We have

$$\begin{aligned}\varphi_{n*}^{(a)}(u,v)&=\inf_{s,t>0}\frac{us+vt}{\varphi_n^{(a)}(s,t)}\\&=\inf_{s\le at}\frac{us+vt}{t\varphi_n(a,1)}\wedge\inf_{at\le s\le a^{-1}t}\frac{us+vt}{\varphi_n(s,t)}\wedge\inf_{s\ge a^{-1}t}\frac{us+vt}{s\varphi_n(1,a)}\\&=\frac{v}{\varphi_n(a,1)}\wedge\inf_{at\le s\le a^{-1}t}\frac{us+vt}{\varphi_n(s,t)}\wedge\frac{u}{\varphi_n(1,a)}.\end{aligned}$$

Since

$$\frac{\varphi_n(s,t)}{s+t}\underset{n\to\infty}{\longrightarrow}\frac{\varphi(s,t)}{s+t}$$

uniformly on the set $\{s,t>0\ :at\le s\le a^{-1}t\}$, we have

$$\inf_{at\le s\le a^{-1}t}\frac{us+vt}{\varphi_n(s,t)}\underset{n\to\infty}{\longrightarrow}\inf_{at\le s\le a^{-1}t}\frac{us+vt}{\varphi(s,t)},$$

whence $\varphi_{n*}^{(a)}(u,v)\to\varphi_*^{(a)}(u,v)$, hence $\liminf_{n\to\infty}\varphi_{n*}\ge\varphi_*^{(a)}$, for every $0<a<1$. But we have

$$\frac{v}{\varphi(a,1)}\ge\frac{v}{ua+v}\varphi_*(u,v)\quad\text{and}\quad\frac{u}{\varphi(1,a)}\ge\frac{u}{u+av}\varphi_*(u,v),$$

hence

$$\varphi_*^{(a)}(u,v)\ge\left(\frac{v}{ua+v}\wedge\frac{u}{u+av}\right)\varphi_*(u,v)\underset{a\to0}{\longrightarrow}\varphi_*(u,v).\ \blacksquare$$

COROLLARY 1.2. *The conjugates of the elements of Γ_φ (resp. $\Gamma_\varphi^{l,\infty}$, $\Gamma_\varphi^{r,\infty}$) are proportional to elements of Γ_{φ_*} (resp. $\Gamma_{\varphi_*}^{r,\infty}$, $\Gamma_{\varphi_*}^{l,\infty}$).*

Proof. If $\psi\in\Gamma_\varphi^f$, i.e. $\psi(u,v)=\varphi(au,bv)/\varphi(a,b)$ for $u,v>0$, we have

$$\begin{aligned}\psi_*(u,v)&=\inf_{s,t>0}\frac{us+vt}{\varphi(as,bt)}\varphi(a,b)\\&=\inf_{s,t>0}\frac{ua^{-1}s+vb^{-1}t}{\varphi(s,t)}\varphi(a,b)\\&=\varphi_*\left(\frac{u}{a},\frac{v}{b}\right)\varphi(a,b).\end{aligned}$$

In particular, $\psi_*(1,1)=\varphi_*(1/a,1/b)\varphi(a,b)$, hence

$$\psi_*(u,v)=\frac{\varphi_*(u/a,v/b)}{\varphi_*(1/a,1/b)}\psi_*(1,1).$$

Now if $\psi\in\Gamma_\varphi$, $\psi(u,v)=\lim_{n\to\infty}\varphi(a_nu,b_nv)/\varphi(a_n,b_n)$, then by Lemma 1.1 we have

$$\psi_*(u,v)=\lim_{n\to\infty}\frac{\varphi_*(u/a_n,v/b_n)}{\varphi_*(1/a_n,1/b_n)}\psi_*(1,1).\ \blacksquare$$

2. The Ψ-functional and the norm on $\varphi(X,Y)^*$. Let X and Y be two Köthe function spaces over the same measure space $(\Omega,\mathcal{A},\mu)$, and φ a normalized Calderón–Lozanovskiĭ function. We set once for all $Z=\varphi(X,Y)$. We identify $L_\infty(\Omega)^*$ with $L_1(S,\Sigma,m)$.

DEFINITION 2.1. For every $x^*\in X_+^*$, $y^*\in Y_+^*$ and $z\in Z_+$, set

$$\Psi(x^*,y^*)(z)=\inf\{\langle x^*,x\rangle+\langle y^*,y\rangle : x\in X_+,\ y\in Y_+,\ z\le\varphi(x,y)\}.$$

PROPOSITION 2.2. *The map $\Psi(x^*,y^*)$ extends to a bounded positive linear form over Z.*

Proof. We have to prove that $\Psi(x^*,y^*)$ is positively linear over Z_+. Let $z=z_1+z_2$, with $z_i\in Z_+$.

Let $\varepsilon>0$, and let $x_i\in X_+$ and $y_i\in Y_+$ be such that $z_i\le\varphi(x_i,y_i)$ and $\Psi(x^*,y^*)(z_i)\ge\langle x^*,x_i\rangle+\langle y^*,y_i\rangle-\varepsilon$. Then (by concavity and homogeneity of φ)

$$z_1+z_2\le\varphi(x_1,y_1)+\varphi(x_2,y_2)\le\varphi(x_1+x_2,y_1+y_2),$$

whence

$$\begin{aligned}\Psi(x^*,y^*)(z_1+z_2)&\le\langle x^*,x_1+x_2\rangle+\langle y^*,y_1+y_2\rangle\\&\le\Psi(x^*,y^*)(z_1)+\Psi(x^*,y^*)(z_2)+2\varepsilon.\end{aligned}$$

Conversely, let $x\in X_+$ and $y\in Y_+$ be such that $z\le\varphi(x,y)$ and $\Psi(x^*,y^*)(z)\ge\langle x^*,x\rangle+\langle y^*,y\rangle-\varepsilon$. We may write $z_i=h_iz$ with $h_i\in L_\infty(\Omega)$ and $h_1+h_2=\mathbf{1}_\Omega$. Set $x_i=h_ix$ and $y_i=h_iy$. We have

$$z_i\le h_i\varphi(x,y)=\varphi(h_ix,h_iy)=\varphi(x_i,y_i),$$

thus

$$\Psi(x^*,y^*)(z_i)\le\langle x^*,x_i\rangle+\langle y^*,y_i\rangle$$

and

$$\Psi(x^*,y^*)(z_1)+\Psi(x^*,y^*)(z_2)\le\langle x^*,x_1+x_2\rangle+\langle y^*,y_1+y_2\rangle$$
$$=\langle x^*,x\rangle+\langle y^*,y\rangle\le\Psi(x^*,y^*)(z)+\varepsilon.$$

To prove the boundedness of $\Psi(x^*,y^*)$, choose for any $z\in Z_+$ two elements $x\in X_+$ and $y\in Y_+$ with $z\le\varphi(x,y)$ and $\|x\|\vee\|y\|\le(1+\varepsilon)\|z\|$. Then

$$\Psi(x^*,y^*)(z)\le\langle x^*,x\rangle+\langle y^*,y\rangle\le(1+\varepsilon)\|z\|(\|x^*\|+\|y^*\|).$$

Thus $\Psi(x^*,y^*)\in Z_+^*$ and $\|\Psi(x^*,y^*)\|\le\|x^*\|+\|y^*\|$. ■

PROPOSITION 2.3. *For every $x_0\in X_+$, $y_0\in Y_+$ and $z^*\in Z_+^*$, we have*

$$\inf\{\langle x^*,x_0\rangle+\langle y^*,y_0\rangle:\Psi(x^*,y^*)\ge z^*\}=\limsup_{\substack{x\to x_0\\ y\to y_0\\ x\ge0,y\ge0}}\langle z^*,\varphi(x,y)\rangle.$$

(Note that we set $\inf\emptyset=+\infty$.)

Proof. For all $x\in X_+$, $y\in Y_+$ and $z^*\in Z_+^*$, we have for all $x^*\in X^*$ and $y^*\in Y^*$ such that $\Psi(x^*,y^*)\ge z^*$,

$$\langle z^*,\varphi(x,y)\rangle\le\langle\Psi(x^*,y^*),\varphi(x,y)\rangle\le\langle x^*,x\rangle+\langle y^*,y\rangle$$

by the very definition of $\Psi(x^*,y^*)$. Hence

$$\limsup_{\substack{x\to x_0\\ y\to y_0}}\langle z^*,\varphi(x,y)\rangle\le\limsup_{\substack{x\to x_0\\ y\to y_0}}(\langle x^*,x\rangle+\langle y^*,y\rangle)=\langle x^*,x_0\rangle+\langle y^*,y_0\rangle.$$

For the proof of the converse inequality, set

$$h(x_0,y_0):=\limsup_{\substack{x\to x_0\\ y\to y_0\\ x\ge0,y\ge0}}\langle z^*,\varphi(x,y)\rangle.$$

Note that we allow $x=x_0$ or $y=y_0$ in this limsup, thus $h(x_0,y_0)\ge\langle z^*,\varphi(x_0,y_0)\rangle$. The function h is clearly upper semicontinuous (it is the u.s.c. envelope of the function $(x,y)\mapsto\langle z^*,\varphi(x,y)\rangle$). It is also straightforward to verify that h is positively homogeneous and concave over $X_+\times Y_+$. As a consequence of the Hahn–Banach Theorem, for all $(x_0,y_0)\in X_+\times Y_+$ and $\varepsilon>0$, there exists a $F\in(X\times Y)_+^*$ such that

$$\begin{cases}F(x,y)\ge h(x,y), & \forall x\in X_+,\ \forall y\in Y_+,\\ F(x_0,y_0)\le h(x_0,y_0)+\varepsilon.\end{cases}$$

We may write $F(x,y)=\langle x^*,x\rangle+\langle y^*,y\rangle$ for certain $x^*\in X_+^*$ and $y^*\in Y_+^*$. We then have, for every $x\in X_+$ and $y\in Y_+$,

$$\langle z^*,\varphi(x,y)\rangle\le h(x,y)\le F(x,y),$$

hence for every $z\in Z_+$,

$$\langle z^*,z\rangle\le\inf\{F(x,y):z\le\varphi(x,y)\}=\langle\Psi(x^*,y^*),z\rangle,$$

which means that $z^*\le\Psi(x^*,y^*)$. On the other hand,

$$\langle x^*,x_0\rangle+\langle y^*,y_0\rangle=F(x_0,y_0)\le h(x_0,y_0)+\varepsilon.\ ■$$

Remark. If φ satisfies a two-sided reverse Δ_2-condition (see the introduction), then in fact

$$\inf\{\langle x^*,x_0\rangle+\langle y^*,y_0\rangle:\Psi(x^*,y^*)\ge z^*\}=\langle z^*,\varphi(x_0,y_0)\rangle.$$

For, in this case, the map $X_+\times Y_+\to Z_+$, $(x,y)\mapsto\varphi(x,y)$, is continuous.

