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The splitting spectrum differs from the Taylor spectrum

by

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Abstract. We construct a pair of commuting Banach space operators for which the splitting spectrum is different from the Taylor spectrum.

Let $A_1,\ldots,A_n$ be mutually commuting operators in a Banach space X. The Koszul complex of the n-tuple $(A_1,\ldots,A_n)$ is the complex

$$0 \longrightarrow \Lambda^0(X,e) \xrightarrow{\delta_0} \Lambda^1(X,e) \xrightarrow{\delta_1} \ldots \xrightarrow{\delta_{n-1}} \Lambda^n(X,e) \longrightarrow 0$$

where $\Lambda^p(X,e)$ denotes the vector space of all forms of degree p in indeterminates $e_1,\ldots,e_n$ with coefficients in X and the linear mappings δ_p : $\Lambda^p(X,e)\to\Lambda^{p+1}(X,e)$ are defined by

$$\delta^p(xe_{i_1}\wedge\ldots\wedge e_{i_p})=\sum_{j=1}^n A_jxe_j\wedge e_{i_1}\wedge\ldots\wedge e_{i_p}.$$

It is well known that $\delta_{p+1}\delta_p = 0$ for every p. The *Taylor spectrum* $\sigma_{\mathrm T}(A_1,\ldots,A_n)$ is the set of all n-tuples $(\lambda_1,\ldots\lambda_n)$ of complex numbers for which the Koszul complex of $(A_1-\lambda_1,\ldots,A_n-\lambda_n)$ is not exact [5].

Instead of the Taylor spectrum it is sometimes useful to use the following variation (see e.g. [1], [3], [4]). We say that the n-tuple $(A_1,\ldots,A_n)$ is *splitting-regular* if its Koszul complex is exact and the ranges of the operators δ_p are complemented in $\Lambda^{p+1}(X,e)$. Equivalently, there exist operators $\varepsilon_p:\Lambda^{p+1}(X,e)\to\Lambda^p(X,e)$ $(p=0,\ldots,n-1)$ such that $\varepsilon_p\delta_p+\delta_{p-1}\varepsilon_{p-1}$ is the identity operator on $\Lambda^p(X,e)$ for $p=0,\ldots,n$ (formally we set $\delta_{-1}=\delta_n=0$). The *splitting spectrum* $\sigma_{\mathrm S}(A_1,\ldots,A_n)$ is the set of all $(\lambda_1,\ldots,\lambda_n)\in\mathbb C^n$ such that the n-tuple $(A_1-\lambda_1,\ldots,A_n-\lambda_n)$ is not splitting-regular.

The splitting spectrum has similar properties as the Taylor spectrum. Clearly, $\sigma_{\mathrm T}(A_1,\ldots,A_n)\subset\sigma_{\mathrm S}(A_1,\ldots,A_n)$. For Hilbert space operators these two spectra coincide and the same is true for n-tuples of operators in ℓ_1

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or in ℓ_∞ (cf.[2]). Also, for a single operator A_1 in an arbitrary Banach space, $\sigma_{\mathrm{T}}(A_1) = \sigma_{\mathrm{S}}(A_1)$. Consequently, the polynomially convex hulls of $\sigma_{\mathrm{T}}(A_1,\dots,A_n)$ and of $\sigma_{\mathrm{S}}(A_1,\dots,A_n)$ are equal.

It was generally expected that these two spectra are different for n-tuples of operators on a Banach space but no example was known and it was believed that such an example would be complicated [2]. The aim of this note is to fill this gap in the theory. Surprisingly, the constructed example is rather simple.

We denote by $R(T)$ and $N(T)$ the range and the kernel of an operator T. If X and Y are Banach spaces then $X\oplus Y$ denotes the direct sum endowed with the ℓ_1-norm, $\|(x,y)\|_{X\oplus Y} = \|x\|_X + \|y\|_Y$ $(x\in X,\ y\in Y)$. We use a similar convention for direct sums of more than two Banach spaces.

LEMMA 1. *There exists a Banach space Z and closed subspaces $Y_1, Y_2 \subset Z$ such that $Y_1 + Y_2 = Z$ and the subspace $\{(x,x) : x\in Y_1\cap Y_2\}$ is not complemented in $Y_1\oplus Y_2$.*

Proof. Fix a Banach space Y and a closed subspace $X\subset Y$ which is not complemented in Y.

Clearly, $M = \{(x,-x) : x\in X\}$ is a closed subspace of $Y\oplus Y$. Define $Z = (Y\oplus Y)/M$ and let $\pi : Y\oplus Y\to Z$ be the canonical projection. Clearly, $\pi(x,0) = \pi(0,x)$ for every $x\in X$. Consider operators $J_1, J_2 : Y\to Z$ defined by $J_1 y = \pi(y,0)$ and $J_2 y = \pi(0,y)$ $(y\in Y)$. It is easy to check that J_1 and J_2 are isometries. Let $Y_1 = J_1 Y$ and $Y_2 = J_2 Y$. Clearly, $Z = Y_1 + Y_2$ and $Y_1\cap Y_2 = \{\pi(x,0) : x\in X\} = \{\pi(0,x) : x\in X\}$.