COROLLARY 2.4. *For every $z^*\in Z_+^*$, there exist $x^*\in X_+^*$ and $y^*\in Y_+^*$ such that $z^*\le\Psi(x^*,y^*)$.*

Proof. We have

$$\limsup_{\substack{x\to x_0\\ y\to y_0\\ x\ge0,y\ge0}}\langle z^*,\varphi(x,y)\rangle\le\limsup_{\substack{x\to x_0\\ y\to y_0\\ x\ge0,y\ge0}}\|z^*\|(\|x\|\vee\|y\|)$$
$$=\|z^*\|(\|x_0\|\vee\|y_0\|)<\infty.\ ■$$

THEOREM 2.5. *The norm of any element z^* in Z^* is given by*

$$\|z^*\|=\inf\{\|x^*\|+\|y^*\|:|z^*|\le\Psi(x^*,y^*)\}.$$

Proof. Let $|||z^*|||$ be the right hand side in this relation. The inequality $\|z^*\|\le|||z^*|||$ results from the fact that if $0\le|z|\le\varphi(x,y)$ with $\|x\|\vee\|y\|\le(1+\varepsilon)\|z\|$, and if $|z^*|\le\Psi(x^*,y^*)$, then

$$|\langle z^*,z\rangle|\le\langle\Psi(x^*,y^*),\varphi(x,y)\rangle\le\langle x^*,x\rangle+\langle y^*,y\rangle$$
$$\le(\|x^*\|+\|y^*\|)(\|x\|\vee\|y\|)\le(1+\varepsilon)\|z\|(\|x^*\|+\|y^*\|).$$

Conversely, let $a<|||z^*|||$. Set $H_{z^*}=\{(x^*,y^*)\in X_+^*\times Y_+^*:\Psi(x^*,y^*)\ge z^*\}$. This set is nonempty (by Corollary 2.4), convex and w^*-closed: for we have $H_{z^*}=\bigcap_{x\in X_+,y\in Y_+}H_{z^*,x,y}$, where $H_{z^*,x,y}$ is the w^*-closed hyperplane $\{(x^*,y^*)\in X_+^*\times Y_+^*:\langle x^*,x\rangle+\langle y^*,y\rangle\ge\langle z^*,\varphi(x,y)\rangle\}$.

Set $B_a=\{(x^*,y^*)\in X_+^*\times Y_+^*:\|x^*\|+\|y^*\|\le a\}$. By the Hahn–Banach Theorem, we can separate H_{z^*} from the nonempty w^*-compact set B_a by a w^*-closed hyperplane, i.e. there exists a nonzero couple $(x_0,y_0)\in X\times Y$ such that

$$\inf\{\langle x^*,x_0\rangle+\langle y^*,y_0\rangle:\Psi(x^*,y^*)\ge z^*\}$$
$$\ge\sup\{\langle x^*,x_0\rangle+\langle y^*,y_0\rangle:\|x^*\|+\|y^*\|\le a\}=a\|x_0\|\vee\|y_0\|.$$

We may suppose that $x_0,y_0\ge0$ (replacing these elements by their absolute values). By Proposition 2.3, we deduce that

$$\limsup_{\substack{x\to x_0\\ y\to y_0}}\langle z^*,\varphi(x,y)\rangle\ge a\|x_0\|\vee\|y_0\|.$$

But the left hand side is less than

$$\|z^*\| \limsup_{\substack{x\to x_0\\ y\to y_0}} \|x\| \vee \|y\| = \|z^*\|(\|x_0\| \vee \|y_0\|).$$

Hence $\|z^*\| \ge a$. ∎

In the following proposition we list some important properties of the Ψ-functional.

PROPOSITION 2.6. (a) *The function Ψ is concave, positively homogeneous on $X_+^* \times Y_+^*$ and (w^*, w^*)-upper semicontinuous.*

(b) *The function Ψ is nondecreasing, i.e. $x^* \ge x_1^*$, $y^* \ge y_1^*$ implies $\Psi(x^*, y^*) \ge \Psi(x_1^*, y_1^*)$.*

(c) *The function Ψ is $(L_\infty(S))_+$-homogeneous, i.e. for every $h \in (L_\infty(S))_+$, we have $\Psi(h.x^*, h.y^*) = h.\Psi(x^*, y^*)$.*

(d) *The function Ψ is order continuous, in the sense that for all increasing nets $x_\alpha^* \uparrow x^*$ and $y_\alpha^* \uparrow y^*$, we have $\Psi(x_\alpha^*, y_\alpha^*) \uparrow \Psi(x^*, y^*)$, and similarly for decreasing nets.*

(By (w^*, w^*)-upper semicontinuity of Ψ we mean that for every $z \in Z_+$, the map $(x^*, y^*) \mapsto \langle \Psi(x^*, y^*), z\rangle$ is w^*-upper semicontinuous.)

Proof. (a) For every $z \in Z_+$, the map $(x^*, y^*) \mapsto \langle \Psi(x^*, y^*), z\rangle$ is defined as a g.l.b. of w^*-continuous linear forms. The positive homogeneity of Ψ is straightforward.

(b) is straightforward.

(c) Consider first the case where h is an element of $L_\infty(\Omega, \mathcal{A}, \mu)$. Let $z \in Z_+$ and let $x \in X_+$ and $y \in Y_+$ satisfy $z \le \varphi(x, y)$. Then clearly $hz \le \varphi(hx, hy)$, thus

$$\begin{aligned}\langle h\Psi(x^*, y^*), z\rangle &= \langle \Psi(x^*, y^*), hz\rangle \le \langle x^*, hx\rangle + \langle y^*, hy\rangle\\ &= \langle hx^*, x\rangle + \langle hy^*, y\rangle.\end{aligned}$$

Passing to the infimum with respect to x, y in the last expression, we obtain

$$h\Psi(x^*, y^*) \le \Psi(hx^*, hy^*).$$

Suppose now that $h \le \mathbf{1}_\Omega$, and set $g = \mathbf{1} - h$. We also have

$$g\Psi(x^*, y^*) \le \Psi(gx^*, gy^*).$$

Adding these two inequalities, we obtain

$$\Psi(x^*, y^*) \le \Psi(hx^*, hy^*) + \Psi(gx^*, gy^*) \le \Psi(x^*, y^*)$$

where the last inequality is a consequence of the concavity and the positive homogeneity of Ψ (and the fact that $g + h = \mathbf{1}$). Hence in this relation, the inequality in the middle is an equality, and the same is true for the two preceding relations we added.

If now $h \in L_\infty(S, \Sigma, m)_+$, then there exists, by the lattice version of Helly's theorem (see [K], Theorem 2), a net (h_α) in $(L_\infty(\Omega))_+$ which converges to h for the w^*-topology of $L_\infty(\Omega)^{**}$. Then $h_\alpha.x^* \to h.x^*$, $h_\alpha.y^* \to h.y^*$ and $h_\alpha.\Psi(x^*, y^*) \to h.\Psi(x^*, y^*)$ for the appropriate w^*-topology. Using the upper semicontinuity of Ψ, we obtain the inequality

$$h\Psi(x^*, y^*) \le \Psi(hx^*, hy^*)$$

and we derive equality as before.

(d) The case of decreasing nets is a consequence of the w^*-upper semicontinuity of Ψ (see (a)). Consider increasing nets $x_\alpha^* \uparrow x^*$ and $y_\alpha^* \uparrow y^*$. We have $x_\alpha^* = h_\alpha x^*$ and $y_\alpha = k_\alpha y^*$, with $h_\alpha, k_\alpha \in L_\infty(S)$ and $0 \le h_\alpha \uparrow 1$, $0 \le k_\alpha \uparrow \mathbf{1}$. Then

$$\Psi(h_\alpha x^*, k_\alpha y_\alpha^*) \ge \Psi(h_\alpha \wedge k_\alpha x^*, h_\alpha \wedge k_\alpha y^*) = h_\alpha \wedge k_\alpha \Psi(x^*, y^*) \uparrow \Psi(x^*, y^*).$$ ∎

Remark 2.7. Suppose that the two Köthe spaces are identical: $X = Y =: \Delta$. Then the Ψ-functional is simply given by

$$\forall t_1^*, t_2^* \in \Delta_+, \quad \Psi(t_1^*, t_2^*) = \varphi_*(t_1^*, t_2^*).$$

Proof. Set $z^* = t_1^* + t_2^*$; using Proposition 2.6, we can reduce to the case where $t_i^* = a_i z^*$ for some nonnegative reals a_1, a_2. Then $\varphi_*(x^*, y^*) = \varphi_*(a_1, a_2)z^*$. For every $t, t_1, t_2 \in \Delta_+$ such that $t = \varphi(t_1, t_2)$, we have

$$\begin{aligned}\langle t_1^*, t_1\rangle + \langle t_2^*, t_2\rangle &= \langle z^*, a_1t_1 + a_2t_2\rangle \ge \langle z^*, \varphi_*(a_1, a_2)\varphi(t_1, t_2)\rangle\\ &= \langle \varphi_*(a_1, a_2)z^*, t\rangle,\end{aligned}$$

and conversely: if $\varepsilon > 0$ choose positive reals u_1, u_2 such that $\varphi(u_1, u_2) = 1$ and $u_1a_1 + u_2a_2 \le \varphi_*(a_1, a_2) + \varepsilon$. Set $t_1 = u_1 t$ and $t_2 = u_2 t$. Then $\varphi(t_1, t_2) = t$, and

$$\begin{aligned}\langle z^*, a_1t_1 + a_2t_2\rangle &\le \langle z^*, (\varphi_*(a_1, a_2) + \varepsilon)t\rangle\\ &= \langle \varphi_*(a_1, a_2)z^*, t\rangle + \varepsilon\langle z^*, t\rangle.\end{aligned}$$

Then let $\varepsilon \to 0$. ∎

Remark 2.8. The Ψ-functional is also characterized by the following formula:

$$\langle \Psi(x^*, y^*), z\rangle = \limsup_{\substack{x'\to x^*\ (w^*)\\ y'\to y^*\ (w^*)\\ x'\in X'_+, y'\in Y'_+}} \langle \varphi_*(x', y'), z\rangle$$

for every $x^* \in X_+^*$, $y^* \in Y_+^*$ and $z \in Z$.

This formula is analogous to the formula given in Proposition 2.3 (whose left hand side is related to the dual functional $\Psi_* : X_+^{*\prime} \times Y_+^{*\prime} \to Z_+^{*\prime}$, see §3), except that here the w^*-convergence on X^* and Y^* is involved, not the norm convergence. The better properties of the norm convergence justify that we prefer to consider Ψ_* rather than Ψ in the subsequent sections.

Proof. Consider the map $h : X_+^* \times Y_+^* \to \mathbb{R}_+$ defined by

$$h(x^*, y^*) = \limsup_{\substack{x' \to x^*\ (w^*) \\ y' \to y^*\ (w^*) \\ x' \in X'_+, y' \in Y'_+}} \langle \varphi_*(x', y'), z \rangle.$$

The map h is w^*-u.s.c. and concave. The inequality $h(x^*, y^*) \le \langle \Psi(x^*, y^*), z \rangle$ is easy. Conversely, using the Hahn–Banach theorem, for every couple (x_0^*, y_0^*) in $X_+^* \times Y_+^*$ and $\varepsilon > 0$, we can find a couple (x, y) in $X_+ \times Y_+$ with

$$\begin{cases} \langle x^*, x \rangle + \langle y^*, y \rangle \ge h(x^*, y^*) & \text{for every } (x^*, y^*) \in X_+^* \times Y_+^*, \\ \langle x_0^*, x \rangle + \langle y_0^*, y \rangle \le h(x_0^*, y_0^*) + \varepsilon. \end{cases}$$

We apply the first inequality to a couple $(x', y') \in X'_+ \times Y'_+$; we obtain

$$\langle x', x \rangle + \langle y', y \rangle \ge h(x', y') \ge \langle \varphi_*(x', y'), z \rangle.$$

Let $z' \in Z'_+$. We see that

$$\inf\{\langle x', x \rangle + \langle y', y \rangle : x' \in X'_+,\ y' \in Y'_+,\ \varphi_*(x', y') \ge z'\} \ge \langle z', z \rangle.$$

From Lozanovskiĭ's paper [L3] (Lemma 17), or from Proposition 3.5 below, we know that the left hand side in this last inequality is nothing but $\langle z', \varphi(x, y) \rangle$. Hence $\langle z', \varphi(x, y) - z \rangle \ge 0$ for every $z' \in Z'_+$, which suffices to show that $\varphi(x, y) \ge z$. Thus, by the definition of Ψ,

$$\langle \Psi(x_0^*, y_0^*), z \rangle \le \langle x_0^*, x \rangle + \langle y_0^*, y \rangle \le h(x_0^*, y_0^*) + \varepsilon.$$

Then let $\varepsilon \to 0$. ∎

3. Köthe duality of generalized Calderón–Lozanovskiĭ spaces.

Let E, F and G be three (generalized) Köthe function spaces over the measure space (S, Σ, m), and let $\Psi : E_+ \times F_+ \to G_+$ be a map which is concave, nondecreasing, $L_\infty(S)_+$-homogeneous, order continuous and onto, and suppose that the norm of G is given by the relation

$$\forall g \in G, \quad \|g\| = \inf\{\|e\| + \|f\| : e \in E_+,\ f \in F_+,\ |g| \le \Psi(e, f)\}.$$

Let us call G an *abstract Calderón–Lozanovskiĭ space*. Our main candidates for G and Ψ will be of course $\varphi(X, Y)^*$ and the Ψ-functional of Section 2.

LEMMA 3.1. *For every $(e^*, f^*) \in E_+^* \times F_+^*$ and $g \in G$, set*

$$\Psi_*(e^*, f^*)(g) = \inf\{\langle e^*, e \rangle + \langle f^*, f \rangle : \Psi(e, f) \ge g\}.$$

Then Ψ_* extends to an element of G_+^*. The functional Ψ_*: $E_+^* \times F_+^* \to G_+^*$ is concave, (w^*, w^*)-upper semicontinuous, nondecreasing, $L_\infty(S)_+$-homogeneous, and order continuous. Moreover, the restriction of Ψ_* to $E'_+ \times F'_+$ takes values in G'_+.

Proof. It is clear that $\Psi_*(e^*, f^*)$ extends to an element of G_+^* (same proof as for Proposition 2.2). The properties of Ψ_* are proved like those of Ψ in Proposition 2.6. Let now $e' \in E'_+$ and $f' \in F'_+$. If (g_α) is a decreasing net in G, $g_\alpha \downarrow 0$, then we can write $g_\alpha = h_\alpha \cdot g_0$ with $h_\alpha \in L_\infty(S)$, $h_\alpha \downarrow 0$. Let (e_0, f_0) be such that $\Psi(e_0, f_0) \ge g_0$, and set $e_\alpha = h_\alpha e_0$ and $f_\alpha = h_\alpha f_0$. Then $e_\alpha \downarrow 0$, $f_\alpha \downarrow 0$ and $\Psi(e_\alpha, f_\alpha) \ge g_\alpha$; thus

$$\Psi_*(e', f')(g_\alpha) \le \langle e', e_\alpha \rangle + \langle f', f_\alpha \rangle \to 0,$$

hence $\Psi_*(e', f')$ is order continuous. ∎

Let us denote by π the natural band projection $E^* \to E'$ (resp. $F^* \to F'$, $G^* \to G'$) associating with a linear form its absolutely continuous part.

LEMMA 3.2. *For every $(e^*, f^*) \in E_+^* \times F_+^*$, we have $\pi\Psi_*(e^*, f^*) = \Psi_*(\pi e^*, \pi f^*)$.*

Proof. Represent $L_\infty(S)^*$ as a space $L_1(T, \mathcal{T}, \tau)$. Since $L_1(S)$ is a band in $L_1(T)$, S can be considered as a subset of T. It is easy to see that the natural band projection π coincides with the action of the indicator function $\mathbf{1}_S$ on all duals of Köthe function spaces over (S, Σ, m). The assertion is then a consequence of the $L_\infty(T)_+$-homogeneity of Ψ_*, whose proof is analogous to that of Proposition 2.6(c). ∎

PROPOSITION 3.3. *The map $\Psi_* : E'_+ \times F'_+ \to G'_+$ is onto, and the norm of every $g' \in G'$ is given by*

$$\|g'\| = \inf\{\|e'\| \vee \|f'\| : e' \in E'_+,\ f' \in F'_+,\ g' \le \Psi_*(e', f')\}.$$

Proof. By the reasoning of Corollary 2.4, the map $\Psi_* : E_+^* \times F_+^* \to G_+^*$ is onto; so is the map $\pi\Psi_* : E_+^* \times F_+^* \to G'_+$. Upon using Lemma 3.2, it becomes clear that the map $\Psi_* : E'_+ \times F'_+ \to G'_+$ is onto. Similarly the reasoning of Theorem 2.5 gives the formula

$$\|g'\| = \inf\{\|e^*\| \vee \|f^*\| : e^* \in E_+^*,\ f^* \in F_+^*,\ g' \le \Psi_*(e^*, f^*)\},$$

and an appeal to Lemma 3.2 allows us to replace e^*, f^* by $e' = \pi e^*$, $f' = \pi f^*$ in this formula. ∎

LEMMA 3.4 (Reciprocity formula). *For every $(e_0, f_0) \in E_+ \times F_+$ and every $g' \in G'_+$, the following relation holds:*

$$\langle g', \Psi(e_0, f_0) \rangle = \inf\{\langle e', e_0 \rangle + \langle f', f_0 \rangle : e' \in E'_+,\ f' \in F'_+,\ \Psi_*(e', f') \ge g'\}.$$

Proof. By the proof of Proposition 2.3, we have

$$(**) \quad \inf\{\langle e^*, e_0 \rangle + \langle f^*, f_0 \rangle : e^* \in E'_+,\ f^* \in F'_+,\ \Psi_*(e^*, f^*) \ge g'\} = \limsup_{\substack{e \to e_0 \\ f \to f_0 \\ e \ge 0, f \ge 0}} \langle g', \Psi(e, f) \rangle.$$

But by Lemma 3.2, $\Psi_*(e^*, f^*) \ge g'$ implies $\Psi_*(\pi e^*, \pi f^*) \ge g'$, hence in $(**)$ the left hand side equals $\inf\{\langle e', e_0 \rangle + \langle f', f_0 \rangle : e' \in E'_+,\ f' \in F'_+,\ \Psi_*(e', f') \ge g'\}$. Let (e_n) and (f_n) be such that the right hand side in $(**)$ equals

$\lim_{n\to\infty}\langle g', \Psi(e_n, f_n)\rangle$. We may suppose that $e_n \geq e_0$, $f_n \geq f_0$ and that these sequences are nonincreasing (by assuming that $\|e_n - e_0\| \leq 2^{-n}$, $\|f_n - f_0\| \leq 2^{-n}$ and replacing (e_n), (f_n) by $\overline{e}_n = \bigvee_{p\geq n} e_p$, $\overline{f}_n = \bigvee_{p\geq n} f_p$). Then $\Psi(e_n, f_n) \downarrow \Psi(e_0, f_0)$ (order convergence), and since g' is order continuous, $\langle g', \Psi(e_n, f_n)\rangle \to \langle g', \Psi(e_0, f_0)\rangle$. ■

Let us now give a description of the Ψ_*- and Ψ-functionals in terms of g.C.-L. functions.

PROPOSITION 3.5. *Let $S_G \in \Sigma$ be the support of G. There exist g.C.-L. functions ψ and φ over $(S_G, \Sigma|_{S_G}, m|_{S_G})$ such that:*

(i) *for every $(e, f) \in E_+ \times F_+$ and $(e', f') \in E'_+ \times F'_+$, we have $\Psi(e, f)(s) = \psi(s, e(s), f(s))$ and $\Psi_*(e', f')(s) = \varphi(s, e'(s), f'(s))$ for m-a.e. $s \in S_G$;*

(ii) *the partial functions ψ_s and φ_s are conjugate C.-L. functions for a.e. $s \in S_G$.*

Proof. Let $A \subset S_G$ be such that the indicator function $\mathbf{1}_A$ belongs to $E \cap F$. Then the support of $\Psi(\mathbf{1}_A, \mathbf{1}_A)$ is A (since every $\mathbf{1}_A\Psi(e, f) = \Psi(\mathbf{1}_A e, \mathbf{1}_A f)$ belongs, by order continuity of Ψ, to the band generated by $\Psi(\mathbf{1}_A, \mathbf{1}_A)$). For every $a, b > 0$, we have $(a \wedge b)\Psi(\mathbf{1}_A, \mathbf{1}_A) \leq \Psi(a\mathbf{1}_A, b\mathbf{1}_A) \leq (a \vee b)\Psi(\mathbf{1}_A, \mathbf{1}_A)$; hence there exists a unique $h_{a,b} \in L_\infty(S_G)$ with support A such that $\Psi(a\mathbf{1}_A, b\mathbf{1}_A) = h_{a,b}\Psi(\mathbf{1}_A, \mathbf{1}_A)$. The G'_+-valued map $(a, b) \mapsto \Psi(a\mathbf{1}_A, b\mathbf{1}_A)$ is concave, positively homogeneous and order continuous. For every couple (r, t) of positive rationals we can choose a measurable representative $s \mapsto h(s, r, t)$ of $h_{r,t}$ such that for all $s \in A$ the map $\mathbb{Q}_+^2 \to \mathbb{R}_+$, $(r, t) \mapsto h(s, r, t)$, is concave, positively homogeneous (for coefficients in $\mathbb{Q}_+$), and continuous at the points of the boundary $(\mathbb{Q}_+ \times \{0\}) \cup (\{0\} \times \mathbb{Q}_+)$. This function $h(s, \cdot, \cdot)$ is nondecreasing, locally lipschitzian in each variable on the rational open quadrant, and so can be extended by continuity to $\mathbb{R}_+^2$ (set e.g.

$$h(s, a, b) = \lim_{\substack{r\to a,\, t\to b\\ r<a,\, t<b}} h(s, r, t)$$

if $a, b > 0$; $h(s, a, 0) = ah(s, 1, 0)$; $h(s, 0, b) = bh(s, 0, 1)$); then for all nonnegative reals a, b, $h(\cdot, a, b)$ is a measurable representative of $h_{a,b}$, and for every $s \in A$ the partial function $h(s, \cdot, \cdot)$ belongs to $\mathcal{C}_1$.

If now $A \subset S_G$ is such that the indicator function $\mathbf{1}_A$ belongs to E and is disjoint from F, then the support of $\Psi(\mathbf{1}_A, 0)$ is A (since for every $e \in E_+$ and $f \in F_+$, $\mathbf{1}_A\Psi(e, f) = \Psi(\mathbf{1}_A e, 0)$, which belongs to the band generated by $\Psi(\mathbf{1}_A, 0)$). We have $\Psi(a\mathbf{1}_A, 0) = a\Psi(\mathbf{1}_A, 0)$.

We have a partition $S_G = S_0 \cup S_1 \cup S_2$, where S_0 is the intersection of S_G with the support of $E \cap F$, while S_1 (resp. S_2) is the part of S_G disjoint from F (resp. E). Finally, let $(A_\alpha)_{\alpha\in J_i}$ be a Σ-measurable partition of S_i $(i = 0, 1, 2)$ with the corresponding indicator functions in $E \cap F$, resp. E, F. Find a family $(h_\alpha)_{\alpha\in J_0}$ of normalized g.C.-L. functions by the preceding construction applied to the sets $(A_\alpha)_{\alpha\in J_0}$; set

$$\psi = \sum_{\alpha\in J_0} \Psi(\mathbf{1}_{A_\alpha}, \mathbf{1}_{A_\alpha})h_\alpha + \sum_{\alpha\in J_1} \Psi(\mathbf{1}_{A_\alpha}, 0)p_1 + \sum_{\alpha\in J_2} \Psi(0, \mathbf{1}_{A_\alpha})p_2$$

where p_1, p_2 are the (constant) C.-L. functions $p_1(u, v) = u$ and $p_2(u, v) = v$.