Suppose on the contrary that the space $D = \{(\pi(x,0),\pi(0,x)) : x\in X\}$ is complemented in $Y_1\oplus Y_2 = \{(\pi(y_1,0),\pi(0,y_2)) : y_1,y_2\in Y\}$. Then D is complemented also in the closed subspace

$$W = \{(\pi(y,0),\pi(0,y)) : y\in Y\} = \{(J_1y, J_2y) : y\in Y\}\subset Y_1\oplus Y_2.$$

Let $J : Y\to W$ be defined by $Jy = (\pi(y/2,0),\pi(0,y/2))$. Clearly, J is an isometry onto W and $JX = D$, so that X is complemented in Y, a contradiction.

THEOREM 2. *There exist a Banach space W and commuting operators $A_1, A_2\in\mathcal{L}(W)$ such that $\sigma_{\mathrm{T}}(A_1,A_2)\neq\sigma_{\mathrm{S}}(A_1,A_2)$.*

Proof. Let Z, Y_1 and Y_2 be the Banach spaces from the previous lemma. For $i,j\in\mathbb{Z}$ set

$$W_{ij} = \begin{cases} Z & (i,j\ge 1),\\ Y_1 & (i\ge 1,\ j\le 0),\\ Y_2 & (i\le 0,\ j\ge 1),\\ Y_1\cap Y_2 & (i,j\le 0).\end{cases}$$

Clearly, $W_{ij}\subset W_{i+1,j}$ and $W_{ij}\subset W_{i,j+1}$. Set $W = \bigoplus_{i,j\in\mathbb{Z}} W_{ij}$ and let $A_1, A_2\in\mathcal{L}(W)$ be the shift operators to the right and upwards:

$$A_1\Big(\bigoplus_{i,j} w_{ij}\Big) = \bigoplus_{i,j} w_{i-1,j},\qquad A_2\Big(\bigoplus_{i,j} w_{ij}\Big) = \bigoplus_{i,j} w_{i,j-1}.$$

Clearly, A_1 and A_2 are commuting isometries. Further, $W_{ij} = W_{i+1,j}\cap W_{i,j+1}$ and $W_{ij} = W_{i-1,j} + W_{i,j-1}$ for all $i,j\in\mathbb{Z}$. So $R(A_1)+R(A_2) = W$.

The Koszul complex of the pair (A_1,A_2) is of the form

$$0\longrightarrow W\xrightarrow{\delta_0} W\oplus W\xrightarrow{\delta_1} W\longrightarrow 0, \tag{1}$$

where $\delta_0 w = (A_1w, A_2w)$ and $\delta(w,z) = -A_2w + A_1z$ $(w,z\in W)$. Clearly, δ_0 is bounded below and $R(\delta_1) = R(A_1)+R(A_2) = W$.

To show the exactness of the Koszul complex (1) it is sufficient to prove $N(\delta_1)\subset R(\delta_0)$. Let $(\bigoplus w_{ij},\bigoplus z_{ij})\in N(\delta_1)$ for some $w_{ij}, z_{ij}\in W_{ij}$. Then, for all $i,j\in\mathbb{Z}$, $w_{i,j-1} = z_{i-1,j}$, so that

$$w_{ij} = z_{i-1,j+1}\in W_{ij}\cap W_{i-1,j+1} = W_{i-1,j}$$

and

$$z_{ij} = w_{i+1,j-1}\in W_{ij}\cap W_{i+1,j-1} = W_{i,j-1}.$$

Set $u = \bigoplus w_{i+1,j} = \bigoplus z_{i,j+1}$. Then $\delta_1 u = (A_1u, A_2u) = (\bigoplus w_{ij},\bigoplus z_{ij})$. Hence $N(\delta_1) = R(\delta_0)$, the Koszul complex (1) is exact and $(0,0)\notin\sigma_{\mathrm{T}}(A_1,A_2)$.

We show that $R(\delta_0)$ is not complemented in $W\oplus W$. Suppose on the contrary that there exists a projection $P\in\mathcal{L}(W\oplus W)$ with range $R(\delta_0)$. Let $Q\in\mathcal{L}(W\oplus W)$ be defined by $Q(\bigoplus w_{ij},\bigoplus z_{ij}) = (w_{1,0}, z_{0,1})\in W_{1,0}\oplus W_{0,1}$. Clearly, $Q^2 = Q$ and $PQP = QP$, so that $(QP)^2 = Q(PQP) = QP$ is also a projection with

$$R(QP) = \{(w_{1,0}, z_{0,1}) : w_{1,0} = z_{0,1}\in W_{0,0}\} = \{(x,x) : x\in Y_1\cap Y_2\}.$$

Clearly, $R(QP)$ is complemented also in $W_{1,0}\oplus W_{0,1} = Y_1\oplus Y_2$, which contradicts Lemma 1.

Hence $(0,0)\in\sigma_{\mathrm{S}}(A_1,A_2)$ and $\sigma_{\mathrm{S}}(A_1,A_2)\neq\sigma_{\mathrm{T}}(A_1,A_2)$.

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