The equality $\Psi(e, f)(s) = \psi(s, e(s), f(s))$ is then verified first for step functions; then for arbitrary e, f, by using the order continuity of Ψ.

Define now φ by $\varphi(s, a, b) = (\psi_s)_*(a, b)$, for every $s \in S_G$. It is in fact measurable, because in the definition of the conjugate functions $(\psi_s)_*$ one can restrict the infimum to the positive rationals (or by Lemma 1.1). For $e' \in E'_+$ and $f' \in F'_+$, define $\Phi(e', f')$ by $\Phi(e', f')(s) = \varphi(s, e'(s), f'(s))$ if $s \in S_G$, and $= 0$ if $s \notin S_G$. Since for all $e \in E_+$ and $f \in F_+$, we have $\varphi(s, e'(s), f'(s))\psi(s, e(s), f(s)) \leq e'(s)e(s) + f'(s)f(s)$ for a.e. s, it is clear that $\Phi(e', f') \leq \Psi_*(e', f')$. Conversely, using a suitable version of the von Neumann measurable selection theorem (as in [Au]), we can find, for every $\varepsilon > 0$, two measurable maps $h, k : S_G \to \mathbb{R}_+$ such that

$$h(s)e'(s) + k(s)f'(s) \leq (1 + \varepsilon)\varphi(s, e'(s), f'(s)), \qquad \psi(s, h(s), k(s)) = 1,$$

for a.e. $s \in S_G$. For all $A \in \Sigma$, $A \subset S_G$, with $m(A) > 0$, there exists $B \in \Sigma$, $B \subset A$, with $m(B) > 0$, such that $\mathbf{1}_B h \in E$ and $\mathbf{1}_B k \in F$. Then $\Psi(\mathbf{1}_B h, \mathbf{1}_B k) = \mathbf{1}_B$, and

$$\int_B \Psi_*(e', f')\, dm \leq \int_B he'\, dm + \int_B kf'\, dm \leq (1 + \varepsilon)\int_B \Phi(e', f')\, dm.$$

This shows $\Psi_*(e', f') \leq \Phi(e', f')$ (since $\Psi_*(e', f')$ is supported by S_G). ■

Note that by the reciprocity formula of Lemma 3.4, we would obtain the same result by constructing first φ from Ψ_*, and then setting $\psi_s = (\varphi_s)_*$.

4. The representation theorem. In this section, we prove the following representation theorem:

THEOREM 4.1. *Let X and Y be two Köthe function spaces over the same measure space $(\Omega, \mathcal{A}, \mu)$, φ a normalized Calderón–Lozanovskiĭ function and $\varphi(X, Y)$ the corresponding Calderón–Lozanovskiĭ space. Given two standard realizations of the duals X^*, Y^* as (generalized) Köthe function spaces over the measure space (S, Σ, m), there is a standard realization of $\varphi(X, Y)^*$ and a generalized Calderón–Lozanovskiĭ function ψ over (S, Σ, m) such that m-almost all nonzero partial functions ψ_s have their conjugate functions in the set Γ_φ and $\varphi(X, Y)^* = \psi(X^*, Y^*)$.*

Note. By Corollary 1.2, the normalized functions $\psi_s/\psi_s(1, 1)$ belong in fact to Γ_{φ_*}.

We postpone the proof of Theorem 4.1 after that of the following Proposition 4.2. Let $\Phi = \Psi_*$ be the conjugate functional $X_+^{*\prime} \times Y_+^{*\prime} \to Z_+^{*\prime}$ (as defined in Section 3) of the Ψ-functional $X_+^* \times Y_+^* \to Z_+^*$ defined in Section 2.

PROPOSITION 4.2. *Let $\xi \in X_+^{*\prime}$ and $\eta \in Y_+^{*\prime}$ be such that $\Phi(\xi,\eta) \neq 0$. There exists a normalized g.C.-L. function $\widetilde{\varphi} = \widetilde{\varphi}_{\xi,\eta}$, defined over the support $S_{\xi,\eta}$ of $\Phi(\xi,\eta)$, with partial functions $\widetilde{\varphi}_s$ belonging to Γ_φ for a.e. $s \in S_{\xi,\eta}$, such that for every $a, b \in \mathbb{R}_+$, we have $\Phi(a\xi, b\eta) = \widetilde{\varphi}(\cdot, a, b)\Phi(\xi,\eta)$. For two such functions $\widetilde{\varphi}$, $\widetilde{\varphi}'$, one has $\widetilde{\varphi}_s = \widetilde{\varphi}'_s$ for a.e. $s \in S_{\xi,\eta}$.*

Proof. The existence and unicity of $\widetilde{\varphi}_{\xi,\eta}$ are clear (see the proof of Proposition 3.5), the point is to prove $\widetilde{\varphi}_{\xi,\eta} \in \Gamma_\varphi$.

We first reduce to the case where $\xi \in i_X(X_+)$ and $\eta \in i_Y(Y_+)$ (where i_X, i_Y are the natural injections $X \to X^{*\prime}$, $Y \to Y^{*\prime}$). For, if the lemma is true in this case, it is then trivially true when $\xi = \sum_i \mathbf{1}_{A_i} i_X(x_i)$ and $\eta = \sum_i \mathbf{1}_{A_i} i_Y(y_i)$, where $(A_i) \subset \Sigma$ is a system of disjoint sets. In the general case, we can find directed nets $\xi_\alpha \uparrow \xi$ and $\eta_\alpha \uparrow \eta$, where $\xi_\alpha \in X^{*\prime}$ and $\eta_\alpha \in Y^{*\prime}$ have the preceding form. Then $\Phi(a\xi_\alpha, b\eta_\alpha) \uparrow \Phi(a\xi, b\eta)$ for every $a, b \geq 0$, whence $\widetilde{\varphi}_{\xi_\alpha,\eta_\alpha}(\cdot, a, b) \to \widetilde{\varphi}_{\xi,\eta}(\cdot, a, b)$ for every $a, b \geq 0$. Hence for a.e. $s \in S_{\xi,\eta}$ we have $\widetilde{\varphi}_{\xi_\alpha,\eta_\alpha}(s, a, b) \to \widetilde{\varphi}_{\xi,\eta}(s, a, b)$, a priori for all rationals, but in fact for all nonnegative reals a, b by a continuity argument. Hence $\widetilde{\varphi}_s \in \Gamma_\varphi$ for a.e. s.

Fix $x_0 \in X_+$ and $y_0 \in Y_+$ (we shall identify x_0, y_0 with their images $i_X(x_0), i_Y(y_0)$). We shall prove the following claim:

CLAIM. *For every $z^* \in Z_+^*$ such that $\langle \Phi(x_0, y_0), z^* \rangle > 0$, and every $\varepsilon > 0$, there exists $A \in \mathcal{A}$ such that $\langle \Phi(x_0, y_0), \mathbf{1}_A z^* \rangle > 0$ and that the map*

$$H_{z^*}^A : (u, v) \mapsto \frac{\langle \Phi(ux_0, vy_0), \mathbf{1}_A z^* \rangle}{\langle \Phi(x_0, y_0), \mathbf{1}_A z^* \rangle}$$

lies in $\mathcal{C}_1$ at a distance from Γ_φ less than ε.

Suppose that the claim is proved. Assume that the function $\widetilde{\varphi} := \widetilde{\varphi}_{x_0,y_0}$ has partial functions $\widetilde{\varphi}_s$ not belonging to Γ_φ for s in a nonnegligible set. Since the function $S_{x_0,y_0} \to \mathbb{R}_+$, $s \mapsto d(\widetilde{\varphi}_s, \Gamma_\varphi)$, is measurable (see §1(b) for the definition of the distance d), there exist $\varepsilon > 0$ and a subset $S_1 \in \Sigma$ such that $d(\widetilde{\varphi}_s, \Gamma_\varphi) > \varepsilon$ for every $s \in S_1$. Since $\mathcal{C}_1$ can be covered by a finite number of d-balls of diameter less than $\varepsilon/3$, we can find $\theta \in \mathcal{C}_1$ with $d(\theta, \Gamma_\varphi) > 2\varepsilon/3$ and a subset $S_2 \subset S_1$ such that $d(\widetilde{\varphi}_s, \theta) < \varepsilon/3$ for every $s \in S_2$. For every $z^* \in Z_+^*$ with support in S_2, the map $H_{z^*} : (u, v) \mapsto \langle \Phi(ux_0, vy_0), z^* \rangle / \langle \Phi(x_0, y_0), z^* \rangle$ satisfies $d(H_{z^*}, \theta) \leq \varepsilon/3$, since

$$H_{z^*}(u, v) = \frac{\int \widetilde{\varphi}(s, u, v) z^*(s)\, dm(s)}{\int z^*(s)\, dm(s)}$$

and the d-balls are convex. Fixing such a z^*, and considering a set $A \in \mathcal{A}$ given by the Claim (with $\varepsilon/3$ in place of ε), we obtain a contradiction for $z_1^* = \mathbf{1}_A z^*$.

Now we prove the claim. It suffices to prove that for every $\varepsilon > 0$, $u_1, \ldots, u_m > 0$, and $v_1, \ldots, v_m > 0$ there exist $\theta \in \Gamma_\varphi$ such that for all $i, j = 1, \ldots, m$, $|H(u_i, v_j) - \theta(u_i, v_j)| < \varepsilon$.

By Proposition 2.3, for every $u, v > 0$, we can find sequences $(x_n)_n \subset X_+$ and $(y_n)_n \subset Y_+$ such that $x_n \to x_0$, $y_n \to y_0$ and

$$(*) \qquad \langle z^*, \varphi(ux_n, vy_n) \rangle \xrightarrow[n\to\infty]{} \langle \Phi(ux_0, vy_0), z^* \rangle.$$

In fact, we may assume that $x_n \geq x_0$ and $y_n \geq y_0$ for all n (since this limit is a lim sup). We can find sequences $(x_n)_n$ and (y_n) which give rise to this limit $(*)$ simultaneously for the $(m+1)^2$ couples (u_i, v_j), $i, j = 0, \ldots, m$, (where we set $u_0 = 1, v_0 = 1$) in place of (u, v): for, we choose for each (u, v) sequences $(x_n^{(u,v)})_n$, $(y_n^{(u,v)})_n$ greater than x_0, resp. y_0, converging to x_0, resp. y_0 and satisfying $(*)$, and then set

$$x_n = \bigvee_{i,j=0}^{m} x_n^{(u_i,v_j)} \quad \text{and} \quad y_n = \bigvee_{i,j=0}^{m} y_n^{(u_i,v_j)}.$$

The point now is that in fact we have

$$\langle t^*, \varphi(u_i x_n, v_j y_n) \rangle \xrightarrow[n\to\infty]{} \langle \Phi(u_i x_0, v_j y_0), t^* \rangle$$

uniformly for all $t^* \in Z_+^*$ with $t^* \leq z^*$. For, we have

$$\begin{aligned}
&\langle \Phi(u_i x_0, v_j y_0), t^* \rangle - \langle t^*, \varphi(u_i x_n, v_j y_n) \rangle \\
&\quad = \limsup_{\substack{x\to x_0\\ y\to y_0}} \langle t^*, \varphi(u_i x, v_j y) - \varphi(u_i x_n, v_j y_n) \rangle \\
&\quad \leq \limsup_{\substack{x\to x_0\\ y\to y_0}} \langle t^*, \varphi(u_i(x \vee x_n), v_j(y \vee y_n)) - \varphi(u_i x_n, v_j y_n) \rangle \\
&\quad \leq \limsup_{\substack{x\to x_0\\ y\to y_0}} \langle z^*, \varphi(u_i(x \vee x_n), v_j(y \vee y_n)) - \varphi(u_i x_n, v_j y_n) \rangle \xrightarrow[n\to\infty]{} 0
\end{aligned}$$

since

$$\begin{aligned}
\limsup_{n\to\infty} \limsup_{\substack{x\to x_0\\ y\to y_0}} \langle z^*, \varphi(u_i x \vee x_n, v_j y \vee y_n) \rangle &= \langle \Phi(u_i x_0, v_j y_0), z^* \rangle \\
&= \lim_{n\to\infty} \langle z^*, \varphi(u_i x_n, v_j y_n) \rangle.
\end{aligned}$$

Hence

$$\langle \Phi(u_i x_0, v_j y_0), t^* \rangle - \langle t^*, \varphi(u_i x_n, v_j y_n) \rangle \leq \varepsilon_n$$

where the sequence $(\varepsilon_n)_n$, converging to zero, can be chosen independent of $t^* \leq z^*$. By applying this also to $z^* - t^*$ in place of z^*, we obtain the desired uniform convergence.

Define a pseudometric on $\mathcal{C}_1$ by

$$\forall\,\theta_1,\theta_2\in\mathcal{C}_1,\quad \delta(\theta_1,\theta_2)=\sup_{i,j=0,\ldots,m}|\theta_1(u_i,v_j)-\theta_2(u_i,v_j)|.$$

Consider now a finite covering $(\Gamma_1,\ldots,\Gamma_N)$ of Γ_φ by Borel subsets of δ-diameter less than $\varepsilon/2$. For every $(x,y)\in X_+\times Y_+$, define $\varphi_{x,y}:\Omega\to\Gamma_\varphi$ by

$$\varphi_{x,y}(\omega)(u,v)=\frac{\varphi(ux(\omega),vy(\omega))}{\varphi(x(\omega),y(\omega))}$$

when $\varphi(x(\omega),y(\omega))\neq 0$ and $=\varphi(u,v)$ if not. Set $A_p^n=\{\omega\in\Omega:\varphi_{x_n,y_n}(\omega)\in\Gamma_p\}$ for $p=1,\ldots,N$. Note that $\bigcup_{p=1}^N A_p^N=\Omega$. Then we have

$$\forall p=1,\ldots,N,\ \forall i,j=0,\ldots,m,$$
$$\lim_{n\to\infty}\langle z^*,\mathbf{1}_{A_p^n}\varphi(u_ix_n,v_jy_n)\rangle=\lim_{n\to\infty}\langle\Phi(u_ix_0,v_jy_0),\mathbf{1}_{A_p^n}z^*\rangle$$

by the uniform convergence result proved above (up to taking a subsequence we may suppose that all these limits do exist). There is a p_0 such that $\lim_{n\to\infty}\langle\Phi(x_0,y_0),\mathbf{1}_{A_{p_0}^n}z^*\rangle>0$. We have

$$\forall i,j=1,\ldots,m,$$
$$\lim_{n\to\infty}\frac{\langle z^*,\mathbf{1}_{A_{p_0}^n}\varphi(u_ix_n,v_jy_n)\rangle}{\langle z^*,\mathbf{1}_{A_{p_0}^n}\varphi(x_n,y_n)\rangle}=\lim_{n\to\infty}\frac{\langle\Phi(u_ix_0,v_jy_0),\mathbf{1}_{A_{p_0}^n}z^*\rangle}{\langle\Phi(x_0,y_0),\mathbf{1}_{A_{p_0}^n}z^*\rangle}.$$

In other words, for sufficiently large n, the map $H=H_{z^*}^{(A_{p_0}^n)}$ has the property that $\delta(\theta-H)\le\varepsilon/2$ for some θ in the closed convex hull (in $\mathcal{C}_1$) of Γ_{p_0} (set $\theta(u,v):=\langle z^*,\mathbf{1}_{A_{p_0}^n}\varphi(ux_n,vy_n)\rangle/\langle z^*,\mathbf{1}_{A_{p_0}^n}\varphi(x_n,y_n)\rangle$). By convexity of δ, the function θ lies at a δ-distance of Γ_φ less than or equal to $\varepsilon/2$. Hence there is a $\theta_0\in\Gamma_\varphi$ with $\delta(H,\theta_0)\le\varepsilon$, which finishes the proof of the Claim, and of Proposition 4.2. ∎

Remark. When φ satisfies a two-sided reverse Δ_2-condition, we have $\Phi(x_0,y_0)=\varphi(x_0,y_0)$ (see the Remark following Proposition 2.3). In this case the ratio $\Phi(ux_0,vy_0)/\Phi(x_0,y_0)$ defines an element $h_{u,v}$ of $L_\infty(\Omega)\subset L_\infty(S)$ (the embedding here comes from the embedding $L_\infty(\Omega)\subset L_\infty(\Omega)^{**}$, which is *not* the conjugate of the band projection $L_\infty(\Omega)^*\to L_1(\Omega)$). Viewed in $L_\infty(\Omega)$, these $h_{u,v}$ define an element of $L_0(\Omega;\Gamma_\varphi^f)$; but viewed in $L_\infty(S)$, they define an element of $L_0(S;\Gamma_\varphi)$ (not $L_0(S;\Gamma_\varphi^f)$ in general).

Proof of Theorem 4.1. An appeal to Proposition 3.5 shows that Z^* and $Z^{*\prime}$ are identified with generalized Calderón–Lozanovskiĭ spaces $\psi(X^*,Y^*)$ and $\psi_*(X^{*\prime},Y^{*\prime})$ for some conjugate g.C.-L. functions ψ and ψ_*; the functionals Ψ and Φ are then related to ψ and ψ_* by the formulas $\Psi(x^*,y^*)(s)=\psi(s,x^*(s),y^*(s))$ and $\Phi(\xi,\eta)(s)=\psi_*(s,\xi(s),\eta(s))$ (for a.e. $s\in S$). It remains to show that $\psi_*(s)\in\Gamma_\varphi$ (when nonzero).

Given a standard realization of X^* and Y^* as (generalized) Köthe function spaces over (S,Σ,m), we can realize Z^* in such a way that for all indicator functions $\mathbf{1}_A\in X^{*\prime}$ and $\mathbf{1}_B\in Y^{*\prime}$, the element $\Phi(\mathbf{1}_A,\mathbf{1}_B)$ of $Z^{*\prime}$ is an indicator function. For, let S_{X^*} and S_{Y^*} be the supports of X^* and Y^*. It suffices to show that $\Phi(\mathbf{1}_A,\mathbf{1}_A)$, $\Phi(\mathbf{1}_B,0)$ and $\Phi(0,\mathbf{1}_C)$ are realized as indicator functions, for every $A\subset S_{X^*}\cap S_{Y^*}$, $B\subset S_{X^*}\setminus S_{Y^*}$ and $C\subset S_{Y^*}\setminus S_{X^*}$; and this can be obtained by a simple change of density.

We now perform the construction of the proof of Proposition 3.5, but starting from the Φ-functional (and considering Ψ as the conjugate Φ_*, by Lemma 3.4). Proposition 4.2 shows that the resulting g.C.-L. function has partial functions a.e. in Γ_φ (when nonzero). ∎

5. Refinement of the representation theorem. In this section we make more precise the set of partial functions ψ_s of the g.C.-L. function ψ which describes $\varphi(X,Y)^*$, according to the position of s in S_{Z^*} (Theorem 5.12 at the end of the section).

We can already treat the case where $s\in S_{X^*}\setminus S_{Y^*}$, resp. $s\in S_{Y^*}\setminus S_{X^*}$: in this case it follows from the proof of Proposition 3.5 that necessarily ψ_s is linear, and depends only on the first, resp. second variable; that is, $\psi_s(u,v)=u$, resp. $\psi_s(u,v)=v$. Hence the band generated in Z^* by $S_{Z^*}\cap S_{X^*}\setminus S_{Y^*}$, resp. $S_{Z^*}\cap S_{Y^*}\setminus S_{X^*}$ coincides with that generated by the same set in X^*, resp. Y^*.

Denote by Δ the intersection $X\cap Y$ (equipped with its natural norm). Let X_Δ^* be the band in X^* whose elements are normal extensions of their restrictions to Δ, in the sense that

$$\forall x\in X_+,\quad \langle|x^*|,x\rangle=\sup\{\langle|x^*|,t\rangle:t\in\Delta,\ 0\le t\le x\},$$

and let $(X_\Delta^*)^\perp$ be the complementary band in X^*. The latter band is simply the band of elements $x^*\in X^*$ having zero restriction to Δ. Let X_0 be the closure of Δ in X; then there is a canonical isometric order isomorphism $x^*\mapsto\mathrm{ext}(x^*)$ from X_0^* onto X_Δ^*, defined for nonnegative $x^*\in X_0^*$ by

$$\forall x\in X_+,\quad \langle\mathrm{ext}(x^*),x\rangle=\sup\{\langle x^*,t\rangle:t\in\Delta,\ 0\le t\le x\}.$$

If r is the restriction map $X^*\to X_0^*$, then $\mathrm{ext}\circ r$ is the band projection from X^* onto X_Δ^* (see [VL]).

The relation between the respective restrictions of X^*, Y^* and Z^* to Δ (which we denote indifferently by π_Δ) is given by the following proposition.

PROPOSITION 5.1. *For every $x^*\in X_+^*$ and $y^*\in Y_+^*$, we have*

$$\pi_\Delta\Psi(x^*,y^*)=\Psi(\pi_\Delta x^*,\pi_\Delta y^*)=\varphi_*(\pi_\Delta x^*,\pi_\Delta y^*).$$

Note that the last member of these equalities is well defined, since $\pi_\Delta x^*$ and $\pi_\Delta y^*$ belong to the same Köthe space Δ^*.

Proof. Let $x^*\in X_+^*$, $y^*\in Y_+^*$ and $t\in\Delta_+^*$. We have

$$\begin{aligned}\langle\pi_\Delta\Psi(x^*,y^*),t\rangle &= \inf\{\langle x^*,x\rangle+\langle y^*,y\rangle : x\in X_+,\ y\in Y_+,\ t\le\varphi(x,y)\}\\ &\le \inf\{\langle x^*,t_1\rangle+\langle y^*,t_2\rangle : t_1,t_2\in\Delta_+,\ t\le\varphi(t_1,t_2)\}\\ &= \langle\Psi(\pi_\Delta x^*,\pi_\Delta y^*),t\rangle.\end{aligned}$$

So it remains to prove the reverse inequality. Let $\varepsilon>0$, $x\in X_+$ and $y\in Y_+$ be such that $t\le\varphi(x,y)$ and

$$\langle\pi_\Delta\Psi(x^*,y^*),t\rangle\ge\langle x^*,x\rangle+\langle y^*,y\rangle-\varepsilon.$$

Suppose first that $\lim_{M\to\infty}\varphi(M,1)=\infty=\lim_{M\to\infty}\varphi(1,M)$. Then for every $\delta>0$, we can find M such that $\varphi(M,\delta)\ge1$ and $\varphi(\delta,M)\ge1$. Then we have

$$t\le\varphi((x\wedge Mt)\vee\delta t,(y\wedge Mt)\vee\delta t).$$

Then $t_1=(x\wedge Mt)\vee\delta t$ and $t_2=(y\wedge Mt)\vee\delta t$ belong to Δ_+ and satisfy $t_1\le x+\delta t$ and $t_2\le y+\delta t$, whence

$$\begin{aligned}\langle x^*,t_1\rangle+\langle y^*,t_2\rangle &\le \langle x^*,x\rangle+\langle y^*,y\rangle+\delta(\|x^*\|\cdot\|t\|_X+\|y^*\|\cdot\|t\|_Y)\\ &\le\langle\pi_\Delta\Psi(x^*,y^*),t\rangle+2\varepsilon\end{aligned}$$

for sufficiently small δ. Since $t\le\varphi(t_1,t_2)$, we obtain the desired inequality.

Suppose now that $\lim_{M\to\infty}\varphi(M,1)=\infty>\lim_{M\to\infty}\varphi(1,M)$. By the same trick as before, we find $t_0\in\Delta_+$ and $y_1\in Y_+^*$ with $t\le\varphi(t_0,y_1)$ and

$$\langle x^*,t_0\rangle+\langle y^*,y_1\rangle\le\langle\pi_\Delta\Psi(x^*,y^*),t\rangle+2\varepsilon.$$

For all $\varepsilon>0$, there exist M_0 such that $\sup_M\varphi(1,M)\le(1+\varepsilon)\varphi(1,M_0)$. Then

$$\varphi(t_0,y_1)\le(1+\varepsilon)\,\varphi(t_0,y_1\wedge M_0t_0).$$

Set $t_1=(1+\varepsilon)\,t_0$ and $t_2=(1+\varepsilon)\,y_1\wedge M_0t_0$. We obtain $t\le\varphi(t_1,t_2)$ and

$$\langle x^*,t_1\rangle+\langle y^*,t_2\rangle\le(1+\varepsilon)\,(\langle\pi_\Delta\Psi(x^*,y^*),t\rangle+2\varepsilon).$$

Finally, if $\lim_{M\to\infty}\varphi(M,1)<\infty$ and $\lim_{M\to\infty}\varphi(1,M)<\infty$ we apply the second trick above simultaneously to x and y.

For the second equality in Proposition 5.1, see Remark 2.7. ■

COROLLARY 5.2. (a) *The spaces* $Z_0=\varphi(X,Y)_0$ *and* $\varphi(X_0,Y_0)_0$ *are identical, with the same norm.*

(b) *For every* $x^*\in X_+^*$ *and* $y^*\in Y_+^*$, *we have*

$$r\Psi(x^*,y^*)=r\Psi(rx^*,ry^*).$$

Proof. (a) The assertion means that the spaces $Z=\varphi(X,Y)$ and $\varphi(X_0,Y_0)$ induce the same norm on Δ. It is clear from the definitions that $\varphi(X_0,Y_0)\subset Z$, with a norm one inclusion map. However, this inclusion map is perhaps not an isometry, nor is Δ necessarily dense in $\varphi(X_0,Y_0)$. One can directly show the identity of these two norms on Δ, or deduce it from Proposition 5.1 as follows:

Let $t\in\Delta_+$ and let $z_0^*\in\varphi(X_0,Y_0)^*$ of norm one be such that $\|t\|_{\varphi(X_0,Y_0)}=\langle z_0^*,t\rangle$. We have, by Theorem 2.5, $z_0^*=\Psi(x_0^*,y_0^*)$ with $x_0^*\in(X_0^*)_+$, $y_0^*\in(Y_0^*)_+$, $\|x_0^*\|+\|y_0^*\|\le1+\varepsilon$. Let x^*, y^* be the normal extensions of x_0^*, resp. y_0^* to X, resp. Y, and $z^*=\Psi(x^*,y^*)$. We have $\|z^*\|\le\|x^*\|+\|y^*\|=\|x_0^*\|+\|y_0^*\|\le1+\varepsilon$. By Proposition 5.1, z^* and z_0^* have the same restriction to Δ. Hence $\|t\|_{\varphi(X_0,Y_0)}=\langle z^*,t\rangle\le(1+\varepsilon)\|t\|_Z$.

(b) Note that the asserted equality makes sense, since by the above, $r\Psi(x^*,y^*)$ and $r\Psi(rx^*,ry^*)$ are members of the dual of the same space. To check their equality, it suffices to check the equality of their restrictions to Δ, and this is a trivial consequence of Proposition 5.1, and the fact that $\pi_\Delta rx^*=\pi_\Delta x^*$ and $\pi_\Delta ry^*=\pi_\Delta y^*$. ■

LEMMA 5.3. (a) *If* $x\in X_{\Delta+}^*$, *then for every* $x\in X_+$, *we have* $\pi_xx^*=\sup\{\pi_tx^*:t\in\Delta_+,\ t\le x\}$.

(b) *For every* $x^*\in X_{\Delta+}^*$, *we have* $\operatorname{Supp}x^*=\operatorname{Supp}\pi_\Delta x^*$. *Consequently*, $\operatorname{Supp}X_\Delta^*=\operatorname{Supp}\pi_\Delta X^*$.

Proof. (a) Let $\nu=\sup\{\pi_tx^*:t\in\Delta_+,\ t\le x\}$. It is clear that $\pi_xx^*\ge\pi_tx^*$ for every $t\le x$, hence $\pi_xx^*\ge\nu$. Conversely, let $h\in L^\infty(\Omega)$ be a step function. For every $\varepsilon>0$, we have $\langle\pi_xx^*,h\rangle=\langle x^*,hx\rangle\le\langle x^*,t_0\rangle+\varepsilon$ for some $t_0\in\Delta_+$ with $t_0\le hx$. Setting $t=h^{-1}t_0$ ($t=0$ where $h=0$), we have $t\in\Delta_+$, $t\le x$ and $\langle\pi_xx^*,h\rangle\le\langle\pi_tx^*,h\rangle+\varepsilon$. Thus $\langle\pi_xx^*,h\rangle\le\sup\{\langle\pi_tx^*,h\rangle:t\in\Delta_+,\ t\le x\}=\langle\nu,h\rangle$ (the last equality because $(\pi_tx^*)_{t\in\Delta_+,t\le x}$ is an upwards directed set). This remains true for every $h\in L_\infty(\Omega)$, by approximation, hence $\pi_xx^*\le\nu$.

(b) For every $\nu\in L_\infty(\Omega)^*$, we have $\nu\perp x^*$ iff $\nu\perp\pi_xx^*$ for all $x\in X_+$; by the above, this is equivalent to saying that $\nu\perp\pi_tx^*$ for all $t\in\Delta_+$, i.e. that $\nu\perp\pi_\Delta x^*$. ■

COROLLARY 5.4. (a) *If* $\lim_{M\to\infty}\varphi(M,1)=\infty=\lim_{M\to\infty}\varphi(1,M)$, *then* $\operatorname{Supp}Z_\Delta^*=\operatorname{Supp}X_\Delta^*\cap\operatorname{Supp}Y_\Delta^*$.

(b) *If* $\lim_{M\to\infty}\varphi(M,1)=\infty>\lim_{M\to\infty}\varphi(1,M)$, *then* $\operatorname{Supp}Z_\Delta^*=\operatorname{Supp}X_\Delta^*$.

(c) *If* $\lim_{M\to\infty}\varphi(M,1)<\infty$ *and* $\lim_{M\to\infty}\varphi(1,M)<\infty$, *then* $\operatorname{Supp}Z_\Delta^*=\operatorname{Supp}X_\Delta^*\cup\operatorname{Supp}Y_\Delta^*$.

Proof. We use the elementary equivalence $\varphi_*(\lambda,1)\sim(\varphi(\lambda^{-1},1))^{-1}$.

(a) In the first case, we obtain $\varphi_*(0,v)=0=\varphi_*(u,0)$ (for every $u,v>0$). By Proposition 5.1, we have $\pi_\Delta Z^*=\varphi_*(\pi_\Delta X^*,\pi_\Delta Y^*)$, which clearly implies $\operatorname{Supp}\pi_\Delta Z^*=\operatorname{Supp}\pi_\Delta X^*\cap\operatorname{Supp}\pi_\Delta Y^*$, and we conclude by Lemma 5.3.

(b) In the second case, we have $\varphi_*(0,v) = 0 < \varphi_*(u,0)$ for all $u, v > 0$, whence $\operatorname{Supp} \pi_\Delta Z^* = \operatorname{Supp} \pi_\Delta X^*$.

(c) In this case, $\varphi_*(0,v) > 0$ and $\varphi_*(u,0) > 0$ for all $u, v > 0$; hence $\operatorname{Supp} \pi_\Delta Z^* = \operatorname{Supp} \pi_\Delta X^* \cup \operatorname{Supp} \pi_\Delta Y^*$. ∎

PROPOSITION 5.5. *There are standard realizations of X^*, Y^*, Z^* for which $Z^* = \psi(X^*, Y^*)$, and the g.C.-L. function ψ satisfies $\psi_s = \varphi_*$ a.e. on the support of Z^*_Δ.*

Proof. Consider the restriction maps $\pi_{X,\Delta} : X^* \to \Delta^*$ and $\pi_{Y,\Delta} : Y^* \to \Delta^*$; these order continuous order homomorphisms have injective restrictions to X^*_Δ, resp. Y^*_Δ.

Suppose first that we are in the case (a) of Corollary 5.4. Consider a complete system $(t^*_\alpha)_\alpha$ of local units of the band V of Δ^* generated by $\pi_\Delta X^* \cap \pi_\Delta Y^*$. We may assume that for all α, $t^*_\alpha \in \pi_\Delta X^* \cap \pi_\Delta Y^*$; then $t^*_\alpha = \pi_{X,\Delta}(x^*_\alpha) = \pi_{Y,\Delta}(y^*_\alpha)$ for uniquely determined elements $x^*_\alpha \in X^*_\Delta$, $y^*_\alpha \in Y^*_\Delta$, which have same supports as the t_α (Lemma 5.3); then $(x^*_\alpha)_\alpha$ and $(y^*_\alpha)_\alpha$ are complete systems of local units of the bands of X^*, Y^* whose supports are both $\operatorname{Supp} Z^*_\Delta$. Finally, set $z^*_\alpha = \Psi(x^*_\alpha, y^*_\alpha)$. Then $z^*_\alpha \in Z^*_\Delta$ (since its support is included in the common support of x^*_α, y^*_α). By Proposition 5.1, we have $\pi_\Delta z^*_\alpha = t^*_\alpha$. For the same reason, we obtain $\pi_\Delta \Psi(hx^*_\alpha, ky^*_\alpha) = \varphi_*(h,k)t^*_\alpha = \pi_\Delta[\varphi_*(h,k)z^*_\alpha]$ (for every h, k in $L_\infty(S)_+$). But $\Psi(hx^*_\alpha, ky^*_\alpha) \le (\|h\|_\infty \vee \|k\|_\infty) z^*_\alpha$ also belongs to Z^*_Δ. Hence $\Psi(hx^*_\alpha, ky^*_\alpha) = \varphi_*(h,k)z^*_\alpha$ (by the injectivity of π_Δ over Z^*_Δ).

Using the order continuity of Ψ, we conclude that $\Psi(x^*, y^*)$ is realized as $\varphi_*(x^*, y^*)$ for every x^*, y^* with support included in $\operatorname{Supp} Z^*_\Delta$.

In the case (b) of Corollary 5.4, we complete the system (t^*_α) by some system (t'^*_β), to obtain a complete system of local units in $\pi_\Delta X^*$; then consider $x'^*_\beta \in X^*_\Delta$, with $\pi_\Delta x'^*_\beta = t'^*_\beta$, and set $z'^*_\beta = \Psi(x'^*_\beta, 0)$. We proceed analogously in the case (c) of Corollary 5.4. ∎

Remark 5.6. (a) The preceding realization of X^*, Y^*, Z^* induces a realization of $X^{*\prime}$, $Y^{*\prime}$, $Z^{*\prime}$ for which $Z^{*\prime} = \widetilde{\varphi}(X^{*\prime}, Y^{*\prime})$ and $\widetilde{\varphi}_s = \varphi$ for a.e. $s \in \operatorname{Supp} Z^*_\Delta$.

(b) If we start from arbitrary standard realizations of X^*, Y^*, and apply the procedure of the proof of Theorem 4.5, we only find that $\widetilde{\varphi}_s \in \Gamma^f_\varphi$ (see §2 for the definition of this set) for a.e. $s \in \operatorname{Supp} Z^*_\Delta$.

For (b), note that by different changes of density on X^*, Y^*, the space $\psi(X^*, Y^*)$ becomes $\widehat{\psi}(X^*, Y^*)$ with $\widehat{\psi}_s \in \Gamma^f_{\psi_s}$ for a.e. s.

The following corollary is a slight improvement of Theorem 1 of [L3].

COROLLARY 5.7. *Suppose that Δ is dense in X and Y. Then the dual Z^* of $Z = \varphi(X,Y)$ can be identified with $V \oplus \varphi_*(X^*, Y^*) \oplus W$, where V, resp. W, is a band in X^*, resp. Y^*.*

Proof. This is an immediate consequence of Proposition 5.5, Corollary 5.4 and the remark at the beginning of this section. ∎

We now investigate the range of values of the g.C.-L. function ψ of Theorem 4.1 outside $\operatorname{Supp} Z^*_\Delta$.

LEMMA 5.8. *The bands $X^{*\perp}_\Delta$ and $Y^{*\perp}_\Delta$ have disjoint supports.*

Proof. This means that for every $x^* \in X^{*\perp}_\Delta$, $y^* \in Y^{*\perp}_\Delta$, $x \in X_+$ and $y \in Y_+$, we have $\pi_x x^* \perp \pi_y y^*$. But this is evident, since, setting $t = x \wedge y$, we have $\pi_t x^* = 0 = \pi_t y^*$, hence $\pi_x x^* = \pi_{x-t} x^*$, $\pi_y y^* = \pi_{y-t} y^*$, and $(x-t) \perp (y-t)$. ∎

PROPOSITION 5.9. *The partial functions of the conjugate g.C.-L. function to the g.C.-L. function ψ of Theorem 4.1 belong to the set $\Gamma^{l,\infty}_\varphi$ for a.e. $s \in \operatorname{Supp} Z^* \cap \operatorname{Supp} X^{*\perp}_\Delta$, and to the set $\Gamma^{r,\infty}_\varphi$ for a.e. $s \in \operatorname{Supp} Z^* \cap \operatorname{Supp} Y^{*\perp}_\Delta$.*

In view of the proof of Theorem 4.1 (and of Lemma 5.8), this assertion is a consequence of the following proposition.

PROPOSITION 5.10. *If $\xi \in X^{*\perp\prime}_{\Delta+}$ and if $\eta \in Y^{*\prime}_{\Delta+}$ has the same support, or $\eta = 0$, then the g.C.-L. function $\widetilde{\varphi}_{\xi,\eta}$ (defined by $\widetilde{\varphi}(\cdot,u,v) = \Phi(u\xi, v\eta)/\Phi(\xi,\eta)$) has a.e. partial functions in the set $\Gamma^{l,\infty}_\varphi$.*

We first prove a lemma.

LEMMA 5.11. *Let $t_0 \in \Delta_+$, $x_0 \in X_+$ and $z^* \in Z^*_+$ with $z^* \perp X^*_\Delta$. Then for every $M \ge 0$, we have*

$$\langle \Phi(x_0, t_0), z^* \rangle = \limsup_{\substack{x \to x_0 + Mt_0 \\ y \to t_0 \\ x \ge My}} \langle z^*, \varphi(x,y) \rangle.$$

Proof. We remark that, for every $M \ge 0$,

$$\langle \Phi(x_0, t_0), z^* \rangle = \langle \Phi(x_0 + Mt_0, t_0), z^* \rangle.$$

For, we have

$$\langle \Phi(x_0, t_0), z^* \rangle = \inf\{\langle x^*, x_0 \rangle + \langle y^*, t_0 \rangle : \Psi(x^*, y^*) \ge z^*\}.$$

We may assume that the support of the element x^* appearing in this infimum is included in that of z^*, hence $x^* \in X^{*\perp}_\Delta$; thus $\langle x^*, t_0 \rangle = 0$, and

$$\begin{aligned}\langle \Phi(x_0, t_0), z^* \rangle &= \inf\{\langle x^*, x_0 + Mt_0 \rangle + \langle y^*, t_0 \rangle : \Psi(x^*, y^*) \ge z^*\} \\ &= \langle \Phi(x_0 + Mt_0, t_0), z^* \rangle.\end{aligned}$$

In the formula asserted in Lemma 5.11, the right hand side is certainly less than

$$\limsup_{\substack{x \to x_0 + My_0 \\ y \to t_0}} \langle z^*, \varphi(x,y) \rangle = \langle \Phi(x_0 + Mt_0, t_0), z^* \rangle = \langle \Phi(x_0, t_0), z^* \rangle.$$

Conversely, let $x_n \to x_0$ and $y_n \to t_0$ be such that

$$\langle z^*, \varphi(x_n, y_n)\rangle \xrightarrow[n\to\infty]{} \langle \Phi(x_0, t_0), z^*\rangle.$$

We may assume that $y_n \ge t_0$. Set

$$y'_n = (y_n - t_0) \wedge \frac{1}{M} x_n + t_0 \quad\text{and}\quad x'_n = x_n + Mt_0.$$

We have clearly $x'_n \to x_0 + Mt_0$, $x'_n \ge My'_n$ and $t_0 \le y'_n \le y_n$, hence $y'_n \to t_0$. We now check that replacing (x_n, y_n) by (x'_n, y'_n) can only increase (hence does not change) the preceding limit. We have (by right subadditivity of φ)

$$\begin{aligned}0 &\le \varphi(x_n, y_n) - \varphi(x_n, y'_n) \le \mathbf{1}_{\{y'_n < y_n\}} \varphi(x_n, y_n - y'_n)\\ &\le \mathbf{1}_{\{x_n \le M(y_n - t_0)\}} \varphi(x_n, y_n - t_0) \le \varphi(x_n \wedge M(y_n - t_0), y_n - t_0).\end{aligned}$$

Observe that $t_n := x_n \wedge M(y_n - t_0) \in \Delta$; hence writing $z^* \le \Psi(x^*, y^*)$ with $x^* \in X_\Delta^{*\perp}$, we obtain

$$\langle z^*, \varphi(t_n, y_n - t_0)\rangle \le \langle y^*, y_n - t_0\rangle \xrightarrow[n\to\infty]{} 0,$$

hence

$$\langle z^*, \varphi(x_n, y'_n)\rangle \xrightarrow[n\to\infty]{} \langle \Phi(x_0, t_0), z^*\rangle.$$

Then we have a fortiori

$$\liminf_{n\to\infty} \langle z^*, \varphi(x'_n, y'_n)\rangle \ge \langle \Phi(x_0, t_0), z^*\rangle. \blacksquare$$

Proof of Proposition 5.10. It is sufficient to prove the assertion when $\xi = \mathbf{1}_A x_0$ and $\eta = \mathbf{1}_A y_0$, where $x_0 \in X_+$, $y_0 \in Y_+$, and $A \subset \operatorname{Supp} X_\Delta^{*\perp}$ is such that $\mathbf{1}_A \eta \in Y_\Delta^{*\prime}$. In fact, we may assume that $y \in Y_0$ (since $Y_0^{*\prime} = (Y_\Delta^*)'$), and even that $y_0 \in \Delta$ by density: more precisely, there exists a nondecreasing sequence (t_n) in Δ, with $t_n \to y_0$ in Y-norm; then $(\mathbf{1}_A t_n)$ is nondecreasing, and $\mathbf{1}_A t_n \to \mathbf{1}_A y_0$ in $Y^{*\prime}$-norm; then $\mathbf{1}_A y_0 = \bigvee_n \mathbf{1}_A t_n$, whence $\Phi(u\mathbf{1}_A x_0, v\mathbf{1}_A t_n) \uparrow \Phi(u\mathbf{1}_A x_0, v\mathbf{1}_A y_0)$.

By Lemma 5.11, from $y_0 \in \Delta$ we deduce that for every $z^* \in Z_+^*$, and $u, v > 0$, we have

$$\langle \Phi(u\mathbf{1}_A x_0, v\mathbf{1}_A y_0), z^*\rangle = \limsup_{\substack{x\to x_0 + My_0\\ y\to y_0\\ x\ge My}} \langle \mathbf{1}_A z^*, \varphi(ux, vy)\rangle.$$

Now the reasoning of the proof of Proposition 4.2 shows that for a.e. $s \in A$, the partial function $\widetilde{\varphi}_s$ belongs to $\Gamma_\varphi^{l,M}$. Since this is true for all $M \ge 0$, we conclude that $\widetilde{\varphi}_s \in \Gamma_\varphi^{l,\infty}$ for a.e. $s \in A$. $\blacksquare$

Applying Corollary 1.2, we can sum up the main results of this section in the following theorem:

THEOREM 5.12. *The partial functions of the normalization $\psi/\psi(\cdot, 1, 1)$ of the g.C.-L. function ψ of Theorem 4.1 belong to the set $\Gamma_{\varphi_*}^f$ for a.e. $s \in \operatorname{Supp} Z_\Delta^*$, to $\Gamma_{\varphi_*}^{r,\infty}$ for a.e. $s \in \operatorname{Supp} Z^* \cap \operatorname{Supp} Y_\Delta^* \cap \operatorname{Supp} X_\Delta^{*\perp}$, and to $\Gamma_{\varphi_*}^{l,\infty}$ for a.e. $s \in \operatorname{Supp} Z^* \cap \operatorname{Supp} X_\Delta^* \cap \operatorname{Supp} Y_\Delta^{*\perp}$. Finally, $\psi_s(u, v) = u$ for a.e. $s \in \operatorname{Supp} Z^* \setminus \operatorname{Supp} Y^*$ and $\psi_s(u, v) = v$ for a.e. $s \in \operatorname{Supp} Z^* \setminus \operatorname{Supp} X^*$. Moreover, on $\operatorname{Supp} Z_\Delta^*$, we can obtain $\psi \equiv \varphi_*$ by choosing appropriate realizations of X^* and Y^*.*

Let us remark that when the linear functions $(u, v) \mapsto u$ and $(u, v) \mapsto v$ do effectively appear as possible values of $(\psi_s)_*$, they belong in fact to $\Gamma_\varphi^{r,\infty}$ or $\Gamma_\varphi^{l,\infty}$. For instance, when $s \in \operatorname{Supp} Z^* \cap \operatorname{Supp} X_\Delta^{*\perp} \setminus \operatorname{Supp} Y^*$, we have $(\psi_s)_* \in \Gamma_\varphi^{l,\infty}$, by Proposition 5.10. When $s \in \operatorname{Supp} Z^* \cap \operatorname{Supp} X_\Delta^* \setminus \operatorname{Supp} Y^*$, we have, on the contrary, $(\psi_s)_* \in \Gamma_\varphi^{r,\infty}$. To see that, it is sufficient to note that if $t_0 \in \Delta_+$, and $\operatorname{Supp} z^* \subset \operatorname{Supp} X_\Delta^* \setminus \operatorname{Supp} Y^*$, then for every $M \ge 0$, we have

$$\langle \Phi(t_0, 0), z^*\rangle = \langle \Phi(t_0, Mt_0), z^*\rangle = \limsup_{\substack{y\to t_0\\ y\ge t_0}} \langle z^*, \varphi(t_0, My)\rangle,$$

the first equality because if $z^* \le \Psi(x^*, y^*)$ then in fact $z^* \le \Psi(x^*, 0)$, hence

$$\langle \Phi(t_0, Mt_0), z^*\rangle = \inf\{\langle x^*, t_0\rangle : z^* \le \Psi(x^*, 0)\} = \langle \Phi(t_0, 0), z^*\rangle,$$

and the second one because if $x_n \to t_0$ and $y_n \to Mt_0$ with $x_n \ge t_0$, $y_n \ge Mt_0$, and $\langle z^*, \varphi(x_n, y_n)\rangle \to \langle \Phi(t_0, Mt_0), z^*\rangle$, then, writing again $z^* \le \Psi(x^*, 0)$,

$$\langle z^*, \varphi(x_n, y_n) - \varphi(t_0, y_n)\rangle \le \langle z^*, \varphi(x_n - t_0, y_n)\rangle \le \langle x^*, x_n - t_0\rangle \xrightarrow[n\to\infty]{} 0.$$

6. Examples

(a) *Spaces E_M where E is order continuous.* Denote by M the Orlicz function associated with the C.-L. function φ by $M^{-1}(t) = \varphi(t, 1)$. If E is a Köthe function space over $(\Omega, \mathcal{A}, \mu)$, we denote by E_M the space $\varphi(E, L_\infty)$. We have $E_M = \{f \in L_0(\Omega, \mathcal{A}, \mu) : \exists \lambda > 0,\ M(|f|/\lambda) \in E\}$ and $\|f\|_{E_M} = \inf\{\lambda > 0 : \|f/\lambda\|_E \le 1\}$.

If E is order continuous, we have $E' = E^* = E_\Delta^*$ (the second equality because Δ is dense in E), hence $\operatorname{Supp} E^* = \Omega$ (considering Ω as embedded in $S = \operatorname{Supp} L_\infty^*$) is contained in the support of the dual of every Köthe space, in particular in that of $L_{\infty\Delta}^* = L_{\infty,0}^*$ (where $L_{\infty,0}$ is the closure of $\Delta = E \cap L_\infty$ in L_∞). We then obtain, by Theorem 5.12,

$$E_M^* = \varphi_*(E', L_1) \oplus L$$

where L is (isometrically order isomorphic to) a band in L_∞^*, i.e. an abstract L_1 space. In particular, if $E = L_1$, we have $E_M = L_M$, and we recover the case of Orlicz spaces. If E is the Lorentz space $L_{w,1}$ associated with the weight w, then E_M is a Lorentz–Orlicz space $L_{w,M}$. The description of the dual $L^*_{w,M}$ reduces thus to that of the Köthe dual.

(b) *Regularly varying C.-L. functions*. We say that the Calderón–Lozanovskiĭ function φ is *regularly varying* if the limits

$$\lim_{a\to\infty}\frac{\varphi(au,v)}{\varphi(a,1)}=:\varphi_l(u,v)\quad\text{and}\quad\lim_{a\to 0}\frac{\varphi(au,v)}{\varphi(a,1)}=:\varphi_r(u,v)$$

exist (for every $u,v>0$). Then φ_l and φ_r are necessarily Calderón interpolation functions: $\varphi_l(u,v)=u^{1-\theta_l}v^{\theta_l}$, $\varphi_r(u,v)=u^{1-\theta_r}v^{\theta_r}$ (for some $0\le\theta_l,\theta_r\le 1$). The conjugate functions $\varphi_{l,*},\varphi_{r,*}$ are respectively identical (up to a constant factor) to φ_l,φ_r. Hence we have

$$\varphi(X,Y)^*=\varphi_*(X_0^*,Y_0^*)\oplus U_l^{1-\theta_l}V_l^{\theta_l}\oplus U_r^{1-\theta_r}V_r^{\theta_r}$$

where X_0, Y_0 are the closures of $\Delta=X\cap Y$ in X, resp. Y; U_l,V_l are the bands of X^*, resp. Y^* with common support $S_l=\operatorname{Supp}\varphi(X,Y)^*\cap\operatorname{Supp}X_\Delta^{*\perp}\cap\operatorname{Supp}Y_\Delta^*$; and U_r,V_r are the bands of X^*, resp. Y^* supported by $S_r=\operatorname{Supp}\varphi(X,Y)^*\cap\operatorname{Supp}X_\Delta^*\cap\operatorname{Supp}Y_\Delta^{*\perp}$.

A simple example of a regularly varying C.-L. function is

$$\varphi(u,v)=u^{1-\alpha}v^{\alpha}\wedge u^{1-\beta}v^{\beta}.$$

(c) *Couples* (X,Y) *with nontrivial sets* S_l, S_r. We now give an example of a couple (X,Y) such that $\operatorname{Supp}Y_\Delta^{*\perp}\cap\operatorname{Supp}X_\Delta^*\ne\{0\}$, and moreover this set does intersect $\operatorname{Supp}\varphi(X,Y)^*$ for every C.-L. function φ.

We take $X=\ell_\infty(\ell_2)$, $Y=\ell_\infty(\ell_p)$, with, say, $2<p<\infty$. In this case $\Delta=X$, hence $X_\Delta^*=X^*$. Define sequences $(f_n^*)\subset\ell_2$ and $(g_n^*)\subset\ell_{p_*}$ by

$$f_n^*=\frac{e_1+\ldots+e_n}{\sqrt n},\qquad g_n^*=\frac{e_1+\ldots+e_n}{n^{1/p_*}},$$

where (e_i) denote indifferently the ℓ_r basis (for all r) and $1/p+1/p_*=1$. Define $F^*\in X^*$ and $G^*\in Y^*$ by

$$\langle F^*,(f_n)\rangle=\lim_{n,\mathcal U}\langle f_n^*,f_n\rangle,\qquad\langle G^*,(g_n)\rangle=\lim_{n,\mathcal U}\langle g_n^*,g_n\rangle$$

for every $(f_n)_n\in\ell_\infty(\ell_2)$ and $(g_n)_n\in\ell_\infty(\ell_p)$, where $\mathcal U$ is some nontrivial ultrafilter over $\mathbb N$. Then $G^*|_X=0$, i.e. $G^*\in Y_\Delta^{*\perp}$, since for all $F=(f_n)\in X$,

$$\langle G^*,F\rangle\le\|F\|_X\lim_{n,\mathcal U}\|g_n^*\|_2=0.$$

On the other hand, it is easy to verify that $\operatorname{Supp}G^*\subset\operatorname{Supp}F^*$. For, let $G\ge 0$ in Y with $\langle G^*,G\rangle\ne 0$. We may suppose that $G=(g_n)$, where for every n, g_n is supported by $(e_1,\ldots,e_n)$. Then $\|g_n\|_2\le n^{1/2-1/p}\|g_n\|_p\le n^{1/2-1/p}\|G\|$; so if we set $f_n=n^{1/p-1/2}g_n$, we have $F:=(f_n)\in X$ and

$$\langle f_n^*,f_n\rangle=\langle n^{1/p_*-1/2}g_n^*,n^{1/p-1/2}g_n\rangle=\langle g_n^*,g_n\rangle,$$

hence $\langle F^*,F\rangle=\langle G^*,G\rangle$. More generally, if $h\in\ell_\infty(\mathbb N\times\mathbb N)$, it is easy to see that $\langle F^*,hF\rangle=\langle G^*,hG\rangle$, which means that $\pi_G G^*=\pi_F F^*$. Thus $\operatorname{Supp}\pi_G G^*\subseteq\operatorname{Supp}F^*$ for all $G\in Y$, wich means that $\operatorname{Supp}G^*\subseteq\operatorname{Supp}F^*$.

Let φ be an arbitrary element of $\mathcal C_1$. We have

$$\varphi(\ell_\infty(\ell_2),\ell_\infty(\ell_p))=\ell_\infty(\varphi(\ell_2,\ell_p))$$

isometrically. Let r be such that $1/2=1/p+1/r$. We have $\ell_2=\ell_p.\ell_r$, so that

$$\begin{aligned}\varphi(\ell_2,\ell_p)&=\varphi(\ell_p.\ell_r,\ell_p.\ell_\infty)\\&=\ell_p.\varphi(\ell_r,\ell_\infty)\quad\text{(2-isomorphically)}\\&=\ell_p.\ell_M\quad\text{with } M^{-1}(s)=\varphi(s^{1/r},1).\end{aligned}$$

For every $n\in\mathbb N$, denote by $\lambda(n)$, $\lambda_*(n)$ and $\lambda_M(n)$ the norms of $e_1+\ldots+e_n$ in the spaces $\varphi(\ell_2,\ell_p)$, $(\varphi(\ell_2,\ell_p))^*$ and ℓ_M respectively. Then $\lambda(n)\sim n^{1/p}\lambda_M(n)$, and $\lambda_*(n)=n/\lambda(n)\sim n^{1/p_*}/\lambda_M(n)$. Define $H^*\in Z^*$ by $\langle H^*,H\rangle=\lim_{n,\mathcal U}\langle h_n^*,h_n\rangle$, where

$$h_n^*=\frac{e_1+\ldots+e_n}{\lambda_*(n)}.$$

With each $G=(g_n)\in Y$, we associate $H=(h_n)\in\varphi(X,Y)$, where $h_n=\lambda_M(n)^{-1}g_n$. Then $\langle H^*,H\rangle\sim\langle G^*,G\rangle$, and by the same reasoning as for F^*, we obtain $\operatorname{Supp}G^*\subseteq\operatorname{Supp}H^*$.

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Received January 2, 1996 (3592)
Revised version October 28, 1996

STUDIA MATHEMATICA 124 (1) (1997)

Almost multiplicative functionals

by

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Abstract. A linear functional F on a Banach algebra A is almost multiplicative if

$$|F(ab) - F(a)F(b)| \le \delta\|a\| \cdot \|b\| \quad \text{for } a, b \in A,$$

for a small constant δ. An algebra is called *functionally stable* or *f-stable* if any almost multiplicative functional is close to a multiplicative one. The question whether an algebra is f-stable can be interpreted as a question whether A lacks an *almost corona*, that is, a set of almost multiplicative functionals far from the set of multiplicative functionals.

In this paper we discuss f-stability for general uniform algebras; we prove that any uniform algebra with one generator as well as some algebras of the form $R(K)$, $K \subset \mathbb{C}$, and $A(\Omega)$, $\Omega \subset \mathbb{C}^n$, are f-stable. We show that, for a Blaschke product B, the quotient algebra H^∞/BH^∞ is f-stable if and only if B is a product of finitely many interpolating Blaschke products.

1. Introduction. Let G be a linear and multiplicative functional on a Banach algebra A and let Δ be a linear functional on A with $\|\Delta\| \le \varepsilon$. Put $F = G + \Delta$. We can easily check by direct computation that F is δ-*multiplicative*, that is,

$$|F(ab) - F(a)F(b)| \le \delta\|a\| \cdot \|b\| \quad \text{for } a, b \in A,$$

where $\delta = 3\varepsilon + \varepsilon^2$. The problem we want to discuss here is whether the converse is true; that is, whether an almost multiplicative functional must be near a multiplicative one. We are interested mostly in uniform algebras. We shall call a Banach algebra *functionally stable* or *f-stable* if

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall F \in \mathfrak{M}_\delta(A)\ \exists G \in \mathfrak{M}(A) \quad \|F - G\| \le \varepsilon,$$

where we denote by $\mathfrak{M}(A)$ the set of all linear multiplicative functionals on A, and by $\mathfrak{M}_\delta(A)$ the set of δ-multiplicative functionals on A. We shall

1991 *Mathematics Subject Classification*: Primary 46J10.
Research was supported in part by a grant from the International Research & Exchanges Board, with funds provided by the National Endowment for the Humanities and the U.S. Department of State